# Coding rotations on intervals 

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#### Abstract

We show that the coding of a rotation by $\alpha$ on $m$ intervals with rationally independent lengths can be recoded over $m$ Sturmian words of angle $\alpha$.


## 1 Introduction

The coding of rotations is a tool for the construction of infinite words over a finite alphabet. Consider a rotation $R_{\alpha}$, given by an angle $\alpha$, and defined for a point $x$ by $R_{\alpha}(x)=\{x+\alpha\}$ where $\{y\}$ denotes the fractional part of $y$. Consider next a partition of the unit circle in $m$ half open intervals $\left\{I_{1}, I_{2}, \cdots I_{m}\right\}$. For any starting point $x$ with $0 \leq x<1$, one gets an infinite word $u$ by $I(x) I\left(R_{\alpha}(x)\right) I\left(R_{\alpha}^{2}(x)\right) \cdots I\left(R_{\alpha}^{n}(x)\right) \cdots$, where $I(y)=i$ if $y \in I_{i}$.

In the special case where $\alpha$ is irrational and and the partition is $I_{1}=[0, \alpha[$ and $I_{2}=[\alpha, 1[$, this construction produces exactly the well-known Sturmian words. These words appear in various domains as computer sciences [2], Physics, Mathematical optimization and play a crucial role in this article. It is remarkable that Sturmian words have a combinatorial characterization. Thus, they are exactly aperiodic words with (subword) complexity $p(n)=n+1$ where the complexity function $p: \mathbb{N} \rightarrow \mathbb{N}$ counts the number of distinct factors of length $n$ in the infinite word $u[2]$. The same general construction allows also to compute Rote words with complexity $p(n)=2 n$ by using an irrational rotation and the partition $I_{1}=\left[0, \frac{1}{2}\left[\right.\right.$ and $I_{2}=\left[\frac{1}{2}, 1[\right.$ (see [7]). More generally, one can obtain infinite words with complexity $p(n)=a n+b$, where $a$ and $b$ are real, by coding of rotation (1), [3].

In addition, codings of rotation with an irrational value of $\alpha$ and the partition $I_{1}=\left[0, \beta\left[\right.\right.$ and $I_{2}=[\beta, 1[$ are intimately related to Sturmian words. Indeed, the
first sequence is the difference term by term of two Sturmian words [6]. Didier gives a characterization of the coding of rotation with a partition of $m$ intervals of length greater than $\alpha$ by using Sturmian words and cellular automata [5]. Finally, Blanchard and Kurka study the complexity of formal languages that are generated by coding of rotation ( 4 .

The goal of this article is to show that the coding of a rotation by $\alpha$ on $m$ intervals with rationally independent lengths can be recoded over $m$ Sturmian words of angle $\alpha$. More precisely, for a given $m$ an universal automaton is constructed such that the edge indexed by the vector of values of the $i$ th letter on each Sturmian word gives the value of the $i$ th letter of the coding of rotation (see Figure 11). If the partition is given by [ $\beta_{j}, \beta_{j+1}$ [ where $\beta_{0}=0<\beta_{1}<$ $\beta_{2}<\cdots<\beta_{j}<\cdots<\beta_{m+1}=1$, then the $\ell$ th Sturmian word is given by the partition $I_{1}=\left[\beta_{\ell}, \beta_{\ell}+\alpha \bmod 1\left[\right.\right.$ and the complement of $I_{1}$ on the unit circle.


Figure 1: Automata for $m=1,2,3$.

## 2 Examples

The figure 2 shows a partition of the unit circle by 4 intervals of form [ $\beta_{j}, \beta_{j+1}[$ and the coding by 8 intervals associated with binary vectors (we can find the coding of the interval $\left[\beta_{j}, \beta_{j+1}[)\right.$ by the automaton for $m=3$ applied to the binary vector value.


Figure 2: Partition of the unit circle.

As an example, using the universal automaton for $m=2$, the three following Sturmian words can be recoded on a word on a three letter alphabet.

$$
\begin{aligned}
& 1001010010100101 \cdots \\
& 0100101001010010 \cdots \\
& 0010100101001010 \cdots
\end{aligned}
$$

is recoded on the following word:

$$
0120201202012020 \cdots
$$

## 3 Notation

We will consider subsets of $[0,1[$ that we call intervals. Let $x, y$ be in $[0,1[$. Then we set

$$
\left[x, y\left[\left\{\begin{array}{ll}
\{z \mid x \leq z<y\} & \text { if } x<y \\
\emptyset & \text { if } x=y \\
\{z \mid x \leq z<1\} \cup\{z \mid 0 \leq z<y\} & \text { if } x>y
\end{array}\right.\right.\right.
$$

In particular, $[x, y[=[0, y[\cup[x, 1[$ if $x>y$. This is precisely the notion of an interval on the torus $\mathbb{T}=\mathbb{R} / \mathbb{Z}$.

Let $\alpha, \beta_{1}, \ldots, \beta_{m}$ be numbers in the interval $] 0,1\left[\right.$, with $\beta_{1}<\cdots<\beta_{m}$. It will be convenient to set $\beta_{0}=0$ and $\beta_{m+1}=1$. The $m+1$ intervals

$$
B_{k}=\left[\beta_{k}, \beta_{k+1}[, \quad k=0, \ldots, m\right.
$$

are a partition of $\left[0,1\left[\right.\right.$. We consider the rotation of angle $\alpha$ defined by $R_{\alpha}(x)=$ $x+\alpha \bmod 1$. Define intervals $I_{k}$ by (all values are computed modulo 1)

$$
I_{k}=\left[\beta_{k}, \beta_{k}+\alpha[, \quad k=0, \ldots, m\right.
$$

We will be specially interested in the nonempty intervals

$$
X_{K}=\bigcap_{k \in K} I_{k} \cap \bigcap_{k \notin K} \overline{I_{k}}
$$

Here, $K$ is a subset of $M=\{0, \ldots, m\}$, and $\overline{I_{k}}=\left[0,1\left[\backslash I_{k}\right.\right.$ is the complement of $I_{k}$. Observe that, for any nonempty interval $I=[x, y[$, one gets $\bar{I}=[y, x[$.

## 4 Circular order

We want to compute intersections of intervals. Although the geometric approach is easy to understand, it is error prone because points are usually not in general position. Therefore, we consider a more combinatoric approach.

Given numbers $x_{1}, \ldots, x_{n} \in\left[0,1\left[\right.\right.$, the sequence $\left(x_{1}, \ldots, x_{n}\right)$ is circularly ordered, or $c$-ordered for short, if there exists an integer $h$ with $1 \leq h \leq n$ such that

$$
\begin{equation*}
0 \leq x_{h} \leq x_{h+1} \leq \cdots \leq x_{n} \leq x_{1} \leq \cdots \leq x_{h-1}<1 \tag{1}
\end{equation*}
$$

If (11) holds, then either $x_{1}=\cdots=x_{n}$, or the integer $h$ is unique. Also, if $\left(x_{1}, \ldots, x_{n}\right)$ is $c$-ordered, then clearly $\left(x_{2}, \ldots, x_{n}, x_{1}\right)$ is $c$-ordered. Any subsequence of a $c$-ordered sequence is $c$-ordered. Observe also that if $\left(x_{1}, \ldots, x_{n}\right)$ is $c$-ordered and $x_{1}<x_{n}$ then $x_{1} \leq \cdots \leq x_{n}$. Indeed, if (11) holds for $h \neq 1$, then $x_{n} \leq x_{1}$.

Two rules are useful.

Lemma 4.1 (i) Translation Rule If $\left(x_{1}, \ldots, x_{n}\right)$ is c-ordered and $y_{i} \equiv x_{i}+$ $\alpha \bmod 1$, then $\left(y_{1}, \ldots, y_{n}\right)$ is $c$-ordered.
(ii) Insertion Rule If $\left(x_{1}, \ldots, x_{n}\right)$ and $\left(y_{1}, \ldots, y_{m}\right)$ are $c$-ordered, if furthermore $y_{1} \neq y_{m}$ and $x_{i}=y_{1}, x_{i+1}=y_{m}$, then $\left(x_{1}, \ldots, x_{i}, y_{2}, \ldots, y_{m-1}, x_{i+1}, \ldots, x_{n}\right)$ is c-ordered.

Proof. (i) We may assume $0 \leq x_{1} \leq \cdots \leq x_{n}<1$. The real numbers $x_{i}+\alpha$ satisfy $x_{1}+\alpha \leq \cdots \leq x_{n}+\alpha<1+x_{1}+\alpha$. If $x_{n}+\alpha<1$, then $y_{i}=x_{i}+\alpha$ and $\left(y_{1}, \ldots, y_{n}\right)$ is $c$-ordered. Otherwise, let $h$ be the smallest integer such that $x_{h}+\alpha \geq 1$. Then

$$
x_{1}+\alpha \leq \cdots \leq x_{h-1}+\alpha<1 \leq x_{h}+\alpha \leq \cdots \leq x_{n}+\alpha
$$

If $h=1$, one gets $1<x_{1}+\alpha \leq \cdots \leq x_{n}+\alpha<2$ and clearly ( $y_{1}, \ldots, y_{n}$ ) is $c$-ordered. If $h>1$, then $x_{n}+\alpha-1<x_{1}+\alpha$ implies

$$
y_{h} \leq \cdots \leq y_{n}<y_{1} \leq \cdots \leq y_{h-1}
$$

(ii) There are two cases. If $x_{i}=\max \left\{x_{1}, \ldots, x_{n}\right\}$, then $x_{i+1} \leq \cdots \leq x_{n} \leq x_{1} \leq$ $\cdots \leq x_{i}$. From $x_{i}=y_{1}, x_{i+1}=y_{m}$, it follows that $y_{m}<y_{1}$. Let $h \neq 1$ be the integer such that $0 \leq y_{h} \leq \cdots \leq y_{m}<y_{1} \leq \cdots \leq y_{h-1}$. Then

$$
0 \leq y_{h} \leq \cdots \leq y_{m}=x_{i+1} \leq \cdots \leq x_{n} \leq x_{1} \leq \cdots \leq x_{i}=y_{1} \leq \cdots \leq y_{h-1}
$$

If $x_{i}<\max \left\{x_{1}, \ldots, x_{n}\right\}$, then $x_{i}=y_{1}<y_{m}=x_{i+1}$ and consequently $x_{i}=y_{1} \leq$ $y_{2} \leq \cdots \leq y_{m}=x_{i+1}$.

We observe that the insertion rule does not hold if $y_{1}=y_{m}$. Consider the two $c$-ordered sequences $(x, x, y)$ and $(x, y, x)$, where $0<x<y<1$. Inserting the second into the first give the sequence $(x, y, x, y)$ which is not $c$-ordered.

We prove another useful formula.

Lemma 4.2 Let $\alpha<1 / 2$. If $(x, y, x+\alpha)$ is $c$-ordered, then $(x, y, x+\alpha, y+\alpha)$ is c-ordered.

Proof. The condition $\alpha<1 / 2$ implies that $(x, x+\alpha, x+2 \alpha)$ is $c$-ordered. By the translation rule, we get that $(x+\alpha, y+\alpha, x+2 \alpha)$ is $c$-ordered. The insertion rule shows that $(x, x+\alpha, y+\alpha, x+2 \alpha)$ is $c$-ordered and, again by the insertion rule, one gets that $(x, y, x+\alpha, y+\alpha)$ is $c$-ordered.

## 5 Intersection

Circular order is useful in considering intersections of intervals. Let $I=[x, y[$ be a nonempty interval. Then $x^{\prime} \in\left[x, y\left[\right.\right.$ iff $\left(x, x^{\prime}, y\right)$ is ordered. Let $I=[x, y[$ and $I^{\prime}=\left[x^{\prime}, y^{\prime}\left[\right.\right.$ be nonempty intervals. Then $x^{\prime} \in I$ iff $\left(x, x^{\prime}, y\right)$ is $c$-ordered. Since $I \cap I^{\prime} \neq \emptyset$ iff $x^{\prime} \in I$ or $x \in I^{\prime}$, the intervals $I$ and $I^{\prime}$ are disjoint iff $\left(x, y, x^{\prime}\right)$ and $\left(x^{\prime}, y^{\prime}, x\right)$ are $c$-ordered. Consequently, we have shown

Lemma 5.1 Let $I=\left[x, y\left[\right.\right.$ and $I^{\prime}=\left[x^{\prime}, y^{\prime}\left[\right.\right.$ be nonempty intervals. Then $I \cap I^{\prime}=$ $\emptyset$ if and only if $\left(x, y, x^{\prime}, y^{\prime}\right)$ is c-ordered.

The length $s$ of an interval $I=[x, y[$ is the number $s=y-x$ if $x \leq y$, and is $s=1-(x-y)$ if $y<x$. In both cases, $y \equiv x+s \bmod 1$ so that, knowing the length, we may write $I=[x, x+s[$.

Lemma 5.2 Let $I=\left[x, y\left[\right.\right.$ and $I^{\prime}=\left[x^{\prime}, y^{\prime}[\right.$ be intervals of the same length $0<\alpha<1 / 2$. If $I$ and $I^{\prime}$ intersect, then $\left(x, x^{\prime}, y, y^{\prime}\right)$ or $\left(x^{\prime}, x, y^{\prime}, y\right)$ is c-ordered. In the first case, $I \cap I^{\prime}=\left[x^{\prime}, y\left[\right.\right.$, in the second case, $I \cap I^{\prime}=\left[x, y^{\prime}[\right.$.

Observe that if the length of $I$ and $I^{\prime}$ is greater than $1 / 2$, then the intersection needs not to be an interval.

Proof. The discussion before Lemma 5.1 shows that $I$ and $I^{\prime}$ intersect if and only if $\left(x, x^{\prime}, y\right)$ or $\left(x^{\prime}, x, y^{\prime}\right)$ are $c$-ordered. From Lemma 4.2 , it follows that $\left(x, x^{\prime}, y, y^{\prime}\right)$ or $\left(x^{\prime}, x, y^{\prime}, y\right)$ is $c$-ordered. Moreover, $y \neq x^{\prime}$ and $x \neq y^{\prime}$ since otherwise $\left(x, y, x^{\prime}, y^{\prime}\right)$ is $c$-ordered and the intervals are disjoint by Lemma 4.2. If $x=x^{\prime}$ (or equivalently if $y=y^{\prime}$ ), then $I=I^{\prime}$. Thus, we may assume that the numbers $x, y, x^{\prime}, y^{\prime}$ are distinct.

Assume the first ordering holds. The formula for the intersection is straightforward if $0 \leq x<x^{\prime}<y<y^{\prime}<1$. If $0 \leq x^{\prime}<y<y^{\prime}<x<1$, then $I=\left[0, y\left[\cup\left[x, 1\left[\right.\right.\right.\right.$ and $I \cap I^{\prime}=\left[x^{\prime}, y[\right.$. The two other cases are proved in the same way.

The previous lemma will be applied to the intervals $I_{k}=\left[\beta_{k}, \beta_{k}+\alpha[\right.$. They all have same length $\alpha$. We write the conclusion for further reference.

Lemma 5.3 Let $\alpha<1 / 2$. Let $I_{k}=\left[\beta_{k}, \beta_{k}+\alpha\left[\right.\right.$ and $I_{\ell}=\left[\beta_{\ell}, \beta_{\ell}+\alpha[\right.$ be two intervals. If $I_{k}$ and $I_{\ell}$ intersect then $\left(\beta_{k}, \beta_{\ell}, \beta_{k}+\alpha, \beta_{\ell}+\alpha\right)$ or $\left(\beta_{\ell}, \beta_{k}, \beta_{\ell}+\right.$ $\left.\alpha, \beta_{k}+\alpha\right)$ is c-ordered. Moreover, $I_{k} \cap I_{\ell}=\left[\beta_{\ell}, \beta_{k}+\alpha[\right.$ in the first case, and $I_{k} \cap I_{\ell}=\left[\beta_{k}, \beta_{\ell}+\alpha[\right.$ in the second case.

The following observation is the basic step for analyzing the coding induced by a rotation. Recall that for $K \subset\{0, \ldots, m\}$,

$$
X_{K}=\bigcap_{k \in K} I_{k} \cap \bigcap_{k \notin K} \overline{I_{k}}
$$

We assume from now on that $\alpha<1 / 2$.

Proposition 5.4 Assume $X_{K} \neq \emptyset$ for some $K \subset\{0, \ldots, m\}$ and assume $\left(\beta_{i_{1}}, \beta_{i_{2}}, \beta_{i_{3}}, \beta_{i_{4}}\right)$ is a $c$-ordered sequence. If $i_{1}, i_{3} \in K$, then $i_{2} \in K$ or $i_{4} \in K$.

Proof. Arguing by contradiction, suppose that $i_{2}, i_{4} \notin K$. Since $X_{K} \neq \emptyset$, the interval $I_{i_{1}} \cap I_{i_{3}}$ is not empty, therefore by Lemma $5.3\left(\beta_{i_{1}}, \beta_{i_{3}}, \beta_{i_{1}}+\alpha, \beta_{i_{3}}+\alpha\right)$ or ( $\beta_{i_{3}}, \beta_{i_{1}}, \beta_{i_{3}}+\alpha, \beta_{i_{1}}+\alpha$ ) is $c$-ordered (or the sequence obtained by exchanging $i_{1}$ and $i_{3}$ ). Consider the first case, the second is the same by exchanging $i_{2}$ and $i_{4}$. Since $\left(\beta_{i_{1}}, \beta_{i_{2}}, \beta_{i_{3}}\right)$ is $c$-ordered, the translation rule shows that $\left(\beta_{i_{1}}+\right.$ $\left.\alpha, \beta_{i_{2}}+\alpha, \beta_{i_{3}}+\alpha\right)$ is $c$-ordered which gives, applying twice the insertion rule, that $\left(\beta_{i_{1}}, \beta_{i_{2}}, \beta_{i_{3}}, \beta_{i_{1}}+\alpha, \beta_{i_{2}}+\alpha, \beta_{i_{3}}+\alpha\right)$ is $c$-ordered. From this, we get that
$\left(\beta_{i_{1}}, \beta_{i_{2}}, \beta_{i_{1}}+\alpha, \beta_{i_{2}}+\alpha\right)$ is $c$-ordered. From Lemma 5.1, we know that $I_{i_{1}} \cap I_{i_{3}}=$ $\left[\beta_{i_{3}}, \beta_{i_{1}}+\alpha\left[\right.\right.$, and this is then disjoint from $\overline{I_{i_{2}}}=\left[\beta_{i_{2}}+\alpha, \beta_{i_{2}}[\right.$.

Proposition 5.5 If $X_{K}$ is not empty, then there exist integers $k, \ell$ with $0 \leq$ $k<\ell \leq m$ such that $\{K, M \backslash K\}=\{\{k, \ldots, \ell-1\},\{\ell, \ldots m, 0, \ldots, k-1\}\}$

Proof. This is a direct consequence of the preceding discussion.

It follows that there are only $(m+1)(m+2)$ intervals $X_{K}$ to be considered. In fact, consider the numbers $0, \beta_{1}, \ldots, \beta_{m}, 1$ and $\alpha, \beta_{1}+\alpha, \ldots, \beta_{m}+\alpha$. They partition $[0,1$ [ into exactly $2 m+2$ intervals. Each of these intervals is contained in one and only one of the $X_{K}$ (but $X_{\emptyset}$ may be scattered over several of the small intervals). This means that, among the $(m+1)(m+2)$ possible intervals $X_{K}$, there are only $2 m+2$ that are used in a particular setting of the values of $\alpha, \beta_{1}, \ldots, \beta_{m}$.

Theorem 5.6 Assume $K \neq \emptyset, M$, and $X_{K} \neq \emptyset$. Then $X_{K}=\left[\beta_{\ell-1}, \beta_{k}+\right.$ $\alpha\left[\cap\left[\beta_{k-1}+\alpha, \beta_{\ell}[\right.\right.$.

If $K=\{k\}$ is a singleton, then the formula still holds with $\ell-1=k$.

Proof. Suppose that $K=\{k, \cdots, \ell-1\}$ with $k<\ell$. The other case is symmetric. We first prove that $\bigcap_{n \in K} I_{n}=\left[\beta_{\ell-1}, \beta_{k}+\alpha\left[\right.\right.$. Set $Y_{K}=\bigcap_{n \in K} I_{n}$.

Since $X_{K} \neq \emptyset$, the interval $I_{k} \cap I_{\ell-1}$ is not empty. By Lemma 4.3 there are two cases: either $\left(\beta_{k}, \beta_{\ell-1}, \beta_{k}+\alpha, \beta_{\ell-1}+\alpha\right)$ is $c$-ordered, or $\left(\beta_{\ell-1}, \beta_{k}, \beta_{\ell-1}+\right.$ $\left.\alpha, \beta_{k}+\alpha\right)$ is $c$-ordered.

We show that this second case cannot happen. Indeed in this case, $Y_{K} \subset I_{k} \cap$ $I_{\ell-1}=\left[\beta_{k}, \beta_{\ell-1}+\alpha\left[\right.\right.$. Moreover for each $n \in M \backslash K$, the sequence $\left(\beta_{\ell-1}, \beta_{n}, \beta_{k}\right)$ is $c$-ordered. By translation and insertion, the sequence $\left(\beta_{\ell-1}, \beta_{n}, \beta_{k}, \beta_{\ell-1}+\right.$ $\left.\alpha, \beta_{n}+\alpha, \beta_{k}+\alpha\right)$ is $c$-ordered. This shows that $I_{n} \supset I_{k} \cap I_{\ell-1}$, and consequently $I_{k} \cap I_{\ell-1} \cap \overline{I_{n}}=\emptyset$ for each $n$ in $M \backslash K$, contradicting the assumption that $X_{K} \neq \emptyset$.

Thus, $\left(\beta_{k}, \beta_{\ell-1}, \beta_{k}+\alpha, \beta_{\ell-1}+\alpha\right)$ is $c$-ordered. This implies that $I_{k} \cap I_{\ell-1}=$ $\left[\beta_{\ell-1}, \beta_{k}+\alpha\left[\right.\right.$. If $i \in K$ then $\left(\beta_{k}, \beta_{i}, \beta_{\ell-1}\right)$ is $c$-ordered. By translation, $\left(\beta_{k}+\right.$ $\left.\alpha, \beta_{i}+\alpha, \beta_{\ell-1}+\alpha\right)$ is $c$-ordered. By insertion of $\left(\beta_{k}, \beta_{i}, \beta_{\ell-1}\right)$ into $\left(\beta_{k}, \beta_{\ell-1}, \beta_{k}+\right.$ $\left.\alpha, \beta_{\ell-1}+\alpha\right)$ one gets $\left(\beta_{k}, \beta_{i}, \beta_{\ell-1}, \beta_{k}+\alpha, \beta_{\ell-1}+\alpha\right)$ is $c$-ordered. Again by insertion of $\left(\beta_{k}+\alpha, \beta_{i}+\alpha, \beta_{\ell-1}+\alpha\right)$, the sequence $\left(\beta_{k}, \beta_{i}, \beta_{\ell-1}, \beta_{k}+\alpha, \beta_{i}+\right.$ $\left.\alpha, \beta_{\ell-1}+\alpha\right)$ is $c$-ordered. Thus $Y_{K}=\left[\beta_{\ell-1}, \beta_{k}+\alpha[\right.$.

The second part of the proof deals with $\bigcap_{n \in M \backslash K} \overline{I_{n}}$. In this intersection the index $n$ runs through the set $\{0, \cdots, k-1, \ell, \cdots, m\}$. The set $M \backslash K$ is partitioned into three possibly empty subsets as follows: $n \in N$ iff $\overline{I_{n}} \supset Y_{K}$, $n \in P$ iff $\overline{I_{n}} \cap Y_{K}=\left[\beta_{\ell-1}, \beta_{n}\left[\right.\right.$ and finally $n \in Q$ iff $\overline{I_{n}} \cap Y_{K}=\left[\beta_{n}+\alpha, \beta_{k}+\alpha[\right.$. Of course,

$$
X_{K}=\bigcap_{n \in N}\left(\overline{I_{n}} \cap Y_{K}\right) \cap \bigcap_{n \in P}\left(\overline{I_{n}} \cap Y_{K}\right) \cap \bigcap_{n \in Q}\left(\overline{I_{n}} \cap Y_{K}\right)
$$

If one of the sets $N, P, Q$ is empty it does not contribute to the intersection.
Clearly $\bigcap_{n \in N}\left(\overline{I_{n}} \cap Y_{K}\right)=Y_{K} . \quad$ Next $\bigcap_{n \in P}\left(\overline{I_{n}} \cap Y_{K}\right)=\bigcap_{n \in P}\left[\beta_{\ell-1}, \beta_{n}[\right.$. If $P$ is not empty then $\ell$ is in $P$ and $\bigcap_{n \in P}\left(\overline{I_{n}} \cap Y_{K}\right)=\left[\beta_{\ell-1}, \beta_{\ell}\right.$. Finally, $\bigcap_{n \in Q}\left(\overline{I_{n}} \cap Y_{K}\right)=\bigcap_{n \in Q}\left[\beta_{n}+\alpha, \beta_{k}+\alpha[\right.$. If $Q$ is not empty then $k-1$ is in $Q$ and $\bigcap_{n \in Q}\left(\overline{I_{n}} \cap Y_{K}\right)=\left[\beta_{k-1}+\alpha, \beta_{k}+\alpha[\right.$.

To finish the proof, we just have to verify that in each case, $X_{K}=\bigcap_{n \in N}\left(\overline{I_{n}} \cap\right.$ $\left.Y_{K}\right) \cap \bigcap_{n \in P}\left(\overline{I_{n}} \cap Y_{K}\right) \cap \bigcap_{n \in Q}\left(\overline{I_{n}} \cap Y_{K}\right)$ is equal to $\left[\beta_{\ell-1}, \beta_{k}+\alpha\left[\cap\left[\beta_{k-1}+\alpha, \beta_{\ell}[\right.\right.\right.$.

If $P \neq \emptyset$ then $\bigcap_{n \in P}\left(\overline{I_{n}} \cap Y_{K}\right)=\left[\beta_{\ell-1}, \beta_{\ell}\left[\right.\right.$ and the sequence $\left(\beta_{\ell-1}, \beta_{\ell}, \beta_{k}+\alpha\right)$ is $c$-ordered (case $P_{1}$ ). If $P=\emptyset$ then the sequence $\left(\beta_{\ell-1}, \beta_{k}+\alpha, \beta_{\ell}\right)$ is $c$-ordered (case $P_{0}$ ). If $Q \neq \emptyset$ then $\bigcap_{n \in Q}\left(\overline{I_{n}} \cap Y_{K}\right)=\left[\beta_{k-1}+\alpha, \beta_{k}+\alpha[\right.$ and the sequence $\left(\beta_{\ell-1}, \beta_{k-1}+\alpha, \beta_{k}+\alpha\right)$ is $c$-ordered (case $\left.Q_{1}\right)$. If $Q=\emptyset$ then the sequence $\left(\beta_{\ell-1}, \beta_{k}+\alpha, \beta_{k-1}+\alpha\right)$ is $c$-ordered (case $\left.Q_{0}\right)$.

Case $\left(P_{1} Q_{1}\right)$. If $P$ and $Q$ are nonempty then $X_{K}=\left[\beta_{\ell-1}, \beta_{k}+\alpha\left[\cap\left[\beta_{\ell-1}, \beta_{\ell}\left[\cap\left[\beta_{k-1}+\right.\right.\right.\right.\right.$ $\alpha, \beta_{k}+\alpha\left[\right.$. As the sequences $\left(\beta_{\ell-1}, \beta_{\ell}, \beta_{k}+\alpha\right)$ and $\left(\beta_{\ell-1}, \beta_{k-1}+\alpha, \beta_{k}+\alpha\right)$ are $c$-ordered, by the insertion rule either the sequence $\left(\beta_{\ell-1}, \beta_{\ell}, \beta_{k-1}+\alpha, \beta_{k}+\alpha\right)$ or $\left(\beta_{\ell-1}, \beta_{k-1}+\alpha, \beta_{\ell}, \beta_{k}+\alpha\right)$ is $c$-ordered. The first case is impossible because $X_{K}$ is not empty. The second case implies that $X_{K}=\left[\beta_{\ell-1}, \beta_{k}+\alpha\left[\cap\left[\beta_{k-1}+\alpha, \beta_{\ell}[\right.\right.\right.$.

Case $\left(P_{0} Q_{1}\right)$. If $P=\emptyset$ and $Q \neq \emptyset$ then $X_{K}=\left[\beta_{\ell-1}, \beta_{k}+\alpha\left[\cap\left[\beta_{k-1}+\alpha, \beta_{k}+\alpha[\right.\right.\right.$ and the sequences $\left(\beta_{\ell-1}, \beta_{k}+\alpha, \beta_{\ell}\right),\left(\beta_{\ell-1}, \beta_{k-1}+\alpha, \beta_{k}+\alpha\right)$ are $c$-ordered. By insertion the sequence $\left(\beta_{\ell-1}, \beta_{k-1}+\alpha, \beta_{k}+\alpha, \beta_{\ell}\right)$ is $c$-ordered. Thus $X_{K}=$ $\left[\beta_{\ell-1}, \beta_{k}+\alpha\left[\cap\left[\beta_{k-1}+\alpha, \beta_{\ell}[\right.\right.\right.$.

Case $\left(P_{1} Q_{0}\right)$ is symmetric to case $\left(P_{0} Q_{1}\right)$.
Case $\left(P_{0} Q_{0}\right)$. If $P=\emptyset$ and $Q=\emptyset$ then $X_{K}=\left[\beta_{\ell-1}, \beta_{k}+\alpha\left[\cap\left[\beta_{k-1}+\alpha, \beta_{\ell}[\right.\right.\right.$ and the sequences $\left(\beta_{k}+\alpha, \beta_{k-1}+\alpha, \beta_{\ell-1}\right),\left(\beta_{k-1}+\alpha, \beta_{\ell}, \beta_{\ell-1}\right)$ are $c$-ordered. By insertion rule either the sequence $\left(\beta_{k}+\alpha, \beta_{k-1}+\alpha, \beta_{\ell}, \beta_{\ell-1}\right)$, or $\left(\beta_{k}+\alpha, \beta_{k-1}+\right.$ $\alpha, \beta_{\ell}, \beta_{\ell-1}$ ) is $c$-ordered. The first case is impossible because $X_{K}$ is non empty. The second case implies that $X_{K}=\left[\beta_{\ell-1}, \beta_{k}+\alpha\left[\cap\left[\beta_{k-1}+\alpha, \beta_{\ell}[\right.\right.\right.$.

Remark: As an additional property, the preceding proof shows that $X_{K}$ is an interval and the interior of $X_{K}$ does not contain any $\beta_{i}$ or $\left\{\beta_{i}+\alpha\right\}$.

## 6 Main result

Proposition 6.1 If $x \in B_{i}$ and $x+\alpha \in B_{j} \cap X_{K}$, then $j \equiv i+|K| \bmod m+1$.

Proof If $K \neq \emptyset$ or $K \neq M$ then $X_{K}=\left[\beta_{\ell-1}, \beta_{k}+\alpha\left[\cap\left[\beta_{k-1}+\alpha, \beta_{\ell}[\right.\right.\right.$. As $x \in\left[\beta_{i}, \beta_{i+1}\left[\right.\right.$, by translation rule we have $y=x+\alpha \in\left[\beta_{i}+\alpha, \beta_{i+1}+\alpha[\right.$. Furthermore, $y \in\left[\beta_{j}, \beta_{j+1}[\right.$. By the preceding remark and by identification, the only possibility is $j=\ell-1$ and $i=k-1$. It follows that $j=i+|K| \bmod m+1$.

If $K=\emptyset$ then $X_{K}=\cap_{n \in M} \overline{I_{n}}$. By hypothesis we have $y=x+\alpha \in B_{j}$. But $y \in X_{K}$ implies that $y \notin\left[\beta_{j}, \beta_{j}+\alpha\left[=I_{j}\right.\right.$.

If $\left|I_{j}\right| \geq\left|B_{j}\right|$ then $B_{j} \cap X_{K}$ should be empty in contradiction with the hypothesis. Thus $\left|I_{j}\right|<\left|B_{j}\right|$. The interval $B_{j}$ is equal to $I_{j} \cup\left[\beta_{j}+\alpha, \beta_{j+1}[\right.$. That is $x+\alpha \in\left[\beta_{j}+\alpha, \beta_{j+1}\left[\right.\right.$ and $x+\alpha \in\left[\beta_{i}+\alpha, \beta_{i+1}+\alpha[\right.$. By identification, we have $i=j$.

If $K=M$ then $X_{K}=\cap_{n \in M} I_{n}$. As $x \in I_{n}$ implies $x+\alpha \notin I_{n}$, by contraposition $x+\alpha \in X_{K}$ implies $x \notin I_{n}$ for all $n$. Thus $\left(x, \beta_{n}, x+\alpha\right)$ is $c$-ordered for all $n \in M$. As $x \in B_{i}$ and $x$ is not in $I_{i}$ the sequence $\left(\beta_{i}, \beta_{i}+\alpha, x, \beta_{i+1}\right)$ is $c$-ordered. Thus $\left(x, \beta_{i+1}, \cdots, \beta_{i+m-1}, \beta_{i}, x+\alpha\right)$ is $c$-ordered. Consequently $X_{K}$ is equal to $\left[\beta_{i}, \beta_{i+1}+\alpha\left[\right.\right.$. As $x+\alpha \in\left[\beta_{j}, \beta_{j+1}[\right.$ by identification we find $j=i$.

From this proposition, we get the following automaton $\mathcal{A}$ (Figure 11 gives the automata for $m=1,2,3)$. Its set of states is the set $M$ in bijection with the intervals $B_{k}$. The alphabet is the set of subsets of $M$ corresponding to the nonempty intervals $N_{K}$. As already mentioned, there are $(m+1)(m+2)$ of them. The transitions or edges are given by the proposition: $(i, K, j)$ is an transition if $j \equiv i+|K| \bmod m+1$.

Observe that the automaton is deterministic. Also, it is universal in the following sense : for a particular setting of $\alpha, \beta_{1}, \ldots, \beta_{m}$, if the $\beta_{i}$ and the $\beta_{j}+\alpha$ are two by two distinct, there are only $2 m+2$ of the edges that are used. Indeed they are exactly $2 m+2$ intervals in the partition and between $\beta_{i}$ and $\beta_{i+1}$ the coding is uniquely determined for all $i$.

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