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Preference structures and their numerical representations

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Abstract

This paper presents a selective survey of numerical representations of preference structures from the perspective of the representational theory of measurement. It reviews historical contributions to ordinal, additive, and expected utility theories, then describes recent contributions in these areas. © 1999—Elsevier Science B.V. All rights reserved

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1. Introduction

Decision theory, which traces its philosophical foundations to antiquity, has developed into a mathematically mature subject in recent times. Early evidence of mathematical analysis in decision theory appears in the eighteenth century writings of Bernoulli [7] on rational analysis of risky decisions and of Borda [9] and Condorcet [15] on aggregation of individuals' preferences through voting or algebraic combination for collective action.

The first third of the present century witnessed a new level of mathematical sophistication in Norbert Wiener's [103] axiomatic analysis of what we now refer to as interval orders, Frank P. Ramsey's [83] axiomatization for decision under uncertainty, and Bruno de Finetti's [18, 19] contributions to subject probability and logical decision making. As mid-century approached, these were joined by the monumental treatise on rational choice and the theory of games by John von Neumann and Oskar Morgenstern [100]. Then, in the 1950s, other books that have profoundly influenced mathematical research in decision theory through the rest of the century appeared. These include Kenneth Arrow's [5] work on social choice theory, L.J. Savage's [85] axiomatic foundations for subjective expected utility theory in decision under uncertainty, and Gerard Debreu's [17] axiomatization of preferences for utility-based economic equilibrium analysis.

The central principle for human judgment and choice in the vast majority of these works and their successors is the notion of order, formalized by transitivity, and the related notion of decision-by-maximization. Even when decision paradigms do not transparently involve maximization, as with Nash equilibria in non-cooperative games

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[48, 77] and some ballot aggregation procedures, individuals are often assumed to have ordered preferences.

The emphasis on order and maximization has led to a huge body of work on quantification of preferences, likelihood judgments, and other qualitative aspects of judgment and choice. The obvious reason is that quantification facilitates the search for optimal or near-optimal decisions. A less obvious reason is that many contributors to decision theory have been instrumental in developing the representational theory of measurement, which subsumes but is certainly not limited to representations of preferences and other aspects of decision theory. The representational theory of measurement was formalized in [88] and has received its most complete expression in the three-volume set by Krantz et al. [55], Suppes et al. [91] and Luce et al. [68]. Its defining characteristic is the quantitative representation by analogous numerical structures of qualitative structures that consist of a ground set X and one or more relations or operations on X. The set X may have a variety of structural properties, e.g. as a Cartesian product set or a set of probability distributions, and one of its relations is often assumed to be a binary ordering relation. A familiar operation is the binary concatenation operation $\oplus$ where $x \oplus y$ denotes the joining together of objects $x, y \in X$ by placing them end-to-end for length measurement or putting them in the same balance pan for weight measurement. We often use $\succ$ to denote an asymmetric and transitive binary relation on X, in which case $(X, \succ)$ is a partially ordered set, and we always define $\sim$ as its symmetric complement by

$$x \sim y \quad \text{if neither} \quad x \succ y \quad \text{nor} \quad y \succ x \ .$$

The relation $\succ$ could denote *is preferred to*, or *is more probable than*, or *is longer than*, and so forth. Corresponding interpretations of $\sim$ are *is indifferent to*, *is equally probable as*, and *is the same length as*. However, if $\succ$ is assumed only to be a partial order without $\sim$ also being transitive, in which case $\sim$ is not necessarily an equivalence relation, then $x \sim y$ for $x \neq y$ could signify incomparability rather than comparable equality.

Positive, closed extensive measurement provides a nice example of a qualitative structure represented by an analogous quantitative structure. The qualitative structure is $(X, \succ, \oplus)$ with order relation $\succ$ on X. We assume also that $\sim$ is transitive, $\succ$ is positive [for all $x, y \in X$, $x \oplus y \succ x$], $\oplus$ is closed under $\sim$ [for all $x, y \in X$ there is a $z \in X$ such that $z \sim x \oplus y$], and the structure satisfies an Archimedean condition that is needed for a real valued as opposed to nonstandard or lexicographic representation. The analogous quantitative structure is $(\mathbb{R}^+, >, +)$, where $\mathbb{R}^+$ denotes the positive reals. The representation is: there exists $\psi : X \to \mathbb{R}^+$ such that, for all $x, y \in X$,

$$x \succ y \Leftrightarrow \psi(x) > \psi(y)$$

and

$$\psi(x \oplus y) = \psi(x) + \psi(y) \ .$$

The mapping ψ thus preserves $\succ$ on X by $>$ on $\mathbb{R}^+$, and takes $\oplus$ into $+$.

The representational theory also pays close attention to the uniqueness status of representing functions. In the example, ψ is unique up to multiplication by a positive constant: if ψ satisfies the representation then so does ψ^* if and only if $\psi^* = a\psi$ for some $a \in \mathbb{R}^+$.

I have described the representational theory of measurement because it provides a general framework for most decision-theoretic representations. Other works that emphasize its approach include Pfanzagl [80], Roberts [84] and Narens [76]. Books not cited earlier that adopt the representational tack for decision theory include Fishburn [23, 25, 29] and Wakker [101], and extensive surveys are available in [22, 24, 30, 36]. The sections to follow discuss the representational theory for a variety of preference structures. They are not exhaustive but rather offer a selective survey that illustrates facets of preference theory and includes recent results not found in earlier surveys.

The next section opens with a few definitions of central importance to our subject and then describes basic representations for ordinal utility theory, additive utility theory, and expected utility theory. Section 3 begins our consideration of specific topics with a discussion of cancellation conditions for finite additive measurement. We emphasize recent work on the extent to which such conditions are needed to ensure additivity. Section 4 continues the additivity theme by showing how a general theorem for additive measurement applies to a utility threshold representation for sets of arbitrary cardinality. Section 5 illustrates recent contributions to decision under risk and decision under uncertainty in two areas. The first is a generalization of Savage's subjective expected utility theory in which utilities are real vectors ordered lexicographically and subjective probabilities take the form of real matrices. The second focuses on the role of a binary operation of joint receipt for situations in which holistic alternatives consist of similar but clearly discernible pieces. Section 6 concludes the paper with examples of preference cycles and representations that accommodate cyclic preferences. The representations described in earlier sections assume that preferences are transitive, or at least acyclic.

2. Preference representations

This section uses an array of preference structures and their quantitative representations to illustrate our subject and provide points of departure for later sections. We first outline three factors that differentiate among various representations and contain important definitions.

Factor 1: Cardinality of X. The main distinction is among finite, countable (finite or denumerable), and uncountably infinite X.

Factor 2: Ordering properties of $\succ$. The following four main categories are common. We say that $\succ$ on X is:

acyclic if its transitive closure is irreflexive (we never have $x_1 \succ x_2 \succ \cdots x_t \succ x_1$ for finite t);

a partial order if it is transitive ($x \succ z$ whenever $x \succ y$ and $y \succ z$) and irreflexive (we never have $x \succ x$);

a weak order if it is a partial order for which $\sim$ is transitive;

a linear order if it is a weak order or partial order for which $\sim$ is the identity relation.

Szpilrajn's [92] theorem implies that an acyclic $\succ$ has a linear extension, i.e., is included in some linear order. If $\succ$ is a weak order then $\sim$ is an equivalence relation (reflexive, symmetric, transitive) and the set $X/\sim$ of equivalence classes in X determined by $\sim$ is linearly ordered by $\succ^*$ on $X/\sim$ defined by $a \succ^* b$ if $x \succ y$ for some (hence for all) $x \in a$ and $y \in b$.

Factor 3: Representational uniqueness. Suppose the quantitative structure of the representation uses only one real-valued function u on X. Assume that u satisfies the representation, and let U denote the set of all $v: X \to \mathbb{R}$ that satisfy it. We then say that *u is unique up to*:

(i) *an ordinal transformation* if $U = \{v\colon \text{for all } x, y \in X,\ v(x) > v(y) \Leftrightarrow u(x) > u(y)\}$;

(ii) *a positive affine transformation* if $U = \{v\colon$ there are real numbers $a > 0$ and b such that $v(x) = au(x) + b$ for all $x \in X\}$;

(iii) *a proportionality transformation* if $U = \{v\colon$ there is an $a \in \mathbb{R}^+$ such that $v(x) = au(x)$ for all $x \in X\}$.

When a representation uses more than one real-valued function, the same definitions apply to individual functions although additional restrictions on admissible transformations usually obtain when the functions are considered jointly. For example, if $X = X_1 \times X_2 \times \cdots \times X_n$ and the representation uses $u_i : X_i \to \mathbb{R}$ for $i = 1, 2, \ldots, n$, we say that the u_i *are unique up to similar positive affine transformations* if another set $\{v_1, v_2, \ldots, v_n\}$ of $v_i : X_i \to \mathbb{R}$ also satisfies the representation if and only if there is an $a \in \mathbb{R}^+$ and $b_1, b_2, \ldots, b_n \in \mathbb{R}$ such that $v_i(x_i) = au(x_i) + b_i$ for all $x_i \in X_i$ and all $i \in \{1, 2, \ldots, n\}$.

Other distinguishing factors include special structures for X, the presence or absence of operations like $\oplus$, and whether a representation involves specialized properties for its real-valued functions such as continuity or linearity. Continuity is often associated with topological structures as described, for example, in [23, 30, 36, 101], and it will not play a prominent role in our present discussion, which is primarily algebraic.

The rest of the section outlines traditional topics in preference theory, where u in a representation is usually referred to as a *utility function*. Theorems that link utility representations to qualitative preference structures by means of assumptions or axioms for preference are noted. Most proofs are available in [23] or in references cited in [30, 36]. I include a few proof comments here to illustrate the representations.

2.1. Ordinal measurement

A fundamental result for $(X, \succ)$ says that if X is countable then there is a utility function $u : X \to \mathbb{R}$ such that

$$x \succ y \Leftrightarrow u(x) > u(y) \quad \text{for all } x, y \in X, \tag{1}$$

if and only if $\succ$ on X is a weak order. In this case, u is unique up to an ordinal transformation. Sufficiency of weak order can be seen by enumerating the indifference classes in $X/\sim$ as $a_1, a_2, \ldots$, defining u^* on $X/\sim$ by

$$u^*(a_i) = \sum\{2^{-j}\colon a_i \succ^* a_j\} ,$$

noting that $u^*(a_k) \geqslant u^*(a_h) + 2^{-h}$ if $a_k \succ^* a_h$, and then defining u on X by $u(x) = u^*(a_i)$ whenever $x \in a_i$.

Weak order is not generally sufficient for (1) when $X/\sim$ is uncountable. For example, the linear order $\succ$ on $\mathbb{R}^2$ defined by $(x_1,x_2) \succ (y_1,y_2)$ if $x_1 > y_1$ or $(x_1 = y_1, x_2 > y_2)$ can be represented lexicographically as $(x_1,x_2) \succ (y_1,y_2) \Leftrightarrow (u_1(x_1), u_2(x_2)) >_{\mathrm{L}} (u_1(y_1), u_2(y_2))$, where $u_i(x_i) = x_i$ and $>_{\mathrm{L}}$ denotes lexicographic order. But it cannot be represented by (1): otherwise, since $u(x_1,0) < u(x_1,1) < u(y_1,0) < u(y_1,1)$ whenever $x_1 < y_1$, every interval $[u(x_1,0), u(x_1,1)]$ would contain a different rational number and yield the contradiction that the countable set of rational numbers is uncountable.

To obtain (1) when $X/\sim$ is uncountable, it needs to be assumed also that $X/\sim$ includes a *countable* subset that is $\succ^*$-order dense in $X/\sim$. By definition, $A \subseteq B$ is order dense in $(B, \succ_0)$ if, whenever $a \succ_0 b$ for $a,b \in B\backslash A$, there is a $c \in A$ such that $a \succ_0 c \succ_0 b$. Countable order denseness is often replaced in economic discussions by a sufficient but nonnecessary topological assumption which implies that u in (1) can defined to be continuous in the topology used for X.

Because (1) implies that $\succ$ is a weak order, it cannot hold when $\succ$ is acyclic or a partial order that is not also a weak order. We can, however, continue to use u to preserve $\succ$ one-way in the manner $x \succ y \Rightarrow u(x) > u(y)$. We can also use the same u to fully preserve, by equality, the strong indifference relation $\approx$ on X defined by

$$x \approx y \quad \text{if for all} \quad z \in X \ , \quad x \sim z \Leftrightarrow y \sim z \ ,$$

for $\approx$ on X is an equivalence relation with $x \succ y \approx z \Rightarrow x \succ z$ and $x \approx y \succ z \Rightarrow x \succ z$. Thus, if X is countable, there is a $u : X \to \mathbb{R}$ for which

$$x \succ y \quad \Rightarrow \quad u(x) > u(y) \quad \text{and} \quad x \approx y \Leftrightarrow u(x) = u(y) \quad \text{for all } x, y \in X, \tag{2}$$

if and only if $\succ$ on X is acyclic. Fig. 1 illustrates $\approx$ on a Haase diagram for a partially ordered set in which one point bears $\succ$ to a second if there is a downward sequence of lines from the first to the second.

When $X/\approx$ is uncountable, (2) holds for acyclic $\succ$ if $\succ^*$, defined in the natural way on $X/\approx$, has a linear extension in which some countable subset is order dense. Further discussion along this line is available in [78, 90]

Suppose $(X, \succ)$ is a partially ordered set that is not necessarily weakly ordered. An alternative to (2) of the two-way or *if and only if* variety that replaces u in (1) by another quantitative construct may then apply if $(X, \succ)$ has additional structure. A case of this occurs when $\succ$ is an *interval order*, i.e., when

$$\{x \succ y,\ z \succ w\} \Rightarrow \{x \succ w \text{ or } z \succ y\} \quad \text{for all} \quad x, y, z, w \in X \ .$$

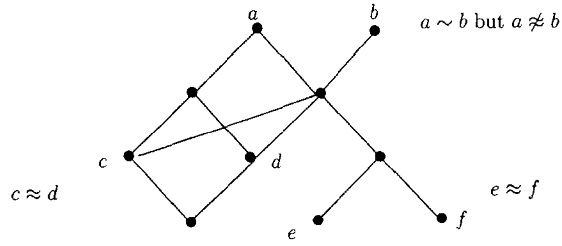

Fig. 1.

The primary two-way representation for an interval order is

$$x \succ y \Leftrightarrow I(x) > I(y) \quad \text{for all } x, y \in X \ , \tag{3}$$

where $I : X \to \boldsymbol{I}$ with $\boldsymbol{I}$ the set of all real intervals, and for $A, B \in \boldsymbol{I}$, $A > B$ means that $a > b$ for all $a \in A$ and all $b \in B$. One basic result (Fishburn [23], Theorem 2.7) is: if $X/\approx$ is countable for an interval order $(X, \succ)$, then (3) holds for a mapping I into nondegenerate closed intervals. Other structures may require open or half-open intervals (consider $X = \boldsymbol{I}$ with $\succ \, = \, >$), and yet others may fail for (3) because there are not enough intervals in $\boldsymbol{I}$ to accommodate the desired representation. Further results are in [10, Ch.7 of 27,79] and other references cited there.

2.2. *Additive measurement*

Several seemingly different types of representations are grouped together under this heading because they have an additive character and can be analyzed by similar mathematical methods. A general theory of additive measurement is presented in [35], where it is applied to a variety of contexts, including positive extensive measurement, additive utility measurement for multiattribute alternatives, difference measurement for strength of preference comparisons, threshold measurement, expected utility, and comparative probability. The paper includes a condition for X of arbitrary cardinality that is necessary and sufficient for the existence of an additive representation. I will describe its approach for threshold measurement in Section 4. The present subsection considers only comparative probability and multiattribute utility to illustrate the additive theme. I include comparative probability under the preference rubric because its relation $\succ$ is often defined from preference comparisons. Suppose x and y are uncertain events. Let g_x be the gamble that pays \$100 if x obtains and \$0 otherwise, and similarly for g_y. Then the approach promoted by de Finetti [19] and Savage [85] defines $x \succ y$ if g_x is preferred to g_y.

We formulate X for the present discussion as a family of subsets of a universal set Ω. For comparative probability, Ω is a set of *states*, members of X are *events*, and X usually includes the empty event $\emptyset$ and universal event Ω. The event set X may or may not be closed under operations like union, intersection, and complementation, and its members can have very different cardinalities.

The set of items to be compared by preference in multiattribute utility theory is a subset A of a Cartesian product set $A_1 \times A_2 \times \cdots \times A_n$ with $n \geqslant 2$. Each A_i is a nonempty set, and we assume without loss of generality that the A_i are mutually disjoint and every $a_i \in A_i$ appears in at least one n-tuple in A. The universal set Ω is defined as $\bigcup_i A_i$, and

$$X = \{\{a_1, a_2, \ldots, a_n\} \colon (a_1, a_2, \ldots, a_n) \in A\} ,$$

so every member of X is an n-element subset of Ω.

Suppose Ω is finite and $\succ$ on X is a weak order. The basic additive representation uses $u : \Omega \to \mathbb{R}$ for

$$x \succ y \Leftrightarrow \sum_{\omega \in x} u(\omega) > \sum_{\omega \in y} u(\omega) \quad \text{for all} \quad x, y \in X . \tag{4}$$

It is common in the multiattribute case to denote the restriction of u on A_i by u_i so, when $x = \{a_1, a_2, \ldots, a_n\}$, $\sum_x u(\omega) = \sum_i u_i(a_i)$. Then, when (4) holds, it remains valid when the origin of each u_i is translated by adding a constant c_i to all u_i values. For comparative probability, we assume $\Omega \succ \emptyset$ and that the union $\succsim$ of $\succ$ and $\sim$ is monotonic, so $x \supset y \Rightarrow x \succsim y$. Then we can take $u \geqslant 0$ and $\sum_\Omega u(\omega) = 1$ when (4) holds, so u becomes a probability distribution on Ω. A necessary and sufficient condition for (4) referred to as cancellation, independence, or additivity, was identified first by Kraft et al. [54]:

Cancellation. For every pair $x^1, x^2, \ldots, x^m$ and $y^1, y^2, \ldots, y^m$ of finite sequences of members of X such that

$$|\{j : \omega \in x^j\}| = |\{j : \omega \in y^j\}| \quad \text{for all} \quad \omega \in \Omega , \tag{5}$$

it is false that $x^j \succsim y^j$ for $j = 1, \ldots, m$ and $x^j \succ y^j$ for some j.

Necessity of Cancellation for (4) follows from the fact that (5) implies

$$\sum_{j=1}^{m} \sum_{\omega \in x^j} u(\omega) = \sum_{j=1}^{m} \sum_{\omega \in y^j} u(\omega) .$$

Hence if (4) holds and if Cancellation is violated by $x^j \succsim y^j$ for all j and $x^j \succ y^j$ for some j, summation over j on the right side of (4) followed by cancellation of identical terms leaves the contradiction that $0 > 0$. Sufficiency of Cancellation for (4) follows from solution theory for finite systems of linear inequalities by way of a solution-existence theorem known by various names, including the separating hyperplane lemma, the theorem of the alternative, Farkas's lemma, and Motzkin's lemma: see, for example [23, 37, 55, 87]. Essentially the same separation lemma applies when $\succ$ is only assumed acyclic or a partial order, with slight modifications in Cancellation. For example, if (4) is to hold when $\Leftrightarrow$ is replaced by $\Rightarrow$, we replace the last line of Cancellation by "it is false that $x^j \succ y^j$ for all j."

When (4) holds for finite Ω under weak order, u is not generally unique in any simple sense. Special conditions that are not necessary for (4) but which yield simple uniqueness forms, such as absolute uniqueness for subjective probabilities, are described in [30, 36, 45]. Additional discussions of Cancellation for finite Ω appears in the next section.

Theories of additive measurement for infinite Ω usually assume nicely structured domains, such as $A = A_1 \times A_2 \times \cdots \times A_n$ for additive utility or $X = 2^\Omega$ for comparative probability. Most also use existence axioms that simplify cancellation conditions, promote representational uniqueness, and facilitate the derivation or assessment of u. Examples for (4) with $A = A_1 \times A_2 \times \cdots \times A_n$ in the multiattribute case of both the algebraic and topological varieties are detailed in [23, 55, 101]. Their cancellation conditions use only $m = 2$ and $m = 3$ in Cancellation, and their u_i functions as defined after (4) are unique up to similar positive affine transformations.

When Ω is infinite for the comparative probability case, the weak order representation (4) is usually replaced by (1) in conjunction with $u(X) \subseteq [0,1], u(\Omega) = 1$, and

$$u(x \cup y) = u(x) + u(y) \quad \text{for all} \quad x, y, x \cup y \in X \quad \text{for which} \quad x \cap y = \emptyset \ ,$$

so that u is a finitely additive probability measure on X. Savage's [85] elegant axiomatization for this representation assumes $X = 2^\Omega$, weak order, $\Omega \succ \emptyset$, $x \succsim \emptyset$ for all x, the $m = 2$ part of Cancellation which says that

$$(x \cup y) \cap z = \emptyset \Rightarrow (x \succ y \Leftrightarrow x \cup z \succ y \cup z) \ ,$$

and an Archimedean axiom involving finite partitions of Ω. The representing measure is unique and satisfies the following *divisibility property*: if $x \succ \emptyset$ then for every $0 < \lambda < 1$ there is an $x_\lambda \subset x$ such that $u(x_\lambda) = \lambda u(x)$. Proof are given in [23, 29] as well as Savage [85]. Survey material on related axiomatizations of comparative probability appears in Section 6 in [36].

2.3. Expected utility

The simplest version of expected utility formulates X as a set of finite-support probability distributions, also called *lotteries*, on a set C of consequences. The consequences in C are viewed as mutually exclusive outcomes of decision. If $x(c_1)+x(c_2)+x(c_3) = 1$ for distinct $c_i \in C$, and if x is chosen over other lotteries, then the unique outcome of the decision is c_i, with probability $x(c_i)$, for $i = 1, 2, 3$. The expected utility representation for weak order involves a utility function $u : C \to \mathbb{R}$ for which

$$x \succ y \Leftrightarrow \sum_C x(c)u(c) > \sum_C y(c)u(c) \quad \text{for all } x, y \in X. \tag{6}$$

It is commonly assumed that X is closed under convex combinations so $\lambda x + (1-\lambda)y$, defined by $(\lambda x + (1-\lambda)y)(c) = \lambda x(c) + (1-\lambda)y(c)$ for all $c \in C$, is in X whenever $x, y \in X$ and $0 \leqslant \lambda \leqslant 1$. This is tantamount to assuming that X is closed under every binary operation $\oplus_\lambda$ in a set of operations for $\lambda \in [0,1]$, with $x \oplus_\lambda y = \lambda x + (1-\lambda)y$.

A straightforward generalization of the simple version under convex closure takes X as a function set closed under a mixture operation which maps each triple (x, λ, y) for

$x, y \in X$ and $0 \leqslant \lambda \leqslant 1$ into another member of X that is usually denoted by (x, λ, y) or $x\lambda y$ or $\lambda x + (1 - \lambda)y$. This is essentially the approach taken in the axiomatization of von Neumann and Morgenstern (1944), whose utility representation is (1) in conjunction with the *linearity property*

$$u(x\lambda y) = \lambda u(x) + (1 - \lambda)u(y) \quad \text{for all} \quad x, y \in X \quad \text{and all} \quad 0 \leqslant \lambda \leqslant 1 . \tag{7}$$

When the simple version applies and u is extended from X to C by $u(c) = u(x_c)$ under the assumption that every degenerate distribution x_c, for which $x_c(c) = 1$, is in X, (6) follows from (1) and (7).

After early confusion about the von Neumann–Morgenstern approach was clarified by Malinvaud [72] (see also [46]), several equivalent axiomatizations for (1) and (7) were developed. My favorite is Jensen's [50] which, in addition to properties for the mixture operation, uses three assumptions for $\succ$ on X. They are weak order and the following for all $x, y, z \in X$ and all $0 < \lambda < 1$:

independence: $x \succ y \Rightarrow x\lambda z \succ y\lambda z$,

continuity: $x \succ y \succ z \Rightarrow x\alpha z \succ y \succ x\beta z$ for some $\alpha, \beta \in (0, 1)$.

Independence says that $\succ$ is preserved under similar mixtures. It is usually defended by an interpretation for $x\lambda z$ and $y\lambda z$, whereby x or y obtains with probability λ and z obtains with probability $1 - \lambda$. Continuity ensures the existence of real-valued utilities as opposed to nonstandard or multidimensional lexicographically ordered utilities. We discuss the lexicographic case in Section 5.

The proof in [23, 25, 29] that (1) and (7) follow from weak order, independence and continuity shows that these axioms imply independence for $\sim$: $x \sim y \Rightarrow x\lambda z \sim y\lambda z$ for all $0 \leqslant \lambda \leqslant 1$ and all $z \in X$. Then, whenever $x \succ y$ and $x \succsim z \succsim y$, we prove that $z \sim x\lambda y$ for a unique $\lambda \in [0, 1]$. We then fix $x_1 \succ x_0$, set $u(x_1) = 1$ and $u(x_0) = 0$, define $u(z) = \lambda$ when $x_1 \succsim z \succsim x_0$ and $z \sim x_1\lambda x_0$, and show that (1) and the linearity property (7) hold for all members of X in the closed preference interval from x_0 and x_1. The representation is then extended to the rest of X in the only way that preserves linearity under indifference. The resulting u is unique up to a positive affine transformation obtained, for example, by choosing values for $u(x_1)$ and $u(x_0)$ other than 1 and 0.

Modifications of (1) with maintenance of linearity (7) when $\succ$ is assumed only to be a partial order are presented in [23, 25, 75, 98], and integral forms for (6) are axiomatizated for certain classes of probability measures in [23, 25].

Beginning around 1980, several investigators developed nonlinear versions of the theory that weaken its independence axiom, which is often inconsistent with actual preferences [3, 51, 70, 96]. Consider an example from Kahneman and Tversky [51] with $z(\$0) = 1$ and $\lambda = 1/4$:

x : \$3000 with probability 1,

y : \$4000 with probability 0.8, nothing otherwise,

$$x' = x\frac{1}{4}z : \$3000 \quad \text{with} \quad \text{probability } 0.25, \quad \$0 \quad \text{otherwise},$$

$$y' = y\frac{1}{4}z : \$4000 \quad \text{with probability} \quad 0.20, \quad \$0 \quad \text{otherwise} .$$

A majority of 94 respondents in their study violated independence with $x \succ y$ and $y' \succ x'$. Fishburn [29, Ch.3] reviews weak order and partial order theories due to Machina [71], Chew [12], Fishburn [26] and Quiggin [82] among others, that accommodate this and other failures of independence. Section 6 describes another theory that also allows violations of transitivity.

The preceding theories are grouped under the heading of *decision under risk* because their consequence probabilities are given as part of the formulation and are not derived from axioms. Theories of expected utility that derive subjective probabilities of uncertain events along with utilities from axioms are grouped under the heading of *decision under uncertainty*. The best known of these is Savage's [85] theory of subjective expected utility. A full account also appears in [23], and a summary is given in [36, Section 7].

Savage's theory is based on a set C of consequences and a set S of *states* of the world that describe the decision maker's areas of uncertainty and are outside his or her control. Subsets of S are uncertain *events*, and the relevant set of events is assumed to be the entire power set 2^S. We take X as the function set C^S of all maps from S to C and refer to each $x \in X$ as an *act*. The *constant act* that assigns consequence c to every state is denoted by $\bar{c}$. If $x \in X$ is the chosen act and state s obtains, then $x(s)$ is the resulting consequence. Savage's representation is (1) and

$$u(x) = \int_S u(x(s))\mathrm{d}\pi(s) \quad \text{for all} \quad x \in X \, , \tag{8}$$

where $u(c)$ on the right denotes $u(\bar{c})$ and π is a finitely additive probability measure on 2^S. The measure π is unique and satisfies the divisibility property at the end of the preceding subsection; u is bounded and unique up to a positive affine transformation.

The representation of (1) and (8) is derived from axioms that include weak order, independence assumptions, and an Archimedean partition axiom. The proof shows first that the axioms for comparative probability on 2^S at the end of the preceding subsection follow from the axioms of preference. This gives π, which is then used to construct lotteries that correspond to acts with finite consequence sets $\{x(s) : s \in S\}$. The natural definition of $\succ$ on the lottery set is shown to satisfy expected utility axioms that yield a linear u on lotteries and establish Savage's representation for all acts with finite consequence sets. The representation is then extended to all acts.

Savage's contribution stimulated a number of alternative axiomatizations of subjective expected utility for decision under uncertainty, including theories in [4, 21, 81]. A comprehensive review is given in [24]. Theories that weaken additivity for subjective probability or transitivity of $\succ$ or $\sim$, including those in [41, 47, 69, 86], are described in Chapter 8 in [29].

Savage's theory has the disadvantage of requiring S to be infinite. The most popular alternatives to his theory that retain weak order and additive subjective probability, and which yield unique subjective probabilities for finite S, use lotteries in their formulations. This is done either by replacing C by the set P_C of all lotteries on C, so that acts map states into lotteries, or by constructing mixed acts as lotteries whose outcomes are Savage acts: see, for example [4, 21, 23]. To illustrate the P_C approach, let X denote the set of all maps from finite S into P_C. State s is said to be *nonnull* if $x \succ y$ for some $x, y \in X$ that differ only in state s. We apply Jensen's axioms of weak order, independence and continuity to $(X, \succ)$ and add a nontriviality condition and a new independence axiom which says that if s and t are nonnull states, and p and q are lotteries in P_C, then, for all $x \in X$, (x with $x(s)$ replaced by p) $\succ$ (x with $x(s)$ replaced by q) $\Leftrightarrow$ (x with $x(t)$ replaced by p) $\succ$ (x with $x(t)$ replaced by q). The axioms imply that there is a unique probability distribution π on S and a linear $u : P_C \to \mathbb{R}$ unique up to a positive affine transformation such that

$$x \succ y \Leftrightarrow \sum_{s \in S} \pi(s)u(x(s)) > \sum_{s \in S} \pi(s)u(y(s)) \quad \text{for all} \quad x, y \in X , \tag{9}$$

with state s null if and only if $\pi(s) = 0$.

Subjective probabilities in (9) arise from two observations. First, Jensen's axioms for $(X, \succ)$ imply additivity over states:

$$x \succ y \Leftrightarrow \sum_{s \in S} u_s(x(s)) > \sum_{s \in S} u_s(y(s)) \quad \text{for all} \quad x, y \in X ,$$

where each u_s on P_C is linear, and the u_s collectively are unique up to similar positive affine transformations. Second, when $\succ_s$ is defined by

$$p \succ_s q \quad \text{if} \quad x \succ y \quad \text{when} \quad x(s) = p,\ y(s) = q\ , \quad \text{and} \quad x(t) = y(t) \text{ otherwise} ,$$

the new independence axiom says that $\succ_s$ is the same for every nonnull state. Since u_s preserves $\succ_s$, it follows for nonnull s and t that u_s is a positive affine transformation of u_t, say $u_s = au_t + b$ with $a > 0$, so $a = \pi(s)/\pi(t)$ for (9). That is, $u_s = \pi(s)u$ and $u_t = \pi(t)u$, and normalization of the $\pi(s)$ then gives (9) with $\sum \pi(s) = 1$.

3. Cancellation conditions

By the late 1960s, weak-order additive representations for infinite X with nice uniqueness properties were well understood [23, 55], but two noticeable gaps existed for finite-X representations. The first concerned conditions that imply nice uniqueness structures comparable to those of some infinite-X representations. This was partly rectified by the late 1980s in a series of papers surveyed in [45].

The second gap concerned Cancellation. To focus this concern, we reformulate Cancellation from Section 2.2 as a sequence of conditions based on the number J of *distinct* pairs (x^j, y^j) involved in (5). The condition for J is denoted by $C(J)$.

$C(J)$: For every sequence $(x^1, y^1), \ldots, (x^J, y^J)$ of *distinct* members of $X \times X$ and corresponding sequence $\alpha_1, \ldots, \alpha_J$ of positive integers such that

$$\sum\{\alpha_j\colon \omega \in x^j\} = \sum\{\alpha_j\colon \omega \in y^j\} \quad \text{for all} \quad \omega \in \Omega ,$$

it is false that $x^j \succsim y^j$ for $j = 1, \ldots, J$ and $x^j \succ y^j$ for some j.

Condition $C(1)$ is vacuous since its hypotheses require $x^1 = y^1$, and $C(2)$ is tantamount to the first-order independence condition which says that if $(x^1, y^1) \neq (x^2, y^2)$ and if every $\omega \in \Omega$ appears in (x^1, x^2) the same number of times it appears in (y^1, y^2), then $x^1 \succ y^1 \;\Leftrightarrow\; y^2 \succ x^2$. An example for $A = \{1,2,3\} \times \{a,b,c\}$ that satisfies $C(2)$ is the linear order

$$3c \succ 3b \succ 2c \succ 1c \succ 3a \succ 2b \succ 2a \succ 1b \succ 1a ,$$

but this violates $C(3)$ because

$$\begin{aligned} 3b &\succ 2c \\ 1c &\succ 3a \\ 2a &\succ 1b . \end{aligned}$$

In this example, $\alpha_1 = \alpha_2 = \alpha_3 = 1$ and $3c = \{3,c\}, 3b = \{3,b\}$, and so forth.

The α_j in $C(J)$ are used for repetitions of the same (x,y) pair in the sequence $(x^1,y^1), \ldots, (x^m,y^m)$ of Cancellation, which is clearly equivalent to the conjunction of $C(2), C(3), \ldots$. Our concern for Cancellation is the smallest J such that every weak-ordered set $A \subseteq A_1 \times A_2 \times \cdots \times A_n$ of a given size has an additive representation if it satisfies $C(2)$ through $C(J)$. We revert here to the product formulation of multiattribute preference, which applies also to comparative probability when $|A_i| = 2$ for all i and an event is characterized by the vector $(a_1, \ldots, a_n)$ which has $a_i = 1$ if state i is in the event and $a_i = 0$ otherwise.

We define the *size* of X, or of $A \subseteq A_1 \times A_2 \times \cdots \times A_n$, as the n-tuple $(\eta_1, \eta_2, \ldots, \eta_n)$ for which $\eta_i = |A_i|$ for each i. To avoid trivial A_i, we assume along with $n \geqslant 2$ that $\eta_i \geqslant 2$ for all i. We then define $f(\eta_1, \eta_2, \ldots \eta_n)$ as the smallest positive integer J^* such that every weak order on X of size $(\eta_1, \eta_2, \ldots, \eta_n)$ that violates Cancellation does so for some $C(J)$ with $J \leqslant J^*$. In other words, if $f(\eta_1, \eta_2, \ldots, \eta_n) = J^*$, then:

(i) there is a weak order $\succ$ on X of size $(\eta_1, \eta_2, \ldots, \eta_n)$ that violates $C(J^*)$ but satisfies $C(J)$ for all $J \in \{2, \ldots, J^* - 1\}$;

(ii) every weak order on an X of size $(\eta_1, \eta_2, \ldots, \eta_n)$ that satisfies $C(J)$ for $J = 2, \ldots, J^*$ also satisfies $C(K)$ for all $K > J^*$ for which $C(K)$ is defined for that size and therefore has an additive representation as in (4).

In the comparative probability setting for weak orders, Kraft et al. [54], proved that X has an additive representation if $n \leqslant 4$ and first-order independence holds, so $f(2,2) = f(2,2,2) = f(2,2,2,2) = 2$. They showed also that $f(2,2,\ldots,2) \geqslant 4$ for all $n \geqslant 5$. In the multiattribute setting, Krantz et al. [55, pp. 427–428] noted that $f(2, \eta_2) = 2$ for all $\eta_2 \geqslant 2$. Little else was known about f until recently.

We summarize here results in [37–39] and note topics for further research. The first two papers focus on $\eta_i = 2$ for all i. Let 2_n denote $(2,2,\dots,2)$ with n entries. The first paper shows that $f(2_5) = 4$ and $f(2_n) \geqslant n - 1$ for $n = 6,7,8$. The latter result is extended to all $n \geqslant 6$ in [38] by explicit constructions based on a theorem in the first paper that is designed to identify structures that violate $C(J)$ for relatively large J but satisfy all $C(J')$ for small J'. Fishburn [37] also shows that for every $n \geqslant 5$ there are weak order cases of comparative probability that violate $C(4)$ but have additive representations whenever one state is deleted, and that there are failures of Cancellation that require $\alpha_i \neq \alpha_j$ for some i and j in any corresponding failure of a $C(J)$. In other words, (4) can have no solution when every applicable $C(J)$ holds under the restriction that $\alpha_1 = \alpha_2 = \cdots = \alpha_J = 1$.

Fishburn [39] considers $\eta_i \geqslant 3$ as well as $\eta_i = 2$ and proves the following upper bound on f:

$$f(\eta_1, \eta_2, \dots, \eta_n) \leqslant \sum_{i=1}^{n} \eta_i - (n-1) \ .$$

This is ineffective for the case of $f(2, \eta_2) = 2$, but shows in conjunction with the lower bound of the preceding paragraph that $n - 1 \leqslant f(2_n) \leqslant n + 1$ for all $n \geqslant 6$. We also prove for $n = 2$ that $\eta_2 \leqslant f(3, \eta_2)$ for all even $\eta_2 \geqslant 4$, and $\eta_2 - 1 \leqslant f(3, \eta_2)$ for all odd $\eta_2 \geqslant 5$. The upper bound for these cases is $f(3, \eta_2) \leqslant \eta_2 + 2$.

Two areas for further research are my conjecture that $f(2_n) = n - 1$ for all $n \geqslant 6$, and derivation of good lower bounds on $f(\eta_1, \eta_2, \dots, \eta_n)$ for general sizes. It seems plausible that $f(\eta_1, \eta_2, \dots, \eta_n)$ is very close to the upper bound $\sum \eta_i - (n-1)$ for most sizes, but this awaits further study.

4. Thresholds

This section describes a general theorem for additive measurement in [35] and applies it to the closed-interval representation

$$x \succ y \Leftrightarrow u(x) > u(y) + \sigma(y) \quad \text{for all} \quad x, y \in X \ ,$$

where $u, \sigma : X \to \mathbb{R}$ and $\sigma \geqslant 0$. A sample of other approaches to interval and more general threshold representations is provided in [1, 2, 6, 8, 11, 20, 74, 91, ch. 16].

Our theorem is a linear separation theorem for arbitrary systems of linear inequalities that have finite numbers of terms. We begin with a nonempty set Y, and let V denote the vector space of all $v : Y \to \mathbb{R}$ for which $\{x \in Y : v(x) \neq 0\}$ is finite. We define λv and $v + v'$ for real λ and $v, v' \in V$ by

$$(\lambda v)(x) = \lambda v(x) \ , \quad (v + v')(x) = v(x) + v'(x) \ .$$

The representation for the theorem consists of distinguished subsets A and B of V and a mapping $\phi : X \to \mathbb{R}$ for which

$$\begin{aligned} &\langle a, \phi \rangle \leqslant 0 \quad \text{for all} \quad a \in A, \\ &\langle b, \phi \rangle > 0 \quad \text{for all} \quad b \in B \ , \end{aligned} \tag{10}$$

where $\langle\alpha,\beta\rangle = \sum \alpha(x)\beta(x)$. We say that (A,B) is *solvable* if there exists a ϕ that satisfies linear system (10). We will state a condition on (A,B) that is necessary and sufficient for solvability. It is assumed, with no loss of generality, that the zero function $\mathbf{0}$ of V is in A and that B is not empty.

A few other definitions are needed. For $V_1, V_2 \subseteq V$, $V_1 + V_2 = \{v + v' \colon v \in V_1,\ v' \in V_2\}$. A subset K of V is a *convex cone* if it is nonempty, closed under convex combinations, and contains λv whenever $\lambda > 0$ and $v \in K$. A convex cone K is without origin if $\mathbf{0} \notin K$. The convex cone generated by nonempty $U \subseteq V$ is denoted by U^*, so

$$U^* = \left\{ \sum_{i=1}^{m} \lambda_i v_i \colon m \in \{1,2,\ldots\}\ ,\ \lambda_i > 0 \text{ and } \ v_i \in U \ \text{ for all } i \right\} .$$

Finally, we say that nonempty $U \subseteq V$ is *Archimedean* if for all $v, v' \in U$, $\lambda v - v' \in U$ for some $\lambda > 0$.

Our separation theorem says that (A,B) is *solvable if and only if* $-A^* + B^*$ *is included in some Archimedean convex cone without origin in* V. Given (10), necessity of the condition on $-A^* + B^*$ is shown by extending ϕ linearly to all of V by $\phi(v) = \langle v, \phi\rangle$ and observing that $\{v \in V \colon \phi(v) > 0\}$ is an Archimedean convex cone without origin that includes $-A^* + B^*$. The sufficiency proof is based on a standard separation theorem discussed, for example, in [52, 53].

To apply the theorem to the opening representation of this section, let X' be a disjoint copy of X with $x' \in X'$ corresponding to $x \in X$, and let $Y = X \cup X'$. The opening representation can then be rewritten as

$$\begin{aligned} &x \succ y \Leftrightarrow \phi(x) - \phi(y) - \phi(y') > 0 && \text{for all } x, y \in X\ , \\ &\phi(x') \geqslant 0 && \text{for all } x' \in X'\ . \end{aligned}$$

Sets A and B for application of the separation theorem are

$$\begin{aligned} A = \{\mathbf{0}\} &\cup \{v \colon v(x) = -v(y) = -v(y') = 1 \text{ and } v = 0 \\ &\qquad \text{otherwise, for all } (x,y) \text{ with } x \sim y \text{ and } x \neq y\} \\ &\cup \{v \colon v(x') = -1 \text{ and } v = 0 \quad \text{otherwise,} \quad \text{for all } x' \in X'\} \\ B = \{v \colon &v(x) = -v(y) = -v(y') = 1 \text{ and } v = 0 \text{ otherwise,} \\ &\text{for all } (x,y) \text{ with } x \succ y\}. \end{aligned}$$

Suppose $-A^* + B^*$ is included in an Archimedean convex cone without origin. Let ϕ satisfy (10). Then, when $x \sim y$ and $x \neq y$, $\phi(x) \leqslant \phi(y) + \phi(y')$ and $\phi(y) \leqslant \phi(x) + \phi(x')$, or $u(x) \leqslant u(y) + \sigma(y)$ and $u(y) \leq u(x) + \sigma(x)$; for $x' \in X'$, $\phi(x') \geqslant 0$, or $\sigma(x) \geqslant 0$; when $x \succ y$, $\phi(x) > \phi(y) + \phi(y')$, or $u(x) > u(y) + \sigma(y)$.

5. Decision under risk and uncertainty

An enormous amount of theoretical and empirical research effort has been devoted to decision under risk and decision under uncertainty during the past few decades.

I comment here on two topics that illustrate very different facets of this work. Both have assumed that $\succ$ on X is a weak order. The first is a theory of subjective expected utility that relaxes continuity or an Archimedean axiom to obtain vector-valued utilities ordered lexicographically along with subjective probabilities characterized by real matrices rather than real numbers. The second departs more radically from traditional theories and considers the role of a binary operation $\ominus$ of joint receipt. The principal investigators are Irving LaValle in the lexicographic domain and Duncan Luce for joint receipt.

The lexicographic story beings with Hausner's [49] lexicographic linear utility theory for a weak order $\succ$ on a mixture space X that can be viewed as a set of lotteries or its generalization for (7) that is closed under a mixture operation. Hausner assumed that $\succ$ and $\sim$ satisfy independence ($x \succ y \Rightarrow x\lambda z \succ y\lambda z$; $x \sim y \Rightarrow x\lambda z \sim y\lambda z$) and proved that $(X, \succ)$ is represented by a linear mapping into a real vector space ordered lexicographically. When the vector space has finite dimension, say $\mathbb{R}^m$, this gives $u : X \to \mathbb{R}^m$ that satisfies (7) along with

$$x \succ y \Leftrightarrow u(x) >_L u(y) \quad \text{for all } \; x, y \in X \; , \tag{11}$$

where $(\alpha_1, \ldots, \alpha_m) >_{\mathrm{L}} (\beta_1, \ldots, \beta_m)$ if the two vectors are not equal and $\alpha_i > \beta_i$ for the smallest i at which they differ. When $u(x) = (u_1(x), \ldots, u_m(x))$, (7) is

$$\begin{aligned} u(x\lambda y) &= \lambda u(x) + (1 - \lambda)u(y) \\ &= (\lambda u_1(x) + (1 - \lambda)u_1(y), \ldots, \lambda u_m(x) + (1 - \lambda)u_m(y)) \\ &= (u_1(x\lambda y), \ldots, u_m(x\lambda y)) \end{aligned}$$

with each u_i a linear functional on X. We say that u is *parsimonious* of dimension m if the representation cannot be satisfied by any linear utility function of smaller dimension. Given that u is parsimonious of dimension m, it is unique up to an affine transformation

$$v = Gu + u_0 \; ,$$

where v, u and u_0 are m-dimensional column vectors, u_0 is fixed, and G is an $m \times m$ lower triangular matrix (0's above the main diagonal) with all diagonal entries positive.

Failures of continuity that force $m \geq 2$ in (7) and (11) are analyzed in detail in [25]. The typical failure occurs when $x \succ y \succ z$ and $y \not\sim x\lambda z$ for all $\lambda \in [0, 1]$, in which case there is a unique $\lambda^* \in [0, 1]$ such that

$$x\alpha z \succ y \succ x\beta z \quad \text{for all } \alpha > \lambda^* > \beta \; , \text{ and}$$

$$\text{either } x\lambda^* z \succ y \text{ or } y \succ x\lambda^* z \; .$$

An example based on marginal probabilities is $x = (x_1, x_2)$ in which x_1 is the probability of dying and x_2 is the probability that you or your heirs will receive \$10. If no increase in x_1 can be compensated for by increasing x_2 to 1, then $\succ$ cannot be represented by a linear unidimensional utility function, but $u(x_1,x_2) = (-x_1,x_2)$ can represent $\succ$ lexicographically.

The extension of linear lexicographic utility to decision under uncertainty in [42, 57–59] formulates X as the set of all finite-support probability distributions, called *mixed acts*, on a set A of acts in C^S with $S = \{1,2,\ldots,n\}$. In the main state-independent version of our theory, we assume that every consequence is relevant for every state in S, that the constant-act set $\{\bar{c}\colon c \in C\}$ is included in A, and that A has some additional structure. We denote by X_C the set of all lotteries on C and let $x_i \in X_C$ denote the marginal distribution in state i of $x \in X$.

The preference relation $\succ$ applies to X, and for p and q in X_C, $p \succ q$ means that $x \succ y$ when $x(\bar{c}) = p(c)$ and $y(\bar{c}) = q(c)$ for every consequence $c \in C$. We assume the following axioms: $\succ$ is a weak order, $\succ$ and $\sim$ satisfy independence,

$$x \sim y \text{ whenever } x_i = y_i \quad \text{for} \quad i = 1,2,\ldots,n \ ,$$

and a relaxed form of Archimedean axiom which implies that the lexicographic hierarchy has only finitely many levels. The axioms imply the existence of linear $\hat{u}\colon X \to \mathbb{R}^J$ and $u\colon X_C \to \mathbb{R}^K$ that preserve lexicographically $\succ$ on X and $\succ$ on X_C, respectively, with parsimonious dimension J of $\hat{u}$ and K of u. Because $\succ$ on X_C is tantamount to the restriction of $\succ$ on X to mixed constant acts, we have $K \leqslant J$, and $K < J$ if preferences between other acts force levels into the hierarchy not accounted for by u on X_C. Uniqueness follows the format described after (11).

Subjective matrix probabilities $\Pi(i)$ for $i = 1,\ldots,n$ rectify $\hat{u}(x)$ with $\{u(x_i)\}$ in the expression

$$\hat{u}(x) = \sum_{i=1}^{n} \Pi(i)u(x_i) + \hat{u}_0 \ ,$$

where $\hat{u}(x)$ and $\hat{u}_0$ are J-dimensional column vectors and $\Pi(i)$ is a $J \times K$ real matrix that premultiplies the K-dimensional column vector $u(x_i)$. Matrix $\Pi(i)$ begins with $\rho(i)$ nonzero columns followed by $K - \rho(i)$ zero columns such that the first nonzero entry in column k for $1 \leqslant k \leqslant \rho(i)$ is a positive number in row $j_k(i)$ for some $1 \leqslant j_1(i) < j_2(i) < \cdots < j_{\rho(i)}(i) \leqslant J$. In addition [59], $j_1(i) = 1$ for some i, $\rho(i) = K$ for some i, and the J rows of the $J \times nK$ matrix $[\Pi(1) : \Pi(2) : \ldots : \Pi(n)]$ are linearly independent. The resulting representation is

$$x \succ y \Leftrightarrow \sum_{i=1}^{n} \Pi(i)u(x_i) >_L \sum_{i=1}^{n} \Pi(i)u(y_i) \quad \text{for all } x, y \in X \ . \tag{12}$$

LaValle and Fishburn [60, 61] show how to assess the vector utilities and matrix probabilities in (12). The latter paper also describes admissible transformations of the matrix-probability distribution Π for any given u that put Π in a standard normalized form. With $\Pi(S) = \Pi(1) + \cdots + \Pi(n)$, we say that Π is a standard matrix distribution if the K columns of $\Pi(S)$ are unit vectors $(0,\ldots,0,1,0,\ldots,0)$ with the 1's in row positions $1 < j_2 < \cdots < j_K \leqslant J$ left to right. Thus, if $K = J$, then $\Pi(S)$ is the $K \times K$

identity matrix. If $K < J$ then standard $\Pi(S)$ has $J - K$ rows of zeros interspersed among the rows below row 1 of the $K \times K$ identity matrix.

Our second topic for decision under uncertainty is motivated by situations with holistic alternatives that consist of similar but clearly identifiable pieces received jointly, such as two checks and a bill in today's mail or the good news and bad news parts of a medical diagnosis. A fundamental behavioral question asks how people evaluate such alternatives for preference comparison or choice. Do they tend to combine similar pieces and then evaluate wholes, or do they evaluate pieces and then combine these evaluations to arrive at holistic assessments? And, in either case, what rules or operations govern the combining process?

To consider these questions, let A_0 denote a nonempty set of basic objects, such as amounts of money or lotteries, and let $\oplus$ denote a binary operation of joint receipt that applies first to A_0 and then to $A_1, A_2, \ldots$ defined recursively by

$$A_{i+1} = \{a \oplus b : a, b \in A_0 \cup A_1 \cup \cdots \cup A_i\}$$

for $i = 0, 1, \ldots$, so that $A_1 \subset A_2 \subset \cdots$ with limit A_∞. We assume that $A_0 \cup A_1 \subseteq X \subseteq A_0 \cup A_\infty$. Then X includes at least one joint-receipt level. We assume also that $\succ$ on X is a weak order.

An elementary case of joint receipt that does not involve decision under risk or uncertainty takes $X = A_0 \cup A_1$ with $A_0 = \mathbb{R}$. We interpret $x \in A_0$ as an amount of money and refer to $x \geqslant 0$ as a gain and to $x \leqslant 0$ as a loss. An early empirical and partly theoretical study of joint receipt for this case is Thaler [93], followed by Thaler and Johnson [94] and Linville and Fischer [62]. They focused in part on the *hedonic editing rule*

$$u(x \oplus y) = \max\{u(x + y),\ u(x) + u(y)\} \quad \text{for all } x, y \in A_0\ , \tag{13}$$

where $u : X \to \mathbb{R}$ is strictly increasing, preserves $\succ$, and has its origin fixed by $u(0) = 0$. This indicates pre-evaluation aggregation in $u(x + y)$ as well as post-evaluation aggregation in $u(x) + u(y)$, with addition as the combining operation in each case. Thaler [93] found that subjects tend to have $x \oplus y \sim x + y$ when x and y are losses, but $x \oplus y \succ x + y$ when x and y are gains, and these agree with (13) if u is convex in losses and concave in gains. Fishburn and Luce [43] provides a complete analysis of (13) under the assumption that u is convex in gains and either convex or concave in losses. The option for (13) is then clear except in the mixed loss and gain region where $x > 0 > y$. Our results for the mixed region, which depend on limiting slopes of u at the origin and $\pm\infty$, show for most cases that there is a continuous curve in $\{(x, y) \colon x \geqslant 0 \geqslant y\}$ that separates $u(x \oplus y) = u(x + y)$ and $u(x \oplus y) = u(x) + u(y)$ when (13) holds.

Research that followed Thaler [93] in [62, 94] and – within a setting of certainty equivalents for monetary lotteries – [13, 65], shows that (13) is not viable in many situations. Part of the difficulty arises in the mixed region, where individuals' assessments of joint receipts are not generally well understood. Another difficulty can be seen in the conjecture of Tversky and Kahneman [97] that $x \oplus y \sim x + y$, which was sustained at least in the loss and gain regions separately in [13]. If $x \oplus y \sim x + y$ everywhere, it would gut (13) by effectively excluding $u(x \oplus y) = u(x) + u(y)$. Also, as Tversky and Kahneman [97] notes, if in fact we assume that $u(x \oplus y) = u(x) + u(y)$, as was done in part of Luce and Fishburn [66], and if $x \oplus y \sim x + y$, then $u(x) = kx$ for some $k > 0$, and this linear form is supported neither by intuition nor by empirical research.

Empirical and theoretical investigations of joint receipt of lotteries or acts in decision under risk or uncertainty include, in addition to the certainty-equivalence approach of Luce [65], Cho and Luce [13], Slovic and Lichtenstein [89], Luce [64], Luce and Fishburn [66, 67] and Cho, Luce and von Winterfeldt [14]. We comment briefly on representational aspects of Luce and Fishburn [66, 67] for a joint-receipt axiomatization of what they refer to as rank- and sign-dependent linear utility. A similar representation without the joint-receipt operation was proposed independently in [97] and axiomatized in [102] under the rubric of cumulative prospect theory.

A central part of the representation in [66] is based on a qualitative structure $(X, \succ, \oplus, e)$ where $\succ$ is a weak order on X, $\oplus$ is a joint receipt operation on X, and e denotes the status quo consequence. Several axioms for the qualitative structure imply that there exists $u : X \to \mathbb{R}$ that satisfies (1) along with $u(e) = 0$ and

$$u(x \oplus y) = \begin{cases} a^+u(x) + b^+u(y) + c^+u(x)u(y) & \text{if} \quad x \succsim e \text{ and } y \succsim e, \\ a^+u(x) + b^-u(y) & \text{if} \quad x \succsim e \succsim y, \\ a^-u(x) + b^+u(y) & \text{if} \quad y \succsim e \succsim x, \\ a^-u(x) + b^-u(y) + c^-u(x)u(y) & \text{if} \quad e \succsim x \quad \text{and} \quad e \succsim y\ . \end{cases} \tag{14}$$

where a^+, a^-, b^+ and b^- are positive constants and c^+ and c^- are constants. If u is unbounded and $\oplus$ is monotonic in the sense that $x \sim y \Rightarrow [x \oplus z \sim y \oplus z$ and $z \oplus x \sim z \oplus y]$ and $x \succ y \Rightarrow [x \oplus z \succ y \oplus z$ and $z \oplus x \succ z \oplus y]$, then $c^+ \geqslant 0$ and $c^- \leqslant 0$. On the other hand, if $x \oplus y \sim x + y$ and if u is bounded and $\oplus$ is monotonic, then [67] $c^+ \leqslant 0$ and $c^- \geqslant 0$. In both cases, if $\oplus$ is commutative and associative, then $a^+ = b^+ = a^- = b^- = 1$. The weighted additive forms in (14) for the mixed cases of $x \succsim e \succsim y$ and $y \succsim e \succsim x$ were adopted as a compromise between $u(x \oplus y) = u(x) + u(y)$ and more complex possibilities.

To expand the formulation to the context of decision under uncertainty, we can take A_0 as the set of all acts in C^S that assign only finite numbers of consequences to the states, define $A_1, A_2, \ldots$ as above, and assume that $\oplus$ is associative so that A_∞ can be replaced by the set A^* of all finite sequences $a_1 \oplus a_2 \oplus \cdots \oplus a_m$ for which $m \geqslant 2$ and $a_i \in A_0$ for all i. Representation (14) applies under this expansion with $X = A_0 \cup A^*$. Given X in this form, Luce and Fishburn [66] describes conditions which imply algebraic forms for the utility of acts that involve subjective probabilities. We illustrate with $C = \mathbb{R}$, as in a monetary context, with $u(\bar{c})$ increasing in c.

Let cEd denote the act in A_0 that yields c if event $E \subseteq S$ obtains and yields d otherwise. Assuming that E is neither empty nor the universal event and that $e = 0$

with $u(0) = 0$, one form is

$$u(cEd) = \begin{cases} \pi^*_>(E)u(\bar{c}) + [1 - \pi^*_>(E)]u(\bar{d}) & \text{if } x \geqslant y, \\ \pi^*_<(E)u(\bar{c}) + [1 - \pi^*_<(E)]u(\bar{d}) & \text{if } x < y, \end{cases}$$

with $0 < \pi^* < 1$. This applies separately to gains, where $* = +$ and $c, d \geqslant 0$, and to losses, where $* = -$ and $c, d \leqslant 0$.

For each $x \in A_0$, let $E^+(x) = \{s \in S\colon x(s) \succ e\}$ and $E^-(x) = \{s \in S\colon e \succ x(s)\}$, and define $x^+, x^- \in A_0$ by

$$x^*(s) = \begin{cases} x(s) & \text{if } s \in E^*(x), \\ e & \text{otherwise,} \end{cases} \quad \text{for } * \in \{+, -\} .$$

A main part of the representation in [66] separates gains from losses in the decomposition

$$u(x) = a^+ u(x^+)\pi^+(E^+(x)) + b^- u(x^-)\pi^-(E^-(x))$$

with $0 \leqslant \pi^*(E) \leqslant 1$ for $* \in \{+, -\}$ and $E \subseteq S$, and with the π^* unique and u unique up to a proportionality transform that fixes $u(e)$ at 0. Additional conditions that allow further refinements for $u(x^+)$ and $u(x^-)$ based on (14) are described in the reference.

Luce and Fishburn [67] focuses on the monetary context with $e = 0$ and considers first the effect of $x \oplus y \sim x + y$ for monetary amounts with $\oplus$ monotonic. We adopt the first and last lines of (14) for joint gains and joint losses respectively, so that

$$\text{for gains,} \quad u(x \oplus y) = u(x) + u(y) - u(x)u(y)/k^* ,$$

$$\text{for losses,} \quad u(x \oplus y) = u(x) + u(y) - u(x)u(y)/k^- .$$

We assume $k^+ > 0$ and $k^- < 0$, which imply $u(x + y) < u(x) + u(y)$ for gains and $u(x + y) > u(x) + u(y)$ for losses, given $x \oplus y \sim x + y$. These inequalities are consistent with studies, including Kahneman and Tversky [51] and others surveyed in [40], that observe increasing marginal utility for losses and decreasing marginal utility for gains. We show in [67] that these hypotheses imply exponential expressions for utility of gains and utility of losses:

$$u(x) = \begin{cases} k^+(1 - \mathrm{e}^{-\alpha x}) \, , \ \alpha > 0 & \text{for gains } (x \geqslant 0), \\ k^-(1 - \mathrm{e}^{\beta x}) \, , \ \ \beta > 0 & \text{for losses } (x \leqslant 0) \, . \end{cases}$$

We also consider lotteries without presuming $x \oplus y \sim x + y$ but maintaining the parts of (14) just noted. Let x denote a lottery on gains $c_1 > c_2 > \cdots > c_m \geqslant 0$ with $x(c_i) = p_i > 0$ and $\sum_1^m p_i = 1$. We identify fairly reasonable assumptions which imply the rank-dependent form [82]

$$u(x) = \sum_{i=1}^{m} u(c_i)[\sigma^+(P_i) - \sigma^+(P_{i-1})]$$

in which $P_i = p_1 + p_2 + \cdots + p_i$ and σ^+ is a continuous and increasing map from $[0, 1]$ onto $[0, 1]$. A similar form with σ^- in place of σ^+ applies to lotteries on losses. The issue of mixed gains and losses is more problematic as described in Section 5 in [67].

6. Nontransitive preferences

A preference relation $\succ$ on a set X is *nontransitive* if there are $x, y, z \in X$ for which $x \succ y, y \succ z$ and not $(x \succ z)$, and is *cyclic* if there are $x_1, x_2, \ldots, x_m \in X$ with $m \geqslant 3$ such that $x_i \succ x_{i+1}$ for all $i < m$ and $x_m \succ x_1$. The representations described thus far assume that $\succ$ is transitive, or at least acyclic. I believe that this reflects a strong attraction of decision theorists and perhaps others to transitivity as an intuitively obvious basis for rational thought and action, an apparently natural desire for order in practical affairs, and the supposed efficiency of optimization based on maximization. Although I find some merit in these contentions, I argue elsewhere [32] against transitivity as an undeniable tenet of rational preferences because I believe that reasonable people sometimes hold nontransitive or cyclic preference patterns that account for their true feelings.

My purpose here is not to recount the arguments for or against transitivity set forth in [32]. Instead, I will illustrate a few situations that might give rise to cyclic patterns and then describe four representations that can account for cyclic preferences by straightforward and elegant generalizations of representations that presume transitivity.

The genesis of cyclic patterns in decision theory may be Condorcet's [15] phenomenon of cyclic majority. The simplest example uses candidates x, y and z and three voters with preference orders $x \succ_1 y \succ_1 z$, $z \succ_2 x \succ_2 y$ and $y \succ_3 z \succ_3 x$. Then, with $\succ$ the strict simple majority relation, $x \succ y \succ z \succ x$. Arrow's [5] famous extension shows that every reasonable rule for aggregating voters' preference orders that makes binary social comparisons without regard to other candidates' positions in voters' orders, and which allows a variety of voter preference profiles, must have profiles for which the social comparison relation is nontransitive. Subsequent contributions on this theme are reviewed in [28].

Multiattribute comparisons provide a source of cyclic preferences for an individual. May [73] asked 62 college students to make binary comparisons between hypothetical marriage partners x, y and z characterized by three attributes, intelligence, looks and wealth:

x: very intelligence, plain, well off,

y: intelligent, very good looking, poor,

z: fairly intelligent, good looking, rich.

Seventeen of the 62 had the 2-to-1 majority cyclic pattern $x \succ y \succ z \succ x$; the other 45 had transitive preferences.

Let (a, p) denote the lottery that pays \$$a$ with probability p and nothing otherwise. Tversky [95] observed that a significant number of people have the cyclic pattern

$$\begin{aligned}(5.00, 7/24) &\succ (4.75, 8/24) \succ (4.50, 9/24) \succ (4.25, 10/24)\\ &\succ (4.00, 11/24) \succ (5.00, 7/24)\ .\end{aligned}$$

In a four-state example with subjective probability of 1/4 for each state, consider four acts with monetary prizes:

		states			
		1	2	3	4
acts	x_1	\$10000	\$9000	\$8000	\$7000
	x_2	\$9000	\$8000	\$7000	\$10000
	x_3	\$8000	\$7000	\$10000	\$9000
	x_4	\$7000	\$10000	\$9000	\$8000

Subjective expected utility theory says that $x_1 \sim x_2 \sim x_3 \sim x_4 \sim x_1$ because every act is equivalent to a lottery with equal chances for the four prizes. However, some people will have $x_1 \succ x_2 \succ x_3 \succ x_4 \succ x_1$ because the first act in each $\succ$ comparison yields a larger prize than the second in three of the four states. Others may have the opposite cycle if they fear that they will experience severe regret if they choose the act with a \$7000 prize in state i over the one with the \$10000 prize in the same state and i turns out to be the state that obtains.

We now describe four representations that accommodate cyclic preferences. The first two apply to multiattribute situations, the third to lottery comparisons in decision under risk, and the fourth to act comparisons in decision under uncertainty.

Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ denote items described by n attributes. Assume that the attribute levels within a given attribute are unambiguously ordered by a weak order $>_i$. The *additive difference representation* is

$$x \succ y \Leftrightarrow \sum_{i=1}^{n} f_i[u_i(x_i) - u_i(y_i)] > 0 \ , \tag{15}$$

where u_i preserves $>_i$ as in (1) and f_i is a strictly increasing functional on its domain with $f_i(\Delta) = -f_i(-\Delta)$. Many possibilities for the f_i allow $x \succ y, y \succ z$ and $z \succ x$ by way of positive sums on the right side of (15) for the three comparisons. Discussions and axioms for (15) and related representations are in [16, 34, 91, 95].

An alternative to (15) is the *nontransitive additive utility representation*

$$x \succ y \Leftrightarrow \sum_{i=1}^{n} \phi_i(x_i, y_i) > 0 \ , \tag{16}$$

in which ϕ_i is a real-valued function on ordered pairs of levels of attribute i with $x_i >_i y_i \Leftrightarrow \phi_i(x_i, y_i) > 0$ and $x_i \sim_i y_i \Leftrightarrow \phi_i(x_i, y_i) = 0$. Axioms for (16) are in [31, 33, 99]. The latter axiomatizations imply that each ϕ_i is *skew symmetric*, i.e.,

$$\phi_i(x_i, y_i) + \phi_i(y_i, x_i) = 0 \ ,$$

and all three imply that the ϕ_i are unique up to proportionality transformations with a common scale multiplier.

Skew symmetry is also used in our other two representations. The representation for lotteries x and y on a set C of consequences is $x \succ y \Leftrightarrow \phi(x, y) > 0$, where ϕ is a real-valued, skew-symmetric and bilinear function on ordered pairs of lotteries. Bilinearity means that $\phi(x\lambda z, y) = \lambda\phi(x, y) + (1 - \lambda)\phi(z, y)$ and $\phi(x, y\lambda z) = \lambda\phi(x, y) + (1 - \lambda)\phi(x, z)$. We refer to the representation as the *SSB representation*, short for skew-symmetric and bilinear. When all individual consequences are in X and it is convex, we have the bilinear expected utility expression

$$\phi(x,y) = \sum_{c\in C}\sum_{d\in C} x(c)y(d)\phi(c,d) .$$

The SSB representation was first described in [56] and is axiomatized in [29], with ϕ unique up to a proportionality transformation. A constant-threshold SSB representation that has $x \succ y \Leftrightarrow \phi(x,y) > 1$ is axiomatized in [44].

Our final representation applies $\succ$ to a Savage act set $X = C^S$ for decision under uncertainty. The representation is

$$x \succ y \quad \Leftrightarrow \quad \int_S \phi(x(s),y(s))\mathrm{d}\pi(s) > 0 , \tag{17}$$

where ϕ is a skew-symmetric functional on $C\times C$ and π is a finitely additive probability measure on 2^S, with π unique and ϕ unique up to a proportionality transformation. When ϕ decomposes as $\phi(c,d) = u(c) - u(d)$, (17) reduces to Savage's subjective expected utility representation. Axioms that imply (17) for all acts that use only finite numbers of consequences are in Chapter 9 in [29]. They are like Savage's axioms in most respects with weak order replaced by an asymmetry condition. Extension to all acts is also discussed in Chapter 9. Other representations that are closely related to (17) appear in [29, 41, 63].

Although cyclic preference patterns have been studied in depth for aggregate relations in voting and social choice theory, they have received very little attention in individual decision theory. Unlike most decision theorists, I think the aversion to cyclic preferences for individuals is unjustified, and I hope that more will be done on the subject in the years ahead.

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