# An algorithm to generate exactly once every tiling with lozenges of a domain* 

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#### Abstract

We first show that the tilings of a domain $\mathcal{D}$ form a lattice (using the same kind of arguments as in Rém99) which we then undertake to decompose and generate without any redundance. To this end, we study extensively the relatively simple case of hexagons and their deformations. We show that general domains can be broken up into hexagon-like parts. Finally we give an algorithm to generate exactly once every element in the lattice of the tilings of a general domain $\mathcal{D}$.

Keywords: tiling, lozenge, lattice, hexagon, seed.


## 1 Introduction

Tilings is an age-old topic for specialists and amateurs alike. Lozenge tilings in particular have intrigued generations of curious people because they can easily be seen as piles of cubes.

In the past few years, mathematics, theoretical physics and computer science have started shedding a new light. We will build on these results to provide an answer to a very natural question: given a tileable domain and sufficiently many lozenges tiles, how can one generate all the tilings of the domain without repeating twice the same tiling?

Conway and Lagarias introduced in CL90 a new, powerful tool to study tilings-related topics: tiling groups, which give a necessary condition for a domain to be tileable, and provide and important bijection in the case of lozenges. Thurston went a step further in Thu90 and showed by a constructive algorithm

[^0]
(a) The two admissible regular triangles

(b) There are only three admissible lozenges

Figure 1: Triangles and lozenges
that one can quickly decide whether a domain is tileable; to this end he uses height functions (see Section 2).

Thurston also hinted that the set of the tilings of a domain, partially ordered with the height functions, should have the structure of a lattice. This was proved in Rém99 by Rémila in 1999 in the case of dominoes. We adapt his proof to the case of lozenges in Section 3 .

We then proceed to the really new material: after a rather extensive study of the case of hexagonal and hexagonal-like domains (see Section 4), we use a geometrical point of view (justified by a bijection between Conway and Lagarias' lozenge group and $\mathbb{Z}^{3}$ ) and the identification of meaningful hexagonal-like sub-domains (see Section 5) to exhibit in the general case a maximal chain of intervals in the lattice of the tilings (see Section (6). This chain is a natural extension to Thurston's minimal and maximal tilings. We finally introduce new minimal tilings which allow us to achieve our goal: (uniquely) generating all the elements of the lattice formed by the tilings (see Section (7).

We believe that the tools introduced in this paper, notably the seeds and their ranges, should prove fruitful in tackling related problems (see Section 8).

## 2 Basic tools and definitions

In this section, we present the definitions of classical objects which will be used in this paper, in an attempt to make it reasonably self-contained. All the new objects are defined later in the paper, at the moment when they are needed.

### 2.1 Tiling with lozenges

First of all, let us define what we mean by a tiling. One needs two regular triangles (see Figure 1 (a)). The whole plane can be covered with a repetition of these figures (see Figure 2 (a)), which gives rise to the triangular grid. A domain is a finite union of triangles in the grid. It is simply connected if it is connected and its complement in the plane is connected. A polygon is a simply connected domain.

We now define three tiles by gluing together two regular triangles; let us call them lozenges (see Figure $1(\mathrm{~b})$ ). A tiling of $\mathcal{D}$ is a set of lozenges included in $\mathcal{D}$ with pairwise disjoint interiors such that the union of the lozenges is $\mathcal{D}$ itself.


Figure 2: A closed path in the triangular grid


Figure 3: A tiling of $\mathcal{D}$

A domain is tileable if it admits at least one tiling. Its boundary or contour is the set of its edges that belong to exactly one of its triangles.

Any finite-length closed path $\mathcal{P}$ whose edges belong to triangles in the triangular grid can be viewed as the boundary path of a domain $\mathcal{D}$ (see Figure 2 (b)).

### 2.2 Tiling groups

Figure ${ }^{3}$ (b) suggests a connection between tilings by lozenges and certain piles of cubes, which we now attempt to clarify. Our tool here is Conway and Lagarias' tiling groups, a formal description of which can be found in CL90, Thu90, so we will restrict ourselves to an intuitive (but rigourous) approach. The idea is to simplify contour words so that those of tileable domains are equivalent to the empty word.

First, label the sides of triangles with letters; in our case, the set $\{a, b, c\}$ is sufficient (see Figure 6 (a)). Our purpose is to use words to partially describe tilings. Let $T$ be a tiling of a polygon and let $P$ be a directed path (in the triangular grid) whose edges belong to lozenges in $T$; that is, $P$ never cuts the interior of a lozenge. Such a path will be called $T$-valid or valid for $T$. Given a starting point, $P$ is completely encoded by a word $x$ on the alphabet $\left\{a, b, c, a^{-1}, b^{-1}, c^{-1}\right\}$.

We want $P$ to be rather straightforward, so that following an edge and then following it in the opposite direction should not change $x$. We thus impose $a a^{-1}=b b^{-1}=c c^{-1}=\varepsilon$ (the empty word) and make all possible simplifications

(a)


(b)

Figure 4: Labelling the sides
in $x$, so that $P$ is now encoded by a word in the free group $F(\{a, b, c\})$. Although this is not a pre-requisite for the rest of this paper, the reader unfamiliar with free groups can look up page 257 in Fra94 for instance.

If $P^{\prime}$ is another $T$-valid path, encoded by a word $y$ in $F(\{a, b, c\})$, with the same starting and ending points as $P$, then $P$ and $P^{\prime}$ define a polygon $\mathcal{D}$ which is tiled by lozenges, and therefore tileable. To reflect this fact, we set the contour word $x y^{-1}$ of $\mathcal{D}$ to be the empty word, $\varepsilon$. In particular, the contour words of each of our three lozenges should be set to $\varepsilon$, that is (see Figure (b)) $b c b^{-1} c^{-1}=\varepsilon$, $a b a^{-1} b^{-1}=\varepsilon$ and $c a c^{-1} a^{-1}=\varepsilon$ by reading counterclockwise (starting from a different vertex only changes the word by a circular permutation). Note that this process is equivalent to removing loops in $P$, or stating that two $T$-valid paths having the same starting and ending points should be described by the same word, as we will see in Corollary 2.4.

The words we consider now belong to the group

$$
L=\left\langle a, b, c \mid b c b^{-1} c^{-1}=a b a^{-1} b^{-1}=c a c^{-1} a^{-1}=\varepsilon\right\rangle
$$

which is Conway and Lagarias' lozenge group.
Definition 2.1 Let $T$ be a tiling of a polygon and let $P=\left(v_{0}, v_{1}, \ldots, v_{p}\right)$ be a valid directed path in $T$. Each edge $\left(v_{i}, v_{i+1}\right)$ can be labelled by an element of the alphabet $\left\{a, b, c, a^{-1}, b^{-1}, c^{-1}\right\}$; the label of $P$ is the word $w$ obtained by the concatenation of the labels of its edges in the order $\left(v_{0}, v_{1}\right), \ldots,\left(v_{p-1}, v_{p}\right)$. The free label $\alpha(w)$ of $P$ is the representative element of $w$ in the free group $F(\{a, b, c\})$. The L-label $\ell(w)$ of $P$ is the representative element of $w$ in the lozenge group $L$.

For the sake of simplicity, the free label and the $L$-label of $P$ will also be written $\alpha(P)$ and $\ell(P)$.

The following lemma gives a necessary condition for a domain to be tileable:
Proposition 2.2 (CL90) If a polygon is tileable, then the $L$-label of its boundary path is trivial.

Proof We make the proof by induction on the surface. (The reader familiar with De Bruijn's worms will readily find a direct proof.) It is enough to prove the result for elementary cycles. Besides, the result holds for single lozenges.


Figure 5: Projection of a cube onto the plane $x+y+z=0$

Let $P$ be a valid path in a tiling $T$ of the polygon $\mathcal{P}$, such that $P$ is distinct from the boundary path $B$ of $\mathcal{P}$. $P$ cuts $\mathcal{P}$ into two tileable polygons $\mathcal{P}_{1}$ and $\mathcal{P}_{2}$. There exist $w$ and $w^{\prime}$ such that the boundary paths of $\mathcal{P}_{1}$ and $\mathcal{P}_{2}$ are $w \alpha(P)$ and $\alpha(P)^{-1} w^{\prime}$, both of which are trivial by the induction hypothesis. Therefore $\ell(B)=\ell\left(w \cdot w^{\prime}\right)=\ell\left(w \alpha(P) \cdot \alpha(P)^{-1} w^{\prime}\right)=\ell(w \alpha(P)) \cdot \ell\left(\alpha(P)^{-1} w^{\prime}\right)=\varepsilon$.

Corollary 2.3 Let $T$ be a tiling of a polygon. The $L$-label of any closed valid path in $T$ is trivial.

Proof Indeed, a valid path delimits a tileable polygon of which it is the boundary path.

Corollary 2.4 Let $T$ be a tiling of a polygon. Two valid paths in $T$ having the same starting and ending points have the same $L$-label.

Proof Indeed, if $w$ and $w^{\prime}$ are the labels of these paths, $w^{-1} \cdot w^{\prime}$ is the label of a valid closed path in $T$.

The relations that appear in the definition of $L$ can be rewritten as $a b=b a$, $a c=c a$ and $b c=c b$, so that $L$ has three generators that commute with each other and is therefore isomorphic to $\mathbb{Z}^{3}$. How can we interpret this nice result?

Let $T$ be a tiling of a polygon $\mathcal{P}$ and let $v$ be a vertex of $\mathcal{P}$. A valid path in $T$ can be associated with a unique word in $L$, which in turn corresponds to a unique path in the 1 -skeleton of a cubical tesselation $\mathcal{T}$ of space. There is therefore a one-to-one correspondence between vertices of $\mathcal{D}$ (resp. segments in $T$ ) and vertices of $\mathbb{Z}^{3}$ (resp. segments in $\mathcal{T}$ ). Following an edge in $T$ is analogous to following one of $\mathbb{Z}^{3}$ 's generators. This bijection allows us to lift each edge of $T$ to an edge in $\mathcal{T}$, each lozenge to a square in $\mathbb{Z}^{3}$, so that $T$ is equivalent to a collection of squares in $\mathcal{T}$. This squares may or may not define the visible parts of cubes, depending on $T$. The squares can be projected along the direction $(1,1,1)$ in $\mathbb{Z}^{3}$ to give back $T$ (see Figures 5 and $3(\mathrm{~b})$ ).


Figure 6: The need for height functions

### 2.3 Height functions

The contour word of a polygon may well have a trivial image in the lozenge group even though the polygon is not tileable. In other words, the condition stated in Proposition 2.2 is not sufficient. Consider for instance Figure 6 (a): the polygon is clearly non-tileable, but its contour word $b c c a b^{-1} c^{-1} c^{-1} a^{-1}$ has a trivial image in $L$ because letters commute. More generally, any closed path in $\mathbb{Z}^{3}$ corresponds to a word in which the number of $a$ 's (resp. $b, c$ ) is equal to the number of $a^{-1}$ 's (resp. $b^{-1}, c^{-1}$ ), so that the image of this word in $L$ is trivial; it would be quite surprising if the projection onto the plane of any closed path in $\mathbb{Z}^{3}$ was tileable. More complicated examples can be exhibited, for instance using Fournier's obstructions (see Fou96). Instead of examining each case, let us develop from this simple example a general idea.

Let $a, b$ and $c$ correspond to $(1,0,0),(0,1,0)$ and $(0,0,1)$ : we can now see Figure (a) as a closed path in $\mathbb{Z}^{3}$ (it looks like a cyclohexane molecule in chair conformation). If the domain was tileable, one could view a tiling as a collection of squares and cubes in $\mathbb{Z}^{3}$. Since the distance between $A$ and $B$ is only one edge in the triangular grid, these points would be linked by either the side or the diagonal of a square in $\mathbb{Z}^{3}$, so that their distance in $\mathbb{Z}^{3}$ would be 1 or $\sqrt{2}$; but if $A$ has coordinates $(0,0,0)$ then $B$ has coordinates $(1,1,2)$ since starting from $A$ we follow $b$ once, $c$ twice and $a$ once.

The distance (in $\mathbb{Z}^{3}$ ) between $A$ and $B$ can easily be made greater: remark that the figure looks like a butterfly, and make the wings bigger without moving $A$ or $B$. This adds 3 to the distance between $A$ and $B$ at each step.

We see that the distance between two points seems important. Two points at distance 1 in the triangular grid should not be distant ones in $\mathbb{Z}^{3}$ if the domain is tileable. What really matters in our butterfly example though is the distance between $A$ and $B$ along the $(1,1,1)$ axis: following an edge in a tiling is like following one of $\mathbb{Z}^{3}$ 's generators, each of which yields the same height increase, say, $i$, along the $(1,1,1)$ axis so that every point in $\mathbb{Z}^{3}$, once orthogonally projected onto the $(1,1,1)$ axis, is at a distance to the origin that
is a multiple of $i$. To each point $p$ of $\mathbb{Z}^{3}$ one can therefore give a value (multiple of $i$ ) which corresponds to the distance between a fixed origin point and the orthogonal projection of $p$ onto the $(1,1,1)$ axis. For convenience, we will forget the geometrical interpretation, place the origin at an arbitrary vertex and set $i=$ 1 (see Figure 6 (b)).

We now proceed to properly define height functions, mainly following Thurston (see Thu90) while preserving the algebraic point of view.

Definition 2.5 We call evaluation function the morphism $\varphi$ from the lozenge group $(L, \cdot)$ with generators $a, b, c$, to the group $(\mathbb{Z},+)$ of integers such that $\varphi(a)=\varphi(b)=\varphi(c)=1$.

Note that this implies $\varphi(\varepsilon)=0$ and $\varphi\left(w \cdot w^{\prime}\right)=\varphi(w)+\varphi\left(w^{\prime}\right)$ for any $w, w^{\prime}$ in $L$, whence $\varphi\left(a^{-1}\right)=\varphi\left(b^{-1}\right)=\varphi\left(c^{-1}\right)=-1$.

Definition 2.6 Let $T$ be a tiling of a polygon $\mathcal{P}$ and let $v$ be a vertex on the boundary path of $\mathcal{P}$. The height function induced by $T$ and $v$ is the function that maps each vertex $x$ of $\mathcal{P}$ to the image by the evaluation function of any valid path from $v$ to $x$.

The correctness of this definition stems from Corollary 2.4 .
Lemma 2.7 On the boundary path of $\mathcal{P}$, the heights of the vertices do not depend on $T$.

Proof Indeed, if $x$ is such a vertex, there exists a path from $v$ to $x$ that lies on the boundary path of $\mathcal{P}$, so the result follow by induction.

Note that only the vertices on the boundary path have fixed heights; those of inner vertices do depend on $T$ (see for instance Figure 8 (a)).

Since changing the reference vertex $v$ only changes the height function by a constant, we will often refer to a height function without mentionning $v$. Thus we will write "the height function associated with the tiling".

### 2.4 Thurston's algorithm

We have seen (see Section 2.3) that Proposition 2.2 does not give a sufficient condition for the tileability of a polygon. Thurston's height functions provides a constructive algorithm, outlined in Thu90, to determine whether a polygon can be tiled, and exhibit a tiling if it can be done.

We will build the minimal tiling of the polygon, in a sense that will be made clear in Section 3. There is a natural (partial) order on the height functions as the tiling changes and this is interpreted as an order on the tilings.

The following algorithm, defined by Thurston in Thu90, is based on this simple result (see Proposition 3.7 and Definition 3.8 in Section 3 to know more about flips):

Lemma 2.8 Let $v$ be a vertex of a polygon $\mathcal{P}$ such that the height function associated with the minimal tiling of $\mathcal{P}$ is maximal on $v$. This vertex cannot belong to the interior of $\mathcal{P}$, otherwise it could be flipped down: therefore $v$ lies on the boundary path of $\mathcal{P}$.

## Algorithm 2.9

- Input: A polygon $\mathcal{P}$.
- Output: The minimal tiling of $\mathcal{P}$ if the polygon is tileable, untileability otherwise.
- Initialization: Initialize the list $L$ to $\emptyset$.
- Step 1: If $\mathcal{P}$ is a single point, return $L$.
- Step 2: Compute the height function on the boundary path of $\mathcal{P}$. If a vertex is given two different heights, return untileability.
- Step 3: Let $v$ be a vertex of $\mathcal{P}$ of maximal height. There exist $u$ and $w$ on the boundary path $B$ of $\mathcal{P}$ such that $(u, v)$ and $(v, w)$ are segments of $B$. Place a lozenge $\ell$ on $\mathcal{P}$ so that $u, v$ and $w$ belong to it. Add $\ell$ and its position to $L$. Update $\mathcal{P}$ to $\mathcal{P} \backslash \ell$. Go back to step 1 .

For the proof of the algorithm, the reader can refer to Thu90. Its complexity is linear in the number of triangles in $\mathcal{P}$. From this algorithm one can easily deduce an algorithm to build the maximal tiling.

### 2.5 Lattices

We will see in Section 3.2 that the height functions associated with the tilings of a polygon define a partial order on this tilings; this order has the interesting property of being a lattice, which we now define. See Figure 9 for an example of a graphical representation of a lattice.

Definition 2.10 (Lattice) A set $S$ partially ordered by a relation $\preccurlyeq$ is a lattice if for any $a$ and $b$ in $S$ there exist $i$ and $s$ in $S$ such that

- $i \preccurlyeq a \preccurlyeq s$ and $i \preccurlyeq b \preccurlyeq s$;
- if $x \preccurlyeq a$ and $x \preccurlyeq b$ then $x \preccurlyeq i$ for any $x \in S$;
- if $a \preccurlyeq y$ and $b \preccurlyeq y$ then $s \preccurlyeq y$ for any $y \in S$;

Such elements $i$ and $s$ are called the infimum and the supremum of $a$ and $b$, noted $\inf (a, b)$ and $\sup (a, b)$ (or $a \wedge b$ and $a \vee b$ for short). If $a$ and $b$ happen to be comparable (e.g. $a \preccurlyeq b$ ) then $i$ and $s$ are called the minimum and the maximum of $a$ and $b$, noted $\min (a, b)$ and $\max (a, b)$ (equal respectively to $a$ and $b$ if $a \preccurlyeq b$ ).


Figure 7: Ferrers diagrams and plane partitions

If $S$ is a finite lattice (such as the tiling lattices considered in this paper), then it admits one maximal and one minimal element.

A lattice $(S, \preccurlyeq)$ is distributive if for any $a, b$ and $c$ in $S$ one has

$$
\left\{\begin{array}{l}
a \wedge(b \vee c)=(a \wedge b) \vee(a \wedge c) \\
a \vee(b \wedge c)=(a \vee b) \wedge(a \vee c)
\end{array}\right.
$$

Definition 2.11 (Interval) An interval $[a ; b]$ (with $a \preccurlyeq b$ ) in a lattice ( $\mathcal{L}, \preccurlyeq$ ) is the set of all $x \in \mathcal{L}$ such that $a \preccurlyeq x \preccurlyeq b$.

Note that $a$ and $b$ need be comparable elements.
Proposition 2.12 Let $(\mathcal{L}, \preccurlyeq)$ be a lattice and $\mathcal{I}$ an interval in $\mathcal{L}$. The order $\preccurlyeq$ defines on $\mathcal{I}$ a structure of lattice.

### 2.6 Partitions, Ferrers diagrams and plane partitions

These definitions will be needed in Section 4.2. See also Figure 7.
Definition 2.13 A partition of an integer $n$ is a non-increasing list of positive integers with sum equal to $n$.

Definition 2.14 A Ferrers diagram is a geometrical representation of a partition: given a partition $\left(\lambda_{1}, \ldots, \lambda_{p}\right)$, the corresponding Ferrers diagram is a collection of $p$ consecutive left-justified rows respectively containing $\lambda_{1}, \ldots, \lambda_{p}$ consecutive squares.

Definition 2.15 A plane partition of an integer $n$ is a filling of a Ferrers diagram with integers (of sum $n$ ), such that the values along the rows and the columns is non-increasing.


Figure 8: Our building blocks

## 3 Flips and lattices

In this section, we first show that there is a one-to-one correspondence between tilings and height functions (up to constant) for a given polygon $\mathcal{P}$, which allows us to show that the set of the tilings of $\mathcal{P}$ under the partial order defined by the height functions has the structure of a lattice. We then consider a local operation (called a flip) on a tiling and prove that it has a straightforward translation in terms of the underlying height function.

A basic remark will be the leading idea for most of this section: a hexagon can be tiled in exactly two ways with lozenges (see Figure 8 (a)). Switching between these two tilings is called flipping. This operation allows us to deduce new tilings from a known one. The really challenging idea is that this allows us in fact to navigate between all the tilings of $\mathcal{D}$. All the results and proofs in this section are adapted to the case of lozenges from Rém9g, which deals with dominoes.

### 3.1 From height functions to tilings

We investigate in this section the link between height functions and tilings. An edge will be said positively directed if it is labelled by $a, b$ or $c$. Recall that an edge in the triangular grid is valid for a tiling $T$ if and only if it does not cut the interior of a lozenge in $T$.

We have seen (see Definition 2.6) how a height function $h$ can be defined, starting from a particular tiling $T$ of a polygon $\mathcal{P}$. Along any valid positively directed edge the height changes by 1 ; along any invalid positively directed edge, the height changes by -2 (see Figures $1(\mathrm{~b})$ and $8(\mathrm{~b})$ ). Therefore $h$ is merely an encoding of $T$, and the latter can be reconstructed from the former.

Lemma 3.1 There is a one-to-one correspondence between the tilings of a polygon $\mathcal{P}$ and the associated height functions (up to constant). In other words, if $T$ and $T^{\prime}$ are two tilings of $\mathcal{P}$ and if $h_{T}(v)=h_{T^{\prime}}(v)$ for any $v \in \mathcal{P}$ then $T=T^{\prime}$.

Proof Assume $\mathcal{P}$ and a height function $h_{T}$ given, the latter corresponding to an unknown tiling $T$ of $\mathcal{P}$. To rebuild $T$ from $h_{T}$, it suffices to draw all the triangles inside $\mathcal{P}$ and to erase those edges whose endpoints have a height
difference of 2 in absolute value. These edges lie on the interior of lozenges and completely characterize them so we are done.

Proposition 3.2 Let $T$ and $T^{\prime}$ be two tilings of a polygon $\mathcal{P}$. For each vertex $v$ of $\mathcal{P}, h_{T}(v)-h_{T^{\prime}}(v)$ is a multiple of 3 .

Proof We make the proof by induction on the vertices. The proposition is true on the boundary path of $\mathcal{P}$ since all height functions take the same values there (see Lemma 2.7). Assume that the proposition holds for $v$ and let $v^{\prime}$ be any of its neighbours in the triangular grid. Let $\mathcal{T}$ be either $T$ or $T^{\prime}$. If $\left(v, v^{\prime}\right)$ is a valid positively directed edge in $\mathcal{T}$ then $h_{\mathcal{T}}\left(v^{\prime}\right)-h_{\mathcal{T}}(v)=1$; otherwise, $h_{\mathcal{T}}\left(v^{\prime}\right)-h_{\mathcal{T}}(v)=-2$ and thus $h_{\mathcal{T}}\left(v^{\prime}\right)-h_{\mathcal{T}}(v)$ takes the same value modulo 3 whether $\left(v, v^{\prime}\right)$ is valid or not. Consequently, if $h_{T}(v)-h_{T^{\prime}}(v)$ is a multiple of 3 , then so is $h_{T}\left(v^{\prime}\right)-h_{T^{\prime}}\left(v^{\prime}\right)$.

An intuitive way to consider this proposition is to think in terms of cubes: adding 3 to the value of a height function on a vertex $v$ is equivalent to adding exactly one cube (see Figure 8 (a)).

### 3.2 The lattice structure

In this section, we prove that the set of the tilings of a polygon can be endowed with a lattice structure. To this end, we define a partial order between tilings by using their height functions:

Definition 3.3 Let $h_{T}$ and $h_{T^{\prime}}$ be two height functions associated with the tilings $T$ and $T^{\prime}$ of a polygon $\mathcal{P}$ such that $h_{T}$ and $h_{T^{\prime}}$ take the same values on the boundary path of $\mathcal{P} . h_{T}$ is less than $h_{T^{\prime}}$ (and we note $h_{T} \leqslant h_{T^{\prime}}$ ) if $h_{T}(v) \leqslant h_{T^{\prime}}(v)$ for any vertex $v \in \mathcal{P}$.

The order between height functions allow us to endow the set of the tilings of $\mathcal{P}$ with a natural order: $T \preccurlyeq T^{\prime}$ if and only if $h_{T} \leqslant h_{T^{\prime}}$.

Proposition 3.4 Let $T$ and $T^{\prime}$ be two tilings of polygon $P$ and let $h_{T}$ and $h_{T^{\prime}}$ be their height functions. The functions $h_{\min }=\min \left(h_{T}, h_{T^{\prime}}\right)$ and $h_{\max }=$ $\max \left(h_{T}, h_{T^{\prime}}\right)$ are themselves height functions.

Proof We prove the result only for $h_{\min }$ since the other case is analogous. We make the proof by induction. Since $h_{T}$ and $h_{T^{\prime}}$ take the same values on the boundary path $\mathcal{B}$ of $\mathcal{P}$ (see Lemma 2.7), $h_{\min }(v)=h_{T}(v)=h_{T^{\prime}}(v)$ for every $v \in \mathcal{B}$. Let $\left(v, v^{\prime}\right)$ be any positively directed edge between vertices of $\mathcal{P}$ in the triangular grid. We claim that $h_{\min }\left(v^{\prime}\right)-h_{\min }(v)$ equals either 1 or -2 .

- If $h_{T}(v)=h_{T^{\prime}}(v)$ then $h_{\min }\left(v^{\prime}\right)-h_{\min }(v)$ equals either $h_{T}\left(v^{\prime}\right)-h_{T}(v)$ or $h_{T^{\prime}}\left(v^{\prime}\right)-h_{T^{\prime}}(v)$ and must therefore be equal to either 1 or -2 .


Figure 9: An example of lattice

- We can now assume without loss of generality that $h_{T}(v)<h_{T^{\prime}}(v)$, so that $h_{\min }(v)=h_{T}(v)$. $h_{T}\left(v^{\prime}\right)$ can only be $h_{T}(v)+1$ or $h_{T}(v)-2$, and $h_{T^{\prime}}(v)$ is at least $h_{T}(v)+3$ by Proposition 3.2. Moreover $h_{T^{\prime}}\left(v^{\prime}\right)$ can only be $h_{T^{\prime}}(v)+1$ or $h_{T^{\prime}}(v)-2$, and therefore $h_{T^{\prime}}\left(v^{\prime}\right)$ is at least $h_{T}(v)+1$. Thus $h_{\min }\left(v^{\prime}\right)=$ $h_{T}\left(v^{\prime}\right)$, from which we derive $h_{\min }\left(v^{\prime}\right)-h_{\min }(v)=h_{T}\left(v^{\prime}\right)-h_{T}(v)$ which must be either 1 or -2 .

We have shown that $h_{\text {min }}$ increases by either 1 or -2 along any positively directed edge in the triangular grid. Consequently, the set of heights (for $h_{\min }$ ) modulo 3 of the vertices of any triangle in the triangular grid must exactly be $\{0,1,2\}$. Moreover the height difference along an edge does not depend on which of the two neighbouring triangles was chosen. Erase all edges whose endpoints have a difference of heights of -2 . What is left is a tiling of $\mathcal{D}$ with lozenges; that is, $h_{\min }$ is a height function.

Corollary 3.5 The order $\preccurlyeq$ induces a structure of distributive lattice on the set of the tilings of $\mathcal{D}$.

Proof If $T_{1}$ and $T_{2}$ are two tilings of a domain $\mathcal{D}$ then the height functions $h_{\min }\left(T_{1}, T_{2}\right)$ and $h_{\max }\left(T_{1}, T_{2}\right)$ are clearly the infimum and supremum of the height functions $h_{T_{1}}$ and $h_{T_{2}}$ (see Definition 2.10), and since height functions encode tilings (see Proposition 3.2) we have defined the infimum and supremum of $T_{1}$ and $T_{2}$.

To prove distributivity, it suffices to consider each vertex of $\mathcal{D}$, which brings us back to checking the relation for integers and the usual min and max functions.

Figure 9 provides an example of the graphical representation of a lattice of tilings. Intuitively, one obtains the infimum of two tilings $T_{1}$ and $T_{2}$ by selecting only the cubes which appear in both, and their supremum by selecting the cubes which appear in either one. In other words, $\inf \left(T_{1}, T_{2}\right)$ is encoded by $h_{\min }\left(T_{1}, T_{2}\right)$ and $\sup \left(T_{1}, T_{2}\right)$ is encoded by $h_{\max }\left(T_{1}, T_{2}\right)$.

### 3.3 Flips

In this section, we introduce an elementary operation classically called a flip; we prove that the set of the tilings of a polygon is connected by flips.

Definition 3.6 Let $T$ be a tiling of a polygon $\mathcal{P}$. A local maximum (resp. minimum) of the height function $h_{T}$ is a vertex $v$ in the interior of $\mathcal{P}$ such that $h_{T}(v) \geqslant h_{T}\left(v^{\prime}\right)\left(\right.$ resp. $\left.h_{T}(v) \leqslant h_{T}\left(v^{\prime}\right)\right)$ for any $v^{\prime}$ neighbour of $v$.

Proposition 3.7 Let $T$ be a tiling of a polygon $P$. A vertex $v$ in the interior of $\mathcal{P}$ is a local extremum of the height function $h_{T}$ if and only if it is the center of a hexagon tiled with three lozenges.

Proof Let $v^{\prime}$ and $v^{\prime \prime}$ be two neighbours of $v$ so that $\left(v, v^{\prime}, v^{\prime \prime}\right)$ is a triangle. If ( $v^{\prime}, v$ ) and $\left(v, v^{\prime \prime}\right)$ were both valid edges in $T$, then $h(v)$ would be less than $h\left(v^{\prime}\right)$ and more than $h\left(v^{\prime \prime}\right)$ (or the converse), so it would not be an extremum. Moreover $\left(v^{\prime}, v\right)$ and $\left(v, v^{\prime \prime}\right)$ cannot both be invalid edges, therefore exactly one of them is. Consequently, the hexagon around $v$ is tiled with exactly three lozenges.

The converse part of the proof is obvious (see Figure (a)).
Definition 3.8 (Flip) A flip is the operation by which one switches from one tiling of a hexagon to the other. An up-flip increases the height while a down-flip decreases it.

We now prove that flips allow us to reach any tiling of $\mathcal{P}$ from any other tiling of $\mathcal{P}$. To this end, we need one more definition:

Definition 3.9 If $T$ and $T^{\prime}$ are two tilings of a polygon $\mathcal{P}$, then the distance between them is

$$
\Delta\left(T, T^{\prime}\right)=\sum_{v \in \mathcal{D}}\left|h_{T}(v)-h_{T^{\prime}}(v)\right|
$$

This function is indeed a distance since it is symmetrical, satisfies separation $\left(\Delta\left(T, T^{\prime}\right)=0\right.$ if and only if $T=T^{\prime}$ by Proposition (3.2) and the triangular inequality (since $|\cdot|$ does).

Proposition 3.10 Let $T$ and $T^{\prime}$ be two tilings of a polygon $\mathcal{P}$. If $T \preccurlyeq T^{\prime}$ then there exists a sequence ( $T_{0}=T, T_{1}, \ldots, T_{n}=T^{\prime}$ ) of tilings of $\mathcal{D}$ such that $T_{p+1}$ is deduced from $T_{p}$ by a single up-flip, $0 \leqslant p \leqslant n-1$.

Proof Assume $T \prec T^{\prime}$, let $m=\min \left\{h_{T}(v) \mid v \in \mathcal{P}\right.$ and $\left.h_{T}(v)<h_{T^{\prime}}(v)\right\}$ and let $v$ be a vertex of $\mathcal{P}$ such that $h_{T}(v)=m$. The vertex $v$ cannot belong to the boundary path of $\mathcal{P}$ by Lemma 2.7 and $h_{T^{\prime}}(v)$ must be at least $h_{T}(v)+3$ by Proposition 3.2. Let $v^{\prime}$ be a neighbour of $v$ such that the edge $\left(v, v^{\prime}\right)$ is valid for
$h_{T} . h_{T^{\prime}}\left(v^{\prime}\right)$ is at least $h_{T^{\prime}}(v)-2 \geqslant h_{T}(v)+1>h_{T}(v)$. Because $m$ is minimal, the edge $\left(v, v^{\prime}\right)$ must be directed from $v$ to $v^{\prime}$. Since this is true for any edge anchored in $v$ that is valid in $h_{T}$, we conclude that $v$ is a local minimum of $h_{T}$.

By an up-flip on $v$, we send $T$ to a tiling $T_{1}$ such that $T \prec T_{1} \preccurlyeq T^{\prime}$. Besides, $h_{T_{1}}$ differs from $h_{T}$ only on the vertex $v$, so that $\Delta\left(T, T_{1}\right)=3$. Using induction, we thus build a sequence of ever greater tilings. Since the distance between two tilings must be a multiple of 3 (see Proposition 3.2), there comes a tiling $T_{n}$ such that $\Delta\left(T_{n}, T^{\prime}\right)=0$ and $T_{n} \preccurlyeq T^{\prime}$, and since height functions encode tilings (see Lemma 3.1), $T_{n}=T^{\prime}$.

Theorem 3.11 The set of the tilings of $\mathcal{P}$ is connex: any tiling $T$ of $\mathcal{P}$ can be reached from any other tiling $T^{\prime}$ of $\mathcal{P}$ by using only flips.

Proof Indeed, any tiling of $\mathcal{P}$ can be reached from the minimal one by using only up-flips (see Proposition 3.10) and the minimal tiling of $\mathcal{P}$ can be reached from any other tiling by using only down-flips.

## 4 Hexagons and pseudo-hexagons

The ultimate goal of this paper is to provide an algorithm for uniquely generating all the elements of the lattice of the tilings of a polygon $P$. As we will see in Section 5, some hexagonal-like subsets of $P$ play a important role. We thus start with the rather simple case of $\mathcal{P}$ being a hexagon or a pseudohexagon (see definitions below).

### 4.1 Piles of cubes

It is well known in tilings lore that the tilings of a hexagon of side $n$ are bijectively related to some piles of cubes. Using Conway and Lagarias' lozenge group, we have explicited in Section 2.3 (following Thu90) the algebraic translation of the geometrical intuition. The bijection between the lozenge group and $\mathbb{Z}^{3}$ implies that it is equivalent to consider lozenges in a tiling or 2-cells (squares) in the cubical tesselation of $\mathbb{Z}^{3}$. Until now, we have been interested in manipulating tilings and watching the translation in the figure. For instance, rearranging the three tiles of a hexagon of side 1 is equivalent to raising the height of exactly one point by 3 : this is an up-flip. Now we look at the bijection the other way: we manipulate squares in $\mathbb{Z}^{3}$ to produce new informations on tilings. For instance, it is easily seen from Figure 9 that while using only flips suffices to generate all the tilings of a polygon, it also generates multiple times the same tiling.

Not all collections of squares in $\mathbb{Z}^{3}$ correspond, once projected onto the plane, to tilings in the triangular grid: see for instance Figure 11 (a). So we need conditions on the squares. In fact, we will build our reasoning not on squares but on cubes, because the elementary operation on a tiling, the flip, has a natural, visual and intuitive translation in terms of cubes (see Figure 10). Moreover, fracture lines (see Section 5.2) can be used to get rid of squares that do not correspond to cubes.


Figure 10: An up-flip is equivalent to adding a cube

(a) Neither a tiling nor a compact pile of cubes

(b) A pseudo-hexagon delimited by three Ferrers diagrams

Figure 11: Compact piles and pseudo-hexagons

The minimal tiling of a polygon has no local maximum in the interior of the polygon, otherwise this maximum could be flipped down. In other words, there is no cube associated to such a tiling. It merely provides squares on which one can put cubes. In the case of a hexagon, the association of the squares can be seen as base planes in $\left(\mathbb{Z}_{+}\right)^{3}$ so that any cube can be encoded by the (integer) coordinates of its lowest corner.

Cubes can be piled, but if we want to preserve a tiling after projection, we must pile them in a way that corresponds to flips. Such piles will be called compact.

Definition 4.1 (Compact pile) A pile of cubes is compact if for any cube of the pile at $\left(i_{0}, j_{0}, k_{0}\right)$ there are cubes at $\left(i, j_{0}, k_{0}\right),\left(i_{0}, j, k_{0}\right)$ and $\left(i_{0}, j_{0}, k\right)$ with $i, j$ and $k$ ranging in $\left[0, i_{0}\right],\left[0, j_{0}\right]$ and $\left[0, k_{0}\right]$.

The next lemma easily stems from this definition:
Lemma 4.2 A pile of cubes is compact if and only if any section of the pile by a plane orthogonal to one axis $(O x, O y$ or $O z)$ yields a Ferrers diagram.

It will not be enough to consider only "perfect" hexagons of size $n \times n \times n$. The hexagon-like sub-polygons that appear in tilings require a somewhat more general approach (see Section 5.5):

Definition 4.3 (Pseudo-hexagon) A pseudo-hexagon is the domain obtained by starting from a compact pile of cubes and performing all the possible downflips.


Figure 12: Plane partitions and limited partitions

Note that by Lemma 4.2 a pseudo-hexagon can also be seen as a figure delimited by three Ferrers diagrams fitting neatly (see Figure 11 (b)). It is also tileable by construction.

Proposition 4.4 A pile of cubes in a peudo-hexagon corresponds to a tiling of the pseudo-hexagon if and only if it is compact.

Proof A pile of cubes corresponds to a tiling of a domain $\mathcal{D}$ if and only if it can be generated (starting from the minimal tiling) by using only up-flips (see Corollary 3.5). An up-flip adds a cube that lies on squares that belong either to the minimal tiling or to an already added cube. The result follows by induction on the number of cubes and Lemma 4.2 .

### 4.2 An algorithm to generate the tilings of a pseudo-hexagon

Proposition 4.4 means that generating the tilings of a pseudo-hexagon is equivalent to generating compact arrangements of cubes, which is quite easier. Indeed, the latter are related one-to-one with plane partitions (see Definition 2.15, Figure 12 (a) and Els84).

Definition and notation 4.5 The parts of the partition $p$ associated with a Ferrers diagram $F$ are noted $p[j]$, and $F[j]$ denotes a collection of $p[j]$ linearly adjacent squares of $\mathbb{Z}^{2}$. A partition on $F[j]$ is a non-increasing sequence of integers placed on this squares. Two partitions $p$ and $q$ are comparable and $p \leqslant q$ if $p[j] \leqslant q[j]$ for all $j$.

A plane partition $P$ on a Ferrers diagram $F$ is a non-increasing sequence of partitions on the $F[j]$. The partition on $F[j]$ is noted $P[j]$ and called the $j$-slice of $P$. Two plane partitions $P$ and $Q$ are comparable and $P \leqslant Q$ if $P[j] \leqslant Q[j]$ for all $j$.

The weight of a partition is its sum; the weight of a plane partition is its sum.
The plane partitions we are dealing with correspond to the tilings of a pseudo-hexagon; they are thus limited, in the sense that the corresponding pile must be embedded in the pile that defines the pseudo-hexagon.

Definition 4.6 A $(s, P)$ limited plane partition is a plane partition $A$ of the integer $s$ such that the plane partitions $A$ and $P$ are comparable and $A \leqslant P$.

We will generate all $(s, P)$ limited plane partitions recursively (see Algorithm 4.11), adding one $j$-slice at a time. To this end, we need to generate the partitions that are less than a given partition.

Definition 4.7 A $(s, F)$ limited partition is a partition $a$ of the integer $s$ such that $a \leqslant p$ where $p$ is the partition associated with the Ferrers diagram $F$.

Figure 12 (b) shows of a graphical representation of a limited partition.
The following recursive algorithm uniquely generates all $(s, F)$ limited partition by adding one by one the parts of the final partitions. The average step is thus to add a row with $k$ squares to a yet unfinished Ferrers diagram; the value of this part must at least be permitted by the "geometry" of $F$ and be less than the last added part. The value must also allow the updated $s$ to fit in the remaining "space"; this is the meaning of $r_{q+1}$ below.

## Algorithm 4.8

- Input: Integers $s, p_{1}, \ldots, p_{n}$ such that $s \leqslant p_{1}+\cdots+p_{n}$.
- Output: List $\mathcal{L}$ of all $\left(s,\left(p_{1}, \ldots, p_{n}\right)\right)$ limited partitions.
- Compute $\operatorname{loop}\left(s, \emptyset,\left(p_{1}, \ldots, p_{n}\right)\right)$ where
$\operatorname{loop}\left(t,\left(a_{1}, \ldots, a_{q}\right),\left(p_{q+1}, \ldots, p_{n}\right)\right)=$
- if $t=0, \mathcal{L} \leftarrow(a_{1}, \ldots, a_{q}, \underbrace{0, \ldots, 0}_{(n-q) \text { times }})$;
- else if $n=q+1, \mathcal{L} \leftarrow\left(a_{1}, \ldots, a_{q}, t\right)$;
- else compute

$$
\begin{aligned}
& \operatorname{loop}\left(t-k,\left(a_{1}, \ldots, a_{q}, k\right),\left(\min \left(p_{q+2}, k\right), \ldots, \min \left(p_{n}, k\right)\right)\right) \\
& \text { for } k=\min \left(p_{q+1}, t, a_{q}\right) \text { downto } r_{q+1} \\
& \text { where } r_{q+1}=\min \left\{1 \leqslant j \leqslant \min \left(p_{q+1}, t, a_{q}\right) \mid\right. \\
& \left.\qquad j+\min \left(p_{q+2}, j\right)+\cdots+\min \left(p_{n}, j\right) \geqslant t\right\} .
\end{aligned}
$$

Proof of the algorithm A more basic way to write the algorithm would be to add any number of squares $\left(k=1 . . \min \left(p_{q+1}, t, a_{q}\right)\right)$ at each step: the min simply ensures that the resulting collection of squares is indeed a Ferrers diagram (the new part must be less than its counterpart in the limiting partition $\left(p_{q+1}\right)$, less than the number of remaining squares $(t)$ and less than the last added part $\left.\left(a_{q}\right)\right)$ and this procedure obviously generates all the desired partitions.

But it is also wasteful, in that many such combinations would not satisfy $t \leqslant$ $p_{q+1}+\cdots+p_{n}$, so we restrict the range of $k$ until its minimal value guarantees, by construction, that the following steps will result in a partition limited by $p$.

Since the values we eliminate would not ultimately yield such a partition, we still generate all the desired partitions.

Finally, our algorithm executes a depth-first search on a dynamically generated tree $\mathcal{T}$ of height $n$ and whose $p^{\text {th }}$ level features all the possible combinations for the $p$ first rows (the other rows being empty), so that no partition is obtained twice.

Complexity Since we perform a depth-first search on $\mathcal{T}$, the execution space of the algorithm is bounded by the height of $\mathcal{T}$ ( $n$, the number of parts in the limiting partition) times the number of leaves, which is the number of $\left(s,\left(p_{1}, \ldots, p_{n}\right)\right)$ limited partitions. We believe no closed formula is known for the latter, but generating series techniques (see for instance FS96 and FSS) yield lemma 4.10:

Definition 4.9 The multidegree of a monomial $u_{1}{ }^{i_{1}} \cdots u_{n}{ }^{i_{n}}$ is the sequence $\left(i_{1}, \ldots, i_{n}\right)$. Its cumulated multidegree is the sequence $\left(i_{1}+\cdots+i_{n}, i_{2}+\cdots+\right.$ $\left.i_{n}, \ldots, i_{n}\right)$.

Lemma 4.10 The number of $\left(s, p=\left(p_{1}, \ldots, p_{n}\right)\right)$ limited partitions is the number of monomials whose cumulated multidegree is less than the conjugate partition of $p$ in

$$
P\left(u_{1}, \ldots, u_{n}\right)=\left[z^{s}\right] \prod_{k=1}^{n} \frac{1}{1-u_{k} z^{k}}
$$

Proof (Sketch) The exponent of $u_{k}$ in a monomial $u_{1}{ }^{i_{1}} \cdots u_{p}{ }^{i_{n}}$ in $P\left(u_{1}, \ldots, u_{n}\right)$ counts the number of rows of length $k$ in the corresponding partition (see Figure 13). This monomial encodes a partition limited by $p$ if and only if it satisfies $i_{r}+i_{r+1}+\cdots+i_{n} \leqslant m(r)+m(r+1)+\cdots+m(n)$ for $1 \leqslant r \leqslant n$ where $m(k)$ is the number of occurences of $k$ in the conjugate partition of $p$.

$$
\begin{gathered}
{\left[z^{5}\right]\left(\frac{1}{1-u_{1} z} \times \frac{1}{1-u_{2} z^{2}}\right)=u_{1}^{1} u_{2}{ }^{2}+u_{1}^{3} u_{2}^{1}+u_{1}^{5} u_{2}{ }^{0}} \\
\text { 由 }
\end{gathered}
$$

Figure 13: Generating series and Ferrers diagrams

Finally, the execution time of the algorithm is proportionnal to its execution space, the multiplying factor being the time needed to compute $r_{q+1}$ at each step, which is at most $n \times p_{1}$ (exhaustive search).

Note that since truncated generating series can be computed by standard programs such as Maple, lemma 4.10 provides another (sub-optimal, but handy) way to generate limited partitions.

We now undertake to generate all the tilings of a pseudo-hexagon, or rather all plane partitions limited by a given plane partition $P=\left(P_{1}, \ldots, P_{n}\right)$ where
the $P_{j}$ 's are the $j$-slices of $P$. It is enough to generate all $(s, P)$ limited plane partitions and to let $s$ range from 0 to the weight $w(P)$ of $P$.

## Algorithm 4.11

- Input: An integer $s$ and a plane partition $P=\left(P_{1}, \ldots, P_{n}\right)$ such that $s \leqslant w(P)$.
- Output: The list $\mathcal{L}$ of all $(s, P)$ limited plane partitions.
- Compute $\operatorname{loop}\left(s, \emptyset,\left(P_{1}, \ldots, P_{n}\right)\right)$ where
$\operatorname{loop}\left(t,\left(A_{1}, \ldots, A_{q}\right),\left(P_{q+1}, \ldots, P_{n}\right)\right)=$
- if $t=0, \mathcal{L} \leftarrow(A_{1}, \ldots, A_{q}, \underbrace{0, \ldots, 0}_{(n-q) \text { times }})$;
- else if $n=q+1$,
* generate all $\left(t, \min \left(A_{q}, P_{q+1}\right)\right)$ limited partitions with Algorithm $4.8 ;$
* for each such partition $p, \mathcal{L} \leftarrow\left(A_{1}, \ldots, A_{q}, p\right)$;
- else for all partition $k$ such that

$$
\left\{\begin{array}{l}
w(k) \leqslant t \\
k \leqslant \min \left(A_{q}, P_{q+1}\right) \\
k+w\left(\min \left(k, P_{q+1}\right)\right)+\cdots+w\left(\min \left(k, P_{n}\right)\right) \geqslant t
\end{array}\right.
$$

compute

$$
\operatorname{loop}\left(t-w(k),\left(A_{1}, \ldots, A_{q}, k\right),\left(\min \left(k, P_{q+2}\right), \ldots, \min \left(k, P_{n}\right)\right)\right)
$$

Proof and complexity of the algorithm The analysis is completely analogous to Algorithm 4.8 s. The execution space is at most $n$ times the number of $(s, P)$ limited plane partitions (for which we believe no closed formula is known, but as above a characterization with generating series techniques can be obtained) and the execution time is at most the execution space times $n$ times the number of ( $u, P_{1}$ ) limited partitions for $u=0 . . w\left(P_{1}\right)$ (exhaustive search).

Note that Algorithm 4.8 is a special case of Algorithm 4.11, in the same manner that a partition is a special case of plane partition; we presented both for the sake of clarity.

## 5 Domains, fracture lines and seeds

Up to now, we have considered polygons, which are hole-free (simply connected) domains on which one knows how to define a consistent height function, an essential ingredient of Section 3. The subsequent results in this paper hold provided the results in Section B do, but this does not necessarily mean one has

(a) A path delimiting a domain with a hole

(b) This domain can be tiled even though it has a hole

Figure 14: Domains with holes
to deal only with simply connected domains (see Section 5.1). It would be quite exciting to examine how works like Fou97 or Fou01 extend to the results of this paper.

For the rest of this paper, we consider domains for which the results in Section 3 hold. To reflect this fact, we will not use the word "polygon".

A domain $\mathcal{D}$ is defined by its contour path, which is simply any finite-length closed path $\mathcal{P}$ in the triangular grid. Fracture lines (see Section 5.2) will allow us to break the domain into several parts (called fertile zones) whose tilings can be generated independently. The lattice of the tilings of $\mathcal{D}$ can then be obtained as the product of the lattices of the tilings of the fertile zones. Finally, seeds (see Section 5.5) will allow us to decompose fertile zones into pseudo-hexagons, for which much is already known.

### 5.1 Domains with holes

A domain defined by a finite-length closed path $\mathcal{P}$ in the triangular grid need not be hole-free: it suffices that $\mathcal{P}$ cross itself in a non-trivial way (see Figure 14 (a) for an example). Such a domain may still be tileable (see Figure 14 (b)). The definitions and results in this paper apply to such domains provided the results of Section 3 hold.

### 5.2 Fracture lines, fertile zones and the fracture algorithm

When considering a general (tileable) domain $\mathcal{D}$, it may happen that a given tile is "fixed", in the sense that it cannot be changed by a flip, so we'd like to remove such a tile and concentrate on "interesting" zones (see Section 5.3). The tool for this is the fracture line, made of vertices whose height does not vary between the minimal and maximal tilings. It allows us to "remove" fixed lozenges but there's more: it also allows us to define disjoint sub-domains which can be tiled independently (see Section 5.3), so that the lattice of the tilings of $\mathcal{D}$ can be obtained by computing a product of smaller, simpler lattices (see Section 5.4). This section attempts to clarify all of the above, step by step, until a complete algorithm can be written, proved and analyzed.

Fracture lines have been investigated in the case of dominoes, notably in Fou96]. In all this section, $\mathcal{D}$ is assumed to be a tileable domain such that the
results of Section 3 hold.
Definition 5.1 Let $T$ be any tiling of $\mathcal{D}$ and let $h$ be the associated height function. A path $P$ in $\mathcal{D}$ is $T$-valid if it is a single vertex of $\mathcal{D}$ or if $\left|h\left(v^{\prime}\right)-h(v)\right|=$ 1 for $v, v^{\prime}$ two consecutive vertices of $P$ (i.e. it follows the edges of lozenges in $T$ ).

A path $P$ in $\mathcal{D}$ is a $T$-valid cycle if it is a closed, repetition-free path in $\mathcal{D}$ and a $T$-valid path.

A subset $F$ of $\mathcal{D}$ is a sub-domain of $\mathcal{D}$ if there exists a tiling $T$ of $\mathcal{D}$ and a $T$-valid cycle $C$ such that $C$ is the contour path of $F$.

Definition 5.2 Let $\mathcal{D}$ be any tileable domain and $v$ one of its vertices. We denote by $\Delta h(v)$ the difference between the height of $v$ in the maximal tiling of $\mathcal{D}$ and its height in the minimal tiling. If $\Delta h(v)=0$ then $v$ is said to be solid.

Definition 5.3 (Fracture line) A fracture line in $\mathcal{D}$ is a $T$-valid cycle in $\mathcal{D}$ (for some tiling $T$ of $\mathcal{D}$ ) that is composed only of solid points. Every domain $\mathcal{D}$ has at least one such fracture line, namely its contour path, which we will call the trivial fracture line of $\mathcal{D}$.

We now undertake to prove Theorem 5.15. We first prove (see Proposition 5.8) that a solid point in the interior of $\mathcal{D}$ cannot be isolated: it belongs at least to a solid lozenge.

Lemma 5.4 Let $F$ be a sub-domain of $\mathcal{D}$, delimited by a valid cycle in a tiling $T$ of $\mathcal{D}$. If a triangle $t$ belongs to $F$, then so does the lozenge $\ell$ to which $t$ belongs in $T$.

Proof Let $a, b, c$ be the vertices of $t$, and let $d$ be the fourth vertex of $\ell$. There are exactly two couples made of vertices in $\{a, b, c\}$ which have a height difference equal to 1 in $T$. Let $a$ and $b$ be the two vertices which have a height difference of 2 (in absolute value), so that $[a ; b]$ is not a valid path in $T$. Since $F$ is delimited by a $T$-valid cycle, $[a ; b]$ cannot be part of the contour path of $F$. Therefore $d$ belongs to $F$, whence the triangle $a b d$ belongs to $F$, and since $\ell$ is the union of the triangles $a b c$ and $a b d$, we are done.

Proposition 5.5 Any sub-domain of a tileable domain is itself tileable.
Proof Let $F$, a sub-domain of a tileable domain $\mathcal{D}$, be delimited by a valid cycle $L$ in a tiling $T$ of $\mathcal{D}$.

If $F$ is a single point, we are done (using no tile). Otherwise, let $s$ be a segment of $L$; it belongs to 2 triangles in the triangular grid, one of which must belong to $F$. By Lemma 5.4, the lozenge $\ell$ to which this triangle belongs in $T$ belongs to $F$. If $F=\ell$, we are done.

Otherwise, let $F_{1}$ be the subset of $\mathcal{D}$ obtained by removing $\ell$ from $F$. It is a priori composed of several disjoint subsets of $\mathcal{D}$, but each of these subsets
is delimited by a $T$-valid cycle so that the situation is equivalent to a single domain $F_{1}$, which is then a sub-domain of $\mathcal{D}$.

Let $F_{p}$ be the domain obtained by recursively removing a lozenge until nothing can be done. If $F_{p}$ contained a triangle it would also contain a lozenge by Lemma 5.4 and so it would not be a fixed point. Therefore it contains no triangle, which means that it is a single point and $F$ is a collection of lozenges and therefore tileable.

Corollary 5.6 A fracture line of $\mathcal{D}$ delimits a tileable sub-domain of $\mathcal{D}$.
Proof Indeed, a fracture line is a valid cycle so it delimits a sub-domain.
Lemma 5.7 Let $s_{1}, \ldots, s_{6}$ be six vertices of $\mathcal{D}$ that define a hexagon of side 1 . Let $c$ be the center of this hexagon. If $\Delta h\left(s_{k}\right) \neq 0$ for $k=1 . .6$, then $\Delta h(c) \neq 0$.

In other words, if $c$ is a vertex in the interior of $\mathcal{D}$ that cannot be flipped, then at least one of its neighbours cannot be flipped.
Proof Let $h_{\text {min }}$ and $h_{\max }$ denote the minimal and maximal height functions. Since $c$ and $s_{k}$ belong to the same triangle, $h_{\min }(c) \leqslant h_{\min }\left(s_{k}\right)+2$ (see Section 3.1). Moreover $h_{\max }\left(s_{k}\right) \geqslant h_{\min }\left(s_{k}\right)+3$ since $\Delta h\left(s_{k}\right) \neq 0$, from which we derive $h_{\text {min }}(c) \leqslant h_{\text {max }}\left(s_{k}\right)-1$. In the maximal tiling, no vertex can be a local minimum (see Proposition 3.7 and Definition 3.8), so $h_{\max }(c)$ cannot be less than $h_{\min }\left(s_{k}\right)-1$ for all $k=1 . .6$, whence $h_{\max }(c) \neq h_{\min }(c)$.

Proposition 5.8 Let $\mathcal{D}$ be any tileable domain, not limited to a single point, and let $v$ be a solid vertex of $\mathcal{D}$. Then $v$ belongs to a fracture line that encloses at least a lozenge.

Proof The result is trivial if $v$ is on the boundary path of $\mathcal{D}$. We now assume that we are not in this case. By Lemma 5.7 we know that there exists a solid vertex $v^{\prime}$ at distance 1 (in the triangular grid) of $v$. Let $T$ be any tiling of $\mathcal{D}$ and let $h$ be the associated height function. If $\left|h(v)-h\left(v^{\prime}\right)\right|$ is equal to 2 then the lozenge whose middle segment is $\left[v ; v^{\prime}\right]$ is defined by two fracture points and we are done.

If $\left|h(v)-h\left(v^{\prime}\right)\right|=1$ then $\left(v, v^{\prime}\right)$ is a valid path in $T$. Let now $s_{1}, \ldots, s_{5}$ and $v^{\prime}$ be the (distinct) vertices of the hexagon around $v$. We prove $a b$ absurdo that at least one of the $s_{k}$ is solid: assume that none of them is solid, so that $h_{\text {min }}\left(s_{k}\right) \neq h_{\text {max }}\left(s_{k}\right)$ for $k=1 . .5$.

- Let us assume that $h(v)=h\left(v^{\prime}\right)-1$. We also have $h(v) \leq h_{\min }\left(s_{k}\right)+2 \leqslant$ $\left(h_{\max }\left(s_{k}\right)-3\right)+2 \leqslant h_{\max }\left(s_{k}\right)-1$ so that $h(v)$ would be less than the height of any of its immediate neighbours, i.e. it would be a local minimum, and thus $\Delta h(v)$ could not be 0 .
- Let us now assume $h(v)=h\left(v^{\prime}\right)+1$. We also have $h(v) \geqslant h_{\max }\left(s_{k}\right)-2 \geqslant$ $\left(h_{\min }\left(s_{k}\right)+3\right)-2=h_{\min }\left(s_{k}\right)+1$ so $h(v)$ would be more than the height of any of its immediate neighbours and could be down-flipped.


Figure 15: The union of two pre-fertile zones need not be pre-fertile

Either case brings a contradiction so at least one of the $s_{k}$ is solid; let us note it $v^{\prime \prime}$. If $\left|h(v)-h\left(v^{\prime \prime}\right)\right|=2$ then, as above, we are done. Otherwise, we have shown that $v$ has two immediate neighbours which are solid and linked to $v$ by an arc in $T$. The same result applies to $v^{\prime}$ and $v^{\prime \prime}$ so that by induction, since the number of vertices in $\mathcal{D}$ is finite, one obtains a fracture line of $\mathcal{D}$. Since the domain delimited by this line contains at least a triangle, it contains at least a lozenge by Lemma 5.4.

We have now completed our preliminary study of fracture lines. Before proving Theorem 5.15, we need to know a bit more about the sub-domains containing no fracture lines. There are two cases, according to whether or not all the vertices of the sub-domain are on its contour path. We start with the second case.

Definition 5.9 A sub-domain $\mathcal{D}^{\prime}$ of $\mathcal{D}$ is pre-fertile if:

- it is connected and tileable;
- at least one of its vertices is not on its contour path $P$;
- $\Delta h(v) \neq 0$ for any such vertex;
- two vertices of $P$ are at distance 1 in the triangular grid if and only if they are neighbours in $P$.

A sub-domain is pre-fertile if all its inner points can be flipped at least once. The union of two pre-fertile zones need not be a pre-fertile zone. Consider for instance Figure 15 where $\mathcal{D}_{1}$ is the left-hand side hexagon and $\mathcal{D}_{2}$ is the righthand side one, so that $A$ belongs to the contour path of both. One sees that $A$ cannot be involved in any flip, whence $\Delta h(A)=0$; and yet it is not on the contour path of $\mathcal{D}_{1} \cup \mathcal{D}_{2}$. For this reason we give the following definition (both $\mathcal{D}_{1}$ and $\mathcal{D}_{2}$ are examples of such zones):

Definition 5.10 (Fertile zone) Let $\mathcal{D}_{1}$ and $\mathcal{D}_{2}$ be two pre-fertile sub-domains of $\mathcal{D}$. Their union is fertile if $\mathcal{D}_{1} \cup \mathcal{D}_{2}$ is itself pre-fertile. A sub-domain of $\mathcal{D}$ is fertile if it is pre-fertile and a maximal element for fertile union.

Proposition 5.11 Let $F$ be a pre-fertile sub-domain of $\mathcal{D}$. It is fertile if and only if its contour path is a fracture line of $\mathcal{D}$.

Proof We first prove that the contour path of a fertile sub-domain $F$ is a fracture line in $\mathcal{D}$. Let $a$ be any vertex of the contour path of $F$. If $a$ belongs to the contour path of $\mathcal{D}$, it is solid by Lemma 2.7. Otherwise, $a$ is in the interior of $\mathcal{D}$; let $H_{a}$ be the hexagon (in the triangular grid) of side 1 whose center is $a$. Since $F$ is fertile it is a maximal element for fertile union so that $F \cup H_{a}$ is not pre-fertile, which means that at least one of its inner vertices is solid. Since none of $F$ 's interior is, $a$ must be this vertex. In other words, all the vertices on the contour path of $F$ are solid and we are done.

Conversely, we prove that a pre-fertile sub-domain whose contour path is a fracture line in $\mathcal{D}$ is fertile. Let $F^{\prime}$ be any other pre-fertile sub-domain of $\mathcal{D}$ such that $F \cap F^{\prime}=\emptyset$ and $F \cup F^{\prime}$ is connected. Let $a$ be a vertex that belongs to both $F$ and $F^{\prime}$ (it is on the boundary path of both). If $a$ is not an inner point of $F \cup F^{\prime}$ for all such $a$, then this union is not pre-fertile (see the last part of Definition 5.9). Otherwise, $a$ is in the interior of $F \cup F^{\prime}$, and it is also solid since it belongs to the boundary path of $F$, whence $F \cup F^{\prime}$ is not pre-fertile, from which we deduce that $F$ is indeed a maximal element for fertile union and therefore a fertile sub-domain.

Proposition 5.12 Let $F$ be a sub-domain of $\mathcal{D}$, delimited by a fracture line $L$, such that at least one of its vertices is not on its contour path. Then $F$ is fertile if and only if it contains no non-trivial fracture line of $\mathcal{D}$ other than $L$.

Proof If $F$ is fertile then $\Delta h(v) \neq 0$ for any of its inner vertices, so that it can contain no fracture line other than $L$. Conversely, recall that a fracture line can be made of only one point: therefore if $F$ contains no fracture line other than $L$ then $\Delta h(v) \neq 0$ for any of its inner vertices. Moreover $F$ is tileable (see Corollary 5.6), connected, and two vertices of $L$ that are not neighbours in $L$ cannot be at distance 1 in the triangular grid (this would define a non-trivial fracture line) so it is pre-fertile, and since its contour path is a fracture line of $\mathcal{D}$ it is a fertile zone by Proposition 5.11.

We have completed our study of fertile zones; we now turn to sub-domains whose vertices are all on the contour path.

Lemma 5.13 Let $L$ be a fracture line of $\mathcal{D}$ enclosing a domain $F$, not limited to a single point, such that all the vertices of $F$ belong to $L$. Then either $F$ is a lozenge or it contains a fracture line which defines a lozenge.

Proof Since $F$ is not limited to a single point, it contains at least a triangle. This triangle belongs to a lozenge $x$ in a tiling $T$ of $\mathcal{D}$. The fracture line $L$ is a $T$-valid path, therefore $x$ is embedded in $F$ by lemma 5.4. By hypothesis all its vertices $\left(v_{1}, v_{2}, v_{3}\right.$ and $\left.v_{4}\right)$ are on a fracture line of $\mathcal{D}$, so that $\Delta h\left(v_{k}\right)=0$ for $k=1 . .4$. Thus the closed path $v_{1}-v_{2}-v_{3}-v_{4}-v_{1}$ is a fracture line in $\mathcal{D}$ and we are done.

We need one last definition, which corresponds to the sub-domains that are obtained by recursively looking for fracture lines, and therefore used by Algorithm 5.16 (see also Figures 16 (b) and 17 (b)):

Definition 5.14 A fracture zone in a tileable domain $\mathcal{D}$ is a sub-domain of $\mathcal{D}$ not reduced to a single point, delimited by a fracture line and containing no other fracture line.

Theorem 5.15 (Fracture theorem) Let $\mathcal{D}$ be any tileable domain such that the results of Section 3 hold. A fracture zone in $\mathcal{D}$ is either a fertile zone or a lozenge.

Proof Let $F$ be a fracture zone of $\mathcal{D}$. If all its vertices belong to its contour path, then it is a lozenge by Lemma 5.13. Otherwise $\Delta h(v) \neq 0$ for any inner vertex $v$ of $F$ by Proposition 5.8 since $F$ is a fracture zone, so that $F$ is fertile by Proposition 5.12.

We now have the tools to extract the fertile zones from a domain:

## Algorithm 5.16 (Fracture algorithm)

- Input: A tileable domain $\mathcal{D}$.
- Output: The fertile zones of $\mathcal{D}$.
- Step 1: Use Thurston's Algorithm 2.9 to build the maximal and minimal tilings of $\mathcal{D}$. Store the solid points of $\mathcal{D}$ in a list $S$.
- Step 2: Use $S$ and the minimal tiling of $\mathcal{D}$ to build all the fracture lines except the trivial one; store them in a list $L$.
- Step 3: Remove from $L$ all the fracture lines consisting of exactly 4 vertices.
- Step 4: Return $L$.

Proof of the algorithm Everything has already been proved in Theorem 5.15.
Complexity Steps 1 and 2, which control the execution time of the algorithm, are both linear in the number of vertices of $\mathcal{D}$.

### 5.3 An example of the use of the Fracture Theorem

We now present an example. Let us consider the finite-length closed path of Figure 16 (a), which delimits a tileable domain (see Figure 16 (b) for its minimal and maximal tilings). Using fracture lines, one can disconnect the central part, whose tilings give rise to a trivial lattice (see Figure 17 (a)). Thus only the fertile zones remain (see Figure 17 (b)).


Figure 16: A domain and its extreme tilings


Figure 17: The fracture zones

### 5.4 An application of the Fracture Theorem

Since fertile zones are delimited by fracture lines, flips done inside one of them cannot have any effect on the other fertile zones: therefore one can study independently the tilings of these zones. Since the tilings of $\mathcal{D}$ will be obtained by "gluing" tilings from these disjoint zones, their lattice can be obtained by product. Consider Figure 18, in which the domain contains two fertile zones. We will see later (see Section (7) an algorithm to generate the lattice of the tilings of such a zone; on this example each fertile zone is a pseudo-hexagon, so its tilings can be found using Algorithm 4.11. We build the product lattice of these lattices by connecting every element in the left-hand side lattice with every element in the right-hand side lattice, which finally gives the lattice of the tilings of the initial domain.

### 5.5 Seeds and their fillings

We have seen in Section 5.2 that fertile zones play a major role in the generation of all the tilings of a generic domain $\mathcal{D}$. We have also seen in Section 5.4 that we can restrict our study to these zones and later compute a product lattice. We now show that fertile zones can be decomposed into collections of pseudohexagons (see Figure 19 (a) for a start). The correspondence between tilings and compact piles of cubes allows us to use geometric terms; in other words, an up-flip can be viewed as adding a cube.

Definition 5.17 (Seed) A seed is the minimal tiling of a hexagon of side 1 .


Figure 18: The main steps in computing the lattice

Note that a pseudo-hexagon (see Definition 4.3) contains exactly one seed.
We now investigate the immediate properties of seeds.
Lemma 5.18 The union of two pseudo-hexagons sharing the same seed is again a pseudo-hexagon.

Proof Using Definition 4.3, it is enough to show that the union of two compact piles of cubes, aligned on the same set of axes, is itself a compact pile of cubes. But this immediately follows from Definition 4.1.

(a) A range and the maximal range of a seed

(b) Covering a seed with a cube potentially creates three new seeds

Figure 19: Seeds and ranges

Definition 5.19 The range $r(s)$ of a seed $s$ in a tiling $T$ of $\mathcal{D}$ is the union of the pseudo-hexagons that contain it. The maximum range $R(s)$ of $s$ is the union of its ranges when $T$ ranges among all the tilings of $\mathcal{D}$.

The set of pseudo-hexagons surrounding a seed $s$ is not empty since it contains at least $s$ itself. Thus the range of $s$ is well-defined according to Lemma 5.18.

Definition 5.20 A tiling of $r(s)$ (resp. the maximal, minimal tiling of $R(s)$ ) will be called a filling of $s$ (resp. the maximal filling, minimal filling of $s$ ). We denote by $\operatorname{Max}(s)$ and $\operatorname{Min}(s)$ the maximal and minimal fillings of $s$; by extension, $\operatorname{Max}(\mathcal{D})$ and $\operatorname{Min}(\mathcal{D})$ are the maximal and minimal tilings of $\mathcal{D}$.

The intuitive idea behind the term "filling" is that one gradually fills $r(s)$ with (compact piles of) cubes. Note that a filling of a seed is a compact pile of cubes and that the maximal filling of $s$ is obtained by doing all the possible up-flips in $r(s)$.

Adding a cube to cover a seed $s$ (in other words, flipping $s$ ) generally creates three new seeds $s_{1}, s_{2}$ and $s_{3}$ as shown in Figure 19 (b). We thus see that there is a natural partial order on the ranges of seeds, defined by inclusion. In our example, $r\left(s_{i}\right) \subset r(s)$ for $i \in\{1,2,3\}$. We will be mostly interested in the maximal elements for this partial order and now proceed to define this more precisely.

Definition 5.21 A seed $s$ is a child of a seed $s^{\prime}$ if $R(s)$ is embedded in $R\left(s^{\prime}\right)$. A seed is a proper seed if it is the child of only itself.

Proposition 5.22 Every cube in a tiling of $\mathcal{D}$ belongs to a filling of a proper seed.

Proof Every cube is the maximal tiling of a hexagon of side 1 and it is added when the minimal tiling of this hexagon is flipped. But this minimal tiling is a seed, whose range is included in the range of a proper seed. Therefore the cube belongs to a filling of this proper seed.

Proposition 5.23 The maximal fillings of two distinct proper seeds are disjoint.

Proof Let $C$ be a cube belonging to the maximal filling of two proper seeds $s_{1}$ and $s_{2}$. Since a filling of a seed is a compact pile of cubes, there is an uninterrupted line of cubes connecting $C$ to the base planes defined by $s_{1}$ and $s_{2}$. Since these planes vary two by two by a translation, the two basis are in fact the same, so that $s_{1}=s_{2}$.

Corollary 5.24 The maximal fillings of the proper seeds in a domain $\mathcal{D}$ form a partition of the subset of $\mathbb{Z}^{3}$ defined by the maximal tiling of the fertile zones of $\mathcal{D}$.

We know that the set of the tilings of $\mathcal{D}$ is connected by flips; therefore maximally filling all the seeds that appear in $\operatorname{Min}(\mathcal{D})$ potentially creates new flippable zones: that is, new seeds and new proper seeds (see Figure 20). To clarify the situation, we give the following definition:

Definition 5.25 A proper seed is of order 0 if it appears in the minimal tiling of $\mathcal{D}$. It is of order $n+1$ if it is a proper seed in the tiling of $\mathcal{D}$ obtained when all the proper seeds of order $k \in\{1, \ldots, n\}$ are maximally filled with cubes.


Figure 20: Seeds

### 5.6 Seeds of order 1

Let us examine the basic properties of a seed $s$ of order 1 . Remember that this is the minimum tiling of a hexagon and that it does not appear in the minimal tiling of $\mathcal{D}$. Thus at most two of the three lozenges that make up $s$ can come from $\operatorname{Min}(\mathcal{D})$.

Let us first assume that exactly two of the three lozenges in $s$ come from $\operatorname{Min}(\mathcal{D})$, and exactly one from a proper seed $t$ of order 0 . Then $s$ appears while filling the range of $t$, and it is a flippable zone: therefore it is not a proper seed and should be flipped while filling $t$; it is not a seed of order 1 .

So at least two of the lozenges in $s$ come from fillings of proper seeds of order 0 . For reasons of symmetry one can assume that they are as in Figure 20. Since they come from fillings they belong to cubes, so that the immediate neighbourhood of $s$ looks like the right-most picture in Figure 20.

### 5.7 Seeds of greater order

For a given domain $\mathcal{D}$ the order of any seed is bounded since $\operatorname{Min}(\mathcal{D})$ and $\operatorname{Max}(\mathcal{D})$ vary by a given number of cubes and each proper seed contributes at least one cube. Conversely, if $n$ is any integer, there exist domains in which one can find proper seeds of order $n$. To illustrate this, consider Figure 21. If one maximally fills all the seeds of order 0 (b), a sub-domain closely resembling the first can be outlined (c); indeed, only the size differs.

Thus one can build a family $\left(\mathcal{D}_{n}\right)$ of domains in which covering all the seeds of order 0 of $\mathcal{D}_{n+1}$ exactly yields $\mathcal{D}_{n}$ and therefore all the seeds which are of order $k$ in $\mathcal{D}_{n}$ are of order $k+1$ in $\mathcal{D}_{n+1}$. Using this procedure, one can generate seeds of any order.

## $6 \quad C$-minimal tilings and intervals in $\mathcal{L}(\mathcal{D})$

A close inspection of the lattice $\mathcal{L}(\mathcal{D})$ of the tilings of $\mathcal{D}$ will reveal that it contains interesting intervals, which we will use to generate all the tilings of a domain by performing the required flips in an orderly fashion.


Figure 21: A family of domains

(a) A lozenge in $\operatorname{Max}(\mathcal{D})$ and the cor-
(b) $\operatorname{Min}_{\left(C_{i}\right)}(T)$ with $\#\left\{C_{i}\right\}=2$ responding $C$-minimal tiling

Figure 22: Tilings defined by the presence of a lozenge

## 6.1 $C$-minimal tilings of a pseudo-hexagon

Let $\mathcal{D}$ be a pseudo-hexagon and let $C$ be a cube in a filling of the proper seed $s$ of $\mathcal{D}$; as we will see in Section 6.3, marking such a cube can be very useful. What can one say about the fillings of $s$ in which $C$ appears? Since fillings are compact piles of cubes, each such filling must also contain all the cubes in the parallelepiped defined by the diagonal going from $s$ to $C$ (see Figure 22 (a)).

Definition 6.1 Let $P$ be a pseudo-hexagon and $C$ a cube in a filling of its proper seed $s$. The $C$-minimal tiling of $P$ (which we denote by $\operatorname{Min}_{C}(P)$ ) is the tiling associated with the filling of $s$ obtained by starting from $\operatorname{Min}(T)$ and adding all the cubes in the parallelepiped defined by the diagonal $(s, C)$.

This definition is consistent. First, a $C$-minimal tiling is a tiling since the parallelepiped $(\mathcal{P}(C))$ is a compact pile of cubes. Second, if a filling of $s$ contains $\mathcal{P}(C)$ then it contains $C$. Finally, if a filling of $s$ contains $C$ then it must contain $\mathcal{P}(C)$ since a filling is a compact pile.

Definition 6.2 Let $P$ be a pseudo-hexagon and $\left(C_{i}\right)_{1 \leqslant i \leqslant n}$ a collection of cubes in a filling of the proper seed $s$ of $P$. Let $T_{k}$ be the $C_{k}$-minimal tiling of $P$ for $k \in\{1, \ldots, n\}$. The $\left(C_{i}\right)$-minimal tiling of $T$ (which we denote by $\operatorname{Min}_{\left(C_{i}\right)}(P)$ ) is $\operatorname{Sup}\left(T_{1}, \ldots, T_{n}\right)$.

An example of such a tiling is given in Figure 22 (b).
All of the above can be reformulated in a dual way. Instead of adding cubes from the minimal tiling, remove cubes from the maximal one: this can be viewed


Figure 23: The successive $\mathcal{D}^{k}$ for $k=0 . . d(\mathcal{D})$
as adding "anti-cubes", the rule being that cubes and anti-cubes annihilate each other. We could thus define a $C$-maximal tiling as a tiling of $\mathcal{D}$ which does not contain the cube $C$. The notion can be extended to families, thus defining $\left(C_{i}\right)$-maximal tilings.

### 6.2 The fundamental intervals of $\mathcal{L}(\mathcal{D})$

Let now $\mathcal{D}$ be any domain (such that the results of Section 3 hold) and let $\mathcal{T}(\mathcal{D})$ be the set of its tilings and $\mathcal{L}(\mathcal{D})$ the associated lattice. We have seen that $\operatorname{Max}(\mathcal{D})$, seen as a pile of cubes, can be partitionned according to the maximal fillings of its proper seeds.

Definition 6.3 We denote by $\mathcal{D}^{n}$, $n \geqslant 1$, the tiling of $\mathcal{D}$ obtained by maximally filling the ranges of all the proper seeds of order $k \in\{1, \ldots, n\}$. By convention, $\mathcal{D}^{0}=\operatorname{Min}(\mathcal{D})$. The degree $d(\mathcal{D})$ of $\mathcal{D}$ is the minimum of the integers $n$ such that $\mathcal{D}^{n}=\operatorname{Max}(\mathcal{D})$. A cube $C$ is of order $k$ if it belongs to the filling of a seed of order $k$.

Note that $\operatorname{Max}\left(\mathcal{D}^{k}\right)=\operatorname{Min}\left(\mathcal{D}^{k+1}\right)$ for $0 \leqslant k<d(\mathcal{D})$. See Figure 23 for an example.

Notation 6.4 The set of all the tilings $t$ of $\mathcal{D}$ for which $\mathcal{D}^{n} \preccurlyeq t \preccurlyeq \mathcal{D}^{n+1}$ is an interval in the lattice of the tilings of $\mathcal{D}$; we denote this interval by $\mathcal{T}_{n}(\mathcal{D})$, $0 \leqslant n<d(\mathcal{D}))$. By extension, $\mathcal{T}(\mathcal{D})$ denotes the set of all the tilings of $\mathcal{D}$.

Definition 6.5 The $\mathcal{T}_{n}(\mathcal{D}), 0 \leqslant n<d(\mathcal{D})$, are the fundamental intervals of the lattice $\mathcal{L}(\mathcal{D})$ of the tilings of $\mathcal{D}$.

Proposition $6.6\left(\mathcal{T}_{0}(\mathcal{D}), \ldots, \mathcal{T}_{d(\mathcal{D})-1}(\mathcal{D})\right)$ is a maximal chain of intervals in $\mathcal{L}(\mathcal{D})$.

Proof It follows from the definitions that $\mathcal{D}^{0}=\operatorname{Min}(\mathcal{D}), \mathcal{D}^{d(\mathcal{D})}=\operatorname{Max}(\mathcal{D})$ and $\operatorname{Max}\left(\mathcal{D}^{k}\right)=\operatorname{Min}\left(\mathcal{D}^{k+1}\right)$ for $0 \leqslant k<d(\mathcal{D})$.

A graphical representation of this chain is given in Figure 24.


Figure 24: The fundamental intervals of $\mathcal{T}(\mathcal{D})$


Figure 25: $C$-minimal tilings in $\mathcal{D}$

### 6.3 C-minimal tilings of a fertile zone

Let $\mathcal{D}$ be a domain such that the results of Section 3 hold. Using fracture lines (see Section 5.2), we can suppose that $\mathcal{D}$ is a fertile zone. Let $C$ be a cube that appears in a filling of a proper seed $s$ of order $n$ of $\mathcal{D}$. It is clear from the definitions that $C \in \mathcal{D}^{n}$. $C$ induces a $C$-minimal tiling of $R(s)$, let us denote it by $\mathcal{P}(C)$, whose contour path is a pseudo-hexagon (whose proper seed is $s$ ) and therefore a range of $s$ (see Figure 25 (a)).

Definition 6.7 The $C$-minimal tiling of $\mathcal{D}$, let us denote it by $\operatorname{Min}_{C}(\mathcal{D})$, is the infimum of all the tilings of $\mathcal{D}$ that contain $C$. If $\left(C_{i}\right)_{1 \leqslant i \leqslant n}$ is a family of cubes, the $\left(C_{i}\right)$-minimal tiling of $\mathcal{D}$ is $\operatorname{Min}_{\left(C_{i}\right)}(\mathcal{D})=\operatorname{Sup}\left(\operatorname{Min}_{C_{1}}(\mathcal{D}), \ldots, \operatorname{Min}_{C_{n}}(\mathcal{D})\right)$.

We now sketch a recursive construction of $\operatorname{Min}_{C}(\mathcal{D})$ (see Figure 25 (b)). $\mathcal{P}(C)$ marks cubes that belong to $\mathcal{D}^{n-1}$ and therefore to the maximal fillings of seeds $t_{i}$ of order $k \leqslant n-1$. These cubes give rise to $\left(C_{j}\right)$-minimal tilings in the $t_{i}$, which in turn mark cubes in $\mathcal{D}^{n-2}$, and so on. Since we always choose minimal tilings in the ranges of the proper seeds, the tiling is less than any tiling containing $C$, and it contains $C$, therefore it is $\operatorname{Min}_{C}(\mathcal{D})$.

## 7 An algorithm to generate the lattice of the tilings of a general domain

In Section 4.2 we have proposed an algorithm to generate efficiently the tilings of a pseudo-hexagon. We now proceed to the general case, where no hypothesis is made on the shape of the domain $\mathcal{D}$, except that the results of Section 3 hold.

In order to make use of already found tilings and reduce the computation time, we need the following important fact: if $\left(C_{i}\right)$ is a collection a cubes, all of them of order $n$, the interval of $\mathcal{L}(\mathcal{D})$ between $\operatorname{Min}_{\left(C_{i}\right)}(\mathcal{D})$ and $\mathcal{D}^{n}$ plus the cubes $C_{i}$ is isomorphic to the interval between $\operatorname{Min}_{\left(C_{i}\right)}(\mathrm{D})$ minus the cubes $C_{i}$ and $\mathcal{D}^{n}$. This is obvious because cubes of order $k<n$ can be added to $\operatorname{Min}_{\left(C_{i}\right)}(\mathcal{D})$ whether the cubes of order $n$ are present or not.

Formally stated, there is an isomorphism between the intervals $\left[\operatorname{Min}_{\left(C_{i}\right)}(\mathcal{D})\right.$; $\left.\operatorname{Sup}\left(\operatorname{Min}_{\left(C_{i}\right)}(\mathcal{D}), \mathcal{D}^{n}\right)\right]$ and $\left[\operatorname{Inf}\left(\operatorname{Min}_{\left(C_{i}\right)}(\mathcal{D}), \mathcal{D}^{n}\right) ; \mathcal{D}^{n}\right]$ of $\mathcal{L}(\mathcal{D})$.

## Algorithm 7.1

- Input: A finite-length closed path $\mathcal{P}$, enclosing a domain $\mathcal{D}$, in the triangular grid.
- Output: The lattice $\mathcal{L}(\mathcal{D})$ of the tilings of $\mathcal{D}$.
- Step 1: Use Algorithm 5.16 to break the domain into mutually independant fracture zones (see Definition 5.14). The lattice of the tilings of a single lozenge is trivial. For each of the fertile zones, follow steps 2 to 4 .
- Step $2\left(\mathcal{D}\right.$ is now a fertile zone): Recursively build the tilings $\mathcal{D}^{k}, 0 \leqslant$ $k \leqslant d(\mathcal{D})$. Set the list $L_{\mathcal{D}}$ of the tilings of $\mathcal{D}$ to $\emptyset$.
- Step 3: For $k=0$ to $\operatorname{deg}(\mathcal{D})-1$ do:
- Step 3.1: For each proper seed of order $k$, use Algorithm 4.11 to generate all the fillings of the seed.
- Step 3.2: Compute the cartesian product of these fillings.
- Step 3.3: Each element of this product is a family $\left(C_{i}\right)$ of cubes; for each of these families, compute $\operatorname{Min}_{\left(C_{i}\right)}(\mathcal{D})$.
- Step 3.4: Removing the cubes $C_{i}$ from $\operatorname{Min}_{\left(C_{i}\right)}(\mathcal{D})$ yields a collection of cubes of order at most $k-1$, which we denote by $\operatorname{Min}_{\left(C_{i}\right)}^{k-1}(\mathcal{D})$, or the empty set if $k=0$. Look up in $L_{\mathcal{D}}$ the (already found) tilings between $\operatorname{Min}_{\left(C_{i}\right)}^{k-1}(\mathcal{D})$ and $\mathcal{D}^{k}$; for each of these tilings $t$, add $t+\left(C_{i}\right)$ to $L_{\mathcal{D}}$.
- Step 4: For each tiling $T$ in $L_{\mathcal{D}}$, compute the height function and find the local minima: they correspond to cubes that can be added. Connect $T$ to each of the tilings obtained from it by adding exactly one of these cubes. This generates the lattice of the tilings of the fertile zone $\mathcal{D}$.
- Step 5: Compute the product of the lattices found for each fracture zone.

An practical implementation of this algorithm should update the lattice of the fertile zone at each newly found tiling.
Proof of the algorithm: Let $T$ be any tiling of a fertile zone. It can be associated with a unique pile of cubes. Let $M$ be the maximum of the orders of these cubes. The cubes of order $M$ belong to the fillings of the seeds of order $M$, and all of them have been taken care of by construction. The cubes of order at most $M-1$ in $T$ form a pile of cubes which has been generated in step 3.4. We conclude that $T$ has been encountered at least once by the algorithm. Moreover, all the tilings produced by the algorithm are distinct. Therefore we have generated exactly once each tiling of each fertile zone of $\mathcal{D}$, and hence exactly once each tiling of $\mathcal{D}$.
Space complexity: Let $|\mathcal{T}(\mathcal{D})|$ denote the number of tilings of the domain $D$ (we believe no closed formula is known). Each tiling of $\mathcal{D}$ is generated only once; the number of links in $\mathcal{L}(\mathcal{D})$ starting from a particular vertex is at most the width of the lattice $\mathcal{L}(\mathcal{D})$, of which we believe nothing is known except that it is trivially bounded by $|\mathcal{T}(\mathcal{D})|$; and there is a small overhead to account for the $\mathcal{D}^{k}$. The execution space of the algorithm is thus $\mathrm{O}\left(|\mathcal{T}(\mathcal{D})|^{2}\right)$.

Time complexity: The execution time of the algorithm is controlled by steps 3 and 4 , but since the average (or worst case) number of seeds and size of their maximal ranges is (totally unknown and) highly dependent on the shape of the domain, no non-trivial bound can be given for the time being. A rough analysis can be conducted as follows. In step 3.3, one needs to compute a minimal tiling for a familiy of cubes, of which there is at most the number of cubes in the maximal tiling of $\mathcal{D}$, which in turn is less than the number $\mathcal{T}(\mathcal{D})$ of tilings of $\mathcal{D}$ (since each cube is bijectively related to the associated minimal tiling). The minimal tiling for each cube can be computed in a number of steps that is again at most the number of cubes in the maximal tiling of $\mathcal{D}$. Looking up the already found tilings in step 3.4 can be done in at most $|\mathcal{T}(\mathcal{D})|$ operations. Since steps 3.3 and 3.4 must be conducted for each tiling, the overall cost of step 3 is $O\left(|\mathcal{T}(\mathcal{D})|^{3}\right)$. In step 4, computing the height function requires $|\mathcal{D}|$ operations for each tiling and connecting a tiling to its fathers in the lattice requires at most $\mathcal{T}(\mathcal{D})$ operations, so that the global cost of step 4 is $O\left(|\mathcal{T}(\mathcal{D})|^{2} \times|\mathcal{D}|\right)$.

Thus an upper bound for the time complexity is $O\left(|\mathcal{T}(\mathcal{D})|^{3}\right)$, but the execution time is probably far less in practice.

## 8 Conclusion and perspectives

We have shown that the isomorphism between Conway and Lagarias' lozenge group and $\mathbb{Z}^{3}$ is quite fruitful; it justifies the intuitive geometric interpretation of lozenge tilings and allows us notably to define the proper seeds and the
important intermediate tilings $\mathcal{D}^{k}$, of which Thurston's minimal and maximal tilings of $\mathcal{D}$ are particular cases.

Also, we have made much use of the lattice structure; this seems to indicate that lattice theory is a tool well adapted to the study of tilings.

We believe that the techniques developped in this paper may be extended to study some related questions:

- Enumerating the tilings of $\mathcal{D}$, without generating them, is a difficult question; Algorithm 7.1 can be used to exhibit an exact (but not very useful) formula, which is turn could provide good bounds if one were able to count the number of plane partitions that are both greater and smaller than two given plane partitions. Cruder bounds can be obtained using the lattice structure.
- In the case of tilings with dominoes there is a natural definition of flips, and therefore of seeds. The range of a seed $s$ in a tiling $T$ can be defined as follows: mark all the seeds in $T$ except $s$ as forbidden (they should not be flipped) and perform all the possible up-flips. The range of $s$ is the collection of squares around the vertices that have undergone a flip in the process. It seems therefore likely that our algorithms could be adapted straightforwardly to the case of dominoes. The geometric interpretation might be preserved using Levitov tiles (see RY00).
- Our definition of (the maximal tiling of) a pseudo-hexagon is a natural generalization of a Ferrers diagram since it is merely a geometrical representation of a plane partition. Let us now attribute an integer to each cube in such a way that sequences are non-increasing along each axis: this is a generalization of a Ferrers diagram in dimension 4 (this is called a solid partition in Mac16]). Thus one can easily define a generalization of pseudo-hexagons in dimension $p \geqslant 2$ and may therefore be able to study tilings in this dimension. In particular, the recursive algorithms of Section 4.2 should be readily upgradable.
- If more counting results were known, one could use Algorithms 7.1 and 4.11 to generate tilings uniformly at random.
- The $C$-minimal tilings seem to correspond to the sup-irreducible elements of the lattice of the tilings. It would therefore be interesting to study the link (with Birkhoff's theorem) between these tilings and the order associated with the lattice.


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