On the Expressive Power of First-Order Boolean Functions in PCF*

Riccardo Pucella
Department of Computer Science
Cornell University
Ithaca, NY, 14853, USA

Prakash Panangaden
School of Computer Science
McGill University
Montreal H3A 2K7, Canada

Abstract

Recent results of Bucciarelli show that the semilattice of degrees of parallelism of first-order boolean functions in PCF has both infinite chains and infinite antichains. By considering a simple subclass of Sieber's sequentiality relations, we identify levels in the semilattice and derive inexpressibility results concerning functions on different levels. This allows us to further explore the structure of the semilattice of degrees of parallelism: we identify semilattices characterized by simple level properties, and show the existence of new infinite hierarchies which are in a certain sense natural with respect to the levels.

1 Introduction

In this paper we study the relative definability of first-order boolean functions with respect to Plotkin's language PCF [9], a simply-typed λ-calculus with recursion over the ground types of integers and booleans. Relative definability defines a preorder on continuous boolean functions, and this ordering induces a natural equivalence relation. The object of our study will be the structure of the resulting partially ordered set of equivalence classes of functions (called degrees of parallelism). Work by Trakhtenbrot [16, 17], Sazonov [13], Lichtenthäler [7] and Bucciarelli and Malacaria [2, 5] show that the structure of degrees of parallelism is highly non-trivial: even when restricted to first-order functions, the poset forms a sup-semilattice and contains a "two-dimensional" hierarchy of functions, both infinite chains and infinite antichains of functions.

It is known that the definability ordering is completely characterized by the sequentiality relations of Sieber. The result is a duality of sorts: f can be defined using g if the sequentiality relations under which g is invariant is a subset of the sequentiality relations under which f is invariant. Therefore, it seems worthwhile to try to derive the set of sequentiality relations under which a given function is invariant. As a first step towards this goal we focus our attention in this paper on a simple class of sequentiality relations we call presequentiality relations. Invariance under presequentiality relations induces a coarser ordering on functions than full sequentiality relations, from

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which we cannot infer definability results but can infer strong inexpressibility results. In effect, this coarser ordering is a "skeleton" of the definability preorder.

The main advantage of working with presequentiality relations is that we can completely characterize the set of presequentiality relations under which a given function is invariant. It turns out that a pair of integers is sufficient to completely describe this set. This pair of integers, called the presequentiality level of the function, can straightforwardly be derived from the trace of the function. Well-known functions in the definability preorder, such as Parallel OR, the Berry-Plotkin function, the Gustave function, the Detector function, can be easily characterized in terms of presequentiality levels. We use presequentiality levels to guide our exploration of the definability preorder: we present subsemilattices with natural presequentiality level characterizations, namely the stable, unstable, stable-dominating and monovalued functions. We exhibit natural hierarchies of functions in these lattices, where natural is taken to mean that every function in the hierarchy has a different presequentiality level, thereby making the hierarchy part of the skeleton of the definability preorder.

This paper is structured as follows. In the next section, we review the required mathematical preliminaries, rigorously defining the notions of relative definability, traces, linear coherence, as well as stating useful existing results. In Section 3, we study presequentiality relations, and prove the two main lemmas of this paper: the Reduction Lemma and the Closure Lemma, which allow us to find canonical representatives for the set of presequentiality relations under which a function is invariant. In Section 4, we point out the relationship between the canonical representatives and the trace of the function, and thus define the notion of presequentiality level. Section 5 then investigates the structure of the definability preorder guided by presequentiality levels, as described above.

This work is in the lineage of the work of Bucciarelli in [2] and Bucciarelli and Malacaria in [5]. The main results from this paper were originally reported in [11].

2 Preliminaries

In this section, we review some of the mathematical background to our study of first-order monotone boolean functions and the previous work already done on the subject by Trakhtenbrot, Sazanov, Bucciarelli and Malacaria. We assume knowledge of PCF and its continuous model [9], as well as a passing familiarity with logical relations [10]. Let $\mathcal{B}$ be the flat domain of boolean values. Given $f : \mathcal{B}^k \to \mathcal{B}$ and $x = (x_1, \ldots, x_k)$, then $f(x)$ stands for $f(x_1, \ldots, x_k)$, and given $A = \{x^1, \ldots, x^n\} \subseteq \mathcal{B}^k$, $f(A)$ is defined to be $\{f(x^i) : x^i \in A\}$. As usual, π_1 and π_2 represent the projection functions associated with the cartesian product on sets.

Relative definability refers to the ability to define some function using another function: a function can define another function if there exist some algorithm in some language that uses the former to compute the latter. In our case, algorithms are taken to be PCF-terms: given two continuous functions f and g, we say that f is *PCF-expressible* (or simply expressible) by g, denoted $f \preceq g$, if there exists a PCF-term M such that $f = [\![M]\!] g$. Equivalent terminologies in the literature for $f \preceq g$ are "f is less parallel than g", or f is g-expressible. The $\preceq$ preorder induces an equivalence relation $\equiv$ on continuous function such that $f \equiv g$ iff $f \preceq g$ and $g \preceq f$. The equivalence classes are called *degrees of parallelism*, and two functions f,g with $f \equiv g$ are called *equiparallel*. The degree of parallelism of a continuous function f is denoted $[f]$.

We are interested in studying the structure of first-order degrees of parallelism. Trakhtenbrot [16, 17] and Sazanov [13] first investigated the subject and pointed out finite subposets of degrees (though not necessarily first-order degrees). Some facts are consequences of well-known results. The poset of degrees of parallelism must have a top element, Parallel OR (POR), by Plotkin's full abstraction result for PCF+POR [9]. On the other hand, the poset must have a bottom element, the degree of all M-sequential functions. Indeed, a fundamental property of PCF is that PCF-definable functions are exactly the M-sequential functions. A function $f : \mathcal{B}^k \to \mathcal{B}$ is *M-sequential* [8] (or simply sequential) if it is constant or if there exists an integer i (called an *index of sequentiality*) with $1 \leq i \leq k$ such that $x_i = \perp$ implies that $f(x_1, \ldots, x_k) = \perp$ and such that for any fixed value x_i, the function of the remaining arguments is also M-sequential. In [5], it is proved that first-order degrees of parallelism form a sup-semilattice, which we will denote CONT[1].

Proposition 2.1 *The poset of first-order degrees of parallelism is a sup-semilattice with a bottom element (the set of sequential functions) and a top element (the degree of* POR*).*

The trace of a function is the central notion we use to study boolean functions. The trace is a representation of the minimum inputs needed for the function to produce a result. Formally, given a first-order monotone function $f : \mathcal{B}^k \to \mathcal{B}$, the *trace* of f is

$$\mathrm{tr}(f) = \left\{ (v, b) | v \in \mathcal{B}^k, b \in \mathcal{B}, b \neq \perp, f(v) = b \text{ and } \forall v' < v, f(v') = \perp \right\}$$

For $x, y \in \mathcal{B}$, let $x \uparrow y$ hold if x and y have a common upperbound in $\mathcal{B}$, that is if $x = \perp$ or $y = \perp$ or $x = y$. Extend $\uparrow$ pointwise to tuples in $\mathcal{B}^n$. It is easy to see that a first-order monotone boolean function f is stable (in the sense of Berry [1]) if and only if for all $v_1, v_2 \in \pi_1(\mathrm{tr}(f))$, $v_1 \not\uparrow v_2$. Note that the monotonicity of f insures that if $v_1 \uparrow v_2$ then $f(v_1) = f(v_2)$. For a set of tuples $A \subseteq \mathcal{B}^k$, a set $B \subseteq \mathcal{B}^k$ is an Egli-Milner lowerbound for A if for every $x \in A$, there is a $y \in B$ with $y \leq x$, and for every $y \in B$, there is an $x \in A$ with $y \leq x$.

Linear coherence is used by Bucciarelli and Erhard to study first-order boolean functions in [3, 4, 2]. A subset $A = \{v^1, \ldots, v^n\}$ of $\mathcal{B}^k$ is *linearly coherent* (or simply coherent) if for every coordinate, either a tuple in A contains $\perp$ at that coordinate, or all the tuples in A have the same value at that coordinate, that is

$$\forall j \in \{1, \ldots, k\} \left(\forall l \in \{1, \ldots, n\}, v^l_j \neq \perp \right) \Rightarrow \forall l_1, l_2 \in \{1, \ldots, n\}, v^{l_1}_j = v^{l_2}_j$$

A subset $A = \{v^1, \ldots, v^n\}$ of $\mathcal{B}^k$ is $\perp$*-covering* if for every coordinate a tuple in A contains $\perp$ at that coordinate, that is

$$\forall j \in \{1, \ldots, k\}, \exists i \in \{1, \ldots, k\}, v^i_j = \perp$$

It is easy to see that if A is $\perp$-covering then A is coherent. Abusing the terminology, we will sometimes say that a first-order monotone boolean function f is $\perp$-covering if $\pi_1(\mathrm{tr}(f))$ has the corresponding property.

Monovalued functions are an important class of functions we study. A first-order monotone boolean function f is *monovalued* if $|\pi_2(\mathrm{tr}(f))| = 1$. By another abuse of terminology, we will say that a subset $A \subseteq \pi_1(\mathrm{tr}(f))$ is monovalued if $|f(A)| = 1$. A boolean function which is not monovalued will sometimes be called bivalued[2].

[1] CONT refers to the fact that those functions are continuous: recall that for first-order boolean functions, monotone functions are continuous.

[2] The term "bivalued" refers of course to the fact that there are two non-$\perp$ values in the boolean domain — a function is bivalued if $|\pi_2(\mathrm{tr}(f))| = 2$.

We define two operations on boolean functions. Given a first-order monotone boolean function $f : \mathcal{B}^k \to \mathcal{B}$, let $\mathrm{neg}(f) : \mathcal{B}^k \to \mathcal{B}$ be the function returning *tt* when f returns *ff* and returning *ff* when f returns *tt*. As for the second operation, given two first-order monotone boolean functions $f : \mathcal{B}^k \to \mathcal{B}$ and $g : \mathcal{B}^{k'} \to \mathcal{B}$, (without loss of generality, assume there exists an $l \geq 0$ with $k = k' + l$) define the function $f + g : \mathcal{B}^{\max(k,k')+1} \to \mathcal{B}$ by the following trace:

$$\begin{aligned}\mathrm{tr}(f+g) &= \{((\mathit{tt}, x_1, \ldots, x_k), b) : ((x_1, \ldots, x_k), b) \in \mathrm{tr}(f)\} \bigcup \\ &\quad \{((\underbrace{\mathit{ff}, \ldots, \mathit{ff}}_{l+1}, x_1, \ldots, x_{k'}), b) : ((x_1, \ldots, x_{k'}), b) \in \mathrm{tr}(g)\}\end{aligned}$$

As shown in [5], $f + g$ is equiparallel to the least upperbound of f and g in CONT, in other words $[f + g] = [f] \vee [g]$.

Bucciarelli illustrates the non-trivial structure of the CONT semilattice by exhibiting hierarchies[3] of functions in CONT [2]. He defines the function $\mathrm{BUCC}_{(n,m)}$ via the following description: the trace of $\mathrm{BUCC}_{(n,m)}$ has m elements and each trace element returns *tt*; for any subset of less than n elements (and at least two) of the first projection of the trace, there exists a coordinate which makes that subset incoherent. The Bucciarelli hierarchy is actually a two-dimensional infinite hierarchy of functions.

Generalizing the techniques used in [2], Bucciarelli and Malacaria prove the following proposition in [5], in their attempt to find a characterization of the CONT semilattice in terms of hypergraphs (this proposition is restated so that it does not refer to hypergraphs)

Proposition 2.2 (Bucciarelli, Malacaria) *Let f, g be two first-order monotone boolean functions. If there exists a function $\alpha : \mathrm{tr}(f) \to \mathrm{tr}(g)$ such that*

1. *for all $A \subseteq \mathrm{tr}(f)$, if $\pi_1(A)$ is non-singleton and linearly coherent, then $\pi_1(\alpha(A))$ is non-singleton and linearly coherent.*
2. *for all $A \subseteq \mathrm{tr}(f)$ with $\pi_1(A)$ non-singleton and linearly coherent, and for all $x, y \in A$, we have $\pi_2(x) \neq \pi_2(y) \Rightarrow \pi_2(\alpha(x)) \neq \pi_2(\alpha(y))$.*

then $f \preceq g$.

This property will be used often in this paper to prove definability results between functions.

3 Presequentiality relations

Relative definability for first-order boolean functions is fully characterized by Sieber's sequentiality relations, introduced in [14]. Sequentiality relations are the logical relations [10] under which the constants of PCF are invariant. Recall that an n-ary logical relation R on a λ-model $(D^\tau)_{t\in\mathrm{Type}}$ is a family of relations $R^\tau \subseteq (D^\tau)^n$ such that for all types σ, τ and $f_1, \ldots, f_n \in D^{\sigma\to\tau}$,

$$R^{\sigma\to\tau}(f_1, \ldots, f_n) \Leftrightarrow \forall d_1, \ldots, d_n, R^\sigma(d_1, \ldots, d_n) \Rightarrow R^\tau(f_1 d_1, \ldots, f_n d_n)$$

An element $d \in D^\tau$ is *invariant* under R if $R^\tau(d, \ldots, d)$ holds. We now give the definition of sequentiality relations in a slightly different form than Sieber in [14], distinguishing the simple kind of sequentiality relations which we call presequentiality relations.

[3] A hierarchy is simply an ω-chain in the definability preorder.

Definition 3.1 *For each $n \geq 0$ and each pair of sets $A \subseteq B \subseteq \{1, \ldots, n\}$, the* presequentiality relation $S_n^{A,B} \subseteq (D^\tau)^n$, *$\tau = \iota, o$, is an n-ary logical relation defined by*

$$S_n^{A,B}(d_1, \ldots, d_n) \Leftrightarrow (\exists i \in A. d_i = \perp) \vee (\forall i, j \in B. d_i = d_j)$$

An n-ary logical relation R is called a sequentiality relation *if R is an intersection of presequentiality relations.*

Sieber's relations are defined for full PCF, that is with both integers (type ι) and booleans (type o). For the purposes of this paper, it is sufficient to look at relations over the booleans, that is over $\mathcal{B} = D^o$. For the special case of a first-order boolean function $f : \mathcal{B}^k \to \mathcal{B}$, invariance under $S_n^{A,B}$ means that for tuples $(x_1^1, \ldots, x_n^1), \ldots, (x_1^k, \ldots, x_n^k)$ in $S_n^{A,B}$, we have $\left(f(x_1^1, \ldots, x_1^k), \ldots, f(x_n^1, \ldots, x_n^k)\right)$ also in $S_n^{A,B}$. The following proposition, proved in [14], gives the full characterization of the definability preorder for first-order functions. It is interesting to note that this characterization is effective and Stoughton implemented an algorithm that decides $f \preceq g$ given the functions f and g [15].

Proposition 3.2 (Sieber) *For any first-order monotone boolean functions f and g, $f \preceq g$ if and only if for any sequentiality relation R, if g is invariant under R then f is also invariant under R.*

Proposition 3.2 tells us that a function f is not g-expressible if we can exhibit a sequentiality relation R such that g is invariant under R but f is not. If we restrict our attention to presequentiality relations, it is easy to see that invariance under presequentiality relations induces a coarser ordering than invariance under sequentiality relations, that is it identifies more functions. If two functions are invariant under the same presequentiality relations, then nothing can be said about their relative definability. However, if they are not invariant under the same presequentiality relations, we can derive strong inexpressibility results, since presequentiality relations are a weak class of sequentiality relations. In effect, invariance under presequentiality relations can be viewed as defining the "skeleton" of the relative definability preorder. The advantage of working with presequentiality relations is that they are simpler than full sequentiality relations, and a great deal of structure can be extracted straightforwardly, as we will presently see.

The central problem of this paper is to determine the presequentiality relations under which a given function is invariant. An early restricted form of this may already be found in [2]. The following two lemmas show that it is not necessary to consider every presequentiality relation. The Reduction Lemma tells us that it is sufficient to look at presequentiality relations of a simple form. The Closure Lemma says that if a function is invariant under a presequentiality relation $S_n^{A,B}$, invariance holds under any presequentiality relation with "smaller" A and B. In Section 4, we will see how these lemmas lead to a simple characterization of the set of presequentiality relations under which a function is invariant.

Lemma 3.3 (Reduction Lemma) *Given $f : \mathcal{B}^k \to \mathcal{B}$ a first-order monotone boolean function and $A \subseteq B \subseteq \{1, \ldots, n\}$, one of the following holds:*

1. *$(A = B)$ f is invariant under $S_n^{A,A} \Leftrightarrow f$ is invariant under $S_{|A|}^{\{1,\ldots,|A|\},\{1,\ldots,|A|\}}$*
2. *$(A \subset B)$ f is invariant under $S_n^{A,B} \Leftrightarrow f$ is invariant under $S_{|A|+1}^{\{1,\ldots,|A|\},\{1,\ldots,|A|+1\}}$.*

Lemma 3.4 (Closure Lemma) *Given $f : \mathcal{B}^k \to \mathcal{B}$ a first-order monotone boolean function and m any integer with $m \geq 0$, the following holds:*

1. *f invariant under $S_{m+1}^{\{1,\ldots,m\},\{1,\ldots,m+1\}} \Rightarrow f$ invariant under $S_m^{\{1,\ldots,m\},\{1,\ldots,m\}}$.*
2. *f invariant under $S_{m+1}^{\{1,\ldots,m+1\},\{1,\ldots,m+1\}} \Rightarrow f$ invariant under $S_m^{\{1,\ldots,m\},\{1,\ldots,m\}}$*
3. *f invariant under $S_{m+2}^{\{1,\ldots,m+1\},\{1,\ldots,m+2\}} \Rightarrow f$ invariant under $S_{m+1}^{\{1,\ldots,m\},\{1,\ldots,m+1\}}$*

The proof of these lemmas is much more digestible when split across several technical lemmas (3.5,3.6,3.7) which we now state and prove.

Lemma 3.5 *Let $m(M)$ be the least n such that $M \subseteq \{1, \ldots, n\}$, and let $f : \mathcal{B}^k \to \mathcal{B}$ be a first-order monotone boolean function. The function f is invariant under $S_n^{A,B}$ iff f is invariant under $S_{m(B)}^{A,B}$.*

Proof. $(\Rightarrow)$ We show that if f is invariant under $S_n^{A,B}$, then for all $n' \leq n$ such that $B \subseteq \{1, \ldots, n'\}$, f is invariant under $S_{n'}^{A,B}$.

For the sake of contradiction, assume there exist n, A, B, n' with $n' \leq n$ such that f is invariant under $S_n^{A,B}$ but not under $S_{n'}^{A,B}$. That is, there exist tuples $\left(x_1^1, \ldots, x_{n'}^1\right), \cdots, \left(x_1^k, \ldots, x_{n'}^k\right) \in S_{n'}^{A,B}$ and $(y_1, \ldots, y_{n'}) \notin S_{n'}^{A,B}$ with $y_i = f(x_i^1, \ldots, x_i^k)$.

The tuples

$$(x_1^1, \ldots, x_{n'}^1, \perp, \ldots, \perp), \cdots, (x_1^k, \ldots, x_{n'}^k, \perp, \ldots, \perp)$$

then must be in $S_n^{A,B}$. Since $(y_1, \ldots, y_{n'}) \notin S_{n'}^{A,B}$, we must have $(y_1, \ldots, y_{n'}, \perp, \ldots, \perp) \notin S_n^{A,B}$, contradicting the invariance of f under $S_n^{A,B}$.

$(\Leftarrow)$ We show that if f is invariant under $S_n^{A,B}$, then for all $n' \geq n$, f is invariant under $S_{n'}^{A,B}$.

For the sake of contradiction, assume there exist n, A, B and $n' \geq n$ such that f is invariant under $S_n^{A,B}$ but not under $S_{n'}^{A,B}$. That is, there exist tuples $\left(x_1^1, \ldots, x_{n'}^1\right), \cdots, \left(x_1^k, \ldots, x_{n'}^k\right) \in S_{n'}^{A,B}$ and $(y_1, \ldots, y_{n'}) \notin S_{n'}^{A,B}$ with $y_i = f(x_i^1, \ldots, x_i^k)$. Observe that $(x_1, \ldots, x_{n'}) \in S_{n'}^{A,B} \Leftrightarrow (x_1, \ldots, x_n) \in S_n^{A,B}$. Hence, $\left(x_1^1, \ldots, x_n^1\right), \cdots, \left(x_1^k, \ldots, x_n^k\right) \in S_n^{A,B}$ but $(y_1, \ldots, y_n) \notin S_n^{A,B}$ contradicting the invariance of f under $S_n^{A,B}$. ☐

Lemma 3.6 *Given $f : \mathcal{B}^k \to \mathcal{B}$ a first-order monotone boolean function, f is invariant under $S_n^{A,B}$ iff f is invariant under $S_n^{\{1,\ldots,|A|\},\{1,\ldots,|B|\}}$.*

Proof. We show the following more general result: let A, B, C, D be sets with $A \subseteq B \subseteq \{1, \ldots, n\}, C \subseteq D \subseteq \{1, \ldots, n\}$, and let p be a permutation of $\{1, \ldots, n\}$ into $\{1, \ldots, n\}$ such that $p(A) = C$ and $p(B) = D$. Then f is invariant under $S_n^{A,B} \Leftrightarrow f$ is invariant under $S_n^{C,D}$.

Let us first prove that

$$(x_1, \ldots, x_n) \in S_n^{A,B} \Leftrightarrow (x_{p^{-1}(1)}, \ldots, x_{p^{-1}(n)}) \in S_n^{C,D}. \tag{1}$$

Let $(x_1, \ldots, x_n) \in S_n^{A,B}$, and $y_i = x_{p^{-1}(i)}$. To show $(y_1, \ldots, y_n) \in S_n^{C,D}$, consider the two cases:

1. There is an $i \in A, x_i = \perp$. In which case, let $c = p(i)$, with $c \in C$ since $i \in A$. Moreover, $y_c = x_{p^{-1}(c)} = x_{p^{-1}(p(i))} = x_i = \perp$, so there is a $j \in C, y_j = \perp$.

2. For all $i, j \in B, x_i = x_j$. Assume there are $i, j \in D, y_i \neq y_j$. Then $x_{p^{-1}(i)} \neq x_{p^{-1}(j)}$, hence there are $i', j' \in B, x_{i'} \neq x_{j'}$, a contradiction. Hence for all $i, j \in D, y_i = y_j$.

Hence $(y_1, \ldots, y_n) \in S_n^{C,D}$. The reverse direction follows by symmetry of the permutation p, proving (1).

Now, observe that we need only show one direction of the general result (the reverse direction follows by symmetry of the permutation p).

Consider any tuples $(x_1^1, \ldots, x_n^1), \cdots, \left(x_1^k, \ldots, x_n^k\right) \in S_n^{A,B}$. Let $y_i = f(x_i^1, \ldots, x_i^k)$. Since f is invariant under $S_n^{A,B}$, $(y_1, \ldots, y_n) \in S_n^{A,B}$.

By (1), each tuple $\left(x_1^j, \ldots, x_n^j\right)$ is also in $S_n^{C,D}$ and so is $(y_1, \ldots, y_n) \in S_n^{C,D}$, hence f is invariant under $S_n^{C,D}$.

To prove the lemma, it is sufficient to show that there exists a permutation p of $\{1, \ldots, n\}$ such that $p(A) = \{1, \ldots, |A|\}$, $p(B) = \{1, \ldots, |B|\}$, which is immediate. $\square$

Lemma 3.7 *Given $f : \mathcal{B}^k \to \mathcal{B}$ a first-order monotone boolean function. Then f is invariant under $S_n^{A,B}$, $|B \backslash A| = 1$ iff f is invariant under $S_n^{A,B'}$ for any B' such that $B \subseteq B'$.*

Proof. ($\Rightarrow$) We show that if f is invariant under $S_n^{A,B}$, $|B \backslash A| = 1$, then for any B' such that $B \subseteq B'$, f is invariant under $S_n^{A,B'}$.

By Lemma 3.5 and Lemma 3.6, it is sufficient to show that for any m, if f invariant under $S_{m+1}^{\{1,\ldots,m\},\{1,\ldots,m+1\}}$ then f is invariant under $S_n^{\{1,\ldots,m\},\{1,\ldots,n\}}$ for any $n \geq m+1$.

For the sake of contradiction, assume that for some m and $n \geq m+1$, f is invariant under the presequentiality relation $S_{m+1}^{\{1,\ldots,m\},\{1,\ldots,m+1\}}$ but not under $S_n^{\{1,\ldots,m\},\{1,\ldots,n\}}$. Then there are tuples $(x_1^1, \ldots, x_n^1), \cdots, \left(x_1^k, \ldots, x_n^k\right) \in S_n^{\{1,\ldots,m\},\{1,\ldots,n\}}$ but $(y_1, \ldots, y_n) \notin S_n^{\{1,\ldots,m\},\{1,\ldots,n\}}$, for $y_i = f(x_i^1, \ldots, x_i^k)$. Hence, for all $i \leq m$, $y_i \neq \perp$ and there are I, J such that $y_I \neq y_J$. Without loss of generality, choose I the minimal such index.

We proceed by case analysis on the value of I and J:

1. $(I \leq m)$ Consider the following tuples $(x_1^1, \ldots, x_m^1, x_J^1), \cdots, \left(x_1^k, \ldots, x_m^k, x_J^k\right)$ which are in $S_{m+1}^{\{1,\ldots,m\},\{1,\ldots,m+1\}}$; by assumption of the invariance of f, we have $(y_1, \ldots, y_m, y_J) \in S_{m+1}^{\{1,\ldots,m\},\{1,\ldots,m+1\}}$ Hence, either there is $i \leq m$ such $y_i = \perp$ (a contradiction), or $y_I = y_J$ (also a contradiction).

2. $(J \leq m)$ Same argument.

3. $(I, J > m)$ We further consider 3 subcases.

   (a) $(y_I = \perp)$. Consider the tuples $(x_1^1, \ldots, x_m^1, x_I^1), \ldots, \left(x_1^k, \ldots, x_m^k, x_I^k\right)$ which are in $S_{m+1}^{\{1,\ldots,m\},\{1,\ldots,m+1\}}$; by assumption of the invariance of f we have $(y_1, \ldots, y_m, y_I) \in S_{m+1}^{\{1,\ldots,m\},\{1,\ldots,m+1\}}$. So either there is $i \leq m$ such that $y_i = \perp$ (a contradiction), or $y_I = y_i$ for all $i \leq m$ (also a contradiction)

(b) ($y_J = \bot$) Same argument.

(c) ($y_I, y_J \neq \bot$) By choice of minimal I, we know that $y_1 = \cdots = y_m$ and all are either *tt* or *ff*. On the other hand, $y_I \neq y_J$ and $y_I, y_J \neq \bot$, so let $c = I$ or J, such that $y_c \neq y_1$. Consider the tuples $(x_1^1, \ldots, x_m^1, x_c^1), \cdots, \left(x_1^k, \ldots, x_m^k, x_c^k\right)$, easily seen to be tuples in $S_{m+1}^{\{1,\ldots,m\},\{1,\ldots,m+1\}}$, and by assumption of the invariance of f, we have $(y_1, \ldots, y_m, y_c) \in S_{m+1}^{\{1,\ldots,m\},\{1,\ldots,m+1\}}$. So either there is an $i \leq m$ such that $y_i = \bot$ (a contradiction), or $y_c = y_1$ (also a contradiction)

($\Leftarrow$) We show that if f is invariant under $S_n^{A,B}$, then f is invariant under $S_n^{A,B'}$ for all $A \subseteq B' \subseteq B$.

For the sake of contradiction, assume there exist n, A, B, B' with $A \subseteq B' \subseteq B$ such that f is invariant under $S_n^{A,B}$ but not under $S_n^{A,B'}$. Then there exist tuples $(x_1^1, \ldots, x_n^1), \cdots, \left(x_1^k, \ldots, x_n^k\right) \in S_n^{A,B'}$ such that $(y_1, \ldots, y_n) \notin S_n^{A,B'}$ with $y_i = f(x_i^1, \ldots, x_i^k)$.

Fix an arbitrary $I \in A$. Consider the following tuples: $\left(z_1^j, \ldots, z_n^j\right)$ for $1 \leq j \leq k$, with

$$z_i^j = \begin{cases} x_i^j & \text{if } i \in B' \\ x_I^j & \text{if } i \in B \backslash B' \\ \bot & \text{otherwise} \end{cases}$$

We first verify that these tuples are in $S_n^{A,B}$. For each $j, 1 \leq j \leq k$, consider the original tuple $\left(x_1^j, \ldots, x_n^j\right) \in S_n^{A,B'}$. In other words, either

1. there is an $i \in A$, $x_i^j = \bot$, and for that $i \in A$, we have $z_i^j = x_i^j = \bot$. Hence $\left(z_1^j, \ldots, z_n^j\right) \in S_n^{A,B}$, or
2. For all $i \in A$, $x_i^j \neq \bot$, and for all $i, i' \in B', x_i^j = x_{i'}^j$. Hence, for all $i, i' \in B', z_i^j = z_{i'}^j$. Moreover, for all $i \in B \backslash B', z_i^j = x_I^j$ for $I \in A \subseteq B'$. Hence, for all $i, i' \in B, z_i^j = z_{i'}^j$ and the tuple $\left(z_1^j, \ldots, z_n^j\right) \in S_n^{A,B}$.

By the above construction, we see that for all $i \in B', f(z_i^1, \ldots, z_i^k) = y_i$.

Since $(y_1, \ldots, y_n) \notin S_n^{A,B'}$, we have for all $i \in A, y_i \neq \bot$ and there are $i, j \in B', y_i \neq y_j$. This implies that for all $i \in A, f(z_i^1, \ldots, z_i^k) \neq \bot$ and there are $i, j \in B' \subseteq B$ such that $f(z_i^1, \ldots, z_i^k) \neq f(z_j^1, \ldots, z_j^k)$. In other words, f is not invariant under $S_n^{A,B}$, contracting the assumption. □

The proofs of the Reduction and Closure Lemmas are now immediate.

Proof. (Reduction Lemma)

1. ($A = B$) By Lemma 3.6, we have that f is invariant under $S_n^{A,A}$ iff f is invariant under $S_n^{\{1,\ldots,|A|\},\{1,\ldots,|A|\}}$ and by Lemma 3.5, f is invariant under $S_n^{\{1,\ldots,|A|\},\{1,\ldots,|A|\}}$ iff f is invariant under $S_{|A|}^{\{1,\ldots,|A|\},\{1,\ldots,|A|\}}$.
2. ($A \subset B$) By Lemma 3.6, f is invariant under $S_n^{A,B}$ iff f is invariant under $S_n^{\{1,\ldots,|A|\},\{1,\ldots,|B|\}}$. By Lemma 3.7, f is invariant under $S_n^{\{1,\ldots,|A|\},\{1,\ldots,|B|\}}$ iff f is invariant under $S_n^{\{1,\ldots,|A|\},\{1,\ldots,|A|+1\}}$, and by Lemma 3.5, this happens iff f is invariant under $S_{|A|+1}^{\{1,\ldots,|A|\},\{1,\ldots,|A|+1\}}$. □

Proof. (Closure Lemma)

1. The ($\Leftarrow$) direction in the proof of Lemma 3.7 actually proves this case.

2. Given tuples $(x_1^1, \ldots, x_m^1), \cdots, \left(x_1^k, \ldots, x_m^k\right) \in S_m^{\{1,\ldots,m\},\{1,\ldots,m\}}$ we show $(y_1, \ldots, y_m) \in S_m^{\{1,\ldots,m\},\{1,\ldots,m\}}$ with $y_i = f(x_i^1, \ldots, x_i^k)$.

   By assumption, the tuples $(x_1^1, \ldots, x_m^1, x_1^1), \cdots, \left(x_1^k, \ldots, x_m^k, x_1^k\right)$ are in $S_{m+1}^{\{1,\ldots,m+1\},\{1,\ldots,m+1\}}$.

   By invariance of f under $S_{m+1}^{\{1,\ldots,m+1\},\{1,\ldots,m+1\}}$, we have $(y_1, \ldots, y_m, y_1) \in S_{m+1}^{\{1,\ldots,m+1\},\{1,\ldots,m+1\}}$ which means that either there is $i \leq m$ such that $y_i = \perp$ or for all $i, j \leq m$, $y_i = y_j$. Hence $(y_1, \ldots, y_m) \in S_m^{\{1,\ldots,m\},\{1,\ldots,m\}}$.

3. Same argument as part (2): assume tuples $(x_1^i, \ldots, x_{m+1}^i)$ in $S_{m+1}^{\{1,\ldots,m\},\{1,\ldots,m+1\}}$, and consider the tuples $(x_1^i, \ldots, x_m^i, x_1^i, x_{m+1}^i)$. □

4 Presequentiality levels

The Reduction Lemma and the Closure Lemma of the previous section can be used to show that the set of presequentiality relations under which a function is invariant is characterized by two integers (allowing for ∞). Given f a function invariant under presequentiality relations $\{S_n^{A_i,B_i}\}_{i\in I}$; by the Reduction Lemma, this is equivalent to saying that f is invariant under the presequentiality relations $\{S_{|A_i|}^{\{1,\ldots,|A_i|\},\{1,\ldots,|A_i|\}}\}_{i\in I, A_i=B_i}$ and $\{S_{|A_i|+1}^{\{1,\ldots,|A_i|\},\{1,\ldots,|A_i|+1\}}\}_{i\in I, A_i\subset B_i}$. By the Closure Lemma, there must exist maximal i and j (possibly ∞) such that f is invariant under $S_k^{\{1,\ldots,k\},\{1,\ldots,k\}}$ for all $k \leq i$ and f is invariant under $S_{k+1}^{\{1,\ldots,k\},\{1,\ldots,k+1\}}$ for all $k \leq j$. We will call the pair (i, j) the *presequentiality level* (p-level) of the function f. Clearly, a function with a p-level of (∞, ∞) is invariant under all presequentiality relations. Since every function in a degree of parallelism must be invariant under the same presequentiality relations (by Proposition 3.2), we also talk about the presequentiality level of a degree of parallelism. Alternatively, a function with a p-level of (i, j) is easily seen by applications of the Reduction Lemma and the Closure Lemma to be invariant under a presequentiality relation $S_n^{A,B}$ if and only if either $|A| = |B| \leq i$ or $|A| < |B|$ with $|A| \leq j$.

In view of the discussion following Proposition 3.2, no definability information can be inferred for two functions with the same p-level. However, functions with different p-levels yield immediate inexpressibility results:

Corollary 4.1 *Given f and g first-order monotone boolean functions with p-levels of (i_f, j_f) and (i_g, j_g) respectively. If $i_f > i_g$ or $j_f > j_g$, then $g \not\preceq f$.*

In summary, two integers are sufficient to completely characterize the set of presequentiality relations under which a function is invariant. It turns out that these integers can be derived straightforwardly from the trace of the function. Define the *coefficient of (linear) coherence* of a first-order monotone boolean function f by

$$\mathrm{cc}(f) = \min\{|A| : A \subseteq \pi_1(\mathrm{tr}(f)), |A| \geq 2, A \text{ coherent}\}$$

with $\mathrm{cc}(f)$ defined to be ∞ when $\pi_1(\mathrm{tr}(f))$ has no non-singleton linearly coherent subset. Similarly, define the *bivalued coefficient of (linear) coherence* of a first-order monotone boolean function f by

$$\mathrm{bcc}(f) = \min\{|A| : A \subseteq \pi_1(\mathrm{tr}(f)), |A| \geq 3, A \text{ coherent and bivalued}\}$$

with $\mathrm{bcc}(f)$ is defined to be ∞ when $\pi_1(\mathrm{tr}(f))$ has no non-singleton bivalued linearly coherent subset. We note that $\mathrm{bcc}(f) \geq \mathrm{cc}(f)$ for all f.

The relationship between coefficients of coherence and presequentiality levels is expressed by the following proposition, which provides a mechanical way of determining the presequentiality level of a function, and hence of determining the set of presequentiality relations under which a function is invariant.

Lemma 4.2 *Let $f : \mathcal{B}^k \to \mathcal{B}$ be a first-order monotone boolean function. Then f has a p-level of $(\mathrm{bcc}(f)-1, \mathrm{cc}(f)-1)$ (assuming standard rules for ∞).*

Proof. We prove the result for $\mathrm{cc}(f)$. Consider the three cases:

1. ($\mathrm{cc}(f) = 2$) We show that f is invariant under $S_2^{\{1\},\{1,2\}}$ but not $S_3^{\{1,2\},\{1,2,3\}}$. Assume f is not invariant under $S_2^{\{1\},\{1,2\}}$. Then there exist tuples $(x_1^1, x_2^1), \cdots, \left(x_1^k, x_2^k\right) \in S_2^{\{1\},\{1,2\}}$ such that $(y_1, y_2) \notin S_2^{\{1\},\{1,2\}}$, with $y_i = f(x_i^1, \ldots, x_i^k)$. This means that $y_1 \neq \bot$ and $y_1 \neq y_2$. It is easy to see that $(x_1^1, \ldots, x_1^k) \leq (x_2^1, \ldots, x_2^k)$, since for each $i \leq k$, either $x_1^i = \bot$ or $x_1^i = x_2^i$. So by monotonicity of f, $y_1 \leq y_2$, contradicting $y_1 \neq \bot$, and $y_1 \neq y_2$. So f must be invariant under $S_2^{\{1\},\{1,2\}}$. On the other hand, applying f to the tuples $(x_1^1, x_2^1, \bot), \cdots, \left(x_1^k, x_2^k, \bot\right) \in S_3^{\{1,2\},\{1,2,3\}}$, where the first two coordinates of the tuples are the elements of the first projection of the trace forming a linearly coherent subset of size 2, yields the tuple $(\mathit{tt}, \mathit{tt}, \bot)$ or $(\mathit{ff}, \mathit{ff}, \bot)$, neither of which is in $S_3^{\{1,2\},\{1,2,3\}}$.

2. ($3 \leq \mathrm{cc}(f) < \infty$) We show f is invariant under $S_{\mathrm{cc}(f)}^{\{1,\ldots,\mathrm{cc}(f)-1\},\{1,\ldots,\mathrm{cc}(f)\}}$ but not under $S_{\mathrm{cc}(f)+1}^{\{1,\ldots,\mathrm{cc}(f)\},\{1,\ldots,\mathrm{cc}(f)+1\}}$. Assume f is not invariant under $S_{\mathrm{cc}(f)}^{\{1,\ldots,\mathrm{cc}(f)-1\},\{1,\ldots,\mathrm{cc}(f)\}}$. Then there exist tuples $\left(x_1^1, \ldots, x_{\mathrm{cc}(f)}^1\right), \cdots, \left(x_1^k, \ldots, x_{\mathrm{cc}(f)}^k\right) \in S_{\mathrm{cc}(f)}^{\{1,\ldots,\mathrm{cc}(f)-1\},\{1,\ldots,\mathrm{cc}(f)\}}$ such that $\left(y_1, \ldots, y_{\mathrm{cc}(f)}\right) \notin S_{\mathrm{cc}(f)}^{\{1,\ldots,\mathrm{cc}(f)-1\},\{1,\ldots,\mathrm{cc}(f)\}}$ with $y_i = f(x_i^1, \ldots, x_i^k)$. This means that for all $i \leq \mathrm{cc}(f)-1$, $y_i \neq \bot$ and there are I, J with $y_I \neq y_J$. Let $C \subseteq \pi_1(\mathrm{tr}(f))$ be an Egli-Milner lowerbound of the first $\mathrm{cc}(f)-1$ coordinates of the given tuples, $|C| \leq \mathrm{cc}(f)-1$. We cannot have $|C| = 1$ (say $C = \{v\}$), since that would imply that $v \leq (x_{\mathrm{cc}(f)}^1, \ldots, x_{\mathrm{cc}(f)}^k)$: for each $i \leq k$, either one of $x_j^i = \bot$ for $j \leq \mathrm{cc}(f)-1$ (hence $v_j = \bot$) or $x_j^i = x_{j'}^i$ for all $j, j' \leq \mathrm{cc}(f)-1$ (hence $v_j \leq x_j^i = x_{\mathrm{cc}(f)}^i$). But monotonicity of f would imply that for all i, j, $y_i = y_j$, a contradiction. Hence, $|C| \geq 2$. But since the first $\mathrm{cc}(f)-1$ coordinates of the given tuples form a coherent subset, C being an Egli-Milner lowerbound must also be coherent (by a result in [2]). But this contradicts the fact that the minimal size for a coherent subset of $\pi_1(\mathrm{tr}(f))$ is $\mathrm{cc}(f)$. So, f is invariant under $S_{\mathrm{cc}(f)}^{\{1,\ldots,\mathrm{cc}(f)-1\},\{1,\ldots,\mathrm{cc}(f)\}}$. On the other hand, consider the tuples $\left(x_1^1, \ldots, x_{\mathrm{cc}(f)}^1, \bot\right), \cdots, \left(x_1^k, \ldots, x_{\mathrm{cc}(f)}^k, \bot\right) \in S_{\mathrm{cc}(f)+1}^{\{1,\ldots,\mathrm{cc}(f)\},\{1,\ldots,\mathrm{cc}(f)+1\}}$ where the first $\mathrm{cc}(f)$ coordinates are the elements of a coherent subset of size $\mathrm{cc}(f)$ of $\pi_1(\mathrm{tr}(f))$ (which exists by assumption). Appplying f to these tuples yields a tuple $(y_1, \ldots, y_{\mathrm{cc}(f)}, \bot)$ with $y_i \neq \bot$ for $i \leq \mathrm{cc}(f)$, which cannot be in $S_{\mathrm{cc}(f)+1}^{\{1,\ldots,\mathrm{cc}(f)\},\{1,\ldots,\mathrm{cc}(f)+1\}}$.

3. ($\mathrm{cc}(f) = \infty$) We show that f is invariant under all presequentiality relations of the form $S_{i+1}^{\{1,\ldots,i\},\{1,\ldots,i+1\}}$. Assume that there exists an i such that f is not invariant under $S_{i+1}^{\{1,\ldots,i\},\{1,\ldots,i+1\}}$. The same reasoning as in the previous case leads to a contradiction, although instead of contradicting the minimal size of a coherent subset of $\pi_1(\mathrm{tr}(f))$ being $\mathrm{cc}(f)$, we contradict the fact that there is no coherent subset of $\pi_1(\mathrm{tr}(f))$.

The argument for $\mathrm{bcc}(f)$ is similar. $\square$

We can use Lemma 4.2 to show that presequentiality levels are preserved by the least upper-bound operation on functions in a natural way:

Lemma 4.3 *Given f and g first-order monotone boolean functions with p-levels of (i_f, j_f) and (i_g, j_g) respectively. Then the p-level of $f + g$ is*

$$(\min(i_f, i_g), \min(j_f, j_g))$$

.

Proof. Immediate by Lemma 4.2 and the definition of $f + g$ in terms of f and g. $\square$

It is not hard to check that any first-order monotone boolean function has a p-level (i, j) with $i \geq 2$ and $j \geq 1$ (consider 3 cases: $\mathrm{cc}(f) = \infty, cc(f) < \infty = \mathrm{bcc}(f), \mathrm{bcc}(f) < \infty$). We can easily characterize sequential functions:

Proposition 4.4 *A first-order monotone boolean function has a p-level of (∞, ∞) if and only if it is sequential*

Proof. ($\Rightarrow$) It is sufficient to show that if $\mathrm{cc}(f) = \infty$, then f is sequential. Let us first prove the following auxiliary result: given $f : \mathcal{B}^{k+1} \to \mathcal{B}$ a monotone function and $f' : \mathcal{B}^k \to \mathcal{B}$ defined by

$$f'(x_1, \ldots, x_k) = f(x_1, \ldots, y, \ldots, x_k)$$

for some fixed y as the i^{th} argument of f. Then $\mathrm{cc}(f') \geq \mathrm{cc}(f)$.

Consider the two cases:

1. ($\mathrm{cc}(f) = \infty$) In this case, there is no linearly coherent subset of $\pi_1(\mathrm{tr}(f))$, and hence there can be no linearly coherent subset of $\pi_1(\mathrm{tr}(f'))$ (otherwise, it would yield a linearly coherent subset of $\pi_1(\mathrm{tr}(f))$. Hence, $\mathrm{cc}(f') = \infty \geq \mathrm{cc}(f)$ by definition.

2. ($\mathrm{cc}(f) < \infty$) Given $A \subseteq \pi_1(\mathrm{tr}(f'))$ a coherent subset of size $\mathrm{cc}(f')$. Let B be the following set:

$$\{(x_1, \ldots, x_{k+1}) \in \pi_1(\mathrm{tr}(f)) : (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{k+1}) \in A, x_i \leq y\}\,.$$

   We check that $B \subseteq \pi_1(\mathrm{tr}(f))$ is linearly coherent. First, notice that $|B| = |A|$. Moreover, we see that for all tuples in B, the i$^{\text{th}}$ position is either a $\perp$ or a value y. Added to the fact that A is linearly coherent, we see that B must be linearly coherent, and hence $\mathrm{cc}(f) \leq \mathrm{cc}(f')$.

And this proves the auxiliary result.

We now prove the sufficient condition by induction on the arity of f.

(base case) $f : \mathcal{B} \to \mathcal{B}$. Consider $f(\perp)$. If $f(\perp) \neq \perp$, then by monotonicity f is constant, and hence sequential. if $f(\perp) = \perp$, then consider $f(y)$ for a fixed y. This must be a constant, so f is sequential (by the definition of sequentiality).

(induction step) Assume the result holds for all functions of arity k. Consider $f : \mathcal{B}^{k+1} \to \mathcal{B}$, with $\mathrm{cc}(f) = \infty$.

1. We first need to show that there exists an index of sequentiality. Assume not: for all i, for any fixed $x_j, \forall j \neq i$, $f(x_1, \ldots, \perp, \ldots, x_{k+1}) \neq \perp$. Then $\pi_1(\mathrm{tr}(f))$ must be $\perp$-covering, which contradicts $\mathrm{cc}(f) = \infty$.

2. Given i the index of sequentiality of f, look at the function $f'(z_1, \ldots, z_k) = f(z_1, \ldots, y, \ldots, z_k)$ for a fixed y in position i. By the auxiliary result, $\mathrm{cc}(f') = \infty$, and the induction hypothesis applies to show that f' and therefore f must be sequential. □

($\Leftarrow$) Immediate, since f sequential implies that f is PCF-definable, and hence f must be invariant under all sequentiality relations — including presequentiality relations.

5 Structural results

In this section, we use p-levels to guide our exploration of the CONT semilattice. The approach is roughly as follows: we identify interesting classes of functions (stable functions, unstable functions, stable-dominating functions, monovalued functions), and show that they have a natural characterization in terms of p-levels. We then use the p-level characterization to look for interesting natural hierarchies. A hierarchy is deemed natural if it is made up of functions living on different p-levels. We also show that interesting well-known functions also have a natural characterization in terms of p-levels.

5.1 The STABLE semilattice

Define a *stable degree of parallelism* to be a degree of parallelism containing at least one stable function. We can characterize stable degrees in terms of p-levels:

Proposition 5.1 *A degree of parallelism is stable if and only if its p-level is of the form* (i, j) *with* $i \geq 2$ *and* $j \geq 2$

Proof. ($\Rightarrow$) Given f a stable function. Then $\mathrm{cc}(f) \geq 3$,and by Lemma 4.2, f must have a p-level of the form (i, j) with $j \geq \mathrm{cc}(f) - 1 \geq 2$. Since f is monotone, $i \geq 2$.

($\Leftarrow$) Given f with a p-level (i, j) with $j \geq 2$. By Lemma 4.2, $\mathrm{cc}(f) - 1 \geq 2$, so that $\mathrm{cc}(f) \geq 3$. Hence, f must be stable. □

As a consequence, every function in a stable degree of parallelism must be stable. Let STABLE be the subposet of CONT consisting of all stable degrees of parallelism.

Proposition 5.2 STABLE *is a subsemilattice of* CONT.

Proof. It is easy to see that the least upperbound of two stable degrees of parallelism is itself a stable degree of parallelism. The degree of sequential functions is the bottom element of the semilattice and the Berry-Plotkin function (BP) is its top element, as noted by Plotkin and reported by Curien in [6]. $\square$

The Berry-Plotkin function is defined by the following trace:

$$\begin{array}{|ccc|c|}\hline \perp & tt & ff & tt \\ tt & ff & \perp & ff \\ ff & \perp & tt & ff \\ \hline \end{array}$$

We can in fact completely characterize the degree of parallelism of BP via presequentiality levels:

Proposition 5.3 *Given f a first-order monotone boolean function. Then f has a p-level of $(2,2)$ iff $f \equiv \mathrm{BP}$.*

Proof. ($\Rightarrow$) Given f with a p-level of $(2,2)$. This means that $\mathrm{bcc}(f) = 3$, in other words, there exists an $A \subseteq \pi_1(\mathrm{tr}(f))$ bivalued and linearly coherent, with $|A| = 3$. We can assume without loss of generality that one element of A returns *tt* and the remaining two return *ff* (otherwise, consider $\mathrm{neg}(f)$ which is equiparallel to f and has the desired property). Define $g : \mathrm{tr}(\mathrm{BP}) \to \mathrm{tr}(f)$ by sending the first trace element of BP (the one returning *tt*) to the element of A returning *tt*, and the remaining elements of BP to the elements of A returning *ff*. Since A is linearly coherent, it is clear that g satisfies the condition of Proposition 2.2, and $\mathrm{BP} \preceq f$, Hence by Proposition 5.1, f is stable, so $f \preceq \mathrm{BP}$.

($\Leftarrow$) Given $f \equiv \mathrm{BP}$. Then f must be invariant under the same sequentiality relations, hence the p-level of f is the same as the p-level of BP, namely $(2,2)$. $\square$

5.2 The Gustave hierarchy

The structure of STABLE is non-trivial. Since the functions $\mathrm{BUCC}_{(n,m)}$ are easily seen to be stable, the whole Bucciarelli hierarchy is in STABLE. We can identify a subhierarchy of the Bucciarelli hierarchy derived from the Gustave function [1]. The Gustave function GUST is given by the following trace (in matrix form):

$$\begin{array}{|ccc|c|}\hline \perp & tt & ff & tt \\ tt & ff & \perp & tt \\ ff & \perp & tt & tt \\ \hline \end{array}$$

Definition 5.4 *Let $\mathrm{GUST}_i : \mathcal{B}^{2i+1} \to \mathcal{B}$ ($i \geq 1$) be defined by the following trace (in matrix form):*

$$\begin{array}{|cccccc|c|}\hline \perp & tt & ff & \cdots & tt & ff & tt \\ ff & \perp & tt & \cdots & ff & tt & tt \\ tt & ff & \perp & \cdots & tt & ff & tt \\ & \vdots & & & & & \vdots \\ ff & tt & ff & \cdots & \perp & tt & tt \\ tt & ff & tt & \cdots & ff & \perp & tt \\ \hline \end{array}$$

Note that GUST_1 is just GUST. It is easy to verify the following:

Proposition 5.5 $\mathrm{GUST}_i \equiv \mathrm{BUCC}_{(2i+1,2i+1)}$.

Proof. First note that a monovalued first-order monotone boolean function with $|\mathrm{tr}(f)| = \mathrm{cc}(f) = n$ is such that $f \equiv \mathrm{BUCC}_{(n,n)}$, by an application of Proposition 2.2, and note that $|\mathrm{tr}(\mathrm{GUST}_i)| = \mathrm{cc}(\mathrm{GUST}_i) = 2i + 1$. $\square$

By Lemma 4.2, the functions GUST_i have a p-level of $(\infty, 2i)$. This characterization allows us to derive the following result:

Proposition 5.6 *There is no minimal stable non-sequential function.*

Proof. Assume g is a stable non-sequential function that is minimal, i.e. for all f, f stable, non-sequential, $g \preceq f$.

Since g is not sequential, by Proposition 4.4, there must be some A, B, n such that g is not invariant under $S_n^{A,B}$.

Consider $\mathrm{GUST}_{|A|}$. By the p-level of GUST_i functions, since $|A| \leq 2\,|A|$, $\mathrm{GUST}_{|A|}$ is invariant under $S_n^{A,B}$.

Hence $g \not\preceq \mathrm{GUST}_{|A|}$, a contradiction. $\square$

On the other hand, we can show that the Gustave hierarchy is co-final in the non-sequential functions, that is any non-sequential function must dominate one of the functions in the hierarchy.

Proposition 5.7 *Given f a stable non-sequential first-order monotone boolean function. Then there exists an integer i such that* $\mathrm{GUST}_i \preceq f$.

Proof. The function f being non-sequential implies that $\mathrm{cc}(f) < \infty$ by Propositions 4.4 and 4.2. Moreover, f being stable implies that $\mathrm{cc}(f) \geq 3$ (by Lemma 4.2 and Proposition 5.1). Let A be a linearly coherent subset of $\pi_1(\mathrm{tr}(f))$ of size $\mathrm{cc}(f)$. Define a arbitrary function $g : \mathrm{tr}(\mathrm{GUST}_{\mathrm{cc}(f)}) \to \mathrm{tr}(f)$ with $\pi_1(g(\mathrm{tr}(\mathrm{GUST}_{\mathrm{cc}(f)}))) = A$. It is easy to see that the conditions of Proposition 2.2 are satisfied, so that $\mathrm{GUST}_{\mathrm{cc}(f)} \preceq f$. $\square$

Note that Propositions 5.6 and 5.7 can be derived directly from Bucciarelli's result. We merely identify a natural subset of the Bucciarelli hierarchy that is sufficient for our purpose.

5.3 The Bivalued-Gustave hierarchy

Functions in the Gustave hierarchy (and indeed, in the Bucciarelli hierarchy) are all monovalued. We return to monovalued functions in Section 5.6. For now, let us extend the Gustave hierarchy to a hierarchy of bivalued functions, the Bivalued-Gustave hierarchy.

Definition 5.8 *Let* $\mathrm{BGUST}_i^j : \mathcal{B}^{2i+1} \to \mathcal{B}$ $(j \leq i)$ *be the function defined by the following trace (in matrix form):*

$$
\begin{array}{|cccccc|c|}
\hline
\perp & tt & ff & \cdots & tt & ff & r_1 \\
ff & \perp & tt & \cdots & ff & tt & r_2 \\
tt & ff & \perp & \cdots & tt & ff & r_3 \\
 & \vdots & & & & & \vdots \\
ff & tt & ff & \cdots & \perp & tt & r_{2i} \\
tt & ff & tt & \cdots & ff & \perp & r_{2i+1} \\
\hline
\end{array}
$$

with

$$r_l = \begin{cases} \textit{ff} & \textit{if } 1 \le l \le j \\ \textit{tt} & \textit{otherwise} \end{cases}$$

Let us first show that the j parameter in BGUST_i^j is unnecessary: we may pick BGUST_i^1 as a representative of the class of BGUST_i^j functions, and drop the superscript to refer to the function as BGUST_i.

Lemma 5.9 *Given* $j, j' \le i$, $\mathrm{BGUST}_i^j \equiv \mathrm{BGUST}_i^{j'}$.

Proof. We prove by induction on j that for all j,$\mathrm{BGUST}_i^j \equiv \mathrm{BGUST}_i^1$. The case $j = 1$ is trivial. For the induction step ($j \ge 2$), assume that $\mathrm{BGUST}_i^{j-1} \equiv \mathrm{BGUST}_i^1$ and consider BGUST_i^j. We show $\mathrm{BGUST}_i^j \equiv \mathrm{BGUST}_i^{j-1}$. Define the following terms:

$$\begin{aligned} M_1 &= \lambda f \lambda x_1 \dots x_{2i+1}.\text{if } f(x_1, \dots, x_{2i+1}) \\ &\quad \text{then } f(x_2, \dots, x_{2i+1}, x_1) \text{ else } \textit{ff} \text{ fi} \\ M_2 &= \lambda f \lambda x_1 \dots x_{2i+1}.\text{if } f(x_1, \dots, x_{2i+1}) \\ &\quad \text{then } \textit{tt} \text{ else } f(x_{2i+1}, x_1, \dots, x_{2i}) \text{ fi} \end{aligned}$$

It is not hard to see that $\mathrm{BGUST}_i^j = [\![M_1]\!] \mathrm{BGUST}_i^{j-1}$ and $\mathrm{BGUST}_i^{j-1} = [\![M_2]\!] \mathrm{BGUST}_i^j$, thereby showing $\mathrm{BGUST}_i^j \equiv \mathrm{BGUST}_i^{j-1} \equiv \mathrm{BGUST}_i^1$ by the induction hypothesis. □

It remains to show that the functions BGUST_i actually form a hierarchy. First note that by Lemma 4.2 BGUST_i has a p-level of $(2i, 2i)$.

Proposition 5.10 $\mathrm{BGUST}_i \preceq \mathrm{BGUST}_j$ *iff* $i \ge j$.

Proof. ($\Leftarrow$) A straightforward application of Proposition 2.2: consider any surjective function $g : \mathrm{tr}(\mathrm{BGUST}_i) \to \mathrm{tr}(\mathrm{BGUST}_j)$ sending the unique trace element returning *tt* to the unique trace element returning *tt*, and any trace element returning *ff* to any trace element returning *ff*. It is easy to see that all conditions of Proposition 2.2 are satisfied, and $\mathrm{BGUST}_i \preceq \mathrm{BGUST}_j$.

($\Rightarrow$) Assume $i < j$. The p-level of BGUST_i is $(2i, 2i)$ and the p-level of BGUST_j is $(2j, 2j)$. By Corollary 4.1, $\mathrm{BGUST}_i \not\preceq \mathrm{BGUST}_j$. □

The following result is immediate:

Proposition 5.11 *For all* i, $\mathrm{GUST}_i \preceq \mathrm{BGUST}_i$.

Proof. Via Proposition 2.2. □

Combining functions in the Gustave hierarchy and the Bivalued-Gustave hierarchy via the least upperbound operation produces a two-dimensional hierarchy, with functions of the form $\mathrm{BGUST}_i + \mathrm{GUST}_j$. A trivial application of Lemma 4.3 gives a p-level of $(2i, 2\min(i,j))$ for $\mathrm{BGUST}_i + \mathrm{GUST}_j$. This allows us to derive the following governing equations describing the structure of the hierarchy:

Proposition 5.12 $\mathrm{BGUST}_i + \mathrm{GUST}_j \preceq \mathrm{BGUST}_{i'} + \mathrm{GUST}_{j'}$ *iff* $i' \le i$ *and* $\min(i', j') \le \min(i, j)$.

Proof. ($\Rightarrow$) We prove the contrapositive. If $i < i'$ or $\min(i, j) < \min(i', j')$, then by Corollary 4.1 and the p-level of functions in the hierarchy, $\mathrm{BGUST}_i + \mathrm{GUST}_j \not\preceq \mathrm{BGUST}_{i'} + \mathrm{GUST}_{j'}$.

($\Leftarrow$) Since $i' \leq i$, Proposition 5.10 tells us that $\mathrm{BGUST}_i \leq \mathrm{BGUST}_{i'} \leq \mathrm{BGUST}_{i'} + \mathrm{GUST}_{j'}$. We then consider three cases:

1. ($\min(i, j) = i$) Proposition 5.11 implies that

$$\mathrm{GUST}_j \preceq \mathrm{BGUST}_j \preceq \mathrm{BGUST}_i \preceq \mathrm{BGUST}_{i'} + \mathrm{GUST}_{j'}$$

   Hence, $\mathrm{BGUST}_i + \mathrm{GUST}_j \preceq \mathrm{BGUST}_{i'} + \mathrm{GUST}_{j'}$.

2. ($\min(i, j) = j$, $\min(i', j') = i'$) By assumption, $i' \leq j$, and hence by Proposition 5.11, $\mathrm{GUST}_j \preceq \mathrm{BGUST}_j \preceq \mathrm{BGUST}_{i'} \preceq \mathrm{BGUST}_{i'} + \mathrm{GUST}_{j'}$. Hence $\mathrm{BGUST}_i + \mathrm{GUST}_j \preceq \mathrm{BGUST}_{i'} + \mathrm{GUST}_{j'}$.

3. ($\min(i, j) = j$, $\min(i', j') = j'$) By assumption, $j' \leq j$, and hence

$$\mathrm{GUST}_j \preceq \mathrm{GUST}_{j'} \preceq \mathrm{BGUST}_{i'} + \mathrm{GUST}_{j'}$$

   Hence $\mathrm{BGUST}_i + \mathrm{GUST}_j \preceq \mathrm{BGUST}_{i'} + \mathrm{GUST}_{j'}$. □

5.4 The UNSTABLE semilattice

Define an *unstable degree of parallelism* to be a degree of parallelism containing no stable function. It is easy to show that a degree of parallelism is unstable if and only if it has a p-level of the form $(i, 1)$ with $i \geq 2$, by Proposition 5.1. Let UNSTABLE be the subposet of CONT consisting of all unstable degrees of parallelism. Define the Detector function (DET) to simply return *tt* if one of its two inputs has a value (*tt* or *ff* indifferently). For various reasons, it is simpler to work with the following function ttDET which is easily seem to be equiparallel to DET:

tt	$\perp$	*tt*
$\perp$	*tt*	*tt*

Proposition 5.13 UNSTABLE *is a subsemilattice of* CONT.

Proof. It is easy to see that the least upperbound of two unstable degrees of parallelism is unstable. The top element of UNSTABLE is the degree of POR and its bottom element is the degree of the Detector function. This last fact is an application of Proposition 2.2: given f an unstable first-order monotone boolean function; since f is unstable, there must exist $A \subseteq \pi_1(\mathrm{tr}(f))$ with A coherent and $|A| = 2$. Define a function

$$g : \mathrm{tr}(\mathrm{ttDET}) \rightarrow \mathrm{tr}(f)$$

with the only constraint that each element of the trace of ttDET goes to a distinct element of the trace of f corresponding to the subset A. It is easy to see that all the conditions of Proposition 2.2 are met, hence $\mathrm{ttDET} \preceq f$. □

Detector first appeared in the context of asynchronous dataflow networks. Rabinovich shows in [12] that DET is minimal among unstable functions in that context.

A degree of parallelism is unstable if and only if it is not stable, so we see that the STABLE and the UNSTABLE semilattices form a partition of the full CONT semilattice. We presently identify one hierarchy of functions in UNSTABLE (another will be presented in Section 5.5); functions in this hierarchy are derived from POR:

Definition 5.14 *Let* $\mathrm{POR}_i : \mathcal{B}^i \to \mathcal{B}$ *(*$i \geq 2$*) be defined by the following trace (in matrix form):*

$$\begin{array}{|cccccc|c|}
\hline
tt & tt & tt & \cdots & tt & \perp & tt \\
tt & tt & tt & \cdots & \perp & tt & tt \\
 & \vdots & & & & & \vdots \\
tt & tt & \perp & \cdots & tt & tt & tt \\
tt & \perp & tt & \cdots & tt & tt & tt \\
\perp & tt & tt & \cdots & tt & tt & tt \\
ff & ff & ff & \cdots & ff & ff & ff \\
\hline
\end{array}$$

Note that POR_2 is just POR. POR_i takes i inputs and returns *tt* if at least $i-1$ are *tt*, and *ff* if all are *ff*. These functions span the whole range of allowable p-levels for unstable functions as the next proposition shows:

Proposition 5.15 POR_i *has a p-level of* $(i, 1)$.

Proof. Since POR_i is monotone and unstable, it must have a p-level of the form $(j, 1)$ for some $j \geq 2$, by the characterization of p-levels of monotone and stable functions.

By inspection, we see that the only bivalued coherent subset of $\pi_1(\mathrm{tr}(\mathrm{POR}_i))$ is $\pi_1(\mathrm{tr}(\mathrm{POR}_i))$ itself. Hence, $\mathrm{bcc}(f) = i+1$ and by Lemma 4.2, $j = \mathrm{bcc}(f) - 1 = i$. □

These functions indeed form a hierarchy:

Proposition 5.16 $\mathrm{POR}_i \preceq \mathrm{POR}_j$ *iff* $i \geq j$.

Proof. ($\Leftarrow$) Consider the following PCF-term:

$$M = \lambda f.\lambda x_1 \ldots x_{i+1}.ALLEQ(t_1(x_1, \ldots, x_{i+1}), \ldots, t_{i+1}(x_1, \ldots, x_{i+1}))$$

where

$$ALLEQ = \lambda x_1 \ldots x_{i+1}.\text{if } (x_1 = \ldots = x_{i+1}) \text{ then } x_1 \text{ else } \perp \text{ fi}$$

which returns the value v if and only if all the arguments have the value v.

Each t_j is an application of POR_i to a subset of i inputs out of the $i+1$ possible inputs. Since $\binom{i+1}{i} = i+1$, there are $i+1$ such terms. We claim this term is such that $\mathrm{POR}_{i+1} = [\![M]\!]\mathrm{POR}_i$.

1. The t_j functions all return *tt* iff at least i *tt*'s appear in their arguments
   (a) (at least i *tt*'s) Each subset of size i has at least $i+1$ *tt*'s, so each t_j function returns *tt*.
   (b) (less then i *tt*'s) There exists one subset of size i with less than $i-1$ *tt*'s, so the corresponding t_j function returns $\perp$.

2. The t_j functions all return *ff* iff all inputs are *ff*.

   (a) (all *ff*'s) Every t_j returns *ff*.

   (b) (not all *ff*'s) There exists a subset of size i with not all inputs being *ff*. The corresponding t_j does not return *ff*.

(⇒) Assume $i < j$. The result is immediate by Corollary 4.1 and Proposition 5.15.

5.5 The SDOM semilattice

It is clear that unstable functions are strictly more powerful than stable functions, in the sense that no stable function can implement an unstable function, but unstable functions can implement stable functions. In this section, we characterize the unstable functions that can implement all stable functions, and show that they form a subsemilattice of UNSTABLE.

Definition 5.17 *Let f be an unstable first-order monotone boolean function. We say f is* stable-dominating *if for any stable first-order monotone boolean function g, we have $g \preceq f$.*

Since the STABLE semilattice has a top element BP, a necessary and sufficient condition for an unstable function f to be stable-dominating is to have $\mathrm{BP} \ \preceq f$. Since any stable-dominating function must also dominate DET (the bottom element of UNSTABLE), we have that f is stable-dominating if and only if $\mathrm{BP} + \mathrm{DET} \ \preceq f$. This allows us to derive the following characterization of stable-dominating functions:

Proposition 5.18 *Given f an unstable first-order monotone boolean function. Then f is stable-dominating iff f has a p-level of $(2, 1)$.*

Proof. (⇒) Assume f is stable-dominating. Then by previous argument, $\mathrm{BP} + \mathrm{DET} \preceq f$. Since BP has p-level $(2, 2)$ and DET has p-level $(\infty, 1)$, $\mathrm{BP} + \mathrm{DET}$ has p-level $(2, 1)$ by Lemma 4.3. Assume f does not have a p-level of $(2, 1)$. By Proposition 4.4, f must have a p-level of (i, j) with $i \geq 2$, $j \geq 1$ and $i \neq 2$ or $j \neq 1$. But by Corollary 4.1, we get that $\mathrm{BP}+\mathrm{DET} \not\preceq f$, a contradiction.

(⇐) Given f with p-level $(2, 1)$. By the characterization of the p-level of stable functions, f is unstable. We need only check that $\mathrm{BP} \preceq f$. By Lemma 4.2, $\mathrm{bcc}(f) = 3$. Let A be the subset of $\pi_1(\mathrm{tr}(f))$ of size 3. Assume without loss of generality that A has one element returning *tt* and two elements returning *ff* (if not, consider $\mathrm{neg}(f)$ which is equiparallel to f). Define a function $g : \mathrm{tr}(\mathrm{BP}) \to \mathrm{tr}(f)$ sending the element of the trace of BP returning *tt* to the element of A returning *tt* and the elements of the trace of BP returning *ff* to the elements of A returning *ff*. It is easy to see that all the conditions of Proposition 2.2 hold, and hence we have $\mathrm{BP} \preceq f$. So f is stable-dominating. □

Define a *stable-dominating degree of parallelism* to be a degree of parallelism containing a stable-dominating function. By Proposition 5.18, every function in a stable-dominating degree of parallelism is stable-dominating. Let SDOM be the subposet of CONT (in fact, of UNSTABLE) consisting of all stable-dominating degrees of parallelism.

Proposition 5.19 SDOM *is a subsemilattice of* UNSTABLE.

Proof. It is easy to see by the above characterization that the least upperbound of two stable-dominating degrees of parallelism is itself stable-dominating. The bottom element of SDOM is the degree of $\mathrm{BP}+\mathrm{DET}$, and its top element is the degree of POR. $\square$

To show this subsemilattice is non-trivial, we exhibit an hierarchy of functions in SDOM. Note however that because stable-dominating functions are all in the same p-level, we cannot show inexpressibility using presequentiality relations. Consider the functions $\mathrm{BP}+\mathrm{POR}_i$, which are easily seen to be stable-dominating. Note that $\mathrm{BP}+\mathrm{POR}_2 \equiv \mathrm{POR}_2 \equiv \mathrm{POR}$. These functions form a hierarchy:

Proposition 5.20 $\mathrm{BP}+\mathrm{POR}_i \preceq \mathrm{BP}+\mathrm{POR}_j$ *iff* $i \geq j$.

Proof. ($\Leftarrow$) We know $\mathrm{BP} \preceq \mathrm{BP}+\mathrm{POR}_j$ for all $j \geq 2$. Similarly, by Proposition 5.16, $\mathrm{POR}_i \preceq \mathrm{POR}_j \preceq \mathrm{BP}+\mathrm{POR}_j$. Hence, by the property of least upperbounds, we get that $\mathrm{BP}+\mathrm{POR}_i \preceq \mathrm{BP}+\mathrm{POR}_j$.

($\Rightarrow$) Assume $i < j$. Define the following sequentiality relation of arity j

$$R = S_j^{\{1,2\},\{1,2\}} \cap \cdots \cap S_j^{\{1,\ldots,j\},\{1,\ldots,j\}}$$

By Proposition 3.2, it is sufficient to show that $\mathrm{BP}+\mathrm{POR}_j$ is invariant under R, but $\mathrm{BP}+\mathrm{POR}_i$ is not.

1. ($\mathrm{BP}+\mathrm{POR}_j$ invariant) Going back to the definition of $+$, without loss of generality we can take

$$(\mathrm{BP}+\mathrm{POR}_j)(tt, x_1, \ldots, x_j) = \mathrm{POR}_j(x_1, \ldots, x_j)$$

For the sake of contradiction, assume $\mathrm{BP}+\mathrm{POR}_j$ is not invariant under R. Then there exists tuples $\left(x_1^1, \ldots, x_j^1\right), \cdots, \left(x_1^k, \ldots, x_j^k\right) \in R$. Let $y = (y_1, \ldots, y_j)$, with $y_m = \mathrm{BP}+\mathrm{POR}_j(x_m^1, \ldots, x_m^k)$, and $y \notin R$.

By induction on $2 \leq m \leq j$, we show $\mathrm{BP}+\mathrm{POR}_j$ must be invariant under $S_j^{\{1,\ldots,m\},\{1,\ldots,m\}}$. For $m = 2$, $\mathrm{BP}+\mathrm{POR}_j$ is invariant under $S_j^{\{1,2\},\{1,2\}}$ by the Closure Lemma and Proposition 4.4.

For the induction step, assume for the sake of contradiction that $\mathrm{BP}+\mathrm{POR}_j$ is not invariant under $S_j^{\{1,\ldots,m+1\},\{1,\ldots,m+1\}}$. Then there is no $\perp$ in $y_1, \ldots, y_{m+1}$, and there exists I, J with $y_I \neq y_J$. By the induction hypothesis, $\mathrm{BP}+\mathrm{POR}_j$ is invariant under $S_j^{\{1,\ldots,m\},\{1,\ldots,m\}}$, so we must have $y_1 = \cdots = y_m$, and hence the only possibility is that $y_{m+1} \neq y_1$. Since no $\perp$ appears in the resulting tuple, the first tuple above must all be *tt* or all be *ff*, by the definition of $+$. If it is all *ff*, then the columns of the tuples must come from the trace of BP, but since the first m columns are linearly coherent and return the same result, this would mean that the Egli-Milner lowerbound of the first m column has only one element, and since it is also coherent with the last column (which returns a different result), this contradicts BP being stable. Hence, the first tuple must be all *tt*, and the columns must come from the trace of POR_j. But the $m+1$ columns form a linearly coherent set of size less than or equal to j, and we can easily show that they cannot contain the trace element of POR_j that returns false. So we must have $y_{m+1} = y_1$.

Therefore, $\mathrm{BP}+\mathrm{POR}_j$ is invariant under $S_j^{\{1,\ldots,m\},\{1,\ldots,m\}}$ for $2 \leq m \leq j$, hence $\mathrm{BP}+\mathrm{POR}_j$ is invariant under R.

2. ($\mathrm{BP}+\mathrm{POR}_i$ not invariant) Again without loss of generality, we can take

$$(\mathrm{BP}+\mathrm{POR}_i)(\mathit{tt}, x_1, \ldots, x_i) = \mathrm{POR}_i(x_1, \ldots, x_i)$$

We show that $\mathrm{BP}+\mathrm{POR}_i$ is not invariant under $S_j^{\{1,\ldots,i+1\},\{1,\ldots,i+1\}}$, implying it is not invariant under R. Consider the following tuples of length j:

$$(\mathit{tt} \ldots, \mathit{tt}), \left(x_1^1, \ldots, x_{i+1}^1, \perp, \ldots, \perp\right), \cdots, \left(x_1^i, \ldots, x_{i+1}^i, \perp, \ldots, \perp\right)$$

where $\{(\mathit{tt}, x_m^1, \ldots, x_m^i)\}$ ($m \leq i+1$) is the subset of the first projection of the trace of $\mathrm{BP}+\mathrm{POR}_i$ corresponding to POR_i. It is easy to see that all those tuples are in $S_j^{\{1,\ldots,i+1\},\{1,\ldots,i+1\}}$. Applying $\mathrm{BP}+\mathrm{POR}_i$ to the columns of the tuples yields the tuple $(\underbrace{\mathit{tt}, \ldots, \mathit{tt}}_{i}, \mathit{ff}, \perp, \ldots, \perp)$, which is not in $S_j^{\{1,\ldots,i+1\},\{1,\ldots,i+1\}}$. □

5.6 The MONO semilattice

Up to this point all the semilattices we have introduced were related in some way to the partitioning of functions according to whether or not they were stable. We now consider a different characteristic and derive a corresponding semilattice. Define a *monovalued degree of parallelism* to be a degree of parallelism containing at least one monovalued function. We can characterize monovalued degrees of parallelism by their p-level:

Proposition 5.21 *A degree of parallelism is monovalued if and only if its p-level is of the form* (∞, j) *with* $j \geq 1$.

Proof. If f is monovalued then $\mathrm{bcc}(f) = \infty$, since there can be no bivalued coherent subset of $\pi_1(\mathrm{tr}(f))$. Moreover, since f is monotone, it must have a p-level of the form (i, j) with $i \geq 2$ and $j \geq 1$. We know $i = \infty$ (since $\mathrm{bcc}(f) = \infty$), so f must have a p-level of the form (∞, j) with $j \geq 1$. □

Let MONO be the subposet of CONT containing all monovalued degrees of parallelism.

Proposition 5.22 MONO *is a subsemilattice of* CONT.

Proof. The least upperbound of two monovalued degrees of parallelism is itself monovalued. The bottom element of MONO is the degree of all sequential functions, and its top element is the degree of DET, the Detector function. To show this, consider f a monovalued first-order monotone boolean function. Without loss of generality, assume f always returns *tt* (if not, consider $\mathrm{neg}(f)$ which is equiparallel to f). Let ttDET_n be the function of arity n that returns *tt* if one of its arguments is *tt*. It is not hard to show that for all n, $\mathrm{ttDET}_n \preceq \mathrm{ttDET}$. Let $n = |\mathrm{tr}(f)|$. Consider the following PCF-term:

$$M = \lambda p \lambda x_1 \ldots x_k . p(t_1(x_1, \ldots, x_k), \ldots, t_n(x_1, \ldots, x_k))$$

where t_j is a term checking if its arguments agree with the $\mathrm{j^{th}}$ element of $\pi_1(\mathrm{tr}(f))$ — and returning *tt* if they do and blocking if they don't. For example, for the Gustave function GUST, the terms look like:

$$\begin{array}{rcl} t_1 & = & \lambda x_1 x_2 x_3.(x_2 \wedge \neg x_3) \\ t_2 & = & \lambda x_1 x_2 x_3.(x_1 \wedge \neg x_2) \\ t_3 & = & \lambda x_1 x_2 x_3.(x_3 \wedge \neg x_1) \end{array}$$

It is easy to see that $f = [\![M]\!] \mathrm{ttDET}_n$, and since $\mathrm{ttDET}_n \preceq \mathrm{ttDET}$, $f \preceq \mathrm{ttDET}$. □

We note that the MONO semilattice contains the Bucciarelli hierarchy.

We can fully characterize the degree of parallelism of DET via p-levels, as we did with BP:

Proposition 5.23 *Given f a first-order monotone boolean function. Then f has a p-level of $(\infty, 1)$ iff $f \equiv \mathrm{DET}$.*

Proof. (⇒) If f has a p-level of $(\infty, 1)$, then f must be both monovalued and unstable. By minimality of DET in the UNSTABLE semilattice, $\mathrm{DET} \preceq f$. Since DET is the top element for monovalued functions and f monovalued, $f \preceq \mathrm{DET}$. Hence $f \equiv \mathrm{DET}$.

(⇐) Given $f \equiv \mathrm{DET}$. Then f must be invariant under the same sequentiality relations, hence the p-level of f is the same as the p-level of DET, namely $(\infty, 1)$. □

Since a function is unstable if and only if its p-level is $(i, 1)$ for some $i \geq 2$, and it is monovalued if and only if its p-level is (∞, j) for some $j \geq 1$, $[\mathrm{DET}]$ is the only unstable and monovalued degree of parallelism.

We will mention a final interesting result concerning monovalued degrees of parallelism. We can further characterize monovalued degrees of parallelism, a notion involving the description of a function, via extensional properties of the corresponding functions. A function f is *subsequential* if there exists a sequential function g that extends f, that is that dominates f in the extensional ordering on $\mathcal{B}^k$.

Proposition 5.24 *A function f is subsequential if and only if $[f]$ is monovalued.*

Proof. The proof is a corollary of the proposition in [5] which in our terminology states that given f a first-order monotone boolean function, then f is subsequential iff $\mathrm{bcc}(f) = \infty$. By this proposition, f is subsequential iff $\mathrm{bcc}(f) = \infty$. By Lemma 4.2, f is subsequential iff f has p-level (∞, j) for some $j \geq 1$. By Proposition 5.21, f is subsequential iff $[f]$ is monovalued. □

Therefore, every subsequential function is expressible by DET and conversely, DET can only express subsequential functions.

6 Conclusion

In this paper, we set out to explore the structure of CONT, the semilattice of degrees of parallelism of first-order monotone boolean functions. It is known that Sieber's sequentiality relations fully characterize the ordering on the semilattice. By turning our attention to presequentiality relations, a simple class of sequentiality relations, we were able to focus on the skeleton of the definability preorder. The advantage of looking at presequentiality relations is that we were able to completely characterize the set of presequentiality relations under which a given function is invariant via their p-level, a pair of integers which can be extracted from the trace of the function.

We showed that interesting classes of functions have natural characterizations in terms of p-levels, namely stable functions, unstable functions, stable-dominating functions and monovalued functions, and moreover exhibited natural hierarchies within those classes of functions, hierarchies that make up the skeleton of the definability preorder. We were also able to completely charaterize various well-known functions in terms of p-levels: any function with a p-level of $(2, 2)$ is equiparallel to BP, any function with a p-level of $(\infty, 1)$ is equiparallel to DET, any function with a p-level of $(2, 1)$ is equiparallel to POR.

The keys to the p-level characterization are clearly the Reduction and Closure Lemmas, which allow us to derive canonical representatives for large classes of presequentiality relations. The characterization itself is based on the fact that only two canonical presequentiality relations are needed to describe the full set of presequentiality relations under which a function is invariant. The next obvious step in the investigation is to extend this result to full sequentiality realtions. The question becomes: can we find canonical representatives of classes of sequentiality relations? A look at more complicated examples of sequentiality relations (for example, the ones used in the proof in [2], or in the proof of the strictness of the $\mathrm{BP} + \mathrm{POR}_i$ hierarchy in Proposition 5.20) indicates that canonical representatives for full sequentiality relations are far less nicely characterized than their presequentiality counterparts. This is an area of future work, along the lines of the hypergraph approach of [2, 5]. Another area of future work is a study of unstable functions (or unstable degrees of parallelism). The structure of p-levels for stable functions is richer than for unstable functions. Moreover, Bucciarelli's original hierarchy fully lives in the STABLE semilattice. It would be interesting to see if the structure of the $\mathsf{UNSTABLE}$ semilattice is equivalently complicated, or simpler in some respect.

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