# Restriction categories III: colimits, partial limits, and extensivity 

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#### Abstract

A restriction category is an abstract formulation for a category of partial maps, defined in terms of certain specified idempotents called the restriction idempotents. All categories of partial maps are restriction categories; conversely, a restriction category is a category of partial maps if and only if the restriction idempotents split. Restriction categories facilitate reasoning about partial maps as they have a purely algebraic formulation.

In this paper we consider colimits and limits in restriction categories. As the notion of restriction category is not self-dual, we should not expect colimits and limits in restriction categories to behave in the same manner. The notion of colimit in the restriction context is quite straightforward, but limits are more delicate. The suitable notion of limit turns out to be a kind of lax limit, satisfying certain extra properties.

Of particular interest is the behaviour of the coproduct both by itself and with respect to partial products. We explore various conditions under which the coproducts are "extensive" in the sense that the total category (of the related partial map category) becomes an extensive category. When partial limits are present, they become ordinary limits in the total category. Thus, when the coproducts are extensive we obtain as the total category a lextensive category. This provides, in particular, a description of the extensive completion of a distributive category.


## 1 Introduction

In a category $\mathscr{C}$ with a suitable class $\mathscr{M}$ of monomorphisms, one can define a category $\operatorname{Par}_{\mathscr{M}}(\mathscr{C})$, of partial maps in $\mathscr{C}$ whose domain of definition lies in $\mathscr{M}$. The resulting category has further structure which determines, among other things, the extent of the partiality involved. This is necessary as an abstract category can arise as a category of partial maps in more than one way. For example, any category $\mathbf{X}$ can be regarded as the category of partial maps in $\mathbf{X}$ where the class $\mathscr{M}$ consists only of the isomorphisms (the "total subobjects"); thus if $\mathbf{X}=\operatorname{Par}_{\mathscr{M}}(\mathscr{C})$, we have $\operatorname{Par}_{\mathscr{M}}(\mathscr{C})=\operatorname{Par}_{\text {Iso }}(\mathbf{X})$.

[^0]To describe this extra structure a variety of techniques have been employed. We recall below four possible approaches to capturing this further structure, indicating how the "trivial case" where $\mathscr{M}$ is just the isomorphisms can be identified.
A. Given partial maps $f, g: A \rightarrow B$, we define $f \leq g$ if $g$ is defined whenever $f$ is, and they then agree. This makes $\operatorname{Par}_{\mathscr{M}}(\mathscr{C})$ into a bicategory, and is the approach taken by Carboni in [2]; it is also closely related to Freyd's notion of allegory [12. The trivial case is characterized by the fact that the partial order is discrete, in the sense that $f \leq g$ only if $f=g$.
B. If the category $\mathscr{C}$ of total maps has finite products, then this induces a symmetric monoidal structure on $\operatorname{Par}_{\mathscr{M}}(\mathscr{C})$, given on objects by the product in $\mathscr{C}$. The trivial case can be characterized by the fact that this symmetric monoidal structure on the category of partial maps is in fact cartesian (that is, given by the categorical product). This approach was taken by Robinson and Rosolini [16] and by Curien and Obtulowicz [10].
C. If $\mathscr{C}$ has a strict initial object, and the unique map out of the initial object is in $\mathscr{M}$, then $\operatorname{Par}_{\mathscr{M}}(\mathscr{C})$ has zero maps, given by the "nowhere defined" partial maps. These were fundamental in the approach of di Paola and Heller [11]. The presence of these zero maps means that only when the category itself is trivial can the partiality be trivial.
D. To every partial map $f: A \rightarrow B$ we can associate the partial map $\bar{f}: A \rightarrow A$ which is defined whenever $f$ is, in which it acts as the identity. This operation is taken as fundamental in the notion of restriction category studied in the earlier instalments 6, 7] of this sequence of papers and again here. The maps of the form $\bar{f}$ are always idempotents, and are called restriction idempotents. This time the trivial case is characterized by the fact that the restriction idempotents are just the identity maps.

The assignment of $\bar{f}: A \rightarrow A$ to $f: A \rightarrow B$ mentioned above satisfies four axioms:
[R.1] $f \bar{f}=f$ for all $f: A \rightarrow B$;
[R.2] $\bar{f} \bar{g}=\bar{g} \bar{f}$ for all $f: A \rightarrow B$ and $g: A \rightarrow C$;
[R.3] $\overline{g \bar{f}}=\bar{g} \bar{f}$ for all $f: A \rightarrow B$ and $g: A \rightarrow C$;
[R.4] $\bar{g} f=f \overline{g f}$ for all $f: A \rightarrow B$ and $g: B \rightarrow C$.
A key property of restriction categories, not shared by the axiomatics of $11,16,10,2$, is that any full subcategory of a restriction category has an induced restriction structure; in fact the restriction categories are precisely the full subcategories of categories of partial maps. Conversely, a restriction category is a category of partial maps if and only if the restriction idempotents split. Restriction categories facilitate reasoning about partial maps as they have a purely algebraic formulation, which does not involve having any structure on the types.

In this paper we consider the structure on a restriction category arising from limits and colimits on the category of total maps. As the notion of restriction category is not self-dual, we should not expect colimits and limits in restriction categories to behave in the same manner. The notion of coproduct in the restriction context is quite straightforward: a restriction category with restriction coproducts is just a cocartesian object in the 2-category rCat of restriction categories. This means
that the diagonal $\mathbf{X} \rightarrow \mathbf{X} \times \mathbf{X}$ and the unique map $\mathbf{X} \rightarrow \mathbf{1}$ to the terminal restriction category both have left adjoints in the 2 -category $r$ Cat. This is described in more concrete terms in Section 2 it means that the category $\mathbf{X}$ has coproducts which satisfy certain conditions involving the restriction structure.

On the other hand a cartesian object in rCat necessarily has a trivial restriction structure. The suitable notion of a restriction category with restriction products turns out to be a cartesian object in a 2-category rCatl with the same objects and 1-cells as rCat, but with a certain type of "lax natural transformation" as 2-cells. This time the underlying category of a restriction category with restriction products does not in general have products, although the category of total maps does so; a concrete description is given in Section (4. The resulting structure turns out to be equivalent to the p-categories of Robinson and Rosolini [16], and indeed to other different formulations by a variety of authors. A restriction category can also have products which are entirely independent of the restriction structure. The presence of such products does have the slightly surprising effect of ensuring that the lattices of restriction idempotents have finite joins over which the meets distribute.

More generally, the suitable notion of limit turns out to be a certain type of lax limit, and we briefly explore these in Section 4.4. Once again, restriction limits in a restriction category become ordinary limits in the category of total maps.

Of particular interest is the behaviour of the coproduct both by itself and with respect to partial products. We explore in Section 3 various conditions under which the coproducts are "extensive" in the sense that the total category (of the related partial map category) becomes an extensive category. When partial limits are present, they become ordinary limits in the total category. Thus, when the coproducts are extensive we obtain as the total category a lextensive category. This provides, in particular, an alternative description of the extensive completion of a distributive category to that given in [9. This is described in Section 5.4. But what is the importance of being extensive? Section 2 answers this question for partial map categories very concretely: extensivity means that there is a "calculus of matrices." This is critical to understanding and manipulating the maps in these settings.

## Notation

The identity morphism on an object $A$ is denoted by $A$ or $1_{A}$. We write $\langle f \mid g\rangle: A+B \rightarrow C$ for the morphism induced by $f: A \rightarrow C$ and $g: B \rightarrow C$ and $\langle f, g\rangle: A \rightarrow B \times C$ for the morphism induced by $f: A \rightarrow B$ and $g: A \rightarrow C$. We also write $\left\langle f_{\lambda}\right\rangle: \sum_{\lambda \in \Lambda} A_{\lambda} \rightarrow B$ for the morphism induced by a $\Lambda$-indexed family of morphisms $f_{\lambda}: A_{\lambda} \rightarrow B$. Our notation for coproduct injections is more flexible: sometimes we write $i$ and $j$ for the two injections of a binary coproduct, and sometimes we use $i$ with a suitable subscript. We write $\tau: A+B \rightarrow B+A$ for the canonical isomorphism. The projections of a product are usually denoted by $\pi$ with a suitable subscript.

## 2 Coproducts and matrices

It is well known that in the category of sets and binary relations the disjoint union (of a finite family of sets) serves both as coproduct and product, so that there is a "calculus of matrices": see [3], for example. In this section we consider the extent to which this can be adapted to deal not with relations but with partial functions. We then consider when such a calculus is available in an abstract category of partial maps, or restriction category.

Given finite families $\left(A_{\lambda}\right)_{\lambda \in \Lambda}$ and $\left(B_{\kappa}\right)_{\kappa \in K}$ of sets, and a partial function $f: \sum_{\lambda} A_{\lambda} \rightarrow \sum_{\kappa} B_{\kappa}$, we may define a partial function $f_{\lambda \kappa}: A_{\lambda} \rightarrow B_{\kappa}$ for each $\lambda \in \Lambda$ and $\kappa \in K$, by declaring $f_{\lambda \kappa}(x)$ to be defined if and only if $f(x)$ is defined and lies in $B_{\kappa}$, in which case $f_{\lambda \kappa}(x)=f(x)$. Conversely, a matrix $\left(f_{\lambda \kappa}\right)_{\lambda \in \Lambda, \kappa \in K}$, with $f_{\lambda \kappa}$ a partial function from $A_{\lambda}$ to $B_{\kappa}$ for each $\lambda$ and $\kappa$, determines a relation $f$ from $\sum_{\lambda} A_{\lambda}$ to $\sum_{\kappa} B_{\kappa}$, where if $x \in A_{\lambda}$ and $y \in B_{\kappa}$ we have $f(x)=y$ if and only if $f_{\lambda \kappa}(x)=y$. The relation $f$ is in fact a partial function precisely when, for each $\lambda \in \Lambda$, if $f_{\lambda \kappa}(x)$ and $f_{\lambda \kappa^{\prime}}(x)$ are both defined then $\kappa=\kappa^{\prime}$ : in other words, if for each $x$ and $\lambda$, there is at most one $\kappa$ for which $f_{\lambda \kappa}(x)$ is defined.

Not only can we represent partial functions by matrices, we can represent composition of partial functions by matrix multiplication, in the following sense. If $f: \sum_{\lambda} A_{\lambda} \rightarrow \sum_{\kappa} B_{\kappa}$ and $g: \sum_{\kappa} B_{\kappa} \rightarrow$ $\sum_{k} C_{k}$ are partial functions with matrices $\left(f_{\lambda \kappa}\right)_{\lambda \in \Lambda, \kappa \in K}$ and $\left(g_{\kappa \mu}\right)_{\kappa \in K, \mu \in M}$, then the matrix of $g f$ is $\left(\vee_{\kappa} g_{\kappa \mu} f_{\lambda \kappa}\right)_{\lambda \in \Lambda, \mu \in M}$, where $\vee_{\kappa} g_{\kappa \mu} f_{\lambda \kappa}$ is the partial function $h: A_{\lambda} \rightarrow C_{k}$ with $h(x)=g_{\kappa \mu} f_{\lambda \kappa}(x)$ if the right hand side is defined for some (necessarily unique) $\kappa$, and undefined otherwise.

If $f$ is defined by $t: E \rightarrow \sum_{\kappa} B_{\kappa}$ with domain $m: E \rightarrow \sum_{\lambda} A_{\lambda}$, then $f_{\lambda \kappa}$ can be computed as a pullback, as in


In effect we are composing $f$ with the injection $i_{\lambda}: A_{\lambda} \rightarrow \sum_{\lambda} A_{\lambda}$, seen as a total partial map, and the partial map $i_{\kappa}^{*}: \sum_{\kappa} B_{\kappa} \rightarrow B_{\kappa}$ which is defined as the identity on $B_{\kappa}$ and is undefined elsewhere. More abstractly, $i_{\kappa}^{*}$ is the (unique) map satisfying $i_{\kappa}^{*} i_{\kappa}=1$ and $i_{\kappa} i_{\kappa}^{*}=\overline{i_{\kappa}^{*}}$. (We shall say that $i_{\kappa}^{*}$ is the restriction retraction of $i_{\kappa}$.)

We can recover $f$ from the $f_{\lambda \kappa}$ as the composite

$$
\sum_{\lambda} A_{\lambda} \xrightarrow{\sum_{\lambda} h_{\lambda}} \sum_{\lambda \kappa} A_{\lambda} \xrightarrow{\sum_{\lambda \kappa} f_{\lambda \kappa}} \sum_{\lambda \kappa} B_{\kappa} \xrightarrow{\sum_{\kappa} \nabla} \sum_{\kappa} B_{\kappa}
$$

where $h_{\lambda}: A_{\lambda} \rightarrow \sum_{\kappa} A_{\lambda}$ is defined by $h_{\lambda}(x)=(x, \kappa)$ if $f_{\lambda \kappa}(x)$ is defined for some (necessarily unique) $\kappa$, and undefined otherwise. Once again, there is also a more abstract characterization of $h_{\lambda}$ : it is the unique map satisfying $h_{\lambda}^{\prime} h_{\lambda}=\overline{h_{\lambda}}$ and $h_{\lambda} h_{\lambda}^{\prime}=\overline{h_{\lambda}^{\prime}}$, where $h_{\lambda}^{\prime}: \sum_{\kappa} A_{\lambda} \rightarrow A_{\lambda}$ is $\left\langle\overline{f_{\lambda \kappa}}\right\rangle_{\kappa \in K}$. (We shall say that $h_{\lambda}$ is the restriction inverse of $h_{\lambda}^{\prime}$.)

What structure does a restriction category $\mathbf{X}$ need in order to support such a calculus of matrices? Obviously $\mathbf{X}$ must have finite coproducts, and the coproduct injections must have restriction retractions. Also, given a morphism $f: A \rightarrow \sum_{\kappa} B_{\kappa}$, the map $\left\langle\overline{i_{\kappa}^{*} f}\right\rangle_{\kappa \in K}: \sum_{\kappa} A \rightarrow A$ must have a restriction inverse. This sets up a pair of functions between

- the set of morphisms from $\sum_{\lambda} A_{\lambda}$ to $\sum_{\kappa} B_{\kappa}$, and
- the set of matrices $\left(f_{\lambda \kappa}: A_{\lambda} \rightarrow B_{\kappa}\right)_{\lambda \in \Lambda, \kappa \in K}$ with the property that for each $\lambda$, the map $\left\langle i_{\kappa}^{*} \overline{f i_{\lambda}}\right\rangle_{\kappa \in K}: \sum_{\kappa} A_{\lambda} \rightarrow A_{\lambda}$ has a restriction inverse.

Finally we need these functions to be mutually inverse and to respect composition. We shall investigate when this occurs in the remainder of Section 2

### 2.1 Restriction coproducts

In the previous section we saw that for a restriction category $\mathbf{X}$ with coproducts to admit a calculus of matrices, it is necessary that the coproduct injections be restriction monics, and so in particular be total. In this section we examine the situation in which the coproduct injections are total.

Lemma 2.1 Let $\mathbf{X}$ be a restriction category with coproducts, and suppose that the injections of every binary coproduct $A+B$ are total. Then
(i) the unique arrow $z_{A}: 0 \rightarrow A$ is total for every object $A$;
(ii) the codiagonal $\nabla: A+A \rightarrow A$ is total for every object $A$;
(iii) $\overline{f+g}=\bar{f}+\bar{g}$ for all arrows $f$ and $g$.

Proof: To prove (iii), let $f: A \rightarrow A^{\prime}$ and $g: B \rightarrow B^{\prime}$ and write $i: A \rightarrow A+B, j: B \rightarrow A+B$, $i^{\prime}: A^{\prime}+B^{\prime}$, and $j^{\prime}: B^{\prime} \rightarrow A^{\prime}+B^{\prime}$ for the injections. Then $\overline{(f+g)} i=i \overline{(f+g) i}=i \overline{i^{\prime} f}=i \bar{f}$ since $i^{\prime}$ is total, and similarly $\overline{(f+g)} j=j \bar{g}$; thus $\overline{f+g}=\bar{f}+\bar{g}$. The proof of $(i i)$ is similar, while $(i)$ follows immediately from the fact that $z_{A}$ is an injection of the coproduct $A+0$.

We say that such a restriction category has restriction coproducts. A more abstract point of view is that such a restriction category is just a cocartesian object in the 2-category rCat. Recall that an ordinary category $\mathscr{C}$ has binary coproducts if and only if the diagonal functor $\mathscr{C} \rightarrow \mathscr{C} \times \mathscr{C}$ has a left adjoint. More generally an object $\mathscr{C}$ of a 2-category with finite products is said to be cocartesian if the diagonal $\mathscr{C} \rightarrow \mathscr{C} \times \mathscr{C}$ has a left adjoint in the 2-category. Thus a cocartesian restriction category is a restriction category $\mathbf{X}$ for which the diagonal restriction functor $\mathbf{X} \rightarrow \mathbf{X} \times \mathbf{X}$ and the unique restriction functor $\mathbf{X} \rightarrow \mathbf{1}$ to the terminal restriction category both have left adjoints in the 2 -category rCat . Any 2 -functor takes adjunctions to adjunctions, and a finite-productpreserving 2 -functor takes cocartesian objects to cocartesian objects. For instance, there is a 2-functor Total : rCat $\rightarrow$ Cat which sends a restriction category $\mathbf{X}$ to its category of total maps, and clearly Total preserves finite products. Then again, there is a 2 -functor $K_{r}: \mathrm{rCat} \rightarrow \mathrm{rCat}$ which sends a restriction category $\mathbf{X}$ to the restriction category $K_{r}(\mathbf{X})$ obtained by splitting the restriction idempotents of $\mathbf{X}$.

Proposition 2.2 If $\mathbf{X}$ is a restriction category with restriction coproducts then Total $(\mathbf{X})$ and $\operatorname{Total}\left(K_{r}(\mathbf{X})\right)$ have coproducts. If $F: \mathbf{X} \rightarrow \mathbf{Y}$ is a coproduct-preserving restriction functor between restriction categories with restriction coproducts, then $\operatorname{Total}(F): \mathbf{X} \rightarrow \mathbf{Y}$ and $\operatorname{Total}\left(K_{r}(F)\right)$ : $\operatorname{Total}\left(K_{r}(\mathbf{X})\right) \rightarrow \operatorname{Total}\left(K_{r}(\mathbf{Y})\right)$ preserve coproducts.

Proof: The 2-functors Total and Total $\left(K_{r}\right)$ send cocartesian objects to cocartesian objects, and so send restriction categories with restriction coproducts to categories with coproducts.

Similarly, if $F$ preserves coproducts then it commutes with the left adjoints $\mathbf{1} \rightarrow \mathbf{X}$ and $\mathbf{X} \times$ $\mathbf{X} \rightarrow \mathbf{X}$, and so Total $(F)$ commutes with the induced left adjoints $1 \rightarrow \operatorname{Total}(\mathbf{X})$ and $\operatorname{Total}(\mathbf{X}) \times$ $\operatorname{Total}(\mathbf{X}) \rightarrow \operatorname{Total}(\mathbf{X})$; that is, $\operatorname{Total}(F)$ preserves coproducts. The case of Total $\left(K_{r}(F)\right)$ is entirely analogous.

This proposition has a converse when the restriction category is classified. Recall [6] that an arrow $r: A \rightarrow B$ in a restriction category is said to be a restriction retraction if there is an arrow
$i: B \rightarrow A$ with $r i=1$ and $\bar{r}=i r$; such an $i$ is unique. Recall further [7] that a restriction category $\mathbf{X}$ is classified if the inclusion $\operatorname{Total}(\mathbf{X}) \rightarrow \mathbf{X}$ has a right adjoint $R$, and for each object $A$ the counit $\epsilon_{A}: R A \rightarrow A$ is a restriction retraction. The promised converse is now:

Proposition 2.3 If $\mathbf{X}$ is a classified restriction category and Total( $\mathbf{X}$ ) has coproducts, then $\mathbf{X}$ has restriction coproducts. An arbitrary functor $F: \mathbf{X} \rightarrow \mathscr{C}$ preserves coproducts if and only if its restriction $\operatorname{Total}(\mathbf{X}) \rightarrow \mathscr{C}$ to the total maps preserves coproducts. In particular, for a restriction category $\mathbf{Y}$ with restriction coproducts, a restriction functor $F: \mathbf{X} \rightarrow \mathbf{Y}$ preserves coproducts if and only if $\operatorname{Total}(F): \operatorname{Total}(\mathbf{X}) \rightarrow \operatorname{Total}(\mathbf{Y})$ does so.

Proof: Since $\mathbf{X}$ is classified, the inclusion $\operatorname{Total}(\mathbf{X}) \rightarrow \mathbf{X}$ is a left adjoint, and so preserves all existing colimits. Since it is also bijective on objects, $\mathbf{X}$ has coproducts if Total( $\mathbf{X}$ ) does so; and the injections are clearly total.

Since the inclusion $I: \operatorname{Total}(\mathbf{X}) \rightarrow \mathbf{X}$ is bijective on objects, a functor $G: \operatorname{Total}(\mathbf{X}) \rightarrow \mathscr{C}$ preserves coproducts if and only if $G I$ does so. Since $F I$ is just the composite of $\operatorname{Total}(F)$ and the inclusion $\operatorname{Total}(\mathbf{Y}) \rightarrow \mathbf{Y}$, it follows that $F$ preserves coproducts.

We also have:
Proposition 2.4 If $\mathbf{X}$ is a restriction category with coproducts and a zero object, then $\mathbf{X}$ has restriction coproducts.

Proof: If $\mathbf{X}$ has a zero object then the injection $i: A \rightarrow A+B$ has a retraction $\langle 1 \mid 0\rangle: A+B \rightarrow A$, and so is monic; but monomorphisms are always total.

Example 2.5 If $\mathscr{D}$ is a distributive category, then the endofunctor +1 of $\mathscr{D}$ has a well-known monad structure, and the Kleisli category $\mathscr{D}_{+1}$ of this monad has a restriction structure described in Example 7 of Section 2.1.3 of [6]. Since $\mathscr{D}$ has coproducts and the left adjoint $I: \mathscr{D} \rightarrow \mathscr{D}_{+1}$ is bijective on objects, $\mathscr{D}_{+1}$ has coproducts; the injections are in the image of $I$ and so total. Thus $\mathscr{D}_{+1}$ has restriction coproducts.

It is well known (see [4, 5] for example) that the free completion under (finite) coproducts of a category $\mathscr{C}$ can be formed as the category $\operatorname{Fam}(\mathscr{C})$ of finite families of objects of $\mathscr{C}$. Explicitly, an object of $\operatorname{Fam}(\mathscr{C})$ is a finite family $\left(A_{\lambda}\right)_{\lambda \in \Lambda}$ of objects of $\mathscr{C}$, and a morphism from $\left(A_{\lambda}\right)_{\lambda \in \Lambda}$ to $\left(B_{\kappa}\right)_{\kappa \in K}$ consists of a function $\varphi: \Lambda \rightarrow K$ and a family $\left(f_{\lambda}: A_{\lambda} \rightarrow B_{\varphi \lambda}\right)_{\lambda \in I}$ of morphisms in $\mathscr{C}$. The universal property of $\operatorname{Fam}(\mathscr{C})$ is expressed in terms of the fully faithful functor $J: \mathscr{C} \rightarrow \operatorname{Fam}(\mathscr{C})$ sending an object of $\mathscr{C}$ to the corresponding singleton family.

The observation we wish to make here is:
Remark 2.6 If $\mathbf{X}$ is a restriction category then $\operatorname{Fam}(\mathbf{X})$ has a canonical restriction structure, with $\overline{(\varphi, f)}=\left(1_{\Lambda},\left(\overline{f_{\lambda}}\right)_{\lambda \in \Lambda}\right)$. Then $J: \mathbf{X} \rightarrow \operatorname{Fam}(\mathbf{X})$ is clearly a restriction functor. Furthermore, $\mathbf{X}$ has restriction coproducts if and only if $J: \mathbf{X} \rightarrow \operatorname{Fam}(\mathbf{X})$ has a left adjoint in rCat. A purely formal consequence is that $\operatorname{Fam}(\mathbf{X})$ is the free restriction category with restriction coproducts on $\mathbf{X}$; we leave the precise formulation of the universal property to the reader. Another straightforward observation is that the restriction category $\operatorname{Fam}(\mathbf{X})$ is classified whenever $\mathbf{X}$ is so.

### 2.2 Restriction zero objects

To begin with, we allow $\mathbf{X}$ to be an arbitrary restriction category. Given arrows $f: A \rightarrow B$ and $g: B \rightarrow A$ in $\mathbf{X}$, recall [6] that $g$ is restriction inverse to $f$ (and $f$ to $g$ ) if $g f=\bar{f}$ and $f g=\bar{g}$. A restriction inverse is unique if it exists. In the special case where $f$ is total, we have $g f=\bar{f}=1$; then $f$ is said to be a restriction monic and $g$ its restriction retraction, and we often write $f^{*}$ for $f$.

We say that a zero object 0 in a restriction category is a restriction zero if for every object $A$ the zero map $0_{A A}: A \rightarrow A$ is a restriction idempotent; that is, $\overline{0_{A A}}=0_{A A}$.

Lemma 2.7 For a restriction category $\mathbf{X}$, the following are equivalent:
(i) $\mathbf{X}$ has a restriction zero;
(ii) $\mathbf{X}$ has an initial object 0 and a terminal object 1, and each $z_{A}: 0 \rightarrow A$ is a restriction monic;
(iii) $\mathbf{X}$ has a terminal object 1 and each $t_{A}: A \rightarrow 1$ is a restriction retraction.

Proof: $(i) \Rightarrow(i i)$. If 0 is a restriction zero then it is both initial and terminal, and for any object $A$ there is a unique $z_{A}: 0 \rightarrow A$ and a unique $t_{A}: A \rightarrow 0$. Clearly $t_{A} z_{A}=1$, since 0 is initial, while $z_{A} t_{A}=0_{A A}=\overline{0_{A A}}=\overline{z_{A} t_{A}}$.
(ii) $\Rightarrow$ (iii). Let $z_{1}^{*}: 1 \rightarrow 0$ be the restriction retraction of $z_{1}: 0 \rightarrow 1$, and $t_{A}: A \rightarrow 1$ be the unique map. Then $t_{A} z_{A} z_{1}^{*}=1$, since 1 is terminal; and we must show that $z_{A} z_{1}^{*} t_{A}$ is a restriction idempotent. Now $t_{A}=z_{1} z_{A}^{*}$, since 1 is terminal, and so $z_{A} z_{1}^{*} t_{A}=z_{A} z_{1}^{*} z_{1} z_{A}^{*}=z_{A} z_{A}^{*}$, which is indeed a restriction idempotent.
$($ iiii $) \Rightarrow(i)$. For each object $A$, choose $s_{A}: 1 \rightarrow A$ satisfying $\left(t_{A} s_{A}=1\right.$ and) $\overline{s_{A} t_{A}}=s_{A} t_{A}$. Then $s_{0}: 1 \rightarrow 0$ is inverse to $z_{1}: 0 \rightarrow 1$, and so 0 is a zero object. Finally $\overline{0_{A A}}=\overline{s_{A} t_{A}}=s_{A} t_{A}=0_{A A}$.

We now suppose once again that $\mathbf{X}$ has coproducts.
Proposition 2.8 Let $\mathbf{X}$ be a restriction category with coproducts, in which the coproduct injections are restriction monics. Then the initial object 0 is a restriction zero if and only if the maps $i^{*}: A+B \rightarrow A$ are natural in $B$; they are always natural in $A$.

Proof: The $i^{*}$ can be seen as $1_{A}+z_{B}^{*}: A+B \rightarrow A+0$, which are clearly natural in $A$, and will be natural in $B$ if and only if the $z_{B}^{*}: B \rightarrow 0$ are so. But this will be the case if and only if 0 is not just initial but also terminal, and now the result follows by Lemma 2.7

We now observe that in order to have a calculus of matrices, the category $\mathbf{X}$ must have a restriction zero object. We have already seen that the coproduct injections must be restriction monics, and so in particular that $z_{A}: 0 \rightarrow A$ must be one. To deal with empty coproducts, every map $f: A \rightarrow 0$ should be representable as an "empty matrix", which clearly means that there can be at most one such map. Thus in this case 0 is not just an initial object but a zero object (that is, an initial and a terminal object.) By Lemma 2.7 it follows that the initial object is a restriction zero.

Example 2.9 If $\mathscr{D}$ is a distributive category, then the initial object of $\mathscr{D}$ is a restriction zero in $\mathscr{D}_{+1}$. To see this, observe that the left adjoint $I: \mathscr{D} \rightarrow \mathscr{D}_{+1}$ preserves colimits, so 0 is initial in
$\mathscr{D}_{+1}$. For every object $A$, there is a unique arrow $A \rightarrow 0+1=1$ in $\mathscr{D}$, and so 0 is also terminal in $\mathscr{D}_{+1}$. The zero map $0_{A A}: A \rightarrow A$ in $\mathscr{D}_{+1}$ is

$$
A \xrightarrow{!} 1 \xrightarrow{i_{2}} A+1
$$

and its restriction is

$$
A \xrightarrow{\left\langle 1, i_{2}!\right\rangle} A \times(A+1) \xrightarrow{\delta^{-1}} A \times A+A \xrightarrow{\pi_{1}+!} A+1 .
$$

The fact that these two maps agree is an easy exercise in distributive categories.
Lemma 2.10 If $\mathbf{X}$ is a restriction category with restriction coproducts and a restriction zero, then:
(i) each coproduct injection $i$ : $A \rightarrow A+B$ is a restriction monic, with restriction retraction $i^{*}: A+B \rightarrow A$ equal to $\langle 1 \mid 0\rangle: A+B \rightarrow A$, so that the restriction idempotent $i i^{*}$ is $1+0: A+B \rightarrow A+B ;$
(ii) if $f: C \rightarrow A+B$ is total, and the restriction idempotent $\overline{i^{*} f}$ splits, then the section $k: C_{A} \rightarrow C$ of the splitting is the pullback in $\operatorname{Total}(\mathbf{X})$ of the injection $i: A \rightarrow A+B$ along $f$;
(iii) the natural transformations in $\operatorname{Total}(\mathbf{X})$ whose components are the coproduct injections are cartesian.

Proof: $(i)$ We can regard $i$ as $1_{A}+z_{B}$. Then $\left(1_{A}+z_{B}^{*}\right)\left(1_{A}+z_{B}\right)=1$, while $\left(1_{A}+z_{B}\right)\left(1_{A}+z_{B}^{*}\right)=$ $1_{A}+0_{B B}=\overline{1_{A}}+\overline{0_{B B}}=\overline{1_{A}+0_{B B}}$.
(ii) Suppose that $k: C_{A} \rightarrow C$ and $k^{*}: C \rightarrow C_{A}$ provide the splitting, so that $k^{*} k=1$ and $k k^{*}=\overline{i i^{*} f}$. Then $i i^{*} f k k^{*}=(1+0) f \overline{(1+0) f}=(1+0) f=\overline{1+0} f=f \overline{(1+0) f}=f k k^{*}$, and so $i i^{*} f k=f k$. We claim that the commutative square

is in fact a pullback in $\operatorname{Total}(\mathbf{X})$. Since $i$ and $k$ are monic, it will suffice to show that a total map $u: D \rightarrow C$ factorizes through $k$ if $f u$ factorizes through $i$. But if $f u$ factorizes through $i$ then $i i^{*} f u=f u$, and now $k k^{*} u=\overline{i i^{*} f} u=u \overline{i i^{*} f u}=u \overline{f u}=u$.
(iii) We are to show that the square

is a pullback in Total( $\mathbf{X}$ ). Since $\overline{i^{\prime *}(f+g)}=\overline{f i^{*}}=\overline{i^{*}}=i i^{*}$, the result follows by part (ii).
As we saw above, in the category of sets and relations a coproduct $A+B$ is also a product, but in the case of sets and partial functions this is no longer the case. We now describe the trace which
remains of this product structure. In a restriction category $\mathbf{X}$ with restriction coproducts and a restriction zero, we have a functor $+: \mathbf{X} \times \mathbf{X} \rightarrow \mathbf{X}$, and natural transformations $i^{*}: A+B \rightarrow A$ and $j^{*}: A+B \rightarrow B$. If there were a natural diagonal $\Delta: A \rightarrow A+A$ satisfying the triangle equations, this would exhibit $A+B$ as the product of $A$ and $B$. Although there is not such a $\Delta$, we shall see that there are various maps which "try" to be the diagonal; we shall call them decisions.

### 2.3 The calculus of matrices

In this section we consider a restriction category $\mathbf{X}$ with restriction coproducts and a restriction zero. The main aim of this section is to establish, under further conditions still to be determined, a bijection between arrows $f: \sum_{\lambda} A_{\lambda} \rightarrow \sum_{\kappa} B_{\kappa}$ and matrices $\left(f_{\lambda \kappa}\right)$ with the property that for each $\lambda$ the map $\left(\overline{f_{\lambda \kappa}}\right)_{\kappa}: \sum_{\kappa} A_{\lambda} \rightarrow A_{\lambda}$ has a restriction inverse $h_{\lambda}$. This bijection should send $f$ to $\left(i_{\kappa}^{*} f i_{\lambda}\right)_{\lambda, \kappa}$ and $\left(f_{\lambda \kappa}\right)_{\lambda, \kappa}$ to the composite

$$
\sum_{\lambda} A_{\lambda} \xrightarrow{\sum_{\lambda} h_{\lambda}} \sum_{\lambda \kappa} A_{\lambda} \xrightarrow{\sum_{\lambda \kappa} f_{\lambda \kappa}} \sum_{\lambda \kappa} B_{\kappa} \xrightarrow{\nabla} \sum_{\kappa} B_{\kappa} .
$$

The universal property of the coproduct $\sum_{\lambda} A_{\lambda}$ reduces this to the case where $\Lambda$ is a singleton. Thus we are to establish a bijection between the set of morphisms $f: A \rightarrow \sum_{\kappa} B_{\kappa}$ and the set of those $K$-tuples $\left(f_{\kappa}: A \rightarrow B_{\kappa}\right)$ for which $\left(\overline{f_{\kappa}}\right)_{\kappa}: \sum_{\kappa} A \rightarrow A$ has a restriction inverse $h$. For any $f: A \rightarrow \sum_{\kappa} B_{\kappa}$ the induced map $\left\langle\overline{i_{\kappa}^{*} f}\right\rangle_{\kappa}: \sum_{\kappa} A \rightarrow A$ will clearly need to have a restriction inverse $h$. Moreover, $\bar{h}$ will have to be $\bar{f}$. For if $h$ is restriction inverse to $\left\langle\overline{i_{\kappa}^{*}}\right\rangle_{\kappa}$ then

$$
\left.\bar{h}=\overline{\left\langle i_{\kappa}^{*} f\right.}\right\rangle_{\kappa} h=\nabla \sum_{\kappa} \overline{i_{k}^{*} f} h=\nabla \overline{\sum_{\kappa} i_{\kappa}^{*} f} h=\nabla h \overline{\left(\sum_{\kappa} i_{\kappa}^{*} f\right) h}
$$

but for our bijection we need $\left(\sum_{\kappa} i_{\kappa}^{*} f\right) h=f$, so that $\bar{h}=\nabla h \bar{f}$. But then $\bar{h}=\overline{\nabla h \bar{f}}=\overline{h \bar{f}}=\bar{h} \bar{f}=$ $\bar{h} \overline{\left(\sum_{\kappa} i_{\kappa}^{*} f\right) h}=\overline{\left(\sum_{\kappa} i_{\kappa}^{*} f\right) h}=\bar{f}$, as claimed.

If $\left\langle i_{\kappa}^{*} f\right\rangle_{\kappa}$ does have a restriction inverse $h$ and $\bar{h}$ is $\bar{f}$, then we write $[f]$ for $h$, and call it a decision for $f$ or $f$-decision, for reasons which will become clearer below.

Proposition 2.11 An arrow $h: A \rightarrow \sum_{\kappa} A$ is the decision of $f: A \rightarrow \sum_{\kappa} B_{\kappa}$ if and only if $\nabla h=\bar{f}$ and the square

commutes.
We defer to the next section the proof of the proposition. Observe, however, that it helps to explain the name "decision". Since $\overline{[f]}=\overline{\nabla[f]}=\bar{f}$, the decision $[f]$ is defined whenever $h$ is, and the effect of $[f]$ is "to send an element $a \in A$ to the element in the component of $\sum_{\kappa} A$ corresponding to the component of $f(a) \in \sum_{\kappa} B_{\kappa}{ }^{\prime}$.

Theorem 2.12 Let $\mathbf{X}$ be a restriction category with restriction coproducts and a restriction zero, in which every map $f: A \rightarrow \sum_{\kappa} B_{\kappa}$ has a decision. Then there is a bijection between the set
of all maps $f: \sum_{\lambda} A_{\lambda} \rightarrow \sum_{\kappa} B_{\kappa}$ and the set of those matrices $\left(f_{\lambda \kappa}: A_{\lambda} \rightarrow B_{\kappa}\right)_{\lambda, \kappa}$ for which $\left\langle\overline{f_{\lambda \kappa}}\right\rangle_{\kappa}: \sum_{\kappa} A_{\lambda} \rightarrow A_{\lambda}$ has a restriction inverse for every $\lambda$. The bijection sends $f$ to the matrix $\left(i_{\kappa}^{*} f i_{\lambda}\right)_{\lambda, \kappa}$.

Proof: Write $\Phi$ for the function computing the matrix of a map $f: \sum_{\lambda} A_{\lambda} \rightarrow \sum_{\kappa} B_{\kappa}$, and $\Psi$ for the purported inverse, which sends $\left(f_{\lambda \kappa}\right)_{\lambda, \kappa}$ to the composite

$$
\sum_{\lambda} A_{\lambda} \xrightarrow{\sum_{\lambda} h_{\lambda}} \sum_{\lambda \kappa} A_{\lambda} \xrightarrow{\sum_{\lambda \kappa} f_{\lambda \kappa}} \sum_{\lambda \kappa} B_{\kappa} \xrightarrow{\nabla} \sum_{\kappa} B_{\kappa}
$$

where $h_{\lambda}$ is restriction inverse to $\left\langle\overline{f_{\lambda \kappa}}\right\rangle_{\kappa}: \sum_{\kappa} A_{\lambda} \rightarrow A_{\lambda}$.
Starting with $f: \sum_{\lambda} A_{\lambda} \rightarrow \sum_{\kappa} B_{\kappa}$ we get the matrix $\Phi(f)=\left(i_{\kappa}^{*} f i_{\lambda}: A_{\lambda} \rightarrow B_{\kappa}\right)$; and then $\Psi(\Phi(f))$ is the composite

$$
\sum_{\lambda} A_{\lambda} \xrightarrow{\sum_{\lambda}\left[f i_{\lambda}\right]} \sum_{\lambda \kappa} A_{\lambda} \xrightarrow{\sum_{\lambda \kappa} i_{\kappa}^{*} f i_{\lambda}} \sum_{\lambda \kappa} B_{\kappa} \xrightarrow{\nabla} \sum_{\kappa} B_{\kappa} .
$$

To see that this is just $f$, observe that in the diagram

the large upper parallelogram commutes by Proposition [2.11, the upper triangle commutes since $i_{\kappa}^{*} i_{\kappa}=1$, the lower triangle by one of the triangle equations, and the large lower rectangle by naturality of $\nabla$. Thus the entire diagram commutes and $\Psi(\Phi(f))=f$.

Suppose on the other hand that we are given $f_{\lambda \kappa}: A_{\lambda} \rightarrow B_{\kappa}$ for each $\lambda \in \Lambda$ and $\kappa \in K$, and that $\left\langle\overline{f_{\lambda \kappa}}\right\rangle_{\kappa}: \sum_{\kappa} A_{\lambda} \rightarrow A_{\lambda}$ has a restriction inverse $h_{\lambda}$ for each $\lambda \in \Lambda$. Then $\Phi \Psi$ sends the matrix $\left\langle f_{\lambda \kappa}\right\rangle_{\lambda, \kappa}$ to the composite

$$
A_{\lambda} \xrightarrow{i_{\lambda}} \sum_{\lambda} A_{\lambda} \xrightarrow{\sum_{\lambda} h_{\lambda}} \sum_{\lambda \kappa} A_{\lambda} \xrightarrow{\sum_{\lambda \kappa} f_{\lambda \kappa}} \sum_{\lambda \kappa} B_{\kappa} \xrightarrow{\nabla} \sum_{\kappa} B_{\kappa} \xrightarrow{i_{\kappa}^{*}} B_{\kappa}
$$

which, by the naturality of $i_{\lambda}^{*}$ and the definition of $\nabla: \sum_{\lambda} \kappa B_{\kappa} \rightarrow B_{\kappa}$, is just

$$
A_{\lambda} \xrightarrow{h_{\lambda}} \sum_{\kappa} A_{\lambda} \xrightarrow{\sum_{\kappa} f_{\lambda_{\kappa}}} \sum_{\kappa} B_{\kappa} \xrightarrow{i_{\kappa}^{*}} B_{\kappa} .
$$

Naturality of $i_{\kappa}^{*}$ gives $i_{\kappa}^{*}\left(\sum_{\kappa} f_{\lambda \kappa}\right)=f_{\lambda \kappa} i_{\kappa}^{*}$, thus we must show that $f_{\lambda \kappa} i_{\kappa}^{*} h_{\lambda}=f_{\lambda \kappa}$.
Now $i_{\kappa}^{*}$ is restriction inverse to $i_{\kappa}$, and $h_{\lambda}$ is restriction inverse to $\left\langle f_{\lambda \kappa}\right\rangle_{\kappa}$, so $i_{\kappa}^{*} h_{\lambda}$ is restriction inverse to $\left\langle\overline{f_{\lambda \kappa}}\right\rangle_{\kappa} i_{\kappa}$, which is just $\overline{f_{\lambda \kappa}}$. But restriction idempotents are their own restriction inverses, so $i_{\kappa}^{*} h_{\lambda}=\overline{f_{\lambda \kappa}}$. Thus $f_{\lambda \kappa} i_{\kappa}^{*} h_{\lambda}=f_{\lambda \kappa} \overline{f_{\lambda \kappa}}=f_{\lambda \kappa}$, and so $\Phi \Psi$ is indeed the identity, and the bijection is established.

We end this section by showing how to "multiply" matrices:

Proposition 2.13 Under the hypotheses of Theorem 2.12, if $f: \sum_{\lambda} A_{\lambda} \rightarrow \sum_{\kappa} B_{\kappa}$ has matrix $\left(f_{\lambda \kappa}\right)_{\lambda, \kappa}$, and $g: \sum_{\kappa} B_{\kappa} \rightarrow \sum_{\mu} C_{\mu}$ has matrix $\left(g_{\kappa \mu}\right)_{\kappa, \mu}$, then the composite $g f$ has matrix $\left(\vee_{\kappa} g_{\kappa \mu} f_{\lambda \kappa}\right)_{\lambda, \mu}$, where $\vee_{\kappa} g_{\kappa \mu} f_{\lambda \kappa}: A_{\lambda} \rightarrow C_{\mu}$ is given by

$$
A_{\lambda} \xrightarrow{h_{\lambda}} \sum_{\kappa} A_{\lambda} \xrightarrow{\left\langle g_{\kappa \mu} f_{\lambda \kappa}\right\rangle_{\kappa}} C_{\mu}
$$

and $h_{\lambda}$ is the restriction inverse of $\left\langle\overline{f_{\lambda \kappa}}\right\rangle_{\kappa}: \sum_{\kappa} A_{\lambda} \rightarrow A_{\lambda}$.
Proof: We must show that

$$
i_{\mu}^{*} g f i_{\lambda}=\left\langle g_{\kappa \mu} f_{\lambda \kappa}\right\rangle_{\kappa} h_{\lambda} .
$$

By the theorem $f i_{\lambda}=\sum_{\kappa}\left(i_{\kappa}^{*} f i_{\lambda}\right) h_{\lambda}$, so $i_{\mu}^{*} g f i_{\lambda}=i_{\mu}^{*} g \sum_{\kappa}\left(i_{\kappa}^{*} f i_{\lambda}\right) h_{\lambda}=\left\langle i_{\mu}^{*} g i_{\kappa} i_{\kappa}^{*} f i_{\lambda}\right\rangle_{\kappa} h_{\lambda}=\left\langle g_{\kappa \mu} f_{\lambda \kappa}\right\rangle_{\kappa} h_{\lambda}$ as required.

### 2.4 Decisions

In this section we further explore decisions in a restriction category $\mathbf{X}$ with restriction coproducts and restriction zero; the main goal is to prove Proposition 2.11. Recall that $h: A \rightarrow \sum_{\kappa} A$ is the decision of $f: A \rightarrow \sum_{\kappa} B_{\kappa}$ if it is restriction inverse to $\left\langle\overline{i_{\kappa}^{*} f}\right\rangle_{\kappa}: \sum_{\kappa} A \rightarrow A$ and $\bar{h}=\bar{f}$. We say that $h: A \rightarrow \sum_{\kappa} A$ is a decision if it is the decision of some map $f: A \rightarrow \sum_{\kappa} B_{\kappa}$.

## Example 2.14

(i) If $K$ is a singleton, so that we have a single map $f: A \rightarrow B$, a decision for $f$ is a map $h: A \rightarrow A$ which is restriction inverse to $\bar{f}$ : this is just $\bar{f}$ itself.
(ii) If $K$ is empty, so that $f$ is the unique map $A \rightarrow 0$, a decision for $f$ is a map $h: A \rightarrow 0$ which is restriction inverse to the unique map $z_{A}: 0 \rightarrow A$ : then $f=h=z_{A}^{*}$.
(iii) Let $f$ be a coproduct injection $i_{\lambda}: A_{\lambda} \rightarrow \sum_{\lambda} A_{\lambda}$. Then $\left\langle\overline{i_{\kappa}^{*} i_{\lambda}}\right\rangle_{\kappa}: \sum_{\kappa} A_{\lambda} \rightarrow A_{\lambda}$ is $i_{\lambda}^{*}$, which has restriction inverse $i_{\lambda}$. Thus $i_{\lambda}$ is its own decision.

Proposition 2.15 For a map $h: A \rightarrow \sum_{\kappa \in K} A$ the following are equivalent:
(i) $h$ is its own decision;
(ii) $h$ is a decision;
(iii) $h$ has a restriction inverse $g: \sum_{\kappa} A \rightarrow A$ and $g i_{\kappa}: A \rightarrow A$ is a restriction idempotent for each $\kappa$.

Proof: The downward implications are trivial; we must show that given restriction idempotents $e_{\kappa}: A \rightarrow A$ for each $\kappa \in K$, if $\left\langle e_{\kappa}\right\rangle_{\kappa}: \sum_{\kappa} A \rightarrow A$ has a restriction inverse $h$ then $h$ is its own decision.

Since $h$ is restriction inverse to $\left\langle e_{\kappa}\right\rangle_{\kappa}$ and $i_{\kappa}^{*}$ is restriction inverse to $i_{\kappa}$, we see that $i_{\kappa}^{*} h$ is restriction inverse to $\left\langle e_{\kappa}\right\rangle_{\kappa} i_{\kappa}$; but the latter is just $e_{\kappa}$ which is its own restriction inverse. Thus $i_{\kappa}^{*} h=e_{\kappa}$. and so $\overline{i_{\kappa}^{*} h}=e_{\kappa}$. But then $h$ is restriction inverse to $\left\langle\overline{i_{\kappa}^{*} h}\right\rangle_{\kappa}$, which is just to say that $h$ is its own decision.

The next result says that we can "conjugate" decisions by restriction inverses:

Corollary 2.16 If $h: A \rightarrow \sum_{\kappa} A$ is a decision, and $f: A \rightarrow B$ a map with restriction inverse $g: B \rightarrow A$, then

$$
B \xrightarrow{g} A \xrightarrow{h} \sum_{\kappa} A \xrightarrow{\sum_{\kappa} f} \sum_{\kappa} B
$$

is a decision and $\overline{\left(\sum_{\kappa} g\right) h f}=\overline{h f}$.
Proof: Since $h$ is a decision it is restriction inverse to $\left\langle\overline{i_{\kappa}^{*} h}\right\rangle_{\kappa}: \sum_{\kappa} A \rightarrow A$. Since $g$ is restriction inverse to $f$, and $\sum_{\kappa} f$ is restriction inverse to $\sum_{\kappa} g$, also $\left(\sum_{\kappa} f\right) h g$ is restriction inverse to $f\left\langle\overline{i_{\kappa}^{*} h}\right\rangle_{\kappa}\left(\sum_{\kappa} g\right)$. Now $f\left\langle\overline{i_{\kappa}^{*} h}\right\rangle_{\kappa}\left(\sum_{\kappa} g\right) i_{\kappa}=f\left\langle\overline{i_{\kappa}^{*} h}\right\rangle_{\kappa} i_{\kappa} g=f \overline{i_{\kappa}^{*} h} g=f g \overline{i_{\kappa}^{*} h g}=\bar{g} \overline{i_{\kappa}^{*} h g}$ which is a restriction idempotent, thus $\left(\sum_{\kappa} f\right) h g$ is a decision by the Proposition.

Finally, $\overline{\left(\sum_{\kappa} g\right) h f}=\overline{\nabla\left(\sum_{\kappa} g\right) h f}=\overline{g \nabla h f}=\overline{g \bar{h} f}=\overline{g f \overline{h f}}=\overline{\bar{f} \overline{h f}}=\bar{f} \overline{h f}=\overline{h f}$.
Corollary 2.17 If $h: \sum_{\lambda} A_{\lambda} \rightarrow \sum_{\kappa} \sum_{\lambda} A_{\lambda}$ is a decision then so is $k_{\lambda}=\left(\sum_{\kappa} i_{\lambda}^{*}\right) h i_{\lambda}$ for each $\lambda$, and $h$ is the composite

$$
\sum_{\lambda} A_{\lambda} \xrightarrow{\sum_{\lambda} k_{\lambda}} \sum_{\lambda} \sum_{\kappa} A_{\lambda} \xrightarrow{\sigma} \sum_{\kappa} \sum_{\lambda} A_{\lambda}
$$

where $\sigma$ is the canonical isomorphism.
Proof: The fact that $k_{\lambda}$ is a decision is immediate from the previous corollary. On the other hand $\sigma\left(\sum_{\lambda} k_{\lambda}\right) i_{\lambda}=\sigma i_{\lambda} k_{\lambda}=\left(\sum_{\kappa} i_{\lambda}\right) k_{\lambda}=\left(\sum_{\kappa} i_{\lambda}\right)\left(\sum_{\kappa} i_{\lambda}^{*}\right) h i_{\lambda}=\left(\sum_{\kappa} \overline{i_{\kappa}^{*}}\right) h i_{\lambda}=h i_{\lambda} \overline{\left(\sum_{\kappa} i_{\kappa}^{*}\right) h i_{\lambda}}=h i_{\lambda} \overline{h i_{\lambda}}=$ $h i_{\lambda}$ for each $\lambda$, where the penultimate step uses the previous corollary. Thus $\sigma\left(\sum_{\lambda} k_{\lambda}\right)=h$ as claimed.

We are now ready to prove Proposition [2.11] We shall make frequent use of the naturality of $i_{\kappa}^{*}$ :

Proof of Proposition 2.11: First we simplify the condition for $h: A \rightarrow \sum_{\kappa} A$ to be the decision of $f: A \rightarrow \sum_{\kappa} B_{\kappa}$. This will be the case if $h\left\langle\overline{i_{\kappa}^{*} f}\right\rangle_{\kappa}=\overline{\left\langle\overline{i_{\kappa}^{*} f}\right\rangle_{\kappa}}$ and $\left\langle\overline{i_{\kappa}^{*} f}\right\rangle_{\kappa} h=\bar{h}$. Now $\left\langle\overline{i_{\kappa}^{*} f}\right\rangle_{\kappa}=\nabla\left(\sum_{\kappa} \overline{i_{\kappa}^{*} f}\right)=\nabla \overline{\sum_{\kappa}\left(i_{\kappa}^{*} f\right)}$ and so the first condition becomes

$$
h \overline{i_{\kappa}^{*} f}=i_{\kappa} \overline{i_{k}^{*} f}
$$

Suppose that $\left(\sum_{\kappa} f\right) h=\left(\sum_{\kappa} i_{\kappa}\right) f$ and $\nabla h=\bar{f}$. Then $\bar{h}=\overline{\nabla h}=\bar{f}$. Now $\left(\sum_{\kappa} \bar{f}\right) h=\overline{\sum_{\kappa} f} h=$ $h \overline{\left(\sum_{\kappa} f\right) h}=h \overline{\left(\sum_{\kappa} i_{\kappa}\right) h}=h \bar{h}=h$, and so

$$
\overline{i_{\kappa}^{*} h}=\overline{i_{\kappa}^{*}\left(\sum_{\kappa} \bar{f}\right) h}=\overline{\bar{f} i_{\kappa}^{*} h}=\overline{f i_{\kappa}^{*} h}=\overline{i_{\kappa}^{*}\left(\sum_{\kappa} f\right) h}=\overline{i_{\kappa}^{*}\left(\sum_{\kappa} i_{\kappa}\right) f}=\overline{i_{\kappa} i_{\kappa}^{*} f}=\overline{i_{\kappa}^{*} f}
$$

but now

$$
h \overline{i_{\kappa}^{*} f}=h \overline{i_{\kappa}^{*} h}=\overline{i_{k}^{*}} h=i_{\kappa} i_{\kappa}^{*} h=i_{\kappa} \nabla i_{\kappa} i_{\kappa}^{*} h=i_{\kappa} \nabla \overline{i_{k}^{*}} h=i_{\kappa} \nabla h \overline{i_{\kappa}^{*} h}=i_{\kappa} \bar{h} \overline{i_{\kappa}^{*} h}=i_{\kappa} \overline{i_{\kappa}^{*} h}
$$

giving the first condition. As for the second

$$
\begin{aligned}
&\left\langle\overline{i_{\kappa}^{*} f}\right\rangle_{\kappa} h=\nabla\left(\sum_{\kappa} \overline{i_{\kappa}^{*} f}\right) h=\nabla \overline{\sum_{\kappa}\left(i_{\kappa}^{*} f\right)} h=\nabla h \overline{\left(\sum_{\kappa} i_{\kappa}^{*} f\right) h} \\
&=\bar{h} \overline{\left(\sum_{\kappa} i_{\kappa}^{*}\right)\left(\sum_{\kappa} f\right) h}=\bar{h} \overline{\left(\sum_{\kappa} i_{\kappa}^{*}\right)\left(\sum_{\kappa} i_{\kappa}\right) f}=\bar{h} \bar{f}=\bar{h}
\end{aligned}
$$

and so $h$ is the decision of $f$.
Suppose conversely that $h$ is the decision of $f$. Then

$$
\overline{\sum_{\kappa} i_{\kappa}^{*} f} h=h \overline{\left(\sum_{\kappa} i_{\kappa}^{*} f\right) h}=h \overline{\overline{\sum_{\kappa} i_{\kappa}^{*} f} h}=h \overline{\left\langle\left\langle i_{\kappa}^{*} f\right\rangle_{\kappa} h\right.}=h \bar{h}=h
$$

and so $\nabla h=\nabla \overline{\sum_{\kappa} i_{\kappa}^{*} f} h=\nabla\left(\sum_{\kappa} \overline{i_{\kappa}^{*} f}\right) h=\left\langle\overline{i_{\kappa}^{*} f}\right\rangle_{\kappa} h=\bar{h}=\bar{f}$. On the other hand

$$
\left(\sum_{\kappa} f\right) h=\left(\sum_{\kappa} f\right) \overline{\sum_{\kappa} i_{\kappa}^{*} f} h=\sum_{\kappa}\left(f \overline{i_{\kappa}^{*} f}\right) h=\sum_{\kappa}\left(\overline{i_{\kappa}^{*}} f\right) h=\left(\sum_{\kappa} \overline{i_{\kappa}^{*}}\right)\left(\sum_{\kappa} f\right) h
$$

and so

$$
\begin{aligned}
& \left(\sum_{\kappa} i_{\kappa}\right) f=\left(\sum_{\kappa} i_{\kappa}\right) f \bar{h}=\left(\sum_{\kappa} i_{\kappa}\right) f \nabla \overline{\sum_{\kappa} i_{\kappa}^{*} f} h=\left(\sum_{\kappa} i_{\kappa}\right) \nabla\left(\sum_{\kappa} f\right) \overline{\left(\sum_{\kappa} i_{\kappa}^{*}\right)\left(\sum_{\kappa} f\right) h} \\
= & \left(\sum_{\kappa} i_{\kappa}\right) \nabla \overline{\sum_{\kappa} i_{\kappa}^{*}}\left(\sum_{\kappa} f\right) h=\left(\sum_{\kappa} i_{\kappa}\right) \nabla\left(\sum_{\kappa} i_{\kappa} i_{\kappa}^{*} f\right) h=\left(\sum_{\kappa} i_{\kappa}\right)\left(\sum_{\kappa} i_{\kappa}^{*} f\right) h=\left(\sum_{\kappa} \overline{i_{\kappa}^{*}}\right)\left(\sum_{\kappa} f\right) h=\left(\sum_{\kappa} f\right) h .
\end{aligned}
$$

We saw in Example 2.14 that a decision for $f: A \rightarrow \sum_{\kappa \in K} B_{\kappa}$ always exists if $K$ is empty or a singleton. We end this section by proving that all decisions exist provided that binary ones do.

Proposition 2.18 A restriction category $\mathbf{X}$ with restriction coproducts and a restriction zero has all decisions provided that it has a decision for each $f: A \rightarrow B+C$.

Proof: Let $f: A \rightarrow \sum_{\kappa \in K} B_{\kappa}$ be given, where $K$ is a finite set of cardinality greater than 2 . Choose $\lambda \in K$, and regard $\sum_{\kappa \in K} B_{\kappa}$ as the coproduct of $B_{\lambda}$ and $\sum_{\kappa \neq \lambda} B_{\kappa}$ with injections $i$ and $j$. By assumption, $f: A \rightarrow B_{\lambda}+\left(\sum_{\kappa \neq \lambda} B_{\kappa}\right)$ has a decision $h_{\lambda}: A \rightarrow A+A$. Suppose by way of inductive hypothesis that $j^{*} f: A \rightarrow \sum_{\kappa \neq \lambda} B_{\kappa}$ has a decision $h^{\prime}: A \rightarrow \sum_{\kappa \neq \lambda} A$. We shall show that

$$
A \xrightarrow{h_{\lambda}} A+A \xrightarrow{1+h^{\prime}} A+\sum_{\kappa \neq \lambda} A=\sum_{\kappa \in K} A
$$

is a decision for $f: A \rightarrow \sum_{\kappa \in K} B_{\kappa}$.
Commutativity of

gives one of the conditions in Proposition [2.11] it remains to show that $\nabla\left(1+h^{\prime}\right) h_{\lambda}=\bar{f}$. Commutativity of

reduces this to proving that $\overline{f\left(1+j^{*} f\right) h_{\lambda}}=\bar{f}$.
To do so, first observe that $\overline{\left(i^{*} f+1\right)\left(1+j^{*} f\right) h_{\lambda}}=\overline{\left(i^{*} f+j^{*} f\right) h_{\lambda}}=\overline{\left\langle i^{*} f\right|\left|j^{*} f\right\rangle h_{\lambda}}=\overline{h_{\lambda}}=\bar{f}$ so that $\overline{f\left(1+j^{*} f\right) h_{\lambda}}=f \bar{f} \overline{\left(1+j^{*} f\right) h_{\lambda}}=f \overline{\left(i^{*} f+1\right)\left(1+j^{*} f\right) h_{\lambda}} \overline{\left(1+j^{*} f\right) h_{\lambda}}=\sqrt{\overline{\left(i^{*} f+1\right)\left(1+j^{*} f\right) h_{\lambda}}}=$ $\overline{f\left(i^{*} f+j^{*} f\right) h_{\lambda}}=\bar{f}$ as required.

Finally, we record the following result which will be needed below:
Proposition 2.19 If $f: A \rightarrow B+C$ and $f^{\prime}: A^{\prime} \rightarrow B^{\prime}+C^{\prime}$ have decisions $h$ and $h^{\prime}$ then $(1+\tau+1)\left(f+f^{\prime}\right): A+A^{\prime} \rightarrow\left(B+B^{\prime}\right)+\left(C+C^{\prime}\right)$ has decision $(1+\tau+1)\left(h+h^{\prime}\right)$.

Proof: Let $i: B \rightarrow B+C, j: C \rightarrow B+C, i^{\prime}: B^{\prime} \rightarrow B^{\prime}+C^{\prime}$, and $j^{\prime}: C^{\prime} \rightarrow B^{\prime}+C^{\prime}$ be the various injections. Then the injection $k: B+B^{\prime} \rightarrow B+B^{\prime}+C+C^{\prime}$ is given by $(1+\tau+1)\left(i+i^{\prime}\right)$. Similarly, write $l$ for the injection $(1+\tau+1)\left(j+j^{\prime}\right): C+C^{\prime} \rightarrow \underline{B}+\underline{B^{\prime}+C}+\underline{C^{\prime}}$.

First observe that $\overline{(1+\tau+1)\left(h+h^{\prime}\right)}=\overline{h+h^{\prime}}=\bar{h}+\overline{h^{\prime}}=\bar{f}+\overline{f^{\prime}}=\overline{f+f^{\prime}}=\overline{(1+\tau+1)\left(f+f^{\prime}\right)}$. Now $h$ is restriction inverse to $\left\langle\overline{i^{*} f} \mid \overline{j^{*} f}\right\rangle$ and $h^{\prime}$ is restriction inverse to $\left\langle\overline{i^{* *} f^{\prime}} \mid \overline{j^{* *} f^{\prime}}\right\rangle$, thus $h+h^{\prime}$ is restriction inverse to $\left\langle\overline{i^{*} f} \mid \overline{j^{*} f}\right\rangle+\left\langle\overline{i^{\prime *} f^{\prime}} \mid \overline{j^{\prime *} f^{\prime}}\right\rangle$, and $(1+\tau+1)\left(h+h^{\prime}\right)$ is restriction inverse to $\left(\left\langle\overline{i^{*} f} \mid \overline{j^{*} f}\right\rangle+\left\langle\overline{i^{* *} f^{\prime}} \mid \overline{j^{\prime *} f^{\prime}}\right\rangle\right)(1+\tau+1)$. But

$$
\begin{aligned}
\left(\overline{\left\langle i^{*} f\right.}\left|\overline{j^{*} f}\right\rangle+\left\langle\overline{i^{\prime *} f^{\prime}} \mid \overline{j^{\prime *} f^{\prime}}\right\rangle\right)(1+\tau+1) & =(\nabla+\nabla)\left(\overline{i^{*} f}+\overline{j^{*} f}+\overline{i^{* *} f^{\prime}}+\overline{j^{\prime *} f^{\prime}}\right)(1+\tau+1) \\
& =(\nabla+\nabla)(1+\tau+1) \overline{\left(\overline{i^{*} f}+\overline{i^{\prime *} f^{\prime}}+\overline{j^{*} f}+\overline{j^{\prime *} f^{\prime}}\right)} \\
& =\nabla\left(\overline{i^{*} f+i^{\prime *} f^{\prime}}+\overline{j^{*} f+j^{\prime *} f^{\prime}}\right) \\
& =\nabla\left(\overline{\left(i+i^{\prime}\right)^{*}\left(f+f^{\prime}\right)}+\overline{\left(j+j^{\prime}\right)^{*}\left(f+f^{\prime}\right)}\right) \\
& =\left\langle\overline{\left\langle\left(i+i^{\prime}\right)^{*}\left(f+f^{\prime}\right)\right.} \mid \overline{\left(j+j^{\prime}\right)^{*}\left(f+f^{\prime}\right)}\right\rangle
\end{aligned}
$$

so that $(1+\tau+1)\left(h+h^{\prime}\right)$ is the decision of $(1+\tau+1)\left(f+f^{\prime}\right)$ as claimed.

## 3 Extensive restriction categories

### 3.1 Extensivity

In the previous section we saw that a restriction category $\mathbf{X}$ admits a calculus of matrices if it has restriction coproducts, a restriction zero, and decisions. In this section we relate this structure to the question of when $\operatorname{Total}(\mathbf{X})$ and $\operatorname{Total}\left(K_{r}(\mathbf{X})\right)$ are extensive.

Proposition 3.1 If $\mathbf{X}$ is a restriction category with restriction coproducts and a restriction zero, then Total(X) is extensive if and only if, for every total arrow $f: C \rightarrow A+B$, the restriction idempotent $\overline{(1+0) f}$ splits and an $f$-decision exists. If $\mathbf{X}$ has an object 1 which is terminal in Total(X), then it suffices to consider the case $A=B=1$.

Proof: We know that Total( $\mathbf{X}$ ) has coproducts since $\mathbf{X}$ has restriction coproducts, and we know that the coproduct injections in Total $(\mathbf{X})$ are cartesian, since $\mathbf{X}$ has a restriction zero. Thus Total(X) will be extensive if and only if it has pullbacks along coproduct injections, and coproducts are stable.

Suppose that $\overline{(1+0) f}$ splits for every $f: C \rightarrow A+B$, and that an $f$-decision exists. Let $k: C_{A} \rightarrow C$ and $k^{*} C \rightarrow C_{A}$ provide the splitting for $\overline{(1+0) f}$. Let $l: C_{B} \rightarrow C$ and $l^{*}: C \rightarrow C_{B}$ provide the splitting for $\overline{(0+1) f}$, which exists since $\overline{(0+1) f}=\overline{\tau(0+1) f}=\overline{(1+0) \tau f}$. We are to show that $\langle k \mid l\rangle: C_{A}+C_{B}$ is invertible.

The $f$-decision $h: C \rightarrow C+C$ is restriction inverse to $\left\langle k k^{*} \mid l l^{*}\right\rangle$, so that $h\left\langle k k^{*} \mid l l^{*}\right\rangle=k k^{*}+l l^{*}$ and $\left\langle k k^{*} \mid l l^{*}\right\rangle h=\bar{h}=\bar{f}=1$. Thus $h\langle k \mid l\rangle=k+l$ and so $\left(k^{*}+l^{*}\right) h\langle k \mid l\rangle=\left(k^{*}+l^{*}\right)(k+l)=1$, while $\langle k \mid l\rangle\left(k^{*}+l^{*}\right) h=\left\langle k k^{*} \mid l l^{*}\right\rangle h=1$, as required.

Suppose conversely that $\operatorname{Total}(\mathbf{X})$ is extensive. Then any map $C \rightarrow A+B$ has the form $f+g: A^{\prime}+B^{\prime} \rightarrow A+B$, and now $i f=(f+g) i^{\prime}$ so that $f$ is total, and similarly $g$ is total. Also $\overline{(1+0)(f+g)}=\overline{i^{*}(f+g)}=\overline{f i^{\prime *}}=\overline{i^{*}}$ so that $i^{\prime}$ and $i^{\prime *}$ provide a splitting for $\overline{(1+0)(f+g)}$. Finally the identity $A^{\prime}+B^{\prime} \rightarrow A^{\prime}+B^{\prime}$ is easily seen to be a decision for $f+g$.

If 1 is terminal in $\operatorname{Total}(K(\mathbf{X}))$ then it suffices to show stability of the coproduct $1+1$; see 4] or (5].

Corollary 3.2 If $\mathbf{X}$ is a restriction category with restriction coproducts and a restriction zero, then Total $\left(K_{r}(\mathbf{X})\right)$ is extensive if and only every arrow $f: C \rightarrow A+B$ in $\mathbf{X}$ has a decision map. If $\mathbf{X}$ has an object 1 which is terminal in $\operatorname{Total}\left(K_{r}(\mathbf{X})\right)$, then it suffices to consider the case $A=B=1$.

Proof: Since $\mathbf{X}$ has restriction coproducts, so does $K_{r}(\mathbf{X})$, and since $\mathbf{X}$ has a restriction zero, so does $K_{r}(\mathbf{X})$. All restriction idempotents split in $K_{r}(\mathbf{X})$, so by Proposition 3.1, Total $\left(K_{r}(\mathbf{X})\right)$ will be extensive if and only if every total arrow $f:(C, e) \rightarrow\left(A+B, e_{1}+e_{2}\right)$ has a decision. To say that $f$ is total is to say that $\bar{f}=e$.

A decision $h$ for $f:(C, e) \rightarrow\left(A+B, e_{1}+e_{2}\right)$ is an arrow $h: C \rightarrow C+C$ in $\mathbf{X}$ satisfying $(\bar{f}+\bar{f}) h=h=h \bar{f}, \nabla h=\bar{f}$, and $(f+f) h=(i+j) f$; that is, a decision map for $f$ in $\mathbf{X}$.

In light of the proposition, we say that a restriction category $\mathbf{X}$ is extensive if it has restriction coproducts and a restriction zero, and every map $f: C \rightarrow A+B$ has a decision. By the uniqueness of decisions and the characterization of Proposition 2.11 the existence of these decisions can be viewed as a combinator assigning to each $f: C \rightarrow A+B$ a map $\langle f\rangle: C \rightarrow C+C$ satisfying the decision axioms:
[D.1] $\nabla\langle f\rangle=\bar{f} ;$
[D.2] $(f+f)\langle f\rangle=(i+j) f$.
Thus decision structure is equational, and so can be added freely. It would be interesting to have a description of the free extensive restriction category on a restriction category, or the free such on a mere category.

Of course to say that $\mathbf{X}$ is extensive as a restriction category is quite different to saying that is extensive as a mere category. In fact as an extensive restriction category has a zero object it cannot be an extensive category unless it is the trivial category with a single object and a single arrow. The connection between extensive restriction categories and extensive categories is rather (see Corollary (3.2) that if $\mathbf{X}$ is an extensive restriction category then $\operatorname{Total}\left(K_{r}(\mathbf{X})\right)$ is an extensive category.

Example 3.3 If $\mathscr{D}$ is a distributive category, then $\mathscr{D}_{+1}$ is an extensive restriction category, and so $\operatorname{Total}\left(K_{r}\left(\mathscr{D}_{+1}\right)\right)$ is an extensive category. We have already seen that $\mathscr{D}_{+1}$ has restriction coproducts and a restriction zero, thus we may apply Corollary [3.2. If $f: C \rightarrow A+B+1$ is an arrow in $\mathscr{D}_{+1}$ from $C$ to $A+B$, let $h$ be the composite

$$
C \xrightarrow{\langle C, f\rangle} C \times(A+B+1) \xrightarrow{\delta^{-1}} C \times A+C \times B+C \xrightarrow{\pi_{1}+\pi_{1}+!} C+C+1 .
$$

Verification of the commutativity of the diagrams

is a straightforward exercise in distributive categories; here $t: A+B+1+A+B+1+1 \rightarrow$ $A+B+A+B+1$ is the composite of the twist map $A+B+1+A+B+1+1 \rightarrow A+B+A+B+1+1+1$ and $A+B+A+B+$ !. Thus $h$ is the required decision for $f$.

### 3.2 Extensive maps

As well as considering when $\operatorname{Total}(\mathbf{X})$ or $\operatorname{Total}\left(K_{r}(\mathbf{X})\right)$ is extensive, we can look at subcategories which are extensive. To this end, we say that the map $f: A \rightarrow B$ in $\mathbf{X}$ is extensive if for any decision $h: B \rightarrow B+B$ there is an $h f$-decision $k: A \rightarrow A+A$.

## Lemma 3.4

(i) Restriction isomorphisms are extensive;
(ii) Restriction idempotents are extensive;
(iii) Decision maps are extensive;
(iv) Coproduct injections and codiagonals are extensive.

Proof: First we show that restriction isomorphisms are extensive. If $f: A \rightarrow B$ has restriction inverse $g: B \rightarrow A$, and $h: B \rightarrow B+B$ is a decision, then $(g+g) h f$ is a decision and $\nabla(g+g) h f=$ $\overline{(g+g) h f}=\overline{h f}$ by Corollary 2.16. Thus

$$
\begin{aligned}
(h f+h f)(g+g) h f=(h f g+h f g) h f=(h \bar{g}+h \bar{g}) h f & =(h+h) \overline{(g+g)} h f \\
=(h+h) h f \overline{(g+g) h f} & =(h+h) h f \overline{h f}=(h+h) h f=(i+j) h f
\end{aligned}
$$

and so $(g+g) h f$ is an $h f$-decision, and $f$ is extensive.
Every restriction idempotent is restriction inverse to itself, and is therefore extensive. Similarly, decisions and coproduct injections are restriction isomorphisms and therefore extensive. As for the codiagonal, if $h: A \rightarrow A+A$ is a decision, consider the composite

$$
A+A \xrightarrow{h+h} A+A+A+A \xrightarrow{1+\tau+1} A+A+A+A .
$$

On the one hand we have $\nabla(1+\tau+1)(h+h)=(\nabla+\nabla)(h+h)=\bar{h}+\bar{h}=\overline{h+h}=\overline{\nabla(h+h)}=\overline{h \nabla}$, and on the other, $(h \nabla+h \nabla)(1+\tau+1)(h+h)=(h+h) \nabla(h+h)=(h+h) h \nabla=(i+j) h \nabla$; thus $(1+\tau+1)(h+h)$ is an $h \nabla$-decision.

Proposition 3.5 Let $\mathbf{X}$ be a restriction category with restriction coproducts and a restriction zero. Then the extensive maps in $\mathbf{X}$ form a restriction subcategory $\operatorname{Ex}(\mathbf{X})$ of $\mathbf{X}$ which is closed under finite coproducts, contains the decisions; and $\operatorname{Total}\left(K_{r}(\mathrm{Ex}(\mathbf{X}))\right)$ is extensive. Furthermore, $\mathrm{Ex}(\mathbf{X})$ is maximal among restriction subcategories of $\mathbf{X}$ with these properties.

Proof: By Lemma 3.4 we know that $\operatorname{Ex}(\mathbf{X})$ contains the identities, the restriction idempotents, the coproduct injections and the codiagonals. Thus it will be a restriction subcategory provided that it is closed under composition, and it will be closed under finite coproducts provided that the extensive maps are so. By Lemma 3.4 once again, we know that $\mathrm{Ex}(\mathbf{X})$ contains the decisions; while the fact that $\operatorname{Total}\left(K_{r}(\operatorname{Ex}(\mathbf{X}))\right.$ ) is extensive and the maximality of $\operatorname{Ex}(\mathbf{X})$ will follow from Corollary 3.2. Thus we need only show that the extensive maps are closed under composition and coproducts.

If $f: A \rightarrow B$ and $g: B \rightarrow C$ are extensive, and $h: C \rightarrow C+C$ is a decision, let $k: B \rightarrow B+B$ be an $h g$-decision, and let $l: A \rightarrow A+A$ be a $k f$-decision. Then $\nabla l=\overline{k f}=\overline{\nabla k f}=\overline{\overline{h g} f}=\overline{h g f}$
and

$$
\begin{aligned}
(h g f+h g f) l & =(h g+h g)(f+f) l \\
& =(h g+h g)(\overline{h g}+\overline{h g})(f+f) l \\
& =(h g+h g)(\nabla k+\nabla k))(f+f) l \\
& =(h g \nabla+h g \nabla)(k f+k f) l \\
& =(h g \nabla+h g \nabla)(i+j) k f \\
& =(h g+h g) k f \\
& =(i+j) h g f
\end{aligned}
$$

so that $l$ is also an $h g f$-decision.
Finally, let $f: A \rightarrow B$ and $f^{\prime}: A^{\prime} \rightarrow B^{\prime}$ be extensive, and let $h: B+B^{\prime} \rightarrow B+B^{\prime}+B+B^{\prime}$ be a decision. By Corollary 2.17 $h$ can be written as $(1+\tau+1)\left(k+k^{\prime}\right)$, where $k: B \rightarrow B+B$ and $k^{\prime}: B^{\prime} \rightarrow B^{\prime}+B^{\prime}$ are decisions. Since $f$ and $f^{\prime}$ are extensive, $k f$ and $k^{\prime} f^{\prime}$ have decisions, and so, by Proposition 2.19, $(1+\tau+1)\left(k f+k^{\prime} f^{\prime}\right)$ has a decision; but $(1+\tau+1)\left(k f+k^{\prime} f^{\prime}\right)=h\left(f+f^{\prime}\right)$, and so we have proved that $f+f^{\prime}$ is extensive.

Clearly Total $\left(K_{r}(\mathbf{X})\right)$ is extensive if and only if $\operatorname{Ex}(\mathbf{X})=\mathbf{X}$; that is, if every map is extensive. Note, however, that the construction $\operatorname{Ex}(\mathbf{X})$ is not functorial in $\mathbf{X}$.

## 4 Limits in restriction categories

We saw in Section 2 that cocartesian objects in rCat give a good notion of restriction category with coproducts. We now turn to products, and the first thing to observe is that cartesian objects in $r$ Cat are not a good notion.

If $\mathbf{X}$ is a restriction category, and the unique restriction functor !: $\mathbf{X} \rightarrow \mathbf{1}$ has a right adjoint in rCat , then $\mathbf{X}$ has a terminal object 1 , and for each object $A$, the unique map $t_{A}: A \rightarrow 1$ is total. But if $f: A \rightarrow B$ is any map, then $\bar{f}=\overline{\overline{t_{B}} f}=\overline{t_{B} f}=\overline{t_{A}}=1$, and so $f$ is total. Thus $!: \mathbf{X} \rightarrow \mathbf{1}$ can have a right adjoint in $r$ Cat only if the restriction structure on $\mathbf{X}$ is trivial.

The situation for binary products is much the same. Suppose that $\Delta: \mathbf{X} \rightarrow \mathbf{X} \times \mathbf{X}$ has a right adjoint in rCat . Explicitly, this means that $\mathbf{X}$ has binary products as a mere category, the diagonal and projections are total, and $\overline{f \times g}=\bar{f} \times \bar{g}$ for any maps $f$ and $g$. Let $f: A \rightarrow B$ be any map, and let $p, q: A \times A \rightarrow A$ be the projections. Then $\overline{1_{A} \times f}=\overline{\bar{p}\left(1_{A} \times f\right)}=\overline{p\left(1_{A} \times f\right)}=\bar{p}=1$, and so $1_{A} \times f: A \times A \rightarrow A \times B$ is total; and now $\bar{f}=\bar{f} q \Delta=q\left(1_{A} \times \bar{f}\right) \Delta=q \overline{1_{A} \times f} \Delta=q \Delta=1$, so $f$ is total. Thus once again $\Delta: \mathbf{X} \rightarrow \mathbf{X} \times \mathbf{X}$ can have a right adjoint in $r$ Cat only if the restriction structure on $\mathbf{X}$ is trivial.

We shall now look at other possible notions of products in restriction categories; and, more generally, limits.

### 4.1 Cartesian objects in rCat

One possible approach to the unsatisfactory nature of cartesian objects in $r$ Cat is to change the 2category rCat. In [7] we defined a 2-category rCatl with the same objects and arrows as rCat, namely the restriction categories and restriction functors, but with a larger class of 2-cells. For restriction
functors $F, G: \mathbf{X} \rightarrow \mathbf{Y}$, a 2-cell in rCatl from $F$ to $G$ consists of a total map $\alpha_{X}: F X \rightarrow G X$ in $\mathbf{Y}$ for each object $X$ of $\mathbf{X}$, such that for each $f: X \rightarrow Y$ in $\mathbf{X}$, the diagram

commutes. The reason for the name rCatl is that if one thinks of a restriction category $\mathbf{X}$ as a 2-category (where there is a 2-cell $f \leq g$ if and only if $f=g \bar{f}$ ) and restriction functors as 2 -functors, then a 2 -cell in rCatl is precisely a lax natural transformation from $F$ to $G$ whose components are total.

We now define a restriction terminal object in a restriction category $\mathbf{X}$ to be an object $T$ for which the corresponding restriction functor $\mathbf{1} \rightarrow \mathbf{X}$ is right adjoint in $r$ Catl to the unique functor $\mathbf{X} \rightarrow \mathbf{1}$. In more explicit terms, this amounts to giving, for each object $A$ of $\mathbf{X}$, a total map $t_{A}: A \rightarrow T$, such that $t_{T}=1_{T}$ and for each arrow $f: A \rightarrow B$, we have $t_{B} f=t_{A} \bar{f}$.

Proposition 4.1 A restriction terminal object in $\mathbf{X}$ is terminal in Total( $\mathbf{X}$ ). Conversely, if $\mathbf{X}$ is a classified restriction category, then a terminal object in $\operatorname{Total}(\mathbf{X})$ is restriction terminal in $\mathbf{X}$.

Proof: The first statement follows immediately from the fact that Total : rCatl $\rightarrow$ Cat is a 2 functor, and so preserves adjunctions; alternatively, it is equally easy to verify directly.

The second statement is an instance of [7] Proposition 3.7].
This proposition means in particular that there is no ambiguity in saying " $T$ is a restriction terminal object", since the total maps $t_{A}: A \rightarrow T$ are unique. It also shows that restriction terminal objects are unique up to a unique isomorphism.

Another point of view on restriction terminal objects may be obtained by consideration of the functor RId : $\mathbf{X}^{\mathrm{op}} \rightarrow$ Set, defined in 6. This sends an object $A$ to the set of all restriction idempotents on $A$, and a morphism $f: A \rightarrow B$ to the function sending a restriction idempotent $e: B \rightarrow B$ to $\overline{e f}: A \rightarrow A$.

Proposition 4.2 A restriction terminal object is precisely a representation of the functor RId : $\mathrm{X}^{\mathrm{op}} \rightarrow$ Set.

Proof: If $T$ is a restriction terminal object, then any $f: A \rightarrow T$ determines a restriction idempotent $\bar{f}$ on $A$, while any restriction idempotent $e: A \rightarrow A$ determines a map $t_{A} e: A \rightarrow T$. These processes are inverse, since $\overline{t_{A} e}=\bar{e}=e$, and $t_{A} \bar{f}=t_{T} f=1_{T} f=f$.

On the other hand if $T$ is an object equipped with an isomorphism $\alpha: \mathbf{X}(-, T) \cong \operatorname{RId}$ then for each $A$ there is a unique $t_{A}: A \rightarrow T$ with $\alpha_{A}\left(t_{A}\right)=1_{A}$. For any $f: A \rightarrow B$ we have $\alpha_{A}\left(t_{B} f\right)=$ $\operatorname{RId}(f) \alpha_{B}\left(t_{B}\right)=\operatorname{RId}(f)\left(1_{B}\right)=\bar{f}$ and $\alpha_{A}\left(t_{A} \bar{f}\right)=\operatorname{RId}(\bar{f}) \alpha_{A}\left(t_{A}\right)=\operatorname{RId}(\bar{f})\left(1_{A}\right)=\bar{f} ;$ thus $t_{B} f=$ $t_{A} \bar{f}$, since $\alpha_{A}$ is invertible. It remains to show that $t_{T}=1_{T}$. Let $e$ be the restriction idempotent $\alpha_{T}\left(1_{T}\right)$. Then for any $g: A \rightarrow T$ we have $\alpha_{A}(g)=\alpha_{A}\left(1_{T} g\right)=\operatorname{RId}(g)\left(\alpha_{T}\left(1_{T}\right)\right)=\operatorname{RId}(g)(e)=\overline{e g}$. In particular $\alpha_{T}(e)=\overline{e e}=e=\alpha_{T}\left(1_{T}\right)$, so that $e=1_{T}$; and now $\alpha_{T}\left(t_{T}\right)=1_{T}=e=\alpha_{T}\left(1_{T}\right)$, so that $t_{T}=1_{T}$.

Next we turn to the case of a restriction category $\mathbf{X}$ for which the diagonal restriction functor $\Delta: \mathbf{X} \rightarrow \mathbf{X} \times \mathbf{X}$ has a right adjoint in rCatl . We then say that $\mathbf{X}$ has binary restriction products. Explicitly, this means that there is a restriction functor $\mathbf{X} \times \mathbf{X} \rightarrow \mathbf{X}$ whose value at an object $(A, B)$ we denote $A \times B$ and whose value at an arrow $(f, g)$ we denote $f \times g$; and total maps $\Delta: A \rightarrow A \times A, p: A \times B \rightarrow A$, and $q: A \times B \rightarrow B$ satisfying


Once again, Total : rCatl $\rightarrow$ Cat preserves products and adjunctions, so that Total( $\mathbf{X}$ ) will have binary products whenever $\mathbf{X} \rightarrow \mathbf{X} \times \mathbf{X}$ has a right adjoint in rCatl .

It turns out that if $\Delta: \mathbf{X} \rightarrow \mathbf{X} \times \mathbf{X}$ does have a right adjoint in $r$ Catl, then it automatically satisfies certain further conditions, as the following proposition shows. In particular, the diagonal maps $\Delta: A \rightarrow A \times A$ are not just lax natural, but natural.

Proposition 4.3 If $\mathbf{X}$ is a restriction category, and $\Delta: \mathbf{X} \rightarrow \mathbf{X} \times \mathbf{X}$ has a right adjoint in rCatl , then: $(i) \overline{(f \times g) \Delta}=\bar{f} \bar{g}$ for all $f$ and $g$ with the same domain, and (ii) the maps $\Delta: A \rightarrow A \times A$ are natural in $A$.
Proof: $(i)$ Since $\overline{(f \times g) \Delta}=\overline{\overline{(f \times g)} \Delta}=\overline{(\overline{f \times \bar{g}) \Delta}}$, it will suffice to show that $\overline{\left(e \times e^{\prime}\right) \Delta}=e e^{\prime}$, for all restriction idempotents $e$ and $e^{\prime}$.

First observe that $\overline{\left(e \times e^{\prime}\right) \Delta}=p \Delta \overline{\left(e \times e^{\prime}\right) \Delta}=p \overline{e \times e^{\prime}} \Delta=p\left(e \times e^{\prime}\right) \Delta$, and similarly $\overline{\left(e \times e^{\prime}\right) \Delta}=$ $q\left(e \times e^{\prime}\right) \Delta$. Using lax naturality of $p$, we have $\overline{e\left(e \times e^{\prime}\right) \Delta}=e p\left(e \times e^{\prime}\right) \Delta=p\left(e \times e^{\prime}\right) \Delta=\overline{\left(e \times e^{\prime}\right) \Delta}$, and using lax naturality of $q$, we have $e^{\prime} \overline{\left(e \times e^{\prime}\right) \Delta}=\overline{\left(e \times e^{\prime}\right) \Delta}$. Thus $\overline{\left(e \times e^{\prime}\right) \Delta}=e e^{\prime} \overline{\left(e \times e^{\prime}\right) \Delta}=$ $\overline{\left(e \times e^{\prime}\right) \Delta e e^{\prime}}=\overline{\left(e \times e^{\prime}\right) \Delta e e^{\prime}}=\overline{\left(e e e^{\prime} \times e^{\prime} e e^{\prime}\right) \Delta e e^{\prime}}=\overline{\left(e e^{\prime} \times e e^{\prime}\right) \Delta e e^{\prime}}=\overline{\Delta e e^{\prime}}=\overline{e e^{\prime}}=e e^{\prime}$.
(ii) This follows from (i) and lax naturality of $\Delta$, since $(f \times f) \Delta=(f \times f) \Delta \overline{(f \times f) \Delta}=$ $(f \times f) \Delta \bar{f} \bar{f}=(f \times f) \Delta \bar{f}=\Delta f$.

We say that $\mathbf{X}$ has restriction products if it is a cartesian object in rCatl; that is, if it has binary restriction products and a restriction terminal. If $\mathbf{X}$ and $\mathbf{Y}$ are restriction categories with restriction products, then a restriction functor $F: \mathbf{X} \rightarrow \mathbf{Y}$ is said to preserve restriction products if it commutes with the right adjoints $\mathbf{1} \rightarrow \mathbf{X}$ and $\mathbf{X} \times \mathbf{X} \rightarrow \mathbf{X}$ in $r$ Catl. This definition can be made more explicit. If $T$ and $S$ denote the restriction terminal objects of $\mathbf{X}$ and $\mathbf{Y}$, then there is a unique total map $\varphi: F T \rightarrow S$, and $F$ preserves the restriction terminal object if and only if $\varphi: F T \rightarrow S$ is invertible. Similarly, for any objects $X$ and $Y$ of $\mathbf{X}$ there is a unique total map $\psi_{X, Y}: F(X \times Y) \rightarrow F X \times F Y$ commuting with the projections, and $F$ preserves binary restriction products if and only if each $\psi_{X, Y}$ is invertible. We now have:

Proposition 4.4 If $\mathbf{X}$ is a restriction category with restriction products then $\operatorname{Total}(\mathbf{X})$ and $\operatorname{Total}\left(K_{r}(\mathbf{X})\right)$ have products; if $\mathbf{Y}$ is another such restriction category and $F: \mathbf{X} \rightarrow \mathbf{Y}$ is a restriction functor which preserves restriction products then $\operatorname{Total}(F): \operatorname{Total}(\mathbf{X}) \rightarrow \operatorname{Total}(\mathbf{Y})$ and $\operatorname{Total}\left(K_{r}(F)\right):$ $\operatorname{Total}\left(K_{r}(\mathbf{X})\right) \rightarrow \operatorname{Total}\left(K_{r}(\mathbf{Y})\right)$ preserve products.

## 4.2 p-Categories

Here we recall Robinson and Rosolini's notion of p-category [16, in order to compare it to the various structures considered above.

A p-category is a category $\mathbf{X}$ equipped with a functor $\times: \mathbf{X} \times \mathbf{X} \rightarrow \mathbf{X}$, a natural family of maps $\Delta: A \rightarrow A \times A$, and families $p_{A, B}: A \times B \rightarrow A$ natural in $A$, and $q_{A, B}: A \times B \rightarrow B$ natural in $B$, required to make commutative the following diagrams:

$\quad X \times(Y \times Z) \xrightarrow{\Delta}(X \times(Y \times Z)) \times(X \times(Y \times Z)) \xrightarrow{(1 \times p) \times q}(X \times Y) \times(Y \times Z) \xrightarrow{1 \times q}(X \times Y) \times Z$
$f \times(g \times h) \downarrow$
$X^{\prime} \times\left(Y^{\prime} \times Z^{\prime}\right) \xrightarrow{\Delta}\left(X^{\prime} \times\left(Y^{\prime} \times Z^{\prime}\right)\right) \times\left(X^{\prime} \times\left(Y^{\prime} \times Z^{\prime}\right)\right) \xrightarrow{(1 \times p) \times q}\left(X^{\prime} \times Y^{\prime}\right) \times\left(Y^{\prime} \times Z^{\prime}\right) \xrightarrow{1 \times q}\left(X^{\prime} \times Y^{\prime}\right) \times Z^{\prime}$

for all arrows $f, g$, and $h$. The last two diagrams provide a natural associativity isomorphism $\alpha_{X, Y, Z}: X \times(Y \times Z) \rightarrow(X \times Y) \times Z$ and a natural symmetry isomorphism $\tau_{X, Y}: X \times Y \rightarrow Y \times X$.

Given a map $f: X \rightarrow X^{\prime}$, Robinson and Rosolini define $\operatorname{dom} f: X \rightarrow X$ to be

$$
X \xrightarrow{\Delta} X \times X \xrightarrow{1 \times f} X \times X^{\prime} \xrightarrow{p} X
$$

and their Proposition 1.4 verifies that this makes $\mathbf{X}$ into a restriction category. As Robinson and Rosolini observe (in slightly different terminology), although a p-category structure on a category may not be unique, a p-category structure on a restriction category is. Thus it makes sense to ask which restriction categories are p-categories.

Proposition 4.5 A restriction category is a p-category if and only if it has binary restriction products.

Proof: First suppose that $\mathbf{X}$ is a p-category. It is proved in [16, Proposition 1.4] that $\times: \mathbf{X} \times \mathbf{X} \rightarrow$ $\mathbf{X}$ is a restriction functor, and that each instance of $p, q$, and $\Delta$ is total. The diagonal is natural by assumption, and the "triangle equations" linking $\Delta$ with $p$ and $q$ hold by assumption. Thus it remains only to check that the projections $p$ and $q$ are lax natural. In the case of $p$, lax naturality amounts to the equation $p(f \times g)=f p(\bar{f} \times \bar{g})$ for all arrows $f$ and $g$. Consider first the special case where $f$ is the identity. In the diagram

the left square commutes by naturality of $\Delta$, the triangle by functoriality of $\times$, the right square by (one-sided) naturality of $p$, and the curved region by one of the triangle equations. Thus the exterior commutes, which is to say that $p(1 \times g)=p(1 \times \bar{g})$. As for the general case, in the diagram

the left and top regions commute by functoriality of $\times$, the bottom region commutes by the special case just considered, and the right region by the one-sided naturality of $p$. Commutativity of the exterior is the desired equation $p(f \times g)=f p(\bar{f} \times \bar{g})$.

Lax naturality of $q$ states that $q(f \times g)=g q(\bar{f} \times \bar{g})$; we leave the verification to the reader.
Now suppose conversely that $\mathbf{X}$ has binary restriction products. We must show that $p: X \times Y \rightarrow$ $X$ is natural in $X$, that $q: X \times Y \rightarrow Y$ is natural in $Y$, and that $\alpha_{X, Y, Z}$ and $\tau_{X, Y}$ are natural in all variables. The equations involving only instances of $p, q$, and $\Delta$ all hold because the binary restriction products are actual products in Total(X).

For naturality of $p$, we use lax naturality of $p$ and naturality of $\Delta$ to see that $p(f \times 1)=$ $f p \overline{f \times 1}=f p \overline{f p} \bar{q}=f p$; the case of $q$ is similar.

As for $\tau$, first observe that $\overline{(g \times f) \tau}=\overline{(g \times f)(q \times p) \Delta}=\overline{(g q \times f p) \Delta}=\overline{g q} \overline{f p}=\overline{g \times f}$. Now $(g \times f) \tau=(g \times f) \tau \overline{(g \times f) \tau}=(g \times f) \tau \overline{g \times f}=\tau(f \times g)$.

The case of $\alpha$ is similar but more complicated. Since

$$
\overline{((f \times g) \times h) \alpha}=\overline{((f \times g) \times h)((1 \times p) \times q q) \Delta}=\overline{((f \times g p) \times h q q) \Delta}=\overline{f \times g p} \overline{h q q}=\overline{f p} \overline{g p q} \overline{h q q}
$$

and

$$
\overline{f \times(g \times h)}=\overline{f p} \overline{(g \times h) q}=\overline{f p} \overline{\overline{g \times h} q}=\overline{f p} \overline{\overline{g p} \overline{h q} q}=\overline{f p q \overline{g p q} \overline{h q q}}=\overline{f p} \overline{g p q} \overline{h q q}
$$

we have $\overline{((f \times g) \times h) \alpha}=\overline{f \times(g \times h)}$, and now we deduce

$$
((f \times g) \times h) \alpha=((f \times g) \times h) \alpha \overline{((f \times g) \times h) \alpha}=((f \times g) \times h) \alpha \overline{f \times(g \times h)}=\alpha(f \times(g \times h)) .
$$

If $\mathbf{X}$ is a p-category, Robinson and Rosolini define a one-element object to be an object $T$ with a family $t_{X}: X \rightarrow T$ of maps in $\mathbf{X}$ for which $p: X \times T \rightarrow X$ is invertible, with inverse

$$
X \xrightarrow{\Delta} X \times X \xrightarrow{1 \times t_{X}} X \times T .
$$

Proposition 4.6 If $\mathbf{X}$ is a p-category, an object $T$ of $\mathbf{X}$ is a one-element object if and only if it is a restriction terminal object; the map $t_{X}: X \rightarrow T$ in the definition of one-element object is the unique total map from $X$ to $T$.

Proof: To say that

$$
X \xrightarrow{\Delta} X \times X \xrightarrow{1 \times t_{X}} X \times T \xrightarrow{p} X
$$

is the identity is precisely to say that $t_{X}$ is total. The fact that the $t_{X}$ are lax natural and $t_{T}=1_{T}$ for a one-element object $T$ is part of [16, Theorem 3.3].

Conversely, if $T$ is a restriction terminal object, then we have a family $t_{X}: X \rightarrow T$ of total maps; it remains to show that

$$
X \times T \xrightarrow{p} X \xrightarrow{\Delta} X \times X \xrightarrow{1 \times t_{X}} X \times T
$$

is the identity. But this follows from the fact that restriction products in $\mathbf{X}$ are genuine products in $\operatorname{Total}(\mathbf{X})$.

The relationship between p-categories with one-element object and various other structures is analyzed in some detail in [16]. Translating this into our nomenclature, the restriction categories with restriction products are exactly the partial cartesian categories in the sense of Curien and Obtulowicz [10, and or alternatively the pre-dht-symmetric categories of Hoehnke [13], and they are a special case of the bicategories of partial maps of Carboni [2]. For more details on these correspondences, see [16].

### 4.3 Categories with products and a restriction

Before leaving our discussion of products in restriction categories, it is worth discussing a quite different type of product that sometimes exists. While restriction products are tensor products which are actual products in the total map category, it is also possible that a restriction category could have products in the ordinary sense. A well-known example of this is provided by the category of sets and partial maps. The restriction product of $A$ and $B$ is just their product $A \times B$ as sets; this is not the categorical product in the partial map category, which is given by $A+A \times B+B$. (This is more generally true for the partial map category of any lextensive category [4, 5, where the $\mathscr{M}$-maps are taken to be the coproduct injections.)

Recall that the restriction idempotents associated with a particular object $A$ in a restriction category $\mathbf{X}$ form a meet semi-lattice with $e_{1} \wedge e_{2}=e_{1} e_{2}$, and greatest element $\top=1_{A}$. These lattices sit over each object to give the restriction fibration [6] Section 4] $\partial: \mathscr{R}(\mathbf{X}) \rightarrow \mathbf{X}$ over all the maps: the substitutions preserve the meet but not the greatest element. When this fibration is restricted to the total maps one obtains a meet-semilattice fibration.

If a category has both a restriction structure and a terminal object 1 (and here we emphasize we do not assume any relation between the two structures) then we may consider the restriction $\overline{!_{A}}: A \rightarrow A$ of the unique map $!_{A}: A \rightarrow 1$. Then for any $f: A \rightarrow B$ we have $\overline{!_{A}} \bar{f}=\overline{!_{A} \bar{f}}=\overline{!_{A}}$ and thus $\overline{!}_{A}$ must be the least restriction idempotent in the above ordering: in terms of partial maps, this determines the smallest possible domain. Thus the presence of a terminal object forces each object $A$ to have a least element in its lattice of restriction idempotents. Furthermore, it is clear that the substitution functors of the fibration mentioned above preserve these least elements.

When a category has both a restriction structure and finite products then $\overline{\left\langle e, e^{\prime}\right\rangle}$ is the join of $e$ and $e^{\prime}$ in the lattice of restriction idempotents of an object. To see this, first observe that

$$
e \overline{\left\langle e, e^{\prime}\right\rangle e}=p\left\langle e, e^{\prime}\right\rangle \overline{\left\langle e, e^{\prime}\right\rangle}=p\left\langle e, e^{\prime}\right\rangle=e
$$

so that $e \leq \overline{\left\langle e, e^{\prime}\right\rangle}$, and similarly $e^{\prime} \leq \overline{\left\langle e, e^{\prime}\right\rangle}$. Now if $d$ is a restriction idempotent and $e, e^{\prime} \leq d$, then

$$
\overline{\left\langle e, e^{\prime}\right\rangle} d=\overline{\left\langle e, e^{\prime}\right\rangle d}=\overline{\left\langle e d, e^{\prime} d\right\rangle}=\overline{\left\langle e, e^{\prime}\right\rangle}
$$

and so $\overline{\left\langle e, e^{\prime}\right\rangle} \leq d$.
This proves that the semilattices of restriction idempotents are lattices. In fact they are distributive lattices, since

$$
e \wedge\left(e_{1} \vee e_{2}\right)=\overline{\left\langle e_{1}, e_{2}\right\rangle} e=\overline{\left\langle e_{1}, e_{2}\right\rangle e}=\overline{\left\langle e_{1} e, e_{2} e\right\rangle}=e_{1} e \vee e_{2} e=\left(e \wedge e_{1}\right) \vee\left(e \wedge e_{2}\right)
$$

and

$$
e \wedge \perp=\overline{!_{A}} e=\overline{!_{A} e}=\overline{!_{A}}=\perp
$$

Proposition 4.7 If $\mathbf{X}$ is a category with a restriction structure and (finite) products then the fibration of restriction idempotents

$$
\partial: \mathscr{R}(\mathbf{X}) \rightarrow \mathbf{X}
$$

is a fibred join-semilattice and the fibration

$$
\partial_{t}: \mathscr{R}_{t}(\mathbf{X}) \rightarrow \operatorname{Total}(\mathbf{X})
$$

is a fibred distributive lattice.
Proof: It remains only to check that the inverse image functors preserve the relevant structure. In [6, Section 4.1] it was proved that binary meets are always preserved, while the top element is preserved by the total maps. Thus it will suffice to show that for an arbitrary map $f: X \rightarrow Y$, the induced functor $\operatorname{RId}(f): \operatorname{RId}(Y) \rightarrow \operatorname{RId}(X)$ preserves finite joins. For the bottom element we have $\underline{\operatorname{RId}(f)\left(\overline{!_{Y}}\right)}=\overline{\overline{!_{Y}} f}=\overline{!_{Y} f}=\overline{!_{X}}$; for binary joins we have: $\operatorname{RId}(f)\left(e \vee e^{\prime}\right)=\operatorname{RId}(f) \overline{\left\langle e, e^{\prime}\right\rangle}=\overline{\left\langle e, e^{\prime}\right\rangle f}=$ $\overline{\left\langle e f, e^{\prime} f\right\rangle}=e f \vee e^{\prime} f=\operatorname{RId}(f)(e) \vee \operatorname{RId}(f)\left(e^{\prime}\right)$.

In a split restriction category with products this means that the $M$-subobjects in the total category must already have finite joins which are preserved by pulling back. Thus products in the restriction category lead to colimits in the lattices of $\mathscr{M}$-subobjects.

### 4.4 Restriction limits

We have already discussed products and coproducts in restriction categories. Now we turn briefly to more general notions of limit. Once again, these will be analyzed in terms of adjunctions in rCatl.

Let $\mathbf{X}$ be a restriction category and $\mathscr{C}$ a finite category. We shall define the restriction limit of a functor $S: \mathscr{C} \rightarrow \mathbf{X}$ to be a cone $p_{C}: L \rightarrow S C$ over $S$ with total components, satisfying the following universal property. If $q_{C}: M \rightarrow S C$ is a lax cone over $S$ - that is, $S c . q_{C}=q_{D} \overline{S c . q_{C}}$ for any $c: C \rightarrow D$ - then there is a unique arrow $f: M \rightarrow L$ satisfying $p_{C} f=q_{C} e$, where $e$ is the composite of the restriction idempotents $\overline{q_{C}}$.

It follows immediately from the definition that restriction limits are unique up to unique isomorphism. Equally immediate is the fact that if $S$ takes its values in Total(X), then a restriction limit of $\mathbf{X}$ is a genuine limit in $\operatorname{Total}(\mathbf{X})$.

Example 4.8 The restriction limit of the empty diagram is precisely a restriction terminal object. The restriction limit of a diagram on the discrete category with two objects is the restriction product of the corresponding objects.

The following proposition provides a new example:
Proposition 4.9 The restriction limit of an arrow $f$ is precisely a splitting for the idempotent $\bar{f}$.
Proof: A restriction limit of $f$ amounts to a monomorphism $p: P \rightarrow X$ for which $f p$ is total, having the property that for any arrows $q: Q \rightarrow X$ and $q^{\prime}: Q \rightarrow Y$ satisfying $q^{\prime}=\underline{f q \overline{q^{\prime}}}$, there is a unique $r: Q \rightarrow P$ satisfying $p r=q e$ and $f p r=q^{\prime} e$, where $e=\bar{q} \overline{q^{\prime}}$. In fact $\overline{q q^{\prime}}=\bar{q} f q q^{\prime}=\bar{q} \overline{f q q^{\prime}}=$ $\overline{f q q^{\prime}}=\overline{q^{\prime}}$, and $p r=q e$ implies $f p r=f q e=f q \overline{q^{\prime}}=q^{\prime}=q^{\prime} \overline{q^{\prime}}=q^{\prime} e$, and so the only condition on $r$ is that $p r=q \overline{q^{\prime}}$.

Taking $q=1_{X}$ and $q^{\prime}=f$, we obtain a unique $s: X \rightarrow P$ satisfying $p s=\bar{f}$. Taking $q=p$ and $q^{\prime}=f p$, we obtain a unique $t: P \rightarrow P$ satisfying $p t=p \overline{f p}=p$. Since $p s p=\bar{f} p=p \overline{f p}=p$, we deduce by uniqueness of $t$ that $s p=1$. Thus $p$ and $s$ provide a splitting for $\bar{f}$.

On the other hand, if $p: P \rightarrow X$ and $s: X \rightarrow P$ split $\bar{f}$, while $q$ and $q^{\prime}$ satisfy $q^{\prime}=f q \overline{q^{\prime}}$, then $p s q \overline{q^{\prime}}=\bar{f} q \overline{q^{\prime}}=q \overline{f q q^{\prime}}=q \overline{f q \overline{q^{\prime}}}=q \overline{q^{\prime}}$, and $s q \overline{q^{\prime}}$ is unique with this property, since $p$ is monic. Thus $p: P \rightarrow X$ exhibits $P$ as the restriction limit of $f$.

We now, as promised, analyze these restriction limits in terms of adjunctions in rCatl. We continue to suppose that $\mathbf{X}$ is a restriction category, and now allow $\mathscr{C}$ to be an arbitrary category, not necessarily finite. We shall define a restriction category $\mathbf{X}^{\mathscr{C}}$ and a restriction functor $\Delta: \mathbf{X} \rightarrow$ $\mathbf{X}^{\mathscr{C}}$, and show that if $\mathscr{C}$ is finite then this $\Delta$ has a right adjoint if and only if $\mathbf{X}$ has restriction limits of functors with domain $\mathscr{C}$.

As a category, $\mathbf{X}^{\mathscr{C}}$ consists of functors from $\mathscr{C}$ to $\mathbf{X}$ and lax natural transformations between them. More explicitly, given functors $F, G: \mathscr{C} \rightarrow \mathbf{X}$, an arrow $\alpha: F \rightarrow G$ in $\mathbf{X}^{\mathscr{C}}$ consists of an arrow $\alpha_{A}: F A \rightarrow G A$ in $\mathbf{X}$ for each object $A$ of $\mathscr{C}$, such that $\alpha_{B} . F f=G f . \alpha_{A} \overline{\alpha_{B} . F f}$ for every arrow $f: A \rightarrow B$ in $\mathscr{C}$. Composition is defined pointwise: $(\beta \alpha)_{A}=\beta_{A} \alpha_{A}$. The restriction structure is also defined pointwise: $\bar{\alpha}: F \rightarrow F$ has $\bar{\alpha}_{A}=\overline{\alpha_{A}}$. The only thing to check is that $\bar{\alpha}$ is an fact in arrow of the category. To do this, note that $\overline{\alpha_{B} \cdot F f}=\overline{G f \cdot \alpha_{A} \cdot \overline{\alpha_{B} \cdot F f}=\overline{G f \cdot \alpha_{A}} \cdot \overline{\alpha_{B} \cdot F f}=}$ $\overline{\alpha_{A}} \cdot \overline{G f \cdot \alpha_{A}} \cdot \overline{\alpha_{B} \cdot F f}=\overline{\alpha_{A}} \cdot \overline{\alpha_{B}} \cdot F f$, and now $\overline{\alpha_{B}} \cdot F f=F f \cdot \overline{\alpha_{B} \cdot F f}=F f \cdot \overline{\alpha_{A}} \cdot \overline{\alpha_{B}} \cdot F f=F f \cdot \overline{\alpha_{A}} \cdot \overline{\overline{\alpha_{B}}} \cdot F f$.

The restriction functor $\Delta: \mathbf{X} \rightarrow \mathbf{X}^{\mathscr{C}}$ sends an object $X$ of $\mathbf{X}$ to the functor $\mathscr{C} \rightarrow \mathbf{X}$ constant at $X$, and sends an arrow $f: X \rightarrow Y$ to the family of arrows $X \rightarrow Y$, each of which is just $f$.

Proposition 4.10 If $\Delta: \mathbf{X} \rightarrow \mathbf{X}^{\mathscr{C}}$ has a right adjoint in rCatl , then $\operatorname{Total}(\mathbf{X})$ has ordinary $\mathscr{C}$ limits.

Proof: Applying the 2-functor Total : rCatl $\rightarrow$ Cat to the adjunction gives an adjunction

of categories. The functor $\operatorname{Total}(\Delta)$ lands in the full subcategory Total $(\mathbf{X})^{\mathscr{C}}$ of Total $\left(\mathbf{X}^{\mathscr{C}}\right)$ consisting of the functors $\mathscr{C} \rightarrow \mathbf{X}$ landing in $\operatorname{Total}(\mathbf{X})$. It follows that $\Delta: \operatorname{Total}(\mathbf{X}) \rightarrow \operatorname{Total}(\mathbf{X})^{\mathscr{C}}$ has a right adjoint, and so that $\operatorname{Total}(\mathbf{X})$ has $\mathscr{C}$-limits.

In order to compare the two approaches to restriction limits, we assume that the category $\mathscr{C}$ is finite:

Proposition 4.11 If $\mathscr{C}$ is a finite category and $\mathbf{X}$ a restriction category, then to give a right adjoint in rCatl to $\Delta: \mathbf{X} \rightarrow \mathbf{X}^{\mathscr{C}}$ is precisely to give a restriction limit in $\mathbf{X}$ of each functor $S: \mathscr{C} \rightarrow \mathbf{X}$.

Proof: Let $R: \mathbf{X}^{\mathscr{C}} \rightarrow \mathbf{X}$ be right adjoint in rCatl to $\Delta$. If $S: \mathscr{C} \rightarrow \mathbf{X}$ is an object of $\mathbf{X}^{\mathscr{C}}$, let $L=R(S)$, and let $p: L \rightarrow S$ be the component at $S$ of the counit. Then $p$ is a lax cone, and its components $p_{C}: L \rightarrow S C$ are total. But to say that $p$ is a lax cone is to say, for each $f: C \rightarrow D$, that $p_{D}=S f . p_{C} \overline{p_{D}}$, and since $p_{D}$ is total, this means that $p$ is in fact a cone.

We now show that the cone $p: L \rightarrow S$ is a restriction limit cone. If $q: M \rightarrow S$ is a lax cone; that is, an arrow $\Delta M \rightarrow S$ in $\mathbf{X}^{\mathscr{C}}$, then let $f: M \rightarrow R(S)=L$ be given by $R(q): R \Delta M \rightarrow R(S)$ composed with the unit $n: M \rightarrow R \Delta M$. In the diagram

the left square commutes by a restriction category axiom, the right rectangle by lax naturality of $p$, and the curved region by one of the triangle equations. Commutativity of the exterior amounts to the equation $p_{C} f=q_{C} \cdot \overline{R q \cdot n}=q_{C} \bar{f}$.

We shall now show that $\bar{f}$ is the composite of the $\overline{q_{C}}$, which we henceforth denote $e$. For each $\frac{C \text { we have } \bar{f}=\overline{p_{C} f}}{\underline{R_{C}} \overline{q_{C}}}=\overline{q_{C}} \bar{f}$, and so $\bar{f}=e \bar{f}$. On the other hand $\overline{q_{C}} e=e$, , so $\bar{q} \cdot \Delta e=\Delta e$, and $\overline{R q \cdot R \Delta e}=\overline{R(\bar{q} \cdot \Delta e)}=\overline{R \Delta e}$; thus $e \bar{f}=e \overline{R q \cdot n}=e \overline{R q \cdot n . e}=e \overline{R q \cdot R \Delta e . n . e}=e \overline{R \Delta e . n . e}=e . \overline{n . e}=e$. This proves that $\bar{f}$ is the composite of the $\overline{q_{C}}$, and so that $f$ provides the desired factorization.

Finally we must prove that the factorization $f$ is unique. To do this, we shall show that an arrow $f: M \rightarrow R S$ is determined by the $p_{C} f$ and by $\bar{f}$; then if $p_{C} f=p_{C} f^{\prime}$, we also have
$\bar{f}=\overline{p_{C} f}=\overline{p_{C} f^{\prime}}=\overline{f^{\prime}}$. Now in

the rectangle commutes by lax naturality of $n$, and the triangle by one of the triangle equations; commutativity of the exterior confirms that $f$ is determined by $p . \Delta f$ and $\bar{f}$, that is, by the $p_{C} f$ and by $\bar{f}$. This completes the construction of restriction $\mathscr{C}$-limits in $\mathbf{X}$.

Suppose conversely that $\mathbf{X}$ has restriction $\mathscr{C}$-limits. For each $S: \mathscr{C} \rightarrow \mathbf{X}$, define $R(S)$ to be the restriction limit of $S$, and define the component at $S$ of the counit $\Delta R \rightarrow 1$ to be the restriction limit cone $p: \Delta R(S) \rightarrow S$. If $\sigma: S \rightarrow T$ is an arrow in $\mathbf{X}^{\mathscr{C}}$, then for each object $C$ of $\mathscr{C}$, let $q_{C}=\sigma_{C} p_{C}: R(S) \rightarrow T C$. If $f: C \rightarrow D$ is an arrow of $\mathscr{C}$, in the following diagram

the left square commutes by one of the restriction category axioms, the right rectangle by lax naturality of $\sigma$, and the curved region by naturality of $p$. Finally $\overline{\sigma_{C} \cdot S f \cdot p_{C}}=\overline{\sigma_{D} \cdot p_{D}}$ by naturality of $p$ once again, and so $T f \cdot \sigma_{C} \cdot p_{C} \cdot \overline{\sigma_{D} \cdot p_{D}}=\sigma_{D} \cdot p_{D}$, that is, $T f \cdot q_{C} \cdot \overline{q_{D}}=q_{D}$; and so the $q$ form a lax cone. We now define $R(\sigma): R(S) \rightarrow R(T)$ to be the unique arrow for which $\overline{R(\sigma)}$ is the composite of the restriction idempotents $\overline{\sigma_{C} \cdot p_{C}}$, and the diagram

commutes.
The unit $n: M \rightarrow R \Delta M$ is defined to be the unique arrow satisfying $p_{C} n=1$, for each leg $p_{C}: R \Delta M \rightarrow M$ of the restriction limit cone of $\Delta M$.

We leave to the reader the various straightforward verifications: that $R$ is a restriction functor, that the unit and counit are lax natural, and that the triangle equations hold.

Proposition 4.12 A restriction category $\mathbf{X}$ has all (finite) restriction limits if and only if $\mathbf{X}$ is split as a restriction category and $\operatorname{Total}(\mathbf{X})$ has finite limits.

Proof: We have already seen that $\operatorname{Total}(\mathbf{X})$ has $\mathscr{C}$-limits if $\mathbf{X}$ has restriction $\mathscr{C}$-limits; and that restriction idempotents split in $\mathbf{X}$ if $\mathbf{X}$ has restriction $\mathcal{L}$-limits. Thus it remains to show that if $\mathbf{X}$ is a split restriction category and $\operatorname{Total}(\mathbf{X})$ has finite limits, then $\mathbf{X}$ has restriction limits.

Let $S: \mathscr{C} \rightarrow \mathbf{X}$ be given. Define a new functor $S^{\prime}: \mathscr{C} \rightarrow \mathbf{X}$ as follows. For an object $C$ of $\mathscr{C}$ let $e_{C}$ be the composite of all the restriction idempotents $\overline{S f}$ where $f: C \rightarrow D$ is an arrow in $\mathscr{C}$ with domain $C$. Let $i_{C}: S^{\prime} C \rightarrow S C$ and $r_{C}: S C \rightarrow S^{\prime} C$ be the splitting of $e_{C}$. Given an arrow $f: C \rightarrow D$, we have $e_{D} . S f . e_{C}=S f . e_{C}$, and so $S f$ restricts to an arrow $S^{\prime} f: S^{\prime} C \rightarrow S^{\prime} D$ satisfying $i_{D} \cdot S^{\prime} f=S f . i_{C}$; and this defines a functor $S^{\prime}: \mathscr{C} \rightarrow \mathbf{X}$ with a natural transformation $i: S^{\prime} \rightarrow S$. Since $S^{\prime}$ lands in Total(X), we may form its limit $p_{C}: L \rightarrow S^{\prime} C$ in Total(X), and now $i_{C} p_{C}: L \rightarrow S C$ give a cone over $S$ with total components; we shall show that it is a restriction limit cone.

Let $q_{C}: M \rightarrow S C$ be the components of a lax cone over $S$, and write $d: M \rightarrow M$ for the composite of the restriction idempotents $\overline{q_{C}}$. We must show that there is a unique arrow $f: M \rightarrow L$ satisfying $i_{C} . p_{C} . f=q_{C} . d$. Let $i: M^{\prime} \rightarrow M$ and $r: M \rightarrow M^{\prime}$ be a splitting of $d$. Each composite $q_{C} i$ is total, and for an arrow $f: C \rightarrow D$ in $\mathscr{C}$ we have $q_{D} \cdot i=S f \cdot q_{C} \cdot \overline{q_{D}} i=S f \cdot q_{C} i$; thus the $q_{C} i$ form the components of a cone. For each $f: C \rightarrow D$ we have $\overline{S f} \cdot q_{C} \cdot i=q_{C} \cdot i \cdot \overline{S f . q_{C} \cdot i}=q_{C} \cdot i \cdot \overline{q_{D} \cdot i}=q_{C} \cdot i$, and so $i_{C} \cdot r_{C} \cdot q_{C} \cdot i=q_{C} . i$; but this means that $r_{C} \cdot q_{C} . i$ is total, and forms a cone over $S^{\prime}$. Thus by the universal property of the limit $p_{C}: L \rightarrow S^{\prime} C$, there is a unique total map $g: M^{\prime} \rightarrow L$ satisfying $p_{C} \cdot g=r_{C} \cdot q_{C} \cdot i$. Now $i_{C} \cdot p_{C} \cdot g r=i_{C} \cdot r_{C} \cdot q_{C} \cdot i r=q_{C} \cdot i r=q_{C} \cdot d$, so that $g r$ shows the existence of an $f$.

As for the uniqueness, let $f$ is any map satisfying $i_{C} \cdot p_{C} \cdot f=q_{C} \cdot d$; then $\bar{f}=\overline{i_{C} \cdot p_{C} \cdot f}=\overline{q_{C} \cdot d}=$ $\overline{\overline{q_{C}} \cdot d}=\bar{d}=d$. Now $f i$ satisfies $i_{C} \cdot p_{C} \cdot f i=q_{C} \cdot d i=q_{C} \cdot i$, and $\overline{f i}=\overline{f i}=\overline{d i}=\bar{i}=1$; thus by the universal property of the limit $L$ in $\operatorname{Total}(\mathbf{X})$, we have $f i=g$, and now $f=f \bar{f}=f d=f i r=g r$.

Finally we observe that under a further assumption, restriction products and splitting of restriction idempotents suffice to obtain all (finite) limits in the restriction category.

Let $\mathbf{X}$ be a cartesian restriction category. We say that an object $X$ is separable if the diagonal $\Delta: X \rightarrow X \times X$ is a restriction monic; that is, if there is a map $r: X \times X \rightarrow X$ with $r \Delta=1$ and $\Delta r=\bar{r}$. (Recall that such an $r$ is unique if it exists, and is called the restriction retraction of $\Delta$.)

Proposition 4.13 If $\mathbf{X}$ is a split cartesian restriction category in which every object is separable then $\operatorname{Total}(\mathbf{X})$ has all finite limits.

Proof: We already know that Total( $\mathbf{X}$ ) has finite products; it remains to show that it has equalizers. Suppose then that $f, g: X \rightarrow Y$ are given in $\operatorname{Total}(\mathbf{X})$, let $h: X \rightarrow Y \times Y$ be the induced map, and $r: Y \times Y \rightarrow Y$ the restriction retraction of $\Delta: Y \rightarrow Y \times Y$. Now consider the restriction idempotent $\overline{r h}$, and let $i: E \rightarrow X$ and $s: X \rightarrow E$ be its splitting. We shall show that $i$ is the desired equalizer. First of all $i$ has a retraction, so is a monomorphism, and so in turn is total. We must show that $f i=g i$, and that if $j$ is any total map with $f j=g j$ then $j$ factorizes through $i$.

Write $p, q: Y \times Y \rightarrow Y$ for the projections. Observe first that

$$
p \bar{r}=p \Delta r=r=q \Delta r=q \bar{r}
$$

and now

$$
f i=f i s i=f \overline{r h} i=p h \overline{r h} i=p \bar{r} h i=q \bar{r} h i=q h \overline{r h} i=g i s i=g i .
$$

On the other hand, if $j$ is total and $f j=g j$, then $h j=\Delta k$ for a (unique total) map $k$, and so

$$
i s j=\overline{r h} j=j \overline{r h j}=j \overline{r \Delta k}=j \bar{k}=j
$$

and $s j$ gives the required factorization of $j$ through $i$.

## 5 Counital copy categories

We have seen that there are many different ways of describing the structure which we call a restriction category with restriction products, but we shall actually add one more way to this list: the counital copy categories which we introduce below. (The slightly weaker structure of copy category will not be considered in this paper.)

### 5.1 Restriction products revisited

The starting point is that if $\mathbf{X}$ is a restriction category with restriction products, then as a category, $\mathbf{X}$ has a symmetric monoidal structure, with tensor product given by restriction product. The associativity isomorphism is the $\alpha$ appearing in the definition of p-category, while the symmetry is the $\tau$. The unit is the restriction terminal object, and the unit constraint $X \times T \cong X$ is the projection. In light of the coherence results for monoidal categories [15], we shall allow ourselves to omit explicit mention of the associativity isomorphisms, and write as if the tensor product were strictly associative.

As observed by Carboni [2], for each object $X$ the diagonal map $\Delta: X \rightarrow X \times X$ is coassociative and cocommutative, and has a counit given by $t_{X}: X \rightarrow T$. Thus every object has a canonical cocommutative comonoid structure in the symmetric monoidal category. Furthermore, since the $\Delta$ are natural, every morphism $f: X \rightarrow Y$ is a morphism of cosemigroups, although it may not preserve the counit. It will preserve the counit if it is a total map; conversely, if $f$ preserves the counit, that is, if $t_{Y} f=t_{X}$, then $\bar{f}=\overline{t_{Y} f}=\overline{t_{X}}=1$, and so $f$ is total. Thus the total maps are precisely the counit-preserving ones.

There are two further further conditions which necessarily hold in a restriction category with restriction products: the diagonal $\Delta: T \rightarrow T \times T$ must be inverse to the unit isomorphism $r=l: T \times T \rightarrow T$ of the monoidal structure, and the composite

$$
X \times Y \xrightarrow{\Delta \times \Delta} X \times X \times Y \times Y \xrightarrow{1 \times \tau \times 1} X \times Y \times X \times Y
$$

must be $\Delta: X \times Y \rightarrow X \times Y \times X \times Y$. Together these conditions say that $\Delta: X \rightarrow X \times X$ is not just a natural transformation, but a monoidal natural transformation; it can also be viewed as an instance of the "middle four interchange" law for bicategories. To see that these conditions must hold in a restriction category with restriction products, it suffices to observe that they are all equations in Total( $\mathbf{X}$ ), where the tensor product is a genuine product, and that such equations always hold in a symmetric monoidal category for which the tensor product is the categorical (cartesian) product. We call a symmetric monoidal category equipped with maps $\Delta: X \rightarrow X \times X$ which are monoidally natural, coassociative, cocommutative, and have counits, a counital copy category. (In [8] we shall have cause to look at a slightly weaker structure, called a copy category, in which the assumption that the cosemigroups have counits is dropped.) It will turn out that a symmetric monoidal category can have at most one counital copy structure, and has such a structure if and only if it arises from a restriction category with restriction products.

If $\mathscr{V}$ is an arbitrary symmetric monoidal category, let $\operatorname{Copy}(\mathscr{V})$ be the category whose objects are the cocommutative comonoids in $\mathscr{V}$, and whose morphisms are the homomorphisms of cosemigroups. Then Copy $(\mathscr{V})$ has a canonical symmetric monoidal structure: the tensor product of cocommutative comonoids $(C, \delta: C \rightarrow C \otimes C, \epsilon: C \rightarrow I)$ and $(D, \delta: D \rightarrow D \otimes D, \epsilon: D \rightarrow I)$ has
underlying $\mathscr{V}$-object $C \otimes D$, with comultiplication and counit given by

$$
\begin{gathered}
C \otimes D \xrightarrow{\delta \otimes \delta} C \otimes C \otimes D \otimes D \xrightarrow{1 \otimes \tau \otimes 1} C \otimes D \otimes C \otimes D \\
C \otimes D \xrightarrow{\epsilon \otimes \epsilon} I \otimes I \xrightarrow{r} I .
\end{gathered}
$$

But now the map $\delta: C \rightarrow C \otimes C$ in $\mathscr{V}$ is a map $\Delta:(C, \delta, \epsilon) \rightarrow(C, \delta, \epsilon) \otimes(C, \delta, \epsilon)$ in $\operatorname{Copy}(\mathscr{V})$ which is coassociative, cocommutative, and counital by definition of the objects of $\operatorname{Copy}(\mathscr{V})$, natural by definition of morphisms in $\operatorname{Copy}(\mathscr{V})$, and monoidally natural by definition of the monoidal structure on Copy $(\mathscr{V})$. Thus Copy $(\mathscr{V})$ is a counital copy category. On the other hand, there is an evident forgetful functor $U: \operatorname{Copy}(\mathscr{V}) \rightarrow \mathscr{V}$ which strictly preserves the symmetric monoidal structure, and if $\mathscr{V}$ is a counital copy category, then this $U$ is clearly an equivalence of categories. This proves:

Proposition 5.1 The counital copy categories are precisely the symmetric monoidal categories of the form $\operatorname{Copy}(\mathscr{V})$ for some symmetric monoidal $\mathscr{V}$.

We conclude:
Theorem 5.2 The following structures on a category $\mathbf{X}$ are equivalent:
(i) restriction category with restriction products;
(ii) p-category with a one-element object;
(iii) partial cartesian category in the sense of Curien and Obtulowicz;
(iv) counital copy category;
(v) symmetric monoidal structure with $U$ : $\operatorname{Copy}(\mathbf{X}) \rightarrow \mathbf{X}$ an equivalence;
(vi) symmetric monoidal structure for which there exists some equivalence $\mathbf{X} \simeq \operatorname{Copy}(\mathscr{V})$.

All the structure is determined by either the restriction category structure or the symmetric monoidal structure.

### 5.2 Classified restriction categories and equational lifting categories

In this brief section we revisit the analysis in [7] of classified restriction categories, in particular its connection with the equational lifting monads of [1].

Let $\mathscr{C}$ be a symmetric monoidal category, with tensor product $\otimes$, unit $I$, and symmetry $\tau$. The associativity and unit isomorphisms will be suppressed where possible. A symmetric monoidal monad [14] on $\mathscr{C}$ is a monad $T=(T, \eta, \mu)$ equipped with a natural transformation $\varphi_{A, B}: T A \otimes$ $T B \rightarrow T(A \otimes B)$ satisfying the equations



In fact the structure on $T$ of symmetric monoidal monad can be given either by $\varphi$, or by a natural family of maps $\psi_{A, B}: A \times T B \rightarrow T(A \times B)$ satisfying equations given in [14. One obtains $\psi_{A, B}$ from $\varphi_{A, B}$ by composing with $\eta_{A} \times 1_{T B}$, and one obtains $\varphi_{A, B}$ from $\psi_{A, B}$ as the composite

$$
T A \times T B \xrightarrow{\psi_{T A, B}} T(T A \times B) \xrightarrow{T \tau} T(B \times T A) \xrightarrow{T \psi_{B, A}} T^{2}(B \times A) \xrightarrow{T^{2} \tau} T^{2}(A \times B) \xrightarrow{\mu_{A \times B}} T(A \times B)
$$

When the maps $\psi$ are used rather than $\varphi$, one sometimes speaks of a commutative strong monad rather than a symmetric monoidal monad.

A symmetric monoidal monad $T$ on $\mathscr{C}$ induces a symmetric monoidal structure on the Kleisli category $\mathscr{C}_{T}$. If we regard the objects of $\mathscr{C}_{T}$ as being the objects of $\mathscr{C}$, and arrows in $\mathscr{C}_{T}$ from $A$ to $B$ as being arrows in $\mathscr{C}$ from $A$ to $T B$, then the product of objects $A$ and $A^{\prime}$ is $A \otimes A^{\prime}$, while the product of arrows $f: A \rightarrow T B$ and $f^{\prime}: A^{\prime} \rightarrow T B^{\prime}$ is the composite

$$
A \otimes A^{\prime} \xrightarrow{f \otimes f^{\prime}} T B \otimes T B^{\prime} \xrightarrow{\varphi_{B, B^{\prime}}} T\left(B \otimes B^{\prime}\right)
$$

The left adjoint $I: \mathscr{C} \rightarrow \mathscr{C}_{T}$ strictly preserves the symmetric monoidal structure.
If $\mathscr{C}$ is not just symmetric monoidal, but a counital copy category, then applying $I: \mathscr{C} \rightarrow \mathscr{C}_{T}$ to the cocommutative comonoid structures on objects of $\mathscr{C}$, one obtains a canonical cocommutative structure on each object of $\mathscr{C}_{T}$. If the resulting copy maps $A \rightarrow A \otimes A$ in $\mathscr{C}_{T}$ are natural, then they will certainly be monoidally natural, so that $\mathscr{C}_{T}$ will be a counital copy category. As for the naturality, this amounts to commutativity of the exterior of

for every $f: A \rightarrow T B$ in $\mathscr{C}$. Now the quadrilateral commutes by naturality of the copy maps in $\mathscr{C}$, so the exterior will commute provided that the triangular region does so. We therefore define a symmetric monoidal monad $T$ on a counital copy category $\mathscr{C}$ to be a copy monad if $\varphi_{B, B} \Delta=T \Delta$ for all objects $B$.

Proposition 5.3 If $T$ is a copy monad on a counital copy category $\mathscr{C}$, then $\mathscr{C}_{T}$ is a counital copy category.

Example 5.4 If $\mathscr{D}$ is a distributive category, then the monad +1 on $\mathscr{D}$ is a symmetric monoidal monad, via the maps

$$
(A+1) \times(B+1) \xrightarrow{\delta^{-1}} A \times B+A+B+1 \xrightarrow{A \times B+!} A \times B+1
$$

The fact that +1 is a copy monad amounts to commutativity of the exterior of


It follows that $\mathscr{D}_{+1}$ is a counital copy category, and so that $\operatorname{Total}\left(K_{r}\left(\mathscr{D}_{+1}\right)\right)$ has finite products.
In [1, a symmetric monoidal monad on a category with finite products $\mathscr{C}$ is called an equational lifting monad if $\psi_{A, B}: A \times T B \rightarrow T(A \times B)$ satisfies


We observe that this implies commutativity of

which is to say that $T$ is a copy monad. Thus every equational lifting monad is a copy monad.
Question 5.5 Is there a copy monad on a category with finite products which is not an equational lifting monad?

In [7], we defined the notion of a classifying monad on a category $\mathscr{C}$, and gave various characterizations. To give a monad $T$ the structure of a classifying monad is to give its Kleisli category $\mathscr{C}_{T}$ the structure of a restriction category for which $F_{T}: \mathscr{C} \rightarrow \mathscr{C}_{T}$ takes its values among the total maps, and the components of the counit $\epsilon_{T}: F_{T} U_{T} \rightarrow 1$ are restriction retractions. We now prove:

Proposition 5.6 An equational lifting monad is a classifying monad.
Proof: We have already seen that $\mathscr{C}_{T}$ is a counital copy category, and so in particular a restriction category. The restriction of $f: A \rightarrow T B$ is given by

$$
A \xrightarrow{\langle 1, f\rangle} A \times T B \xrightarrow{\eta_{A} \times 1} T A \times T B \xrightarrow{\varphi_{A, B}} T(A \times B) \xrightarrow{T p} T A .
$$

By [7. Proposition 3.15], $T$ will be a classifying monad if and only if the restriction of $\eta_{B} f: A \rightarrow T B$ is $\eta_{A}$, for each $f: A \rightarrow B$ in $\mathscr{C}$; and the restriction of $1: T A \rightarrow T A$ is $T \eta_{A}:: T A \rightarrow T^{2} A$. The restriction of $\eta_{B} f: A \rightarrow T B$ is given by

$$
A \xrightarrow{\left\langle 1, \eta_{B} f\right\rangle} A \times T B \xrightarrow{\eta_{A} \times 1_{T} B} T A \times T B \xrightarrow{\varphi_{A, B}} T(A \times B) \xrightarrow{T p} T A
$$

and

$$
T(p) \varphi_{A, B}\left(\eta_{A} \times 1_{T B}\right)\left\langle 1, \eta_{B} f\right\rangle=T(p) \varphi_{A, B}\left(\eta_{A} \times \eta_{B}\right)\langle 1, f\rangle=T(p) \eta_{A \times B}\langle 1, f\rangle=\eta_{A} p\langle 1, f\rangle=\eta_{A}
$$

as required. For the latter, the restriction of $1: T A \rightarrow T A$ is

$$
T A \xrightarrow{\Delta} T A \times T A \xrightarrow{\eta_{T A} \times 1} T^{2} A \times T A \xrightarrow{\varphi_{T A, A}} T(T A \times A) \xrightarrow{T p} T^{2} A
$$

and

$$
T(p) \varphi_{T A, A}\left(\eta_{T A} \times 1\right) \Delta=T(p) \psi_{A, A} \Delta=T(p) T\left(\eta_{A} \times 1\right) T(\Delta)=T \eta_{A}
$$

as required.

### 5.3 Distributive copy categories

A counital copy category $\mathbf{X}$ is a restriction category with restriction products; if $\mathbf{X}$ also has restriction coproducts and the canonical maps $\delta: A \times B+A \times C \rightarrow A \times(B+C)$ are invertible for all objects $A, B$, and $C$ then we call $\mathbf{X}$ a distributive copy category.

Proposition 5.7 For a counital copy category $\mathbf{X}$ with restriction coproducts, the following are equivalent:
(i) $\mathbf{X}$ is a distributive copy category;
(ii) Total $(\mathbf{X})$ is a distributive category;
(iii) $K_{r}(\mathbf{X})$ is a distributive copy category;
(iv) $\operatorname{Total}\left(K_{r}(\mathbf{X})\right)$ is a distributive category.

Proof: The equivalence of $(i)$ and (ii) is immediate from the fact that restriction products and restriction coproducts in $\mathbf{X}$ are products and coproducts in Total $(\mathbf{X})$; the equivalence of (iii) and $(i v)$ is a special case of this. The fact that (iii) implies $(i)$ is trivial; it remains only to show that if the canonical map $A \times B+A \times C \rightarrow A \times(B+C)$ is invertible for every object in $\mathbf{X}$, then it is so for every object in $K_{r}(\mathbf{X})$. This follows easily from the fact that the objects of $K_{r}(\mathbf{X})$ are retracts of the objects of $\mathbf{X}$.

Since in a distributive category the unique map $0 \rightarrow A \times 0$ is invertible for any object $A$, the proposition implies that the same is true for a distributive copy category.

Our main result about distributive copy categories is:
Theorem 5.8 If $\mathbf{X}$ is a counital copy category with restriction coproducts, then $\mathbf{X}$ is an extensive restriction category if and only if it is a distributive copy category and has a restriction zero.

Proof: If $\mathbf{X}$ is an extensive restriction category with restriction products then it has a restriction zero by definition of extensivity for restriction categories; and $\operatorname{Total}\left(K_{r}(\mathbf{X})\right)$ is extensive with finite products, thus distributive, so that $\mathbf{X}$ is a distributive copy category by the proposition.

Suppose conversely that $\mathbf{X}$ is a distributive copy category with a restriction zero. We must show that every map $f: C \rightarrow 1+1$ has a decision. Let $h$ be the composite

$$
C \xrightarrow{\Delta} C \times C \xrightarrow{C \times f} C \times(1+1) \xrightarrow{\delta^{-1}} C+C .
$$

Then $\nabla h=p(C \times f) \Delta=p(C \times \bar{f}) \Delta=p \Delta \overline{(C \times \bar{f}) \Delta}=p \Delta \bar{f}=\bar{f}$, giving one condition for $h$ to be an $f$-decision. The second follows from commutativity of:


Thus we have the following examples of distributive copy categories:
Example 5.9 (i) For a distributive category $\mathscr{D}$ we saw in Example 5.4 that $\mathscr{D}_{+1}$ is a counital copy category and in Example 3.3 that it is an extensive restriction category. By the theorem, then, $\mathscr{D}_{+1}$ is a distributive copy category.
(ii) If $\mathscr{V}$ is a symmetric monoidal category then $\operatorname{Copy}(\mathscr{V})$ is a counital copy category. If $\mathscr{V}$ also has coproducts, and the tensor product distributes over the coproducts, then Copy $(\mathscr{V})$ is a distributive copy category.
(iii) The category CRng of commutative rings can of course be regarded as the category of commutative monoids in the monoidal category Ab of abelian groups. We can therefore regard CRng ${ }^{\text {op as }}$ the category of cocommutative comonoids in the monoidal category $A b{ }^{\text {op }}$. Now the tensor product in $A b^{\text {op }}$ distributes over coproducts, and so $\operatorname{Copy}\left(A b^{\circ p}\right)$ is a distributive copy category. An object of $\operatorname{Copy}\left(A b^{\mathrm{op}}\right)$ is a cocommutative comonoid in $A b^{\text {op }}$; that is, a commutative ring. In fact $\operatorname{Copy}\left(\mathrm{Ab}^{\mathrm{op}}\right)$ is just $\mathrm{CRng}_{\times}^{\mathrm{op}}$, where $\mathrm{CRng}_{\times}$is the category whose objects are the commutative rings and whose morphisms are the functions preserving,$+ \times$, and 0 , but not necessarily preserving 1 . Thus $\mathrm{CRng}_{\times}^{\mathrm{op}}$ is a distributive copy category. It is not hard to see that idempotents split in $\mathrm{CRng}_{\times}^{\mathrm{op}}$, and that the category of total maps is just CRng ${ }^{\text {op }}$, and so we recover the well-known fact that $\mathrm{CRng}^{\text {op }}$ is extensive.

### 5.4 The extensive completion of a distributive category

In this section we apply the results obtained above to give a description of the extensive completion of a distributive category. There is a 2-category Dist of distributive categories, functors preserving finite products and coproducts, and natural transformations; and there is a full sub-2-category Ext $_{\text {pr }}$ of Dist consisting of those distributive categories which are also extensive. The inclusion has a left biadjoint, and the value at a distributive category $\mathscr{D}$ of this left biadjoint is what we mean by the extensive completion of the distributive category $\mathscr{D}$. An explicit construction of the extensive completion was given in 9; here we shall give an alternative, more conceptual, description.

Given a distributive category $\mathscr{D}$ we have seen that there is a monad +1 on $\mathscr{D}$ whose Kleisli category $\mathscr{D}_{+1}$ has a restriction structure. We may now split the restriction idempotents in $\mathscr{D}_{+1}$, and then take the total maps in this new restriction category, to give a category $\operatorname{Total}\left(K_{r}\left(\mathscr{D}_{+1}\right)\right)$. The image of the left adjoint $\mathscr{D} \rightarrow \mathscr{D}_{+1}$ lands in $\operatorname{Total}\left(\mathscr{D}_{+1}\right)$, and if we compose the resulting functor $I: \mathscr{D} \rightarrow \operatorname{Total}\left(\mathscr{D}_{+1}\right)$ with the map $\operatorname{Total}(J): \operatorname{Total}\left(\mathscr{D}_{+1}\right) \rightarrow \operatorname{Total}\left(K_{r}\left(\mathscr{D}_{+1}\right)\right)$ induced by the inclusion $J: \mathscr{D}_{+1} \rightarrow K_{r}\left(\mathscr{D}_{+1}\right)$, we obtain a functor $N: \mathscr{D} \rightarrow \operatorname{Total}\left(K_{r}\left(\mathscr{D}_{+1}\right)\right)$. It turns out that $N: \mathscr{D} \rightarrow \operatorname{Total}\left(K_{r}\left(\mathscr{D}_{+1}\right)\right)$ exhibits $\operatorname{Total}\left(K_{r}\left(\mathscr{D}_{+1}\right)\right)$ as the extensive completion of $\mathscr{D}$, as we shall see below.

We saw in Example 3.3 that $\operatorname{Total}\left(K_{r}\left(\mathscr{D}_{+1}\right)\right)$ is extensive, and we saw in Example 5.4 that it has finite products, and so lies in Ext ${ }_{\mathrm{pr}}$. The inclusion $J: \mathscr{D}_{+1} \rightarrow K_{r}\left(\mathscr{D}_{+1}\right)$ preserves restriction products and restriction coproducts, and so the induced map $\operatorname{Total}(J): \operatorname{Total}\left(\mathscr{D}_{+1}\right) \rightarrow \operatorname{Total}\left(K_{r}\left(\mathscr{D}_{+1}\right)\right)$ preserves products and coproducts. The left adjoint $\mathscr{D} \rightarrow \mathscr{D}_{+1}$ preserves coproducts, and the inclusion $\operatorname{Total}\left(\mathscr{D}_{+1}\right) \rightarrow \mathscr{D}_{+1}$ preserves and reflects them, so that $I: \mathscr{D} \rightarrow \operatorname{Total}\left(\mathscr{D}_{+1}\right)$ preserves coproducts. On the other hand the left adjoint $\mathscr{D} \rightarrow \mathscr{D}_{+1}$ sends products to restriction products, and so $I: \mathscr{D} \rightarrow \operatorname{Total}\left(\mathscr{D}_{+1}\right)$ also preserves products. Thus $N: \mathscr{D} \rightarrow \operatorname{Total}\left(K_{r}\left(\mathscr{D}_{+1}\right)\right)$ preserves products and coproducts, and so is a morphism in Dist.

It remains to check the universal property. To do this, we use the theory of effective completions of classifying monads developed in [7. Section 5]. Recall that a restriction category $\mathbf{X}$ is classified if the inclusion $\operatorname{Total}(\mathbf{X}) \rightarrow \mathbf{X}$ has a right adjoint $R$ for which the components $\epsilon_{A}: R A \rightarrow A$ are restriction retractions. The induced comonad on $\mathbf{X}$ is called the classifying comonad. A monad $T$ on a category $\mathscr{C}$ was defined in [7] to be a classifying monad if it is equipped with the requisite structure to make the Kleisli category $\mathscr{C}_{T}$ into a classified restriction category whose classifying comonad is the comonad induced by the Kleisli adjunction. The classifying monad $T$ is said to be effective if the restriction category $\mathscr{C}_{T}$ is split and the left adjoint $F_{T}: \mathscr{C} \rightarrow \mathscr{C}_{T}$ exhibits $\mathscr{C}$ as the category of total maps in $\mathscr{C}_{T}$. In other words, a classifying monad is effective if it is the partial map classifier for a category of partial maps. Various characterizations of effective classifying monads were given in [7, Theorem 5.8].

Given a classifying monad $T$ on a category $\mathscr{C}$, the restriction category $\mathscr{C}_{T}$ is classified; the split restriction category $K_{r}\left(\mathscr{C}_{T}\right)$ need not be classified in general, although it will be if $T$ is an interpreted classifying monad in the sense of [7, Section 4]. The precise details of this definition are unimportant in the present context, but it is important to know that the monad +1 on a distributive category $\mathscr{D}$ is an interpreted classifying monad, as observed in [7, Example 4.15]. For a general classifying monad $T$ there is nonetheless a universal way to obtain a split classified restriction category from $\mathscr{C}_{T}$ : it is obtained by splitting more idempotents than just the restriction ones, and is denoted by $K_{\text {cr }}\left(\mathscr{C}_{T}\right)$; see [7] Section 3.3].

Since $K_{\text {cr }}\left(\mathscr{C}_{T}\right)$ is a split classified restriction category, the induced monad on $\operatorname{Total}\left(K_{\text {cr }}\left(\mathscr{C}_{T}\right)\right)$ is
an effective classifying monad. It is in fact the universal way of associating an effective classifying monad to the classifying monad $T$, in a sense made precise in [7, Section 5], and so is called the effective completion of the classifying monad $T$. In the case where the monad $T$ is interpreted such as the monad +1 on a distributive category - then the effective completion may be described more simply as $\operatorname{Total}\left(K_{r}\left(\mathscr{C}_{T}\right)\right)$. We shall use the universal property of the effective completion to show that $\operatorname{Total}\left(K_{r}\left(\mathscr{D}_{+1}\right)\right)$ is the extensive completion of the distributive category $\mathscr{D}$.

Consider distributive categories $\mathscr{D}$ and $\mathscr{E}$, equipped with the corresponding interpreted classifying monads +1 . A morphism of classifying monads (in the sense of 7 ) from $(\mathscr{D},+1)$ to $(\mathscr{E},+1)$ consists of a functor $H: \mathscr{D} \rightarrow \mathscr{E}$ equipped with a family of maps $\varphi: H(A+1) \rightarrow H A+1$ natural in $A$ and rendering commutative the following diagrams:

for all objects $A$ and all morphisms $f: A \rightarrow B+1$. A straightforward argument shows that the condition involving a morphism $f: A \rightarrow B+1$ holds for all such $f$ if and only if it holds for all $f$ with $B=1$; that is, for all $a: A \rightarrow 1+1$. The resulting diagram is:


Given morphisms $(H, \varphi)$ and $(K, \psi)$ of classifying monads, a transformation from $(H, \varphi)$ to $(K, \psi)$ consists of a natural transformation $\alpha: H \rightarrow K$ rendering commutative

for all $A$. There is now a 2-category Dist ${ }_{c l}$ consisting of the distributive categories, the morphisms of classifying monads, and the transformations of these. It is a full sub-2-category of the 2 -categories icMnd and cMnd defined in [7]. The biadjunctions constructed in [7] Section 5.2] now show that $N: \mathscr{D} \rightarrow \operatorname{Total}\left(K_{r}\left(\mathscr{D}_{+1}\right)\right)$ has a canonical structure of morphism of classifying monads, and exhibits $\operatorname{Total}\left(K_{r}\left(\mathscr{D}_{+1}\right)\right)$ as the bireflection of $\mathscr{D}$ into the full sub-2-category of Dist ${ }_{\text {cl }}$ consisting of the extensive categories with finite products.

We now turn to an analysis of the notion of morphism of classifying monads. Suppose, as above, that $\mathscr{D}$ and $\mathscr{E}$ are distributive.

Lemma 5.10 If $H: \mathscr{D} \rightarrow \mathscr{E}$ preserves finite coproducts then there is a unique $\varphi$ making $H$ into $a$ morphism of classifying monads, namely $H A+!: H A+H 1 \rightarrow H A+1$.

Proof: If $\varphi_{A}: H A+H 1 \rightarrow H A+1$ makes $H$ into such a morphism then it must have the form $\left\langle\alpha_{A} \mid \beta_{A}\right\rangle$ where $\alpha_{A}: H A \rightarrow H A+1$ and $\beta_{A}: H 1 \rightarrow H A+1$ are both natural in $A$. Compatibility of $\alpha_{A}$ with the first injection gives $\alpha_{A}=i_{H A}$, while naturality of $\beta_{A}$ gives commutativity of


But $H 0$ is initial, so $\beta_{0}: H 1 \rightarrow H 0+1$ is the unique map, and $\varphi_{A}$ must be $H A+$ ! as claimed.
Conversely, we must show that $\varphi_{A}=H A+$ ! does satisfy the various conditions. It is clearly natural and satisfies the compatibility conditions with $i_{A}: A \rightarrow A+1$ and $A+\nabla: A+1+1 \rightarrow A+1$, so we need only show compatibility with maps $a: A \rightarrow 1+1$. Commutativity of

wherein $\theta$ denotes the canonical isomorphisms expressing the fact that $H$ preserves coproducts, gives commutativity of

Corollary 5.11 If $H, K: \mathscr{D} \rightarrow \mathscr{E}$ preserve finite coproducts, then any natural transformation $\alpha: H \rightarrow K$ is a transformation of classifying monads.

On the other hand, the coproduct-preserving functors are not the only morphisms of classifying monads:

Example 5.12 If $X$ is any object of $\mathscr{E}$, then the constant functor $\Delta X: \mathscr{D} \rightarrow \mathscr{E}$ at $X$ becomes a morphism of classifying monads if we define $\varphi_{A}: H(A+1) \rightarrow H A+1$ to be the injection $X \rightarrow X+1$ for any $A$.

We may now prove
Theorem 5.13 The functor $N: \mathscr{D} \rightarrow \operatorname{Total}\left(K_{r}\left(\mathscr{D}_{+1}\right)\right)$ exhibits $\operatorname{Total}\left(K_{r}\left(\mathscr{D}_{+1}\right)\right)$ as the extensive completion of the distributive category $\mathscr{D}$.

Proof: We know from [7] that composition with $N$ induces, for any extensive category $\mathscr{E}$ with products, an equivalence between the category of morphisms of classifying monads from $\mathscr{D}$ to $\mathscr{E}$ and the category of morphisms of classifying monads from $\operatorname{Total}\left(K_{r}\left(\mathscr{D}_{+1}\right)\right)$ to $\mathscr{E}$. It remains to prove that if $G: \mathscr{D} \rightarrow \mathscr{E}$ preserves finite products and coproducts, and $(H, \varphi): \operatorname{Total}\left(K_{r}(\mathscr{D}+1)\right) \rightarrow \mathscr{E}$ is the induced morphism of classifying monads, then $H$ preserves finite products and coproducts. But since $H$ may be constructed as the composite of $\operatorname{Total}\left(K_{r}\left(G_{+1}\right)\right): \operatorname{Total}\left(K_{r}\left(\mathscr{D}_{+1}\right)\right) \rightarrow \operatorname{Total}\left(K_{r}\left(\mathscr{E}_{+1}\right)\right)$ and the canonical equivalence $\operatorname{Total}\left(K_{r}\left(\mathscr{E}_{+1}\right)\right) \rightarrow \mathscr{E}$, it will suffice to prove that $\operatorname{Total}\left(K_{r}\left(G_{+1}\right)\right)$ preserves finite products and coproducts. Since $G$ preserves coproducts, $G_{+1}$ preserves restriction coproducts, and so $\operatorname{Total}\left(K_{r}\left(G_{+1}\right)\right)$ preserves coproducts by Proposition [2.2. Similarly $G_{+1}$ preserves restriction products and so $\operatorname{Total}\left(K_{r}\left(G_{+1}\right)\right)$ preserves products by Proposition 4.4

## References

[1] A. Bucalo, C. Führmann, and A. Simpson Equational lifting monads, Theoretical Computer Science, to appear.
[2] A. Carboni, Bicategories of partial maps, Cah. de Top. Geom. Diff., 28:111-126, 1987.
[3] A. Carboni, Matrices, relations, and group representations J. Algebra 136:497-529, 1991.
[4] A. Carboni, Stephen Lack, and R.F.C. Walters, Introduction to extensive and distributive categories, J. Pure Appl. Algebra 84:145-158, 1993.
[5] J.R.B. Cockett, Introduction to distributive categories, Math. Structures Comput. Sci. 3:277307, 1993.
[6] J.R.B. Cockett and Stephen Lack, Restriction categories I: Categories of partial maps, Theoretical Computer Science, 270:223-259, 2002.
[7] J.R.B. Cockett and Stephen Lack, Restriction categories II: Partial map classification, Theoretical Computer Science, 294:61-102, 2003.
[8] J.R.B. Cockett and Stephen Lack, Restriction categories IV: Enriched restriction categories, in preparation.
[9] J.R.B. Cockett and Stephen Lack, The extensive completion of a distributive category, Theory Appl. Categ. 8:541-554, 2001.
[10] P.-L. Curien and A. Obtulowicz, Partiality, Cartesian closedness, and toposes, Inform. and Comput. 80:50-95, 1989.
[11] R.A. Di Paola and A. Heller, Dominical categories: recursion theory without elements, J. Symbolic Logic, 52:595-635, 1987.
[12] Peter J. Freyd and André Scedrov, Categories, allegories North-Holland Mathematical Library, 39, North-Holland Publishing Co., Amsterdam, 1990.
[13] H.-J. Hoehnke, On partial algebras, Colloq. Math. Soc. János Bolyai 29:373-412, 1977.
[14] A. Kock, Strong functors and monoidal monads, Arch. Math., 23:113-120, 1972.
[15] Saunders Mac Lane, Categories for the Working Mathematician, Springer-Verlag, New York-Heidelberg-Berlin, 1971.
[16] E.P. Robinson and G. Rosolini, Categories of partial maps, Information and computation 79:94-130, 1988.


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