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The Effective Riemann Mapping Theorem

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The Effective Riemann Mapping Theorem *

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Abstract

The main results of the paper are two effective versions of the Riemann mapping theorem. The first, uniform version is based on the constructive proof of the Riemann mapping theorem by Bishop and Bridges and formulated in the computability framework developed by Kreitz and Weihrauch. It states which topological information precisely one needs about a nonempty, proper, open, connected, and simply connected subset of the complex plane in order to compute a description of a holomorphic bijection from this set onto the unit disk, and vice versa, which topological information about the set can be obtained from a description of a holomorphic bijection. The second version, which is derived from the first by considering the sets and the functions with computable descriptions, characterizes the subsets of the complex plane for which there exists a computable holomorphic bijection onto the unit disk. This solves a problem posed by Pour–El and Richards. We also show that this class of sets is strictly larger than a class of sets considered by Zhou, which solves an open problem posed by him. In preparation, recursively enumerable open subsets and closed subsets of Euclidean spaces are considered and several effective results in complex analysis are proved.

1 Introduction

One of the most famous classical results in complex analysis is the Riemann Mapping Theorem. It states that the subsets of the complex plane which can be mapped conformally onto the unit disk are exactly the nonempty, proper, open, connected and simply connected subsets (we call a function mapping complex numbers to complex numbers *conformal* if and only if it is holomorphic, i.e. analytic, and injective, i.e. one–to–one, on its domain). While it is easy to see that any conformal image of the unit disk has the listed topological properties, it is remarkable that these topological properties of a subset of the complex plane already guarantee the existence not only of a homeomorphism (a continuous bijection whose inverse is continuous as well) onto the unit disk but even of a conformal homeomorphism (a mapping which at least locally preserves the geometry) onto the unit disk.

The most common proofs are pure existence proofs, mostly based on the extremal principle of Fejér and Riesz [16]. The question how to obtain such a conformal mapping for a given

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proper, simply connected region (a *region* is a nonempty, open, connected subset of the complex plane) has received attention already very early. In fact, the first complete proof of the theorem, given by Carathéodory and Koebe [10] in 1912, is already constructive in a naive sense. The conformal mapping was constructed as the combination of a mapping which maps the region into the unit disk and of a limit function obtained by repeatedly applying dilating mappings which map the region to regions filling the unit disk better and better. A pure, constructive proof in the framework and language of constructive mathematics along the lines of the Koebe proof was given by Bishop and Bridges [2]. We shall come back to this later. Pour–El and Richards [15, Problem 5] posed the problem to characterize the subsets of the complex plane for which there exists a computable conformal mapping onto the unit disk. Here computability of a function is defined in the sense that by using better and better approximations for the input one must be able to compute better and better approximations for the output. This is the computability notion for real functions based on the Turing machine model and studied by Grzegorczyk [4, 5], Lacombe [12], Hauck [6, 7], Pour-El and Richards [15], Kreitz and Weihrauch [19, 11, 23, 20], Ko [9], and others. We shall prove the following answer to the Pour–El/Richards problem:

Theorem *For a subset $U \subseteq \mathbb{C}$ of the complex plane the following two statements are equivalent:*

1. *U is a nonempty, proper, r.e. open, connected, simply connected subset of $\mathbb{C}$ and its boundary ∂U is r.e. closed.*
2. *There exists a computable conformal bijection f from the unit disk onto U.*

Furthermore, there exists an algorithm which computes a program for a conformal bijection of the unit disk onto U if the algorithm is given a program for such a set U as an r.e. open set and a program for its boundary as an r.e. closed set. Also an algorithm performing the inverse task exists.

Hence, besides the known non-effective topological properties a subset of the complex plane must satisfy two effective properties in order to possess a computable conformal bijection onto the unit disk (the inverse of a computable conformal mapping is again a computable conformal mapping): it must be r.e. open and its boundary must be r.e. closed. An open set U is called r.e. open if one can enumerate a set of open spheres with rational centre and rational radius which cover exactly U. A closed set C is called r.e. closed if one can enumerate a set of computable real numbers which form a dense subset of C. It is interesting that an effectivity condition plays an important role whose non-effective analogue is trivially fulfilled, namely that the boundary is r.e. closed. Zhou [24] had considered a class of sets which are called recursively open and asked [24, Problem 5.4] whether every set which is the image of the unit disk under a computable conformal mapping must be recursively open. We give a negative answer to this question.

The result above is a direct corollary of the main result of the paper. This is formulated in the language of Type 2 Theory of Effectivity, developed by Kreitz and Weihrauch [19, 11, 20]. This theory allows the definition and analysis of computability for operators between more general objects than real numbers or vectors: the objects are represented by infinite binary (one may also use larger alphabets) sequences containing certain information about them (in the same way as real number representations contain information about real numbers). Our main result states that from a sequence which describes a proper, simply connected region in a certain, purely topological way corresponding to the two effectivity conditions above one can compute a sequence which describes a conformal bijection from the set onto the unit disk, and that the inverse operation, starting with a description of a conformal mapping and computing equivalent topological information about the set, is also computably possible. Hence, we show which topological information precisely about a proper, simply connected region is equivalent to the standard information about the conformal mapping onto the unit disk: it is the information to enumerate spheres covering exactly the set and to enumerate points forming a dense subset of the boundary. The theorem above is obtained by considering the sets and conformal functions which possess computable names with respect to these representations.

We follow the classical proof by Koebe [10] and additionally use ideas and estimates from the strictly constructive proof of the Riemann Mapping Theorem by Bishop and Bridges [2]. They show that the constructive existence of a conformal mapping of a subset U of the complex plane onto the unit disk is equivalent to certain constructive topological conditions on the set U, whose computability theoretic analogues turn out to be equivalent to the conditions in the theorem above. It is interesting that the constructive/computable analogue of the classically purely topological direction of the Riemann Mapping Theorem — that every conformal image of the unit disk is a proper, simply connected region — seems to require an application of the Koebe $\frac{1}{4}$ theorem. This is due to the additional information about the boundary (see the theorem above), which does not appear in the non-effective formulation.

We would like to mention that the problem to construct a conformal mapping onto the unit disk for a given proper, simply connected region is also of great practical interest, compare Henrici [8]. While in this paper we are interested in the computability theoretic aspects of the Riemann Mapping Theorem, a complexity theoretic analysis would certainly be interesting. Besides the book of Bishop and Bridges [2] we mention the exposition of the Riemann Mapping Theorem by Henrici [8], which contains also estimates for the speed of convergence, i.e. for the complexity of the algorithm. The exposition by Remmert [17] discusses also other variants of proofs and gives a lot of historical background.

In the following section we provide a brief introduction into Type 2 Theory of Effectivity, introducing computability for "infinite" objects via representations and Turing machines. In Section 3 we introduce various representations of open subsets and of closed subsets of Euclidean spaces. For example we define the notions of being r.e. open or r.e. closed or recursively open. Section 4 contains several results from computable complex analysis which we need for the proof of the main result and which seem to be of independent interest. In Section 5 the main result and its corollaries are formulated. Then, in Section 6, we prove the main result. We conclude with a summary and with some open problems.

2 Type 2 Computable Analysis

This section contains a brief introduction into computable analysis based on representations and Type 2 computable functions and provides the necessary terminology. For a more complete and systematic treatment see Kreitz and Weihrauch [11, 20, 21].

If one wishes to perform computations over a countable set of objects, e.g. the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$, on a Turing machine, one can do so by using a notation of these objects. One represents the objects by finite strings over the finite alphabet of the Turing machine and performs the computation on these names. Elements of an uncountable set, e.g. the real numbers $\mathbb{R}$, cannot be identified by finite words. Yet, one would like to define and to speak of computable real number functions. Type 2 Theory of Effectivity, developed by Kreitz and Weihrauch [19, 11, 20], see also Hauck [7], offers the following solution for uncountable sets whose cardinality does not exceed the cardinality of the continuum: one represents the objects by infinite sequences of digits from the Turing machine alphabet and performs the computation on longer and longer prefixes of these sequences. Thus, in finite time only finite prefixes have been read and written and only finite, partial information about the objects has been processed. But it is important that this process can at least in principle be continued ad infinitum. This means that it must be possible to carry the computation out with arbitrary precision. For real number functions this approach leads to a computability notion based on approximations. The approach allows also the definition of computability for more complicated objects and operators, e.g. operators between certain spaces of sets or functions. Besides computable real and complex number functions and sets of real vectors we are interested in the computability of operators which transform topological information about certain open subsets of the complex plane into geometric information describing conformal functions, and vice versa.

We denote the set of natural numbers by $\mathbb{N} = \{0, 1, 2, \ldots\}$, the set of real numbers by $\mathbb{R}$. We write $d(x, y) = |x - y|$ for the usual Euclidean distance of two vectors $x, y \in \mathbb{R}^n$, $\mathrm{S}(x, r) := \{y \in \mathbb{R}^n \mid |x - y| < r\}$ for the open sphere in $\mathbb{R}^n$ with centre x and radius r, and $\mathrm{Sc}(x, c) := \{y \in \mathbb{R}^n \mid |x - y| \leq r\}$ for its closure. By $f :\subseteq X \to Y$ we mean a function whose domain $\operatorname{dom} f$ of definition is a subset of X and whose range is a subset of Y.

We start with the definition of notations and representations. In the whole paper Σ will be a finite set, called the *alphabet*, which contains at least the symbols 0, 1, #, and a blank B. By ε we denote the empty string. Σ^* is the set of finite strings over Σ and $\Sigma^\omega = \{p \mid p : \mathbb{N} \to \Sigma\}$ is the set of infinite sequences over Σ.

Definition 2.1 Let X be a set. A *notation* is a surjective function $\nu :\subseteq \{0,1\}^* \to X$. A *representation* is a surjective function $\delta :\subseteq \Sigma^\omega \to X$.

First we introduce a standard notation of numbers.

Example 2.2 The notation $\nu_{\mathbb{N}} : \{0,1\}^* \to \mathbb{N}$ is the quasi–lexicographical bijection between $\{0,1\}^*$ and $\mathbb{N}$ with $\nu_{\mathbb{N}}(\varepsilon) := 0$, $\nu_{\mathbb{N}}(0) := 1$, $\nu_{\mathbb{N}}(1) := 2$, $\nu_{\mathbb{N}}(00) := 3, \ldots$.

We shall often use the Cantor pairing function $\langle , \rangle : \mathbb{N}^2 \to \mathbb{N}$ with $\langle i, j \rangle := \frac{1}{2}(i+j)\cdot(i+j+1)+j$ and the derived bijection $\langle , \rangle : (\{0,1\}^*)^2 \to \{0,1\}^*$ with $\langle v, w \rangle := \nu_{\mathbb{N}}^{-1}\langle \nu_{\mathbb{N}}(v), \nu_{\mathbb{N}}(w)\rangle$. We define inductively $\langle v \rangle := v$ and $\langle v_1, \ldots, v_{n+1} \rangle := \langle \langle v_1, \ldots, v_n \rangle, v_{n+1} \rangle$.

Examples 2.3

1. The notation $\nu_{\mathbb{Z}} : \{0,1\}^* \to \mathbb{Z}$ of the set of integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ is defined by $\nu_{\mathbb{Z}}\langle v, w \rangle := \nu_{\mathbb{N}}(v) - \nu_{\mathbb{N}}(w)$ for $v, w \in \{0,1\}^*$.

2. The notation $\nu_{\mathbb{D}} : \{0,1\}^* \to \mathbb{D}$ of the set of dyadic rational numbers

$$\mathbb{D} = \{x \in \mathbb{Q} \mid (\exists\, k \in \mathbb{Z}, m \in \mathbb{N})\ x = k/2^m\}$$

is defined by $\nu_{\mathbb{D}}\langle v, w \rangle := \nu_{\mathbb{Z}}(v)/2^{\nu_{\mathbb{N}}(w)}$ for $v, w \in \{0,1\}^*$.

3. Let ν be a notation of a set X. Then the notation ν^n of the set X^n is defined by $\nu^n\langle v_1, \ldots, v_n \rangle := (\nu(v_1), \ldots, \nu(v_n))$ for $v_1, \ldots, v_n \in \{0,1\}^*$.

4. Let $n \geq 1$ be fixed. We define a notation ν_{S^n} of the set of open *dyadic* spheres in $\mathbb{R}^n$ with dyadic center and dyadic radius by $\nu_{S^n}\langle v, w\rangle := \mathrm{S}(\nu^n_{\mathbb{D}}(v), \nu_{\mathbb{D}}(w))$ for $v, w \in \{0,1\}^*$. The notation ν_{Sc^n} of closed *dyadic* spheres is defined accordingly.

For representations the following additional constructions are very useful. We define a mapping $\mathrm{En} : \Sigma^\omega \to \{A \mid A \subseteq \{0,1\}^*\}$ by

$$\mathrm{En}(p) := \{v \in \{0,1\}^* \mid \#v\# \text{ is a subword of } p\}.$$

Also, for each $n \geq 1$ we use the standard tupling function $\langle , \rangle : (\Sigma^\omega)^n \to \Sigma^\omega$ defined by $\langle p \rangle := p$, $\langle p, q\rangle := p(0)q(0)p(1)q(1)\ldots$, and $\langle p^{(1)}, \ldots, p^{(n+1)}\rangle := \langle\langle p^{(1)}, \ldots, p^{(n)}\rangle, p^{(n+1)}\rangle$, for $p = p(0)p(1)p(2)\ldots$, $q = q(0)q(1)q(2)\ldots$, $p^{(1)}, \ldots, p^{(n+1)} \in \Sigma^\omega$. The inverse projections π^n_i are defined by $p = \langle \pi^n_1 p, \ldots, \pi^n_n p\rangle$ for $1 \leq i \leq n$. We can also define a tupling function $\langle , \rangle : (\Sigma^\omega)^\omega \to \Sigma^\omega$ via

$$\langle p^{(0)}, p^{(1)}, p^{(2)}, \ldots\rangle(\langle i, j\rangle) := p^{(i)}(j),$$

for $p^{(0)}, p^{(1)}, p^{(2)}, \ldots \in \Sigma^\omega$. The projections π^∞_i are defined by $p = \langle \pi^\infty_0 p, \pi^\infty_1 p, \pi^\infty_2 p, \ldots\rangle$.

Examples 2.4 1. Let $n \geq 1$ be fixed. We define a representation $\rho^{(n)} :\subseteq \Sigma^\omega \to \mathbb{R}^n$ of real n–vectors as follows: if $p \in \Sigma^\omega$ has the form $p = \#v_0\#v_1\#v_2\#\ldots$ for words $v_i \in \mathrm{dom}\,\nu^n_{\mathbb{D}}$ with $|\nu^n_{\mathbb{D}}(v_i) - \nu^n_{\mathbb{D}}(v_j)| \leq 2^{-\min\{i,j\}}$ for all i, j, then we set $\rho^{(n)}(p) := \lim \nu^n_{\mathbb{D}}(v_i)$. If p does not have this form, then $\rho^{(n)}(p)$ is undefined.

2. We shall need also representations for the set of the extended real numbers $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$, endowed with the obvious order relation and the usual arithmetic operations, e.g. $x + \infty = \infty$ for all $x \in \mathbb{R} \cup \{\infty\}$, etc. We define representations $\overline{\rho_<}$, $\overline{\rho_>}$, and $\overline{\rho} :\subseteq \Sigma^\omega \to \overline{\mathbb{R}}$ as follows:

$$\begin{aligned}
\overline{\rho_<}(p) = x \quad &\text{iff} \quad \mathrm{En}(p) = \{v \in \{0,1\}^* \mid \nu_{\mathbb{D}}(v) < x\},\\
\overline{\rho_>}(p) = x \quad &\text{iff} \quad \mathrm{En}(p) = \{v \in \{0,1\}^* \mid \nu_{\mathbb{D}}(v) > x\},\\
\overline{\rho}\langle p, q\rangle = x \quad &\text{iff} \quad \overline{\rho_<}(p) = x \text{ and } \overline{\rho_>}(q) = x.
\end{aligned}$$

3. We define $\rho_<$, $\rho_>$, and $\rho :\subseteq \Sigma^\omega \to \mathbb{R}$ to be the restrictions of $\overline{\rho_<}$, $\overline{\rho_>}$, and $\overline{\rho}$ to names of real numbers.

4. Let $n \geq 1$ be fixed. If δ is a representation of a set X, then the representation δ^n of the set X^n is defined by $\delta^n\langle p^{(1)}, \ldots, p^{(n)}\rangle := (\delta(p^{(1)}), \ldots, \delta(p^{(n)}))$ for $p^{(1)}, \ldots, p^{(n)} \in \Sigma^\omega$.

In order to use notations and representations for computations we have to define computable functions on strings and infinite sequences. We use the usual notions based on the Turing machine model and explained e.g. in Weihrauch [21]. They are based on the idea that by using a large enough prefix of the input (if it is infinite) one can compute a prefix of the output of any desired length, if it is infinite, and the exact output, if it is finite. We give a precise definition only for the case of computable functions whose input and output are infinite sequences. A function $g :\subseteq \Sigma^* \to \Sigma^*$ is called *monotonic*, iff $g(vw) \in g(v)\Sigma^*$ for all $v, vw \in \mathrm{dom}\,g$. The function $f :\subseteq \Sigma^\omega \to \Sigma^\omega$ *induced* by a monotonic function $g :\subseteq \Sigma^* \to \Sigma^*$ is defined by

1. $\mathrm{dom}\,f = \bigcap_{n\in\mathbb{N}}(g^{-1}(\Sigma^n\Sigma^*)\Sigma^\omega)$ (i.e. $p \in \mathrm{dom}\,f$ iff for all $n \in \mathbb{N}$ there is some prefix $v \in \mathrm{dom}\,g$ of p with $|g(v)| \geq n$),

2. $f(p) \in g(v)\Sigma^\omega$ for any prefix $v \in \operatorname{dom} g$ of p (for $p \in \operatorname{dom} f$).

It is clear that f is well-defined by these conditions.

Definition 2.5 A function $F :\subseteq \Sigma^\omega \to \Sigma^\omega$ is called a *computable functional*, iff there is a computable (in the standard sense), monotonic function $g :\subseteq \Sigma^* \to \Sigma^*$ which induces F.

Before we define computability for functions between represented spaces we use computable functionals in order to compare representations.

Definition 2.6 Let $\gamma :\subseteq \Sigma^\omega \to X$ and $\delta :\subseteq \Sigma^\omega \to Y$ be representations of sets X and Y. We write $\gamma \leq \delta$ iff there is a computable functional F with $\gamma(p) = \delta F(p)$ for all $p \in \operatorname{dom}\gamma$. We write $\gamma \equiv \delta$ and say that γ and δ are *equivalent* iff $\gamma \leq \delta$ and $\delta \leq \gamma$.

Examples 2.7

1. $\rho^{(n)} \equiv \rho^n$ for all $n \geq 1$.
2. $\rho \leq \rho^<$, but $\rho^< \not\leq \rho$.

Every computable functional $F :\subseteq \Sigma^\omega \to \Sigma^\omega$ is continuous where on Σ^ω we consider the usual product topology. A base of this topology is formed by the sets $w\Sigma^\omega = \{p \in \Sigma^\omega \mid w$ is a prefix of $p\}$. In analogy to the standard notation φ of computable functions mapping strings to strings (compare e.g. Rogers [18], Weihrauch [20]) there is a total standard representation η of the set of all continuous functionals $F :\subseteq \Sigma^\omega \to \Sigma^\omega$ whose domains are G_δ–sets, see Weihrauch [20].

Theorem 2.8 *There exists a total representation* $\eta : \Sigma^\omega \to \{F :\subseteq \Sigma^\omega \to \Sigma^\omega \mid F$ *is continuous and* $\operatorname{dom} F$ *is a* G_δ*–set*$\}$ *with the following properties:*

1. *(utm Theorem) The functional* $u :\subseteq \Sigma^\omega \to \Sigma^\omega$ *with* $u\langle p, q\rangle := \eta_p(q)$ *for all* $p, q \in \Sigma^\omega$*, is computable.*
2. *(smn Theorem) For every computable functional* $F :\subseteq \Sigma^\omega \to \Sigma^\omega$ *there exists a total computable functional* $G : \Sigma^\omega \to \Sigma^\omega$ *with* $\eta_{G(p)}(q) = F\langle p, q\rangle$ *for all* $p, q \in \Sigma^\omega$*.*

The following definition of computability for represented spaces follows the idea that computations are not performed on the objects themselves but on their names.

Definition 2.9

1. Let $f :\subseteq X \to Y$ be a function between two sets X and Y and let $\gamma :\subseteq \Sigma^\omega \to X$ and $\delta :\subseteq \Sigma^\omega \to Y$ be representations of sets X and Y. We say that f is (γ, δ)*–tracked* by a functional F iff
$$f\gamma(p) = \delta F(p)$$
for all $p \in \operatorname{dom} f\gamma$. If f is (γ, δ)–tracked by η_p for some $p \in \Sigma^\omega$, then we call p a (γ, δ)–*tracking name* for f.
2. The function f is called (γ, δ)*–computable* iff there exists a computable functional which (γ, δ)–tracks f.

A function $f :\subseteq X \to Y$ is (γ, δ)–tracked by a continuous functional if and only if it is continuous with respect to the topologies on X and Y induced by γ and δ, compare Kreitz and Weihrauch [11, 20]. It is important to note that a function $f :\subseteq \mathbb{R}^m \to \mathbb{R}^n$ possesses a (ρ^m, ρ^n)–tracking name iff it is continuous.

We mention one intuitively clear corollary which says that the composition of functions is computable.

Corollary 2.10 *There exists a computable functional* Comp *with the following property: if $\beta :\subseteq \Sigma^\omega \to X$, $\gamma :\subseteq \Sigma^\omega \to Y$, and $\delta :\subseteq \Sigma^\omega \to Z$ are representations of sets X, Y, Z, and if p is a (β, γ)–tracking name for a function $f :\subseteq X \to Y$ and q is a (γ, δ)–tracking name for a function $g :\subseteq Y \to Z$, then $G\langle q, p\rangle$ exists and is a (β, δ)–tracking name for the composition $g \circ f$.*

Proof. By the utm and smn Theorem for η there exists a computable functional Comp with $\eta_{\mathrm{Comp}\langle q,p\rangle}(r) = \eta_q \eta_p(r)$ for all $p, q, r \in \Sigma^\omega$. This functional Comp has the desired property. $\square$

We also wish to introduce computability for elements, not just for functions. This is done via computable names. A sequence $p \in \Sigma^\omega$ is called *computable* iff the function $g : \Sigma^* \to \Sigma^*$ with $g(\nu_{\mathbb{N}}^{-1}(i)) = p(i)$ for all i, is computable. Let φ be a total standard notation of the set of all computable functions $g :\subseteq \Sigma^* \to \Sigma^*$ (compare Rogers [18], Weihrauch [20]).

Definition 2.11 Let $\gamma :\subseteq \Sigma^\omega \to X$ be a representation of a set X.

1. An element $x \in X$ is called γ*–computable* iff there is a computable sequence $p \in \Sigma^\omega$ with $\gamma(p) = x$.
2. The *notation* ν_γ of γ–computable elements *induced by* γ is defined by $\nu_\gamma(v) = \gamma(\varphi_v)$ if the term on the right side exists.

A ρ–computable real number is simply called *computable.*

Lemma 2.12 (Computable Points Lemma) *Let F be a computable functional. If $p \in \mathrm{dom}\, F$ is a computable sequence, then also $F(p)$ is a computable sequence. Furthermore, there is a computable function on strings which maps any φ–name for a computable sequence $p \in \mathrm{dom}\, F$ to a φ–name for $F(p)$.*

This lemma has important consequences. Let $\gamma :\subseteq \Sigma^\omega \to X$ and $\delta :\subseteq \Sigma^\omega \to Y$ be representations of sets X and Y. If $\gamma \leq \delta$, then every γ–computable element is also δ–computable. If γ and δ are equivalent, then an element of X (which is identical with Y in that case) is γ–computable iff it is δ–computable. If $f :\subseteq X \to Y$ is a (γ, δ)–computable function, then f maps γ–computable elements to δ–computable elements. And, given a ν_γ–name of a γ–computable element in $\mathrm{dom}\, f$, one can compute a ν_δ–name for $f(x)$. Further statements of this kind are obtained by considering also computable functionals whose input or output are (γ, δ)–tracking names for functions. Then also the following corollary of Theorem 2.8 is important.

Lemma 2.13 *A function $F :\subseteq \Sigma^\omega \to \Sigma^\omega$ is a computable functional iff there exists a computable $p \in \Sigma^\omega$ with $\eta_p = F$.*

The following proposition summarizes a few characterizations of computable functions $f :\subseteq \mathbb{R}^m \to \mathbb{R}^n$. This is the computability notion for real functions considered e.g. by the authors cited in the introduction. Especially, for functions whose domain of definition is a computable rectangle as considered by Pour–El and Richards [15], one obtains their computability notion.

Proposition 2.14 *Let $m, n \geq 1$ be fixed. The following conditions are equivalent for a function $f :\subseteq \mathbb{R}^n \to \mathbb{R}^m$:*

1. *f is (ρ^n, ρ^m)–computable.*
2. *f is $(\rho^{(n)}, \rho^{(m)})$–computable.*
3. *f possesses a computable (ρ^n, ρ^m)–tracking name.*
4. *There exists an r.e. set $A \subseteq \{0,1\}^*$ with $f^{-1}(\nu_{S^m}(w)) = \mathrm{dom}\, f \cap \bigcup_{\langle v,w\rangle \in A} \nu_{S^n}(w)$ for all $w \in \{0,1\}^*$.*

We conclude this section with a simple example which we shall need later. Let $U \subseteq \mathbb{R}^n$ be an open set. If a sequence $(f_n)_n$ of continuous functions defined on U converges locally uniformly, then it converges to a continuous limit function f. If all functions f_n are computable on U and for every compact subset of U the convergence is computably fast, then the limit function is computable on every compact subset of U, but it does not need to be computable on U. For a counterexample see Pour–El and Richards [15]. This changes, when one can compute a lower bound for the speed of convergence not only in n but also uniformly in the compact sets. We say that a sequence $p \in \Sigma^\omega$ *describes a modulus of converges* for the sequence $(f_n)_n$ on U iff p has the form

$$p = \#\langle u_0, v_0, w_0\rangle \# \langle u_1, v_1, w_1\rangle \# \ldots$$

where $|f_i(x) - f_j(x)| \leq 2^{-\nu_{\mathbb{N}}(v_k)}$ for all k, for all $x \in U \cap \nu_{S^n}(u_k)$, and for all $i, j \geq \nu_{\mathbb{N}}(w_k)$, and if for each $l \in \mathbb{N}$ the set U is a subset of $\bigcup\{\nu_{S^n}(u_k) \mid \langle u_k, v_k, w_k\rangle \in \mathrm{En}(p) \text{ and } \nu_{\mathbb{N}}(v_k) \geq l\}$.

Proposition 2.15 *Fix numbers $n, m \geq 1$. There exists a computable functional H with the following property: if $U \subseteq \mathbb{R}^n$ is an open set, if p is a sequence such that for each i the sequence $\pi_i^\infty p$ is a (ρ^n, ρ^m)–tracking name for a continuous function $f_i : U \to \mathbb{R}^m$, and if q describes a modulus of convergence for the sequence $(f_i)_i$ on U, then $H\langle p, q\rangle$ exists and is a (ρ^n, ρ^m)–tracking name for the limit function of the sequence $(f_i)_i$ on U.*

Proof. Assume that p and q are as in the proposition, and r is a ρ^n–name for a point $x \in U$. Then for any $l \in \mathbb{N}$ we can find a word $\langle u_k, v_k, w_k\rangle \in \mathrm{En}(q)$ such that $\nu_{\mathbb{N}}(v_k) \geq l + 1$ and $x \in \nu_{S^n}(u_k)$. Then we define $i_0 := \nu_{\mathbb{N}}(w_k)$. Using $\pi_{i_0}^\infty p$ we can compute the value $f_{i_0}(x)$ with precision 2^{-l-1} and thus obtain a 2^{-l}–approximation for $\lim_{i\to\infty} f_i(x)$. Hence, there is a computable functional G with the following property: if p and q are as in the proposition, and r is a ρ^n–name for a point $x \in U$, then $G\langle p, q, r\rangle$ exists and is a ρ^m–name for the value of the limit function at x. An application of the smn Theorem for η gives the desired functional H. □

3 Open Sets and Closed Sets

We present a series of representations of open or closed subsets of Euclidean spaces $\mathbb{R}^n$. The representations for open sets will be grouped into four different equivalence classes, depending on how much information about an open set they deliver. By considering the sets with computable names with respect to these representations we obtain a hierarchy of four computability classes of open subset of $\mathbb{R}^n$. One of them corresponds clearly to the class of recursively enumerable subsets of $\mathbb{N}$, two correspond to the class of recursive subsets of $\mathbb{N}$, and for the last one it seems to be most appropriate to say that it lies in between, though closer to the class of r.e. sets. This last class will be of greatest interest for us in connection with the Effective Riemann Mapping Theorem. We shall define and compare the representations and give typical examples of sets with computable names. In the end we relate the classes of sets with computable names to r.e. or recursive sets of natural numbers.

The first three classes of representations (with further elements) have been studied by Weihrauch, Kreitz [23] and Weihrauch, Brattka [22], who also give equivalent versions of the following Theorems 3.2, 3.4, and 3.6. We give the proofs nevertheless since in [22] the representations are all formulated for closed sets and because seeing the reasons for the equivalences is important for the further chapters. Several of the corollaries about open (or closed) sets with computable names, i.e. about r.e. or recursive or birecursively open sets have also been obtained by authors including Lacombe [13], Weihrauch, Kreitz [23], Ge, Nerode [3], Zhou [24], Mori, Tsujii, and Yasugi [14].

We shall define representations on the one hand by enumerating open spheres or points and on the other hand by supplying information about the distance function or a modified characteristic function. For the motivation, consider a set $U \subseteq \mathbb{N}$ of natural numbers and its characteristic function $\chi_U : \mathbb{N} \to \{0,1\}$ with $\chi_U^{-1}\{1\} = U$. The set U is r.e. iff the function χ_U is approximable from below, U is co-r.e. iff χ_U is approximable from above, and U is recursive iff χ_U is computable. So the characteristic function of U gives us a "yes" or "no" answer if we ask whether a point lies inside of U or not. We are dealing here with open subsets U of the continuous space $\mathbb{R}^n$. For a continuously varying input one cannot expect a discrete "yes" or "no" decision. Instead it makes sense to smooth the characteristic function so that it tells us how close a point is to lying outside of U. This leads to the modified characteristic function or modified distance function $\mathrm{cutdist}_{\mathbb{R}^n\setminus U}$ and also to the distance function $\mathrm{dist}_{\mathbb{R}^n\setminus U}$ itself. Later we shall consider another, more symmetric variant.

We define the usual distance function and a modified distance function for an arbitrary closed subset $C \subseteq \mathbb{R}^n$ by:

$$\mathrm{dist}_C : \mathbb{R}^n \to \overline{\mathbb{R}}, \quad \mathrm{dist}_C(x) := d(C,x) = \inf_{y\in C} d(y,x)\,,$$
$$\mathrm{cutdist}_C : \mathbb{R}^n \to \mathbb{R}, \quad \mathrm{cutdist}_C(x) := \min\{1, d(C,x)\}\,.$$

This means especially $\mathrm{dist}_\emptyset(x) = \infty$ and $\mathrm{cutdist}_\emptyset(x) = 1$ for all $x \in \mathbb{R}^n$. The following four representations are those which contain the least information among all the considered representations for open sets.

Definition 3.1 *Let $n \geq 1$. We define the representations δ_{open}, $\delta_{\mathrm{closed-spheres,open}}$, $\delta_{\mathrm{dist,open}}$, and $\delta_{\mathrm{cutdist,open}}$ of the set of open subsets of $\mathbb{R}^n$ by (where $U \subseteq \mathbb{R}^n$ is open):*

$$\delta_{\mathrm{open}}(p) = U \quad \text{iff} \quad U = \bigcup_{w\in\mathrm{En}(p)} \nu_{S^n}(w)\,,$$
$$\delta_{\mathrm{closed-spheres,open}}(p) = U \quad \text{iff} \quad \mathrm{En}(p) = \{w \in \{0,1\}^* \mid \nu_{Sc^n}(w) \subseteq U\}\,,$$
$$\delta_{\mathrm{dist,open}}(p) = U \quad \text{iff} \quad p \text{ is a } (\rho^n, \overline{\rho^<})\text{–tracking name for the function } \mathrm{dist}_{\mathbb{R}^n\setminus U}\,,$$
$$\delta_{\mathrm{cutdist,open}}(p) = U \quad \text{iff} \quad p \text{ is a } (\rho^n, \rho^<)\text{–tracking name for the function } \mathrm{cutdist}_{\mathbb{R}^n\setminus U}\,.$$

Theorem 3.2 *Let $n \geq 1$. The representations δ_{open}, $\delta_{\mathrm{closed-spheres,open}}$, $\delta_{\mathrm{dist,open}}$, and $\delta_{\mathrm{cutdist,open}}$ are equivalent.*

Proof. "$\delta_{\text{open}} \le \delta_{\text{closed-spheres,open}}$": Let p be a δ_{open}–name for an open set $U \subseteq \mathbb{R}^n$. We have to show that we can construct a $\delta_{\text{closed-spheres,open}}$–name q for U. The name p can be considered as a list of open dyadic spheres covering exactly U. An arbitrary closed, hence compact, dyadic sphere is contained in U iff it is covered by finitely many open spheres in this list. And for a given closed dyadic sphere and a finite set of open dyadic spheres one can decide whether the closed sphere is covered by the open spheres. Therefore, we can, using p, enumerate all closed dyadic spheres contained in U. Hence, we can construct a $\delta_{\text{closed-spheres,open}}$–name q for U.

"$\delta_{\text{closed-spheres,open}} \le \delta_{\text{dist,open}}$": We show that there is a computable functional G with the following property: if p is a $\delta_{\text{closed-spheres,open}}$–name for an open set U and q a ρ^n–name for a point $x \in \mathbb{R}^n$, then $G\langle p, q\rangle$ exists and is a $\overline{\rho^<}$–name for $d(\mathbb{R}^n \setminus U, x)$. Applying the smn Theorem for η to this functional will then yield a computable functional F which transforms any $\delta_{\text{closed-spheres,open}}$–name for any open set into a $\delta_{\text{dist,open}}$–name for the same set.

Given a $\delta_{\text{closed-spheres,open}}$–name p for an open set U and a ρ^n–name q for a point $x \in \mathbb{R}^n$, we have to compute a list containing exactly all dyadic rational numbers ϑ with $\vartheta < d(\mathbb{R}^n \setminus U, x)$. We can construct such a list when we can find arbitrarily good dyadic approximations from below for $d(\mathbb{R}^n \setminus U, x)$. Without any further computation we can already enumerate all negative dyadic rationals. In case $x \notin U$ these already give a $\overline{\rho^<}$–name for $d(\mathbb{R}^n \setminus U, x)$. In order to take also the case $x \in U$ into account we sweep through all closed spheres $\mathrm{Sc}(\zeta, \vartheta)$ listed by p and compute lower bounds for the numbers $\vartheta - d(\zeta, x)$ with higher and higher precision. These lower bounds are always lower bounds for $d(\mathbb{R}^n \setminus U, x)$. In case $x \in U$ the supremum of these bounds is $d(\mathbb{R}^n \setminus U, x)$ because q lists all closed dyadic spheres contained in U, hence also spheres with midpoint ζ arbitrarily close to x and with radius ϑ arbitrarily close to $d(\mathbb{R}^n \setminus U, x)$ if this is finite, and with ϑ arbitrarily large if $U = \mathbb{R}^n$.

"$\delta_{\text{dist,open}} \le \delta_{\text{cutdist,open}}$": This is clear.

"$\delta_{\text{cutdist,open}} \le \delta_{\text{open}}$": Using a $\delta_{\text{cutdist,open}}$–name for an open set U we can approximate $\mathrm{cutdist}(\mathbb{R}^n \setminus U, x)$ for an arbitrary point $x \in \mathbb{R}^n$ from below. We obtain a list of dyadic spheres contained in U whose union covers U by going through all dyadic points $\zeta \in \mathbb{R}^n$ and all dyadic rationals $\vartheta \in (0,1)$ and approximating $\mathrm{cutdist}(\mathbb{R}^n \setminus U, \zeta)$ from below. As soon as ϑ turns out to be a lower bound for $\mathrm{cutdist}(\mathbb{R}^n \setminus U, \zeta)$, the sphere $\mathrm{S}(\zeta, \vartheta)$ is added to the list. $\square$

From the Computable Points Lemma (Lemma 2.12) and from Lemma 2.13 we conclude that for an open subset $U \subseteq \mathbb{R}^n$ the following conditions are equivalent.

1. There is an r.e. set A of binary words with $U = \bigcup_{w \in A} \nu_{S^n}(w)$.
2. The set $\{w \in \{0,1\}^* \mid \nu_{Sc^n}(w) \subseteq U\}$ is recursively enumerable, i.e. the set of all closed dyadic spheres contained in U is recursively enumerable.
3. The function $\mathrm{dist}_{\mathbb{R}^n \setminus U}$ is $(\rho^n, \overline{\rho^<})$–computable.
4. The function $\mathrm{cutdist}_{\mathbb{R}^n \setminus U}$ is $(\rho^n, \rho^<)$–computable.

A set $U \subseteq \mathbb{R}^n$ with one (and then all) of these properties is called a *recursively enumerable open* or *r.e. open* set. Then its complement is called a *co-r.e. closed* set.

We define four corresponding equivalent representations for closed subsets of $\mathbb{R}^n$.

Definition 3.3 *Let $n \ge 1$. We define the representations δ_{closed}, $\delta_{\text{open-spheres,closed}}$, $\delta_{\text{dist,closed}}$, and $\delta_{\text{cutdist,closed}}$ of the set of closed subsets of $\mathbb{R}^n$ by (where $C \subseteq \mathbb{R}^n$ is closed):*

$$\delta_{\text{closed}}(p) = C \quad \text{iff} \quad \pi_i^\infty p \in \mathrm{dom}\,\rho^n \cup \{(\#\#\#\#\ldots)\} \text{ for all } i, \text{ and the set}$$

$$\{\rho^n(\pi_i^\infty p) \mid i \in \mathbb{N},\ \pi_i^\infty p \in \operatorname{dom} \rho^n\} \text{ is a dense subset of } C\,,$$

$$\begin{aligned}
\delta_{\text{open-spheres,closed}}(p) = C \quad &\text{iff} \quad \mathrm{En}(p) = \{w \in \{0,1\}^* \mid \nu_{S^n}(w) \cap C \neq \emptyset\}\,,\\
\delta_{\text{dist,closed}}(p) = C \quad &\text{iff} \quad p \text{ is a } (\rho^n, \overline{\rho^>})\text{–tracking name for the function } \mathrm{dist}_C\,,\\
\delta_{\text{cutdist,closed}}(p) = C \quad &\text{iff} \quad p \text{ is a } (\rho^n, \rho^>)\text{–tracking name for the function } \mathrm{cutdist}_C\,.
\end{aligned}$$

Theorem 3.4 *Let $n \geq 1$. The four representations δ_{closed}, $\delta_{\text{open-spheres,closed}}$, $\delta_{\text{dist,closed}}$, and $\delta_{\text{cutdist,closed}}$ are equivalent.*

Proof. "$\delta_{\text{closed}} \leq \delta_{\text{open-spheres,closed}}$": Given a ρ^n–name q of a point and a word w such that the point lies in the sphere $\nu_{S^n}(w)$, one can detect this by computing the point with sufficient precision. Hence, given a δ_{closed}–name for a closed C, which is a list of ρ^n–names for points in a dense subset of C, we can compute a list which contains all dyadic spheres intersecting C by adding a dyadic sphere $\nu_{S^n}(w)$ to the list as soon as one of the points in the list described by p turns out to lie in $\nu_{S^n}(w)$.

"$\delta_{\text{open-spheres,closed}} \leq \delta_{\text{dist,closed}}$": The proof is similar to the proof of "$\delta_{\text{closed-spheres,open}} \leq \delta_{\text{dist,open}}$" in Theorem 3.2.

"$\delta_{\text{dist,closed}} \leq \delta_{\text{cutdist,closed}}$": This is clear.

"$\delta_{\text{cutdist,closed}} \leq \delta_{\text{closed}}$": Assume that a $\delta_{\text{cutdist,closed}}$–name for a closed set C is given. We have to compute a list of ρ^n–names for points which form a dense subset of C. We obtain this list by doing the following for each dyadic point $\zeta \in \mathbb{R}^n$ and each $m \in \mathbb{N}$. We approximate $\mathrm{cutdist}_C(\zeta)$ from above. If 2^{-m} turns out to be larger than $\mathrm{cutdist}_C(\zeta)$, then we set $\zeta_0 := \zeta$ and compute a sequence $\zeta_1, \zeta_2, \ldots$ of dyadic points ζ_i with $|\zeta_{i-1} - \zeta_i| < 2^{-m-i}$ and $\mathrm{cutdist}_C(\zeta_i) < 2^{-m-i}$ for $i = 1, 2, \ldots$. Such a sequence exists and we can find such a sequence, and from such a sequence we can easily compute a ρ^n–name for its limit. The set of these limits is a dense subset of C. Note that this algorithm works also in case $C = \emptyset$, since in that case we do not find any such limit number, but also obtain a correct output sequence by requiring that the algorithm pads the output appropriately with symbols #. □

A δ_{closed}–computable closed set $C \subseteq \mathbb{R}^n$ is called *r.e. closed*. Its complement is called *co–r.e. open*. In the same way as above one can deduce various characterizations of r.e. closed sets by using the Computable Points Lemma. We do not formulate them explicitly. A set $A \subseteq \mathbb{N}$ of natural numbers is recursive iff it is r.e. and co-r.e. We shall define the notion of recursiveness for open or closed subsets of $\mathbb{R}^n$ such that the same holds for these sets. They will be the sets with computable names with respect to the following representations.

By combining the above representations for open sets and for closed sets we obtain representations which contain more information. The same holds, if we offer more information about the distance functions. For example, we can define the following three equivalent representations for open sets.

Definition 3.5 *Let $n \geq 1$. We define the representations δ_{rec}, $\delta_{\text{dist,rec}}$, $\delta_{\text{cutdist,rec}}$ of the set of open subsets of $\mathbb{R}^n$ by:*

$$\begin{aligned}
\delta_{\text{rec}}\langle p, q\rangle = U \quad &\text{iff} \quad \delta_{\text{open}}(p) = U \text{ and } \delta_{\text{closed}}(q) = \mathbb{R}^n \setminus U\,,\\
\delta_{\text{dist,rec}}(p) = U \quad &\text{iff} \quad p \text{ is a } (\rho^n, \overline{\rho})\text{–tracking name for the function } \mathrm{dist}_{\mathbb{R}^n \setminus U}\,,\\
\delta_{\text{cutdist,rec}}(p) = U \quad &\text{iff} \quad p \text{ is a } (\rho^n, \rho)\text{–tracking name for the function } \mathrm{cutdist}_{\mathbb{R}^n \setminus U}\,.
\end{aligned}$$

Theorem 3.6 *Let $n \geq 1$. The representations $\delta_{\rm rec}$, $\delta_{\rm dist,rec}$, and $\delta_{\rm cutdist,rec}$ are equivalent.*

Proof. This follows from Theorem 3.2, Theorem 3.4, and from the definition of $\overline{\rho}$ and ρ. $\square$

A $\delta_{\rm rec}$–computable open set $U \subseteq \mathbb{R}^n$ is called *recursively open.* Then its complement is called *recursively closed.* By definition, an open set U is recursively open iff it is r.e. open and its complement is r.e. closed, and if and only if its distance function $\mathrm{dist}_{\mathbb{R}^n \setminus U}$ is computable. The last property corresponds to the property of the set $\mathbb{R}^n \setminus U$ being "located" in constructive analysis, compare Bishop and Bridges [2].

In the following we consider two more classes of representations of open subsets of $\mathbb{R}^n$. They are obtained by considering not just information about the open set itself and perhaps about its complement but also information about its closure or its boundary. But note that by the following Lemma the possible information about an open set and its closure is not completely independent.

Lemma 3.7 *Let $n \geq 1$. The operation "map an open subset $U \subseteq \mathbb{R}^n$ to its closure" is $(\delta_{\rm open}, \delta_{\rm closed})$–computable.*

Proof. Given a $\delta_{\rm open}$–name for an open set $U \subseteq \mathbb{R}^n$ we can according to Theorem 3.2 compute a list containing all closed dyadic spheres in U. The centers of these spheres constitute a dense subset of the closure of U. Hence, we can compute a $\delta_{\rm closed}$–name for the closure of U. $\square$

Especially, by the Computable Points Lemma, the closure of an r.e. open set is r.e. closed.

We start with a representation which is obtained by combining $\delta_{\rm open}$–information about an open set with $\delta_{\rm closed}$–information about the boundary. The boundary ∂M of an arbitrary set $M \subseteq \mathbb{R}^n$ is the (closed) set $\{x \in \mathbb{R}^n \mid \text{for every } \varepsilon > 0, \mathrm{S}(x,\varepsilon) \cap M \neq \emptyset \text{ and } \mathrm{S}(x,\varepsilon) \cap (\mathbb{R}^n \setminus M) \neq \emptyset\}$. It turns out that this information can also be expressed via the distance functions. We define the following three equivalent representations for open sets. Right now there does not seem to be a motivation to study this type of information about an open set, but we shall see later that it plays an essential role in connection with the Effective Riemann Mapping Theorem.

Definition 3.8 *Let $n \geq 1$. We define the representations $\delta_{\rm open,\partial}$, $\delta_{\rm dist,open,\partial}$, $\delta_{\rm cutdist,open,\partial}$ of the set of open subsets of $\mathbb{R}^n$ by:*

$$
\begin{array}{rcl}
\delta_{\rm open,\partial}\langle p,q\rangle = U & \text{iff} & \delta_{\rm open}(p) = U \text{ and } \delta_{\rm closed}(q) = \partial U\,, \\
\delta_{\rm dist,open,\partial}\langle p,q\rangle = U & \text{iff} & p \text{ is a } (\rho^n, \overline{\rho^<})\text{–tracking name for the function } \mathrm{dist}_{\mathbb{R}^n\setminus U} \text{ and} \\
 & & q \text{ is a } (\rho^n, \overline{\rho})\text{–tracking name for the restricted function } \mathrm{dist}_{\mathbb{R}^n\setminus U}|_U\,, \\
\delta_{\rm cutdist,open,\partial}\langle p,q\rangle = U & \text{iff} & p \text{ is a } (\rho^n, \rho^<)\text{–tracking name for the function } \mathrm{cutdist}_{\mathbb{R}^n\setminus U} \text{ and} \\
 & & q \text{ is a } (\rho^n, \rho)\text{–tracking name for the restricted function } \mathrm{cutdist}_{\mathbb{R}^n\setminus U}|_U\,.
\end{array}
$$

Theorem 3.9 *Let $n \geq 1$. The representations $\delta_{\rm open,\partial}$, $\delta_{\rm dist,open,\partial}$, and $\delta_{\rm cutdist,open,\partial}$ are equivalent.*

Proof. "$\delta_{\rm open,\partial} \leq \delta_{\rm dist,open,\partial}$": If $\langle p,q\rangle$ is a $\delta_{\rm open,\partial}$–name for an open set U, then by Theorem 3.2 from p we can compute a $(\rho^n, \overline{\rho^<})$–tracking name p' for the function $\mathrm{dist}_{\mathbb{R}^n\setminus U}$. Furthermore we have to show that we can compute a $(\rho^n, \overline{\rho})$–tracking name for the restricted function $\mathrm{dist}_{\mathbb{R}^n\setminus U}|_U$. By the smn Theorem for η it is sufficient to show that given additionally a ρ^n–name r for a point $x \in U$, we can compute a $\overline{\rho}$–name for $\mathrm{dist}_{\mathbb{R}^n \setminus U}|_U(x)$. But using p' we can compute a $\overline{\rho^<}$–name for $\mathrm{dist}_{\mathbb{R}^n \setminus U}(x)$. And according to Theorem 3.4, by using q we can compute a $\overline{\rho^>}$–name for $\mathrm{dist}_{\partial U}(x)$. But because we can assume $x \in U$ we have $\mathrm{dist}_{\mathbb{R}^n \setminus U}|_U(x) = \mathrm{dist}_{\mathbb{R}^n \setminus U}(x) = \mathrm{dist}_{\partial U}(x)$. And from a $\overline{\rho^<}$–name and a $\overline{\rho^>}$–name for this value we can obtain a $\overline{\rho}$–name for it.

"$\delta_{\mathrm{dist,open},\partial} \le \delta_{\mathrm{open},\partial}$": According to Theorem 3.2 we can compute a δ_{open}–name for an open set U if we are given a $\delta_{\mathrm{dist,open},\partial}$–name $\langle p, q\rangle$ for it. We have to show that we can also compute a δ_{closed}–name for ∂U. According to Theorem 3.4 and by the smn Theorem for η it is sufficient to show that, given such a $\langle p, q\rangle$ and a ρ^n–name r for a point $z \in \mathbb{R}^n$, we can compute a $\overline{\rho^>}$–name for $\mathrm{dist}_{\partial U}(z)$. Indeed, using p and Theorem 3.2, we can find a list of dyadic points ζ which is dense in U (the centers of all closed dyadic spheres in U). Using q and r, for each of these ζ we can compute a $\overline{\rho}$–name for

$$d(\zeta, z) + \mathrm{dist}_{\mathbb{R}^n \setminus U}(\zeta) = d(\zeta, z) + \mathrm{dist}_{\partial U}(\zeta)\,.$$

From all these $\overline{\rho}$–names we can also compute a $\overline{\rho^>}$–name for the infimum of these values, taken over all ζ. This infimum is the correct value $\mathrm{dist}_{\partial U}(z)$.

"$\delta_{\mathrm{dist,open},\partial} \le \delta_{\mathrm{cutdist,open},\partial}$": This is clear.

"$\delta_{\mathrm{cutdist,open},\partial} \le \delta_{\mathrm{dist,open},\partial}$": We give only the idea. For a given point z (which can be assumed to be in U) one computes a dyadic complex number $\zeta \in S(z_0, 2^{-n}) \cap U$ and a dyadic rational ϑ with $\vartheta < \mathrm{cutdist}_{\mathbb{R}^n \setminus U}(z) < \vartheta + 2^{-n}$ for some large n. If ϑ is close to 1 or identical to 1, one computes also the value $\mathrm{cutdist}_{\mathbb{R}^n \setminus U}(\zeta)$ for all dyadic points in $S(\zeta_0, \vartheta)$ with high precision. If all these values turn out to be close to 1, one extends the search to dyadic complex number still further away from z_0 but inside U. Repeating this one will find larger and larger lower bounds for $\mathrm{dist}_{\mathbb{R}^n \setminus U}(z)$. Either they converge to infinity (then $U = \mathbb{R}^n$) or in this way we eventually find a ζ with $\mathrm{cutdist}_{\mathbb{R}^n \setminus U}(\zeta)$ strictly smaller than 1. Then we have an upper bound for the distance of z from $\mathbb{R}^n \setminus U$ and can compute it with arbitrary precision by always computing $\mathrm{cutdist}_{\mathbb{R}^n \setminus U}(\zeta)$ for dyadic rationals ζ of which we know that they are in U. □

We do not explore all other possible combinations of information about an open set and its complement, boundary, closure. We formulate only one more class of representations. We shall not need it in the later sections, but it is interesting especially in connection with the last representations. The following representations give different possibilities to provide an amount of information about open sets which is equivalent to combining δ_{rec}–information about the open set and about the complement of its closure. Therefore we shall also use the following symmetric distance functions for an arbitrary closed set $C \subseteq \mathbb{R}^n$:

$$\mathrm{symdist}_C : \mathbb{R}^n \to \overline{\mathbb{R}}, \qquad \mathrm{symdist}_C(x) := \begin{cases} d(C, x) = d(\partial C, x) & \text{if } x \notin C \\ -d(\partial C, x) & \text{if } x \in C\,, \end{cases}$$

$$\mathrm{cutsymdist}_C : \mathbb{R}^n \to \mathbb{R}, \qquad \mathrm{cutsymdist}_C(x) := \begin{cases} \min\{1, \mathrm{symdist}_C(x)\} & \text{if } x \notin C \\ \max\{-1, \mathrm{symdist}_C(x)\} & \text{if } x \in C\,. \end{cases}$$

The function $\mathrm{symdist}_C$ measures for a point x how far it is away from C if it is not in C, and how far it is inside C, if it is inside. On the boundary of C it takes the value zero. We define five equivalent representations for open sets.

Definition 3.10 *Let $n \ge 1$. We define representations δ_{birec}, $\delta_{\mathrm{open},\partial,\mathrm{birec}}$, $\delta_{\mathrm{rec,birec}}$, $\delta_{\mathrm{symdist,birec}}$, $\delta_{\mathrm{cutsymdist,birec}}$ of the set of open subsets of $\mathbb{R}^n$ by:*

$$\delta_{\mathrm{birec}}\langle p, q, r\rangle = U \quad \text{iff} \quad \delta_{\mathrm{open}}(p) = U \text{ and } \delta_{\mathrm{closed}}(q) = \partial U$$

$$\text{and } \delta_{\text{open}}(r) = \mathbb{R}^n \setminus \text{ closure of } U\,,$$
$$\delta_{\text{open},\partial,\text{birec}}\langle p,q\rangle = U \quad \text{iff} \quad \delta_{\text{open},\partial}(p) = U \text{ and } \delta_{\text{open},\partial}(q) = \mathbb{R}^n \setminus \text{ closure of } U\,,$$
$$\delta_{\text{rec},\text{birec}}\langle p,q\rangle = U \quad \text{iff} \quad \delta_{\text{rec}}(p) = U \text{ and } \delta_{\text{rec}}(q) = \mathbb{R}^n \setminus \text{ closure of } U\,,$$
$$\delta_{\text{symdist},\text{birec}}(p) = U \quad \text{iff} \quad p \text{ is a } (\rho^n,\overline{\rho})\text{–tracking name for the function } \text{symdist}_{\mathbb{R}^n\setminus U}$$
$$\delta_{\text{cutsymdist},\text{birec}}(p) = U \quad \text{iff} \quad p \text{ is a } (\rho^n,\rho)\text{–tracking name for the function } \text{cutsymdist}_{\mathbb{R}^n\setminus U}\,.$$

Theorem 3.11 *Let $n \geq 1$. The representations δ_{birec}, $\delta_{\text{open},\partial,\text{birec}}$, $\delta_{\text{rec},\text{birec}}$, $\delta_{\text{symdist},\text{birec}}$, and $\delta_{\text{cutsymdist},\text{birec}}$ are equivalent.*

Proof. "$\delta_{\text{symdist},\text{birec}} \equiv \delta_{\text{cutsymdist},\text{birec}}$": This is clear.

"$\delta_{\text{symdist},\text{birec}} \leq \delta_{\text{rec},\text{birec}}$": From a $(\rho^n,\overline{\rho})$–tracking name for the function $\text{symdist}_{\mathbb{R}^n\setminus U}$ one can easily compute $(\rho^n,\overline{\rho})$–tracking names for the functions $\text{dist}_{\mathbb{R}^n\setminus U}$ and $\text{dist}_{\text{closure of } U}$. According to Theorem 3.6 they can be used to obtain δ_{rec}–names for U and $\mathbb{R}^n\setminus$ closure of U.

"$\delta_{\text{rec},\text{birec}} \leq \delta_{\text{open},\partial,\text{birec}}$": This follows from Theorem 3.6 and Theorem 3.9.

"$\delta_{\text{open},\partial,\text{birec}} \leq \delta_{\text{birec}}$": This is trivial.

"$\delta_{\text{birec}} \leq \delta_{\text{symdist},\text{birec}}$": Assume that a δ_{birec}–name $\langle p,q,r\rangle$ and a ρ^n–name s for a point $x \in \mathbb{R}^n$ are given. We have to compute a $\overline{\rho}$–name for $\text{symdist}_{\mathbb{R}^n\setminus U}(x)$. First, according to Theorem 3.9 we can compute a $(\rho^n,\overline{\rho})$–tracking name p' for the function $\text{dist}_{\mathbb{R}^n\setminus U}|_U$, and according to Lemma 3.7 and Theorem 3.6, we can compute a $(\rho^n,\overline{\rho})$–tracking name r' for the function $\text{dist}_{\text{closure of } U}$. Given a number n, in finitely many steps we can find either a point in ∂U which has distance less than 2^{-n} from x, or we find a dyadic sphere in U which contains x, or we find a dyadic sphere in ($\mathbb{R}^n\setminus$ closure of U) which contains x. In the first case we have determined that $|\text{symdist}_{\mathbb{R}^n\setminus U}(x)|$ is at most 2^{-n}, in the other two cases we can compute a $\overline{\rho}$–name for $\text{symdist}_{\mathbb{R}^n\setminus U}(x)$, using p' or r'.

Applying the smn Theorem for η to this algorithm gives the reduction. $\square$

A δ_{birec}–computable open set $U \subseteq \mathbb{R}^n$ is called *birecursively open.* Then its complement is called *birecursively closed.* Using the Computable Points Lemma we also obtain statements of the type "rec=r.e.+complement r.e.": the following three statements are equivalent for an open set $U \subseteq \mathbb{R}^n$.

1. U is birecursively open.
2. The set U and the complement of the closure of U are r.e. open, and the boundary ∂U of U is r.e. closed (note that these three sets are disjoint and their union is $\mathbb{R}^n$).
3. The open set U and the complement of the closure of U have the following property: they are r.e. open and their boundary is r.e. closed.

Now we compare the different classes of representations with each other. For two representations γ and δ we write $\gamma < \delta$ iff $\gamma \leq \delta$ and $\delta \not\leq \gamma$. For set S and T we write $S \subset T$ iff $S \subseteq T$ and $T \not\subseteq S$.

Theorem 3.12 *Let $n \geq 1$. Then $\delta_{\text{birec}} < \delta_{\text{rec}} < \delta_{\text{open},\partial} < \delta_{\text{open}}$ and*

$$\begin{aligned}
&\{U \subseteq \mathbb{R}^n \mid U \text{ is birecursively open}\}\\
&\quad\subset \{U \subseteq \mathbb{R}^n \mid U \text{ is recursively open}\}\\
&\quad\subset \{U \subseteq \mathbb{R}^n \mid U \text{ is r.e. open and } \partial U \text{ is r.e. closed}\}\\
&\quad\subset \{U \subseteq \mathbb{R}^n \mid U \text{ is r.e. open}\}\,.
\end{aligned}$$

Proof. The reducibilities $\delta_{\rm birec} \leq \delta_{\rm rec} \leq \delta_{{\rm open},\partial} \leq \delta_{\rm open}$ follow immediately from the definitions, from Theorem 3.6, and from Theorem 3.9. Because of the Computable Points Lemma these reducibilities induce the inclusions in the second part of the assertion. The negative statements about the reducibilities follow from the Computable Points Lemma and from the fact that the inclusions are proper. This is shown by the following typical examples of open sets in each class. We formulate them for dimension $n = 1$. For $n > 1$ one can take the product of these sets with $\mathbb{R}^{n-1}$. For completeness sake we also give a birecursively open set. We use an injective total recursive function $a : \mathbb{N} \to \mathbb{N}$ such that its range $\{a_i \mid i \in \mathbb{N}\}$ is not recursive, e.g. equal to the halting problem $\{i \mid \varphi_i(i) \text{ exists}\}$. It is of course recursively enumerable. We set $b_i := \sum_{m\leq i} 2^{-a_m}$ for each $i \in \mathbb{N}$, and $b := \lim_{i\to\infty} b_i$. The real number b is contained in the interval $(0, 2)$. It is $\rho^<$–computable, but not ρ–computable, i.e. not a computable real number.

$$
\begin{aligned}
M_1 &:= (0,2),\\
M_2 &:= \bigcup_{i\in\mathbb{N}} (b_i - 2^{-a_i-i}, b_i),\\
M_3 &:= \bigcup_{i\in\mathbb{N}} (b_i - 2^{-a_i-1}, b_i),\\
M_4 &:= (0,b)\,.
\end{aligned}
$$

The set M_1 is obviously birecursively open.

The set M_2 is recursively open but not birecursively open. It is not birecursively open because otherwise the distance $d(M_2, 2) = 2 - b$ would be a computable real number. But this is not the case since b is $\rho^<$–computable but not ρ–computable. The set M_2 is recursively open because its distance function $\mathrm{dist}_{\mathbb{R}\setminus M_2}$ is a computable function. Namely, for a given ρ–name for a real number x and a given k we can compute $d(\mathbb{R} \setminus \bigcup_{m\leq k}(b_m - 2^{-a_m-m}, b_m), x)$ with arbitrary precision, e.g. with precision 2^{-k-1}, and because of

$$
|\mathrm{dist}_{\mathbb{R}\setminus M_2}(x) - d(\mathbb{R} \setminus \bigcup_{m\leq k} (b_m - 2^{-a_m-m}, b_m), x)| \leq 2^{-k-2}
$$

we obtain $\mathrm{dist}_{\mathbb{R}\setminus M_2}(x)$ with precision $2^{-k-1} + 2^{-k-2} < 2^{-k}$.

The set M_3 is r.e. open and its boundary is r.e. closed, but M_3 is not recursively open. It is clear that M_3 is r.e. open and that its boundary $\partial M_3 = \{b\} \cup \{b_i \mid i \in \mathbb{N}\} \cup \{b_i - 2^{-a_i-1} \mid i \in \mathbb{N}\}$ is r.e. closed. We show that the set M_3 is not recursively open. Let us assume that M_3 is recursively open. Then its distance function $\mathrm{dist}_{\mathbb{R}\setminus M_3}$ is a computable function. We shall show that for any k we can compute b with precision 2^{-k}. This, of course, contradicts the fact that b is not a computable real number.

Indeed, for any $k \in \mathbb{N}$ and any $\zeta \in (0, 2) \cap 2^{-k-3} \cdot \mathbb{N}$ we can compute a dyadic rational $\vartheta_{k,\zeta}$ with $|d(\mathbb{R} \setminus M_3, \zeta) - \vartheta_{k,\zeta}| < 2^{-k-4}$. Let ζ_0 be the either the largest number $\zeta \in (0, 2) \cap 2^{-k-3} \cdot \mathbb{N}$ such that $\vartheta_{k,\zeta} > 2^{-k-4}$ or, if no such ζ exists, set ζ_0 to be any dyadic number in M_3. In any case $\zeta_0 \in M_3$. We can compute the unique n_0 with $\zeta_0 \in (b_{n_0} - 2^{-a_{n_0}-1}, b_{n_0})$. We claim that $a_m \geq k + 1$ for all $m > n_0$. If there were an $m > n_0$ with $a_m \leq k$, then there were also a $\zeta \in (b_m - 2^{-a_m-1}, b_m) \cap 2^{-k-3} \cdot \mathbb{N}$ with

$$
\vartheta_{k,\zeta} > d(\mathbb{R} \setminus M_3, \zeta) - 2^{-k-4} > 2^{-k-3} - 2^{-k-4} = 2^{-k-4}\,.
$$

Since this ζ would be larger than ζ_0 we would have a contradiction. Therefore $a_m \geq k+1$ for all $m > n_0$. We conclude

$$b = b_{n_0} + \sum_{j>n_0} 2^{-a_j} \leq b_{n_0} + 2^{-k}\,.$$

Hence, the computed number b_{n_0} is a 2^{-k}–approximation for b.

Finally, the set M_4 is obviously r.e. open, but its boundary $\partial M_4 = \{0, b\}$ is not r.e. closed. Otherwise we could find computable points in this boundary arbitrarily close to b. This would imply that b itself is a computable real number. □

We conclude this section by relating the introduced notions of being r.e. open, r.e. closed, recursively open or closed, and so on with the classical notions for subsets of $\mathbb{N}$. For a subset $A \subseteq \mathbb{N}$ we consider two operations which map this set to a subset of $\mathbb{R}$. We directly consider the set A as a (closed) subset of $\mathbb{R}$, and we consider the open set $U_A := \bigcup_{n \in A} \mathrm{S}(n, 1/4)$. We see that the following conditions are equivalent for a subset $A \subseteq \mathbb{N}$:

1. $A \subseteq \mathbb{N}$ is r.e.,
2. $A \subseteq \mathbb{R}$ is r.e. closed,
3. U_A is r.e. open,
4. U_A is r.e. open and its boundary ∂U_A is r.e. closed.

Also the following conditions are equivalent for $A \subseteq \mathbb{N}$:

1. $A \subseteq \mathbb{N}$ is recursive,
2. $\mathbb{N} \setminus A$ is recursive,
3. $\mathbb{R} \setminus A$ is r.e. open and its boundary $\partial(\mathbb{R} \setminus A) = A$ is r.e. closed,
4. $\mathbb{R} \setminus A$ is recursively open,
5. $\mathbb{R} \setminus A$ is birecursively open,
6. U_A is recursively open,
7. U_A is birecursively open.

Therefore we consider the notions of being r.e. open or of being r.e. closed as corresponding to recursive enumerability, while the notions of recursive open– or closedness and birecursiveness correspond to recursiveness for subsets of $\mathbb{N}$. For the property of being r.e. open and having an r.e. closed boundary one might say that it lies in between these notions or that it is very strong version of being r.e.

4 Results from Computable Complex Analysis

In this section we show several results from computable complex analysis which we shall need for the proof of the Effective Riemann Mapping Theorem and which are of independent interest. We start with effective versions of two elementary results about analytic functions: computability of the derivative and computability of the analytic branch of the square root function for a simply connected open set and a starting point. Then we prove an effective version of the Open Mapping Theorem and conclude that the inverse of a conformal mapping can be computed if the mapping is given. Afterwards we show a result which does not seem to have a direct counterpart in classical, non-effective complex analysis. It can be considered as a stronger version of the Effective Open Mapping Theorem for conformal mappings. We conclude with an effective version of a lemma by Bishop and Bridges [2] related to Harnack's Theorem.

From now on, in this section and the two following sections, we shall always identify the set of complex numbers $\mathbb{C} = \{z = x + \mathrm{i}y \mid x, y \in \mathbb{R}\}$ with the real plane $\mathbb{R}^2$. Computability on $\mathbb{C}$ is therefore introduced via the representation ρ^2. Following Ahlfors [1] we call a subset $U \subseteq \mathbb{C}$ a *region* iff it is nonempty, open, and connected. We assume that every holomorphic function has an open domain of definition. A holomorphic function is called *conformal* iff it is injective. But note that the term "conformal" has mainly a geometric meaning. For this and other notions from complex analysis the reader is referred to Ahlfors [1] or any other textbook on complex analysis.

At first we show a uniform version of the fact that the derivative of a computable holomorphic function is also a computable function, see Pour–El and Richards [15].

Proposition 4.1 *There exists a computable functional* Der *with the following property: if p is a δ_{open}–name for an open set $U \subseteq \mathbb{C}$ and q is a (ρ^2, ρ^2)–tracking name for a holomorphic function $f : U \to \mathbb{C}$, then* $\mathrm{Der}\langle p, q\rangle$ *exists and is a (ρ^2, ρ^2)–tracking name for the derivative $f' : U \to \mathbb{C}$.*

Proof. We show that there exists a computable functional G with the following property: if p and q are as above and r is a ρ^2–name for a point $z \in U$, then $G\langle p, q, r\rangle$ exists and is a ρ^2–name for $f'(z)$. Applying the smn Theorem for η to G yields the desired functional Der.

Assume that p, q and r are given and describe an open set U, a holomorphic function $f : U \to \mathbb{C}$ and a point $z \in U$. Using p and r we can find a closed dyadic sphere $\mathrm{Sc}(\zeta, \vartheta)$ (that means: $\zeta \in \mathbb{D} + \mathrm{i}\mathbb{D}$ and $\vartheta \in \mathbb{D}$) contained in U which contains z in its interior. Using q we can compute the integral

$$\int_{\partial \mathrm{S}(\zeta,\vartheta)} \frac{f(\tilde{z})}{z - \tilde{z}} d\tilde{z}$$

with arbitrary precision, i.e. we can compute a ρ^2–name for it, compare Weihrauch [21]. Division by $2\pi\mathrm{i}$ is computable. Hence, by Cauchy's integral formula we can compute a ρ^2–name for $f'(z)$. □

We shall need to compute analytic branches of the square root function on simply connected regions. We formulate a uniform version. The same can be done for analytic branches of the logarithm. For a simply connected region U which contains a positive real number z_0 and not the point 0 we denote the uniquely defined analytic branch of the square root function on U which gives the positive square root for z_0 by $\sqrt{}^{U,z_0}$.

Proposition 4.2 *There is a computable functional* Sqrt *with the following property: if p is a $\delta_{\rm open}$–name for a simply connected region U with $0 \notin U$ and q is a ρ^2–name for a positive real number $z_0 \in U$ (that means: q is a ρ^2–name for z_0 as a complex number), then* $\mathrm{Sqrt}\langle p, q\rangle$ *exists and is a (ρ^2, ρ^2)–tracking name for $\sqrt{\ }^{U,z_0}$.*

Proof. Let p, q be given as described and let r be a ρ^2–name for a point $z \in U$. Using p and r we can find a finite list of dyadic complex points $\zeta_1, \ldots, \zeta_l$ such that the piecewise linear path γ leading from z_0 over the sequence of points ζ_i to z is contained in U. Using q we can compute a ρ^2–name for

$$z_1 := \log(z_0) + \int_\gamma \frac{1}{z} dz$$

where log denotes the real logarithm. Then we can also compute a ρ^2–name for $e^{z_1/2}$. This is just the value $\sqrt{z}^{U,z_0}$.

We have shown that there is a computable functional which, given p and q as in the theorem and a ρ^2–name for a point $z \in U$, computes a ρ^2–name for $\sqrt{z}^{U,z_0}$. Applying the smn Theorem for η gives the computable functional Sqrt. □

The Open Mapping Theorem (see e.g. Ahlfors [1, p. 132]) asserts that the image of an open set under a nonconstant holomorphic mapping is open as well.

Theorem 4.3 (Effective Open Mapping Theorem) *There exists a computable functional F with the following property: if p is a $\delta_{\rm open}$–name for an open set $U \subseteq \mathbb{C}$ and q is a (ρ^2, ρ^2)–tracking name for a nonconstant holomorphic function f with $U \subseteq \mathrm{dom}\, f$, then $F\langle p, q\rangle$ is defined and a $\delta_{\rm open}$–name for $f(U)$.*

By applying the Computable Points Lemma we see

Corollary 4.4 *If a set $U \subseteq \mathbb{C}$ is r.e. open and f is a computable holomorphic function with $U \subseteq \mathrm{dom}\, f$, then also $f(U)$ is r.e. open.*

Proof of Theorem 4.3. We assume that a $\delta_{\rm open}$–name p for an open set $U \subseteq \mathbb{C}$ and a (ρ^2, ρ^2)–tracking name q for a nonconstant holomorphic function are given. We have to describe a computable functional F which, using these data, computes a $\delta_{\rm open}$–name r for the set $f(U)$.

By Theorem 3.2 we can compute a $\delta_{\rm closed-spheres,open}$–name for U. By reading this name we obtain a list

$$\mathrm{Sc}(\zeta_0, \vartheta_0), \mathrm{Sc}(\zeta_1, \vartheta_1), \mathrm{Sc}(\zeta_2, \vartheta_2), \ldots$$

containing all closed spheres $\mathrm{Sc}(\zeta, \vartheta)$ with $\zeta \in \mathbb{D} + \mathrm{i} \cdot \mathbb{D}$ and $\vartheta \in \mathbb{D}$ which are contained in U. Note that this list is either infinite or empty, namely empty when the set U is empty. We show how to construct a new list of open spheres which are contained in $f(U)$ and cover $f(U)$. For each pair of numbers (i, n) we shall either add a sphere to the new list (then a ν_{S^2}–name can be appended to the so-far written prefix of r) or we add nothing to the list (then the symbol # can be appended to the so-far written prefix of r). It is clear that r, defined by this process, will be a $\delta_{\rm open}$–name for U.

For each pair (i, n) of numbers we do the following. If $\mathrm{Sc}(\zeta_i, \vartheta_i)$ does not exist, i.e. if the list above is empty, then we add nothing to the new list. Let us assume that $\mathrm{Sc}(\zeta_i, \vartheta_i)$ exists. Using q we can compute the minimum $\min_{z \in \partial \mathrm{S}(\zeta_i, \vartheta_i)} |f(z) - f(\zeta_i)|$ of the continuous function

$z \mapsto |f(z) - f(\zeta_i)|$ on the compact boundary $\partial \mathrm{S}(\zeta_i, \vartheta_i)$ of the sphere $\mathrm{S}(\zeta_i, \vartheta_i)$ with arbitrary precision (compare Weihrauch [21, Theorem 6.8]). We compute a natural number l with

$$\min_{z \in \partial \mathrm{S}(\zeta_i, \vartheta_i)} |f(z) - f(\zeta_i)| \in ((l-1) \cdot 2^{-n}, (l+1) \cdot 2^{-n}) .$$

If $l \leq 6$, then we add nothing to the new list. If $l \geq 7$, then, using q again, we compute a dyadic complex number ξ (that means: $\xi \in \mathbb{D} + \mathrm{i} \cdot \mathbb{D}$) with

$$|\xi - f(\zeta_i)| < 2^{-n} ,$$

we set $\xi_{\langle i,n \rangle} := \xi$ and we add the sphere $\mathrm{S}(\xi_{\langle i,n \rangle}, 2^{1-n})$ to the new list. This ends the description of the algorithm F for computing r.

We have to show that the new list of spheres really covers exactly the set $f(U)$. We shall show:

1. For any i, n, if $\xi_{\langle i,n \rangle}$ exists, then $\mathrm{S}(\xi_{\langle i,n \rangle}, 2^{1-n}) \subseteq f(\mathrm{S}(\zeta_i, \vartheta_i))$.
2. For every point $z_0 \in U$ there exists a pair (i, n) with $f(z_0) \in \mathrm{S}(\xi_{\langle i,n \rangle}, 2^{1-n})$.

Fix a pair (i, n) of numbers and assume that $\xi_{\langle i,n \rangle}$ exists. Let $z_1 \in \mathrm{S}(\xi_{\langle i,n \rangle}, 2^{1-n})$. Then for $z_2 \in \partial \mathrm{S}(\zeta_i, \vartheta_i)$

$$|f(z_2) - z_1| \geq |f(z_2) - f(\zeta_i)| - |f(\zeta_i) - \xi_{\langle i,n \rangle}| - |\xi_{\langle i,n \rangle} - z_1| > 6 \cdot 2^{-n} - 2^{-n} - 2^{1-n} = 3 \cdot 2^{-n} .$$

If there were no point $\tilde{z} \in \mathrm{S}(\zeta_i, \vartheta_i)$ with $f(\tilde{z}) = z_1$, then $z \mapsto \frac{1}{f(z) - z_1}$ would define a holomorphic mapping on an open neighborhood of $\mathrm{Sc}(\zeta_i, \vartheta_i)$. By the maximum principle (e.g. Ahlfors [1, p. 134]) the maximum of the absolute value of this function on $\mathrm{Sc}(\zeta_i, \vartheta_i)$ would be taken on the boundary $\partial \mathrm{S}(\zeta_i, \vartheta_i)$. But we have

$$|f(\zeta_i) - z_1| \leq |f(\zeta_i) - \xi_{\langle i,n \rangle}| + |\xi_{\langle i,n \rangle} - z_1| < 2^{-n} + 2^{1-n} = 3 \cdot 2^{-n} < \min_{z_2 \in \partial \mathrm{S}(\zeta_i, \vartheta_i)} |f(z_2) - z_1| .$$

Hence, there exists a number $\tilde{z} \in \mathrm{S}(\zeta_i, \vartheta_i)$ with $f(\tilde{z}) = z_1$. This proves the first claim.

For the second claim fix a point $z_0 \in U$. Since the function f is holomorphic and nonconstant there exists a dyadic number ϑ' such that the closed sphere $\mathrm{Sc}(z_0, \vartheta')$ is contained in U and does not contain any number $z \neq z_0$ with $f(z) = f(z_0)$. Especially, there is a natural number n with

$$9 \cdot 2^{-n} < \min_{z \in \partial \mathrm{S}(z_0, \vartheta')} |f(z_0) - f(z)| .$$

Let $\delta > 0$ be so small that also $\mathrm{Sc}(z_0, \vartheta' + \delta) \subseteq U$. The function f is uniformly continuous on $\mathrm{Sc}(z_0, \vartheta' + \delta) \subseteq U$. Therefore, there exists a positive $\delta' < \delta$ with $|f(z_1) - f(z_2)| < 2^{-n}$ for all $z_1, z_2 \in \mathrm{Sc}(z_0, \vartheta' + \delta)$ with $|z_1 - z_2| \leq \delta'$. Let ζ' be a complex dyadic number in $\mathrm{S}(z_0, \delta')$. Since $\mathrm{Sc}(\zeta', \vartheta') \subseteq U$ there is an index i with $\zeta_i = \zeta'$ and $\vartheta_i = \vartheta'$. We see

$$7 \cdot 2^{-n} < \min_{z \in \partial \mathrm{S}(\zeta_i, \vartheta_i)} |f(\zeta_i) - f(z)| .$$

Hence, $\xi_{\langle i,n \rangle}$ exists and we obtain

$$|f(z_0) - \xi_{\langle i,n \rangle}| \leq |f(z_0) - f(\zeta_i)| + |f(\zeta_i) - \xi_{\langle i,n \rangle}| < 2^{-n} + 2^{-n} = 2^{1-n} .$$

We have shown $f(z_0) \in \mathrm{S}(\xi_{\langle i,n\rangle}, 2^{1-n})$. $\square$

If a holomorphic function $f : U \to \mathbb{C}$ (where $U \subseteq \mathbb{C}$ is open) is injective, then its inverse $f^{-1} : f(U) \to \mathbb{C}$ is also holomorphic. Hence, then $f : U \to f(U)$ and $f^{-1} : f(U) \to U$ are conformal bijections.

Theorem 4.5 *There exists a computable functional* Inverse *with the following property: if p is a δ_{open}–name for an open set $U \subseteq \mathbb{C}$ and q is a (ρ^2, ρ^2)–tracking name for an injective holomorphic function $f : U \to \mathbb{C}$, then* Inverse$\langle p, q\rangle$ *is defined and a (ρ^2, ρ^2)–tracking name for the inverse function $f^{-1} : f(U) \to \mathbb{C}$.*

Proof. We shall describe a computable functional G which on input $\langle p, q, r\rangle$ computes a ρ^2–name for $f^{-1}(z)$ if p and q are names as in the theorem and r is a ρ^2–name for a number $z \in f(U)$. An application of the smn Theorem for η yields the desired computable functional Inverse.

Let p be a δ_{open}–name for an open set $U \subseteq \mathbb{C}$, let q be a (ρ^2, ρ^2)–tracking name for an injective holomorphic function $f : U \to \mathbb{C}$, and let r be ρ^2–name for a point $z \in f(U)$. It is sufficient to show that for arbitrary $n \in \mathbb{N}$ we can find a dyadic sphere with radius less than 2^{-n} containing $f^{-1}(z)$. Indeed, using p we can by Theorem 3.2 compute a $\delta_{\mathrm{closed-spheres,open}}$–name for U. Using this name we can compute a list $(\mathrm{S}(\zeta_i, \vartheta_i))_i$ of spheres with dyadic center ζ_i and dyadic radius $\vartheta_i < 2^{-n}$ which cover exactly U. By the Effective Open Mapping Theorem, just proved, for each of these spheres $\mathrm{S}(\zeta_i, \vartheta_i)$ we can compute a list $(\mathrm{S}(\zeta_{i,j}, \vartheta_{i,j}))_j$ of open spheres covering exactly $f(\mathrm{S}(\zeta_i, \vartheta_i))$. Eventually, we will find a pair (i, j) such that the sphere $\mathrm{S}(\zeta_{i,j}, \vartheta_{i,j})$ contains z. Then $\mathrm{S}(\zeta_i, \vartheta_i)$ contains $f^{-1}(z)$. Thus, for arbitrary n we can find a sphere with dyadic center and dyadic radius less than 2^{-n} which contains $f^{-1}(z)$. This ends the proof. $\square$

Hence, if U is r.e. open and $f : U \to \mathbb{C}$ is a computable conformal mapping, then also $f^{-1} : f(U) \to \mathbb{C}$ is a computable conformal mapping.

Now we shall prove a stronger version of the Effective Open Mapping Theorem for conformal mappings. First, we formulate a non-effective consequence of the Koebe $\frac{1}{4}$ Theorem about the distance functions $\mathrm{dist}_{\mathbb{C}\setminus U}$ and $\mathrm{dist}_{\mathbb{C}\setminus f(U)}$ of an open set U and its image $f(U)$ under a conformal mapping f.

Lemma 4.6 *Let $U \subset \mathbb{C}$ be a proper, open subset of $\mathbb{C}$ and $f : U \to \mathbb{C}$ be a conformal mapping. Then $f(U) \subset \mathbb{C}$ is also proper and for every $u \in U$*

$$\frac{1}{4} \cdot |f'(u)| \cdot d(\mathbb{C} \setminus U, u) \leq d(\mathbb{C} \setminus f(U), f(u)) \leq 4 \cdot |f'(u)| \cdot d(\mathbb{C} \setminus U, u)\,.$$

Proof. First assume that $f(U) = \mathbb{C}$. Then f would be a conformal bijection between U and $\mathbb{C}$. This would imply $U = \mathbb{C}$ in contradiction to our assumption. Hence, $f(U) \subset \mathbb{C}$.

We fix a point $u \in U$ and define $r := d(\mathbb{C} \setminus U, u)$. Because of $r \neq 0$, $f'(u) \neq 0$ (this holds true because f is injective) and $z \in \mathrm{S}(0, 1) \iff rz + u \in \mathrm{S}(u, r)$ we can define a function $g : \mathrm{S}(0, 1) \to \mathbb{C}$ by

$$g(z) := \frac{1}{r \cdot f'(u)} \cdot (f(rz + u) - f(u))$$

for all $z \in \mathrm{S}(0,1)$. The function g is conformal and satisfies $g(0) = 0$ and $g'(0) = 1$. Hence, the Koebe $\frac{1}{4}$ Theorem (compare Bishop and Bridges [2, Ch. 5, Theorem 7.14], Henrici [8]) implies

$$\mathrm{S}(0, \frac{1}{4}) \subseteq g(\mathrm{S}(0,1)) .$$

This is equivalent to

$$\mathrm{S}(f(u), \frac{r \cdot |f'(u)|}{4}) \subseteq f(\mathrm{S}(u,r)) .$$

Using $\mathrm{S}(u,r) \subseteq U$ we conclude $\mathrm{S}(f(u), \frac{r\cdot|f'(u)|}{4}) \subseteq f(U)$, hence

$$\frac{r \cdot |f'(u)|}{4} \leq d(\mathbb{C} \setminus f(U), f(u)) .$$

That is the left inequality.

For the right inequality we notice that the set $V := f(U)$ is open and the inverse function $f^{-1} : f(U) \to \mathbb{C}$ is a conformal mapping. By substituting V, f^{-1}, and the point $v := f(u)$ for U, f, and u in the left inequality, we read

$$\frac{1}{4} \cdot |(f^{-1})'(v)| \cdot d(\mathbb{C} \setminus V, v) \leq d(\mathbb{C} \setminus f^{-1}(V), f^{-1}(v)) = d(\mathbb{C} \setminus U, u) .$$

Using $(f^{-1})'(v) = \frac{1}{f'(u)}$ we obtain

$$d(\mathbb{C} \setminus V, v) \leq 4 \cdot |f'(u)| \cdot d(\mathbb{C} \setminus U, u) .$$

That is the right inequality. $\square$

Theorem 4.7 *There exists a computable functional F with the following property: if p is a $\delta_{\mathrm{open},\partial}$–name for an open set $U \subseteq \mathbb{C}$ and q is a (ρ^2, ρ^2)–tracking name for an injective holomorphic function f with $U \subseteq \mathrm{dom}\, f$, then $F\langle p, q\rangle$ is defined and a $\delta_{\mathrm{open},\partial}$–name for $f(U)$.*

Applying the Computable Points Lemma yields the following corollary.

Corollary 4.8 *If $U \subseteq \mathbb{C}$ is an r.e. open set with an r.e. closed boundary and $f : U \to \mathbb{C}$ is a computable conformal mapping, then also $f(U)$ is r.e. open and its boundary $\partial f(U)$ is r.e. closed.*

Proof of Theorem 4.7. By Theorem 3.9 it is sufficient to show that a computable functional exists which computes a $\delta_{\mathrm{dist,open},\partial}$–name for $f(U)$, given the input p and q as above. Such a name consists of two components: a $(\rho^2, \overline{\rho_<})$–tracking name for the distance function $\mathrm{dist}_{\mathbb{C}\setminus f(U)}$ and a $(\rho^2, \overline{\rho})$–tracking name for the restricted distance function $\mathrm{dist}_{\mathbb{C}\setminus f(U)}|_{f(U)}$, restricted to $f(U)$. By the Effective Open Mapping Theorem we can compute a δ_{open}–name $r^{(1)}$ for $f(U)$, and hence by Theorem 3.9 we can compute the first component, a $(\rho^2, \overline{\rho_<})$–tracking name $r^{(2)}$ for the distance function $\mathrm{dist}_{\mathbb{C}\setminus f(U)}$. It is now sufficient to show that there is a computable functional G which computes a $(\rho^2, \overline{\rho_>})$–tracking name $r^{(3)}$ for the restricted distance function $\mathrm{dist}_{\mathbb{C}\setminus f(U)}|_{f(U)}$, since using $r^{(2)}$ and $r^{(3)}$ one can easily compute the desired second component, a $(\rho^2, \overline{\rho})$–tracking name for the restricted distance function $\mathrm{dist}_{\mathbb{C}\setminus f(U)}|_{f(U)}$.

We shall show that there is a computable functional H which on input: a $\delta_{\mathrm{open},\partial}$–name p for an open set $U \subseteq \mathbb{C}$, a (ρ^2, ρ^2)–tracking name q for an injective holomorphic function f with

$U \subseteq \operatorname{dom} f$, and a ρ^2–name r for a complex number z, computes a $\overline{\rho_>}$–name for $d(\mathbb{C} \setminus f(U), z)$ if z is an element of $f(U)$. Applying the smn Theorem for η to H gives the desired computable functional G.

We do not need to specify what H does if z is not an element of $f(U)$. So assume that p, q and r are given as described and that $z \in f(U)$. By Theorem 3.2 and using p we can compute an infinite list

$$\mathrm{Sc}(\zeta_0, \vartheta_0), \mathrm{Sc}(\zeta_1, \vartheta_1), \mathrm{Sc}(\zeta_2, \vartheta_2), \ldots$$

containing exactly all closed dyadic spheres contained in U. This list is infinite because we can already assume that $z \in f(U)$ and hence $U \neq \emptyset$. For each i we do the following, using the name q for f:

1. We compute a ρ–name for the absolute value of the derivative $f'(\zeta_i)$, according to Proposition 4.1. Note that $f'(\zeta_i) \neq 0$ since f is injective on U.

2. We compute a ρ-name for $d(f(\zeta_i), z)$.

3. We compute a $\overline{\rho}$–name for $d(\mathbb{C} \setminus U, \zeta_i)$.

4. Using these three names we compute a $\overline{\rho}$–name $q^{(i)}$ for

$$a_i := 4 \cdot |f'(\zeta_i)| \cdot d(\mathbb{C} \setminus U, \zeta_i) + d(f(\zeta_i), z)\,.$$

Using all these names $q^{(i)}$ we can compute a $\overline{\rho_>}$–name s for the infimum $a := \inf_{i \in \mathbb{N}} a_i$ and set $H\langle p, q, r\rangle := s$.

In order to show the correctness of the algorithm we have to show only $a = d(\mathbb{C} \setminus f(U), z)$. We distinguish the cases $U = \mathbb{C}$ and $U \neq \mathbb{C}$. In the first case we have $d(\mathbb{C} \setminus U, \zeta_i) = \infty$ for all i, hence $a_i = \infty$ for all i, hence $a = \infty$. Since $f(U)$ is a conformal image of U we also have $f(U) = \mathbb{C}$, and hence $d(\mathbb{C} \setminus f(U), z) = \infty$. Thus, in the first case $U = \mathbb{C}$ we have $a = \infty = d(\mathbb{C} \setminus f(U), z)$. In the second case $U \neq \mathbb{C}$ each of the distances $d(\mathbb{C} \setminus U, \zeta_i)$ is finite, hence each a_i is finite, hence a is a real number. Also, the conformal image $f(U)$ of $U \neq \mathbb{C}$ is a proper subset of $\mathbb{C}$. Hence, $d(\mathbb{C} \setminus f(U), z)$ is a real number. A triangle inequality and the right inequality of Lemma 4.6 yield

$$d(\mathbb{C} \setminus f(U), z) \leq d(\mathbb{C} \setminus f(U), f(\zeta_i)) + d(f(\zeta_i), z) \leq a_i$$

for all i, hence $d(\mathbb{C} \setminus f(U), z) \leq a$. Fix a real number $\varepsilon > 0$ and a complex number $z_0 \in \mathbb{C} \setminus f(U)$ with $d(\mathbb{C} \setminus f(U), z) = d(z_0, z)$. The set $\{\tilde{z} \in U \mid d(z_0, f(\tilde{z})) < \varepsilon\}$ is open and nonempty. Therefore there is a sphere $\mathrm{Sc}(\zeta_i, \vartheta_i)$ in this set. For this i, a triangle inequality yields

$$d(\mathbb{C} \setminus f(U), z) = d(z_0, z) \geq d(f(\zeta_i), z) - d(z_0, f(\zeta_i)) > d(f(\zeta_i), z) - \varepsilon\,,$$

and the left inequality of Lemma 4.6 yields

$$\begin{aligned} d(f(\zeta_i), z) &= a_i - 4 \cdot |f'(\zeta_i)| \cdot d(\mathbb{C} \setminus U, \zeta_i) \\ &\geq a_i - 4 \cdot 4 \cdot d(\mathbb{C} \setminus f(U), f(\zeta_i)) \\ &\geq a_i - 16 \cdot d(z_0, f(\zeta_i)) \\ &> a_i - 16 \cdot \varepsilon\,. \end{aligned}$$

We have shown that for every $\varepsilon > 0$ there exists an index i with $a_i \le d(\mathbb{C} \setminus f(U), z) + 17 \cdot \varepsilon$. We conclude $d(\mathbb{C} \setminus f(U), z) \ge \inf_{i\in\mathbb{N}} a_i = a$. This ends the proof of $a = d(\mathbb{C} \setminus f(U), z)$ and of the theorem. $\square$

We end this section with a result which will be needed in the last part of the proof of the Riemann mapping theorem. It is related to Harnack's Theorem. For the proof we need the following consequence of Poisson's integral formula, see Bishop and Bridges [2].

Lemma 4.9 (Bishop and Bridges [2, Ch. 5, Proposition 7.2]) *Let R be a positive real number, $z_0 \in \mathbb{C}$, and $f :\subseteq \mathbb{C} \to \mathbb{C}$ be a function which is holomorphic on a neighborhood of* $\mathrm{Sc}(z_0, R)$. *Then for any $z \in \mathrm{S}(z_0, R)$*

$$|f(z)| \le |f(z_0)| \cdot \frac{R - |z|}{R + |z|} + \max\{|f(z)| \mid z \in \mathrm{Sc}(z_0, R)\} \cdot \frac{2 \cdot |z|}{R + |z|}\,.$$

Actually, we use only the following corollary.

Corollary 4.10 *Let R be a positive real number, $z_0 \in \mathbb{C}$, and $f :\subseteq \mathbb{C} \to \mathbb{C}$ be a function which is holomorphic on a neighborhood of* $\mathrm{Sc}(z_0, R)$ *and with* $\max\{|f(z)| \mid z \in \mathrm{Sc}(z_0, R)\} \le 1$. *Then for any $z \in \mathrm{Sc}(z_0, R/2)$*

$$|f(z)| \le \frac{1}{3} \cdot |f(z_0)| + \frac{2}{3}\,.$$

Proof. For $z \in \mathrm{Sc}(z_0, R/2)$ one has by the lemma:

$$\begin{aligned} |f(z)| &\le |f(z_0)| \cdot \frac{R - |z|}{R + |z|} + \frac{2|z|}{R + |z|} \\ &= 1 - (1 - |f(z_0)|) \cdot \frac{R - |z|}{R + |z|} \\ &\le 1 - (1 - |f(z_0)|) \cdot \frac{R - R/2}{R + R/2} \\ &= \frac{1}{3} \cdot |f(z_0)| + \frac{2}{3}\,. \end{aligned}$$

$\square$

We use the following representation $\delta_{\mathcal{K}}^{>}$ of compact subsets of $\mathbb{R}^2$ (more generally, of $\mathbb{R}^n$) taken from Weihrauch and Brattka [22]:

$$\delta_{\mathcal{K}}^{>}(w\#p) = K \quad \text{iff } K \subseteq \nu_{S^2}(w) \text{ and } \delta_{\mathrm{open}}(p) = \mathbb{R}^2 \setminus K.$$

The following proposition is a translation of Corollary 7.3 in Ch. 5 of Bishop and Bridges [2]. It is related to Harnack's Theorem, compare Henrici [8].

Proposition 4.11 *Given a δ_{open}–name p for a region $U \subseteq \mathbb{C}$ and a $\delta_{\mathcal{K}}^{>}$–name q for a compact subset $K \subseteq U$, one can compute a number $n \in \mathbb{N}$ such that*

$$\max\{|f(z)| \mid z \in K\} \le \frac{3^n - 1}{3^n} + \frac{1}{3^n} \cdot \min\{|f(z)| \mid z \in K\}$$

for all holomorphic functions $f : U \to \mathbb{C}$ with $\sup\{|f(z)| \mid z \in U\} \le 1$.

Proof. For a region U we set $\mathcal{F}(U) := \{f : U \to \mathbb{C} \mid f$ is holomorphic and $|f(z)| \le 1\}$ for all $z \in U\}$.

First, we show the following: given a δ_{open}–name p for a region U and two dyadic complex numbers ζ_0 and ξ in U we can compute a number m such that

$$|f(\xi)| \le \frac{3^m - 1}{3^m} + \frac{1}{3^m} \cdot |f(\zeta_0)|$$

for all $f \in \mathcal{F}(U)$. First, we compute a dyadic number ϑ_0 with $\mathrm{Sc}(\zeta_0, 2\vartheta_0) \subseteq U$. Then, since U is connected, we can use p and Theorem 3.2 in order to find a finite list of pairs $(\zeta_1, \vartheta_1), \ldots, (\zeta_l, \vartheta_l)$ such that $\xi \in \mathrm{S}(\zeta_l, \vartheta_l)$ and for each $i \in \{1, \ldots, l\}$

$$\zeta_i \in \mathrm{S}(\zeta_{i-1}, \vartheta_{i-1}) \quad \text{and} \quad \mathrm{Sc}(\zeta_i, 2\vartheta_i) \subseteq U\,.$$

We claim that $m := l + 1$ has the demanded property. By Corollary 4.10 one obtains via induction $|f(\zeta_i)| \le \frac{3^i - 1}{3^i} + \frac{1}{3^i} \cdot |f(\zeta_0)|$ for all $i \in \{1, \ldots, l\}$ and finally also

$$|f(\xi)| \le \frac{3^{l+1} - 1}{3^{l+1}} + \frac{1}{3^{l+1}} \cdot |f(\zeta_0)|\,.$$

This proves the first claim.

Given valid p and q, we can compute a finite set of pairs $(\zeta_1, \vartheta_1), \ldots, (\zeta_l, \vartheta_l)$ such that the union of the spheres $\mathrm{S}(\zeta_j, \vartheta_j)$ covers K and even each of the larger spheres $\mathrm{S}(\zeta_j, 3 \cdot \vartheta_j)$ is contained in U. For any two $i, j \in \{1, \ldots, l\}$ let $m(i, j)$ be a number as computed according to the algorithm described in the first step of the proof. We claim that $n := 2 + \max\{m(i, j) \mid i, j \in \{1, \ldots, l\}\}$ has the desired property.

Indeed, let $z_0, z_1 \in K$ be two numbers in K. Let $i_0, i_1 \in \{1, \ldots, l\}$ be indices with $z_0 \in \mathrm{S}(\zeta_{i_0}, \vartheta_{i_0})$ and $z_1 \in \mathrm{S}(\zeta_{i_1}, \vartheta_{i_1})$. Then $\mathrm{Sc}(z_0, 2 \cdot \vartheta_{i_0}) \subseteq U$. By applying Corollary 4.10 we obtain $|f(\zeta_{i_0})| \le \frac{1}{3}|f(z_0)| + \frac{2}{3}$ and $|f(z_1)| \le \frac{1}{3}|f(\zeta_{i_1})| + \frac{2}{3}$. Together with $|f(\zeta_{i_1})| \le \frac{3^{m(i_0,i_1)} - 1}{3^{m(i_0,i_1)}} + \frac{1}{3^{m(i_0,i_1)}} \cdot |f(\zeta_{i_0})|$ we obtain

$$|f(z_1)| \le \frac{3^n - 1}{3^n} + \frac{1}{3^n} \cdot |f(z_0)|\,.$$

This proves the claim because z_0 and z_1 are arbitrary elements of K. $\square$

5 The Effective Riemann Mapping Theorem

We formulate several effective versions of the Riemann Mapping Theorem as described in the introduction, among them the main result of the paper.

Form now on we denote by $D = S(0, 1) \subseteq \mathbb{C}$ the unit disk in the complex plane. If z is a complex number and we write $z > 0$, then this means that z is a positive real number.

Theorem 5.1 *There exist computable functionals F and G with the following properties:*

1. *If p is a $\delta_{\text{open},\partial}$–name for a simply connected region U which is a proper subset of the complex plane, and q is a ρ^2–name for a point $z_0 \in U$, then $F\langle p, q\rangle$ exists and is a (ρ^2, ρ^2)–tracking name for the unique conformal bijection $f : D \to U$ with $f(0) = z_0$ and $f'(0) > 0$.*

2. *If p is a (ρ^2, ρ^2)–tracking name for a conformal mapping f defined on D, then $G(p)$ is defined, $\pi_1^2 G(p)$ is a $\delta_{\mathrm{open},\partial}$–name for the proper, simply connected region $f(D)$, and $\pi_2^2 G(p)$ is a ρ^2–name for the point $f(0)$.*

This is the main result of the paper. It will be proved in the following section. We have formulated the first part with the conformal bijection mapping D onto U. By Theorem 4.5 we could also formulate it using the inverse mapping. It is also worthwhile to remember that Theorem 3.9 gives other representations of open sets which are equivalent to $\delta_{\mathrm{open},\partial}$.

If we forget about the point which is supposed to correspond to 0 in the unit disk and define two representations $\delta'_{\mathrm{open},\partial}$ and δ_{conf} of proper, simply connected regions by

$$\delta'_{\mathrm{open},\partial} \quad := \quad \text{the restriction of } \delta_{\mathrm{open},\partial} \text{ to names of proper, simply connected regions} \subset \mathbb{C},$$

$$\delta_{\mathrm{conf}}(p) = f(D) \quad \text{iff} \quad p \text{ is a } (\rho^2, \rho^2)\text{–tracking name for a conformal mapping } f \text{ on } D,$$

then we can express the most important aspects of Theorem 5.1 compactly as follows.

Corollary 5.2 *The representations $\delta'_{\mathrm{open},\partial}$ and δ_{conf} of proper, simply connected regions are equivalent.*

Proof. Given a $\delta'_{\mathrm{open},\partial}$–name p of a proper, simply connected region $U \subset \mathbb{C}$ we can immediately compute a name q for a point $z_0 \in U$. Applying the functional F from Theorem 5.1 to p and q we can compute a δ_{conf}–name for U. The other reduction follows immediately from the other part of Theorem 5.1. □

This corollary expresses that the topological information contained in a $\delta'_{\mathrm{open},\partial}$–name for a simply connected region, that is, enumerating a covering sequence of open spheres in the set and a dense sequence of points in its boundary, is equivalent to the geometric information which describes a conformal bijection of the set with the unit disk.

Let us consider the simply connected regions and functions with computable names. We call a finite string v a *program* for an open set U (or a closed set C or a function defined on the unit disk) if v is a $\nu_{\delta_{\mathrm{open}}}$–name for U (resp. a $\nu_{\delta_{\mathrm{closed}}}$–name for C resp. if φ_v is a (ρ^2, ρ^2)–tracking name for the function), compare Definition 2.11.

Corollary 5.3 *If U is a nonempty, proper, r.e. open, connected, and simply connected subset of $\mathbb{C}$, its boundary ∂U is r.e. closed, and $z_0 \in U$ is a computable point, then the unique conformal bijection $f : D \to U$ with $f(0) = z_0$ and $f'(0) > 0$ is computable. Given programs for U as an r.e. open set, for ∂U as an r.e. closed set, and of z_0, one can compute a program of f.*

Proof. This follows from the first part of Theorem 5.1, from Lemma 2.12 and from Lemma 2.13. □

By forgetting the point corresponding to 0 and taking also the inverse statement into consideration we can characterize which subsets of the complex plane are computably isomorphic to the unit disk and obtain the theorem stated in the introduction.

Theorem 5.4 *For a subset $U \subseteq \mathbb{C}$ of the complex plane the following two statements are equivalent:*

1. *U is a nonempty, proper, r.e. open, connected, simply connected subset of $\mathbb{C}$ and its boundary ∂U is r.e. closed.*

2. *There exists a computable conformal bijection f from D onto U.*

Furthermore, there exists an algorithm which computes a program for a conformal bijection of the unit disk onto U if it is given a program for such a set U as an r.e. open set and a program for its boundary as an r.e. closed set. Also an algorithm performing the inverse task exists.

Proof. The assertion follows from Corollary 5.2, from Lemma 2.12 and from Lemma 2.13. □

One should bare in mind that by Theorem 4.5 also the inverse of a computable conformal mapping is computable. Using Theorem 3.9 and the Computable Points Lemma one can obtain further effective topological properties which are equivalent to the two effective properties above that the set is r.e. open and its boundary is r.e. closed. One combination corresponds to constructive conditions formulated by Bishop and Bridges [2]. They formulate their main result with a property called "mappability", but they show that for a simply connected region U it is equivalent to a property called "maximal extent property". This can be translated into the language of computability as the condition that U is a proper subset of $\mathbb{C}$ and that the restricted distance function $\mathrm{dist}_{\mathbb{C}\setminus U}|_U$, restricted to U, is computable (to be precise: for this property they use the distance induced by the embedding of $\mathbb{C}$ into the Riemann sphere). The picture is complete when one additionally interprets constructive openness by r.e. openness.

Theorem 5.4 answers a problem posed by Pour–El and Richards [15, Problem 5]. It is interesting that the answer to their problem is given by a class of sets which lies strictly between the class of proper, r.e. open, simply connected regions and proper, recursively open, simply connected regions, as we shall see immediately.

Zhou [24, Problem 5.4] asked whether the image $f(D)$ of the unit disk D under a computable conformal mapping f on D is a recursively open set. Theorem 5.4 and the following proposition show that this is not the case. The proposition shows that the proper hierarchy considered in Theorem 3.12 for arbitrary dimension $n \geq 1$ is still proper in dimension 2 if we restrict ourselves to proper, simply connected regions. Let $\mathcal{R}$ denote the set of proper, simply connected regions in the complex plane.

Proposition 5.5

$$\begin{aligned}
&\{U \in \mathcal{R} \mid U \text{ is birecursively open}\} \\
&\quad\subset \{U \in \mathcal{R} \mid U \text{ is recursively open}\} \\
&\quad\subset \{U \in \mathcal{R} \mid U \text{ is r.e. open and } \partial U \text{ is r.e. closed}\} \\
&\quad\subset \{U \in \mathcal{R} \mid U \text{ is r.e. open}\}.
\end{aligned}$$

Proof. The inclusions follow from Theorem 3.12. That they are proper is shown by the following counterexamples. For $i = 2, 3, 4$, we define

$$\tilde{M}_i := \{z = x + \mathrm{i}y \mid 0 < x < 2 \ \text{ and } \ 0 < y < 1\} \cup \{z = x + \mathrm{i}y \mid x \in M_i \ \text{ and } \ 0 < y < 10\}$$

where the sets M_2, M_3, and M_4 are the typical one-dimensional examples of sets defined in the proof of Theorem 3.12. The sets $\tilde{M}_i$ are proper, simply connected regions. The set $\tilde{M}_2$ is recursively open but not birecursively open, $\tilde{M}_3$ is r.e. open and has an r.e. closed boundary, but it is not recursively open, and the set $\tilde{M}_4$ is r.e. open but its boundary is not r.e. closed. □

6 Proof of the Effective Riemann Mapping Theorem

In this section we shall prove Theorem 5.1. At the end of the section we make two remarks about the proof.

First we prove the second assertion of Theorem 5.1. It is well known that $f(D)$ is a proper, simply connected region if f is a conformal mapping on D. The work which still needs to be done has been done already in the proof of Theorem 4.7. Let H be a computable functional whose existence was proved in Theorem 4.7, let q be a computable $\delta_{\text{open},\partial}$–name for the unit disk D, and let r be a computable ρ^2–name for the point $0 \in \mathbb{C}$. We define the functional G by

$$G(p) := \langle H\langle q, p\rangle, \eta_p(r)\rangle .$$

The functional G is computable by the utm Theorem for η. It has the properties stated in the second part of Theorem 5.1. Thus, the second part of Theorem 5.1 is proved.

We come to the first part of Theorem 5.1, to the construction of the conformal bijection of the unit disk onto a proper, simply connected region U, given only topological information about U (the uniqueness follows from the Schwarz Lemma).

We split the construction of the functional F into two steps.

1. We show that there is computable functional F_1 with the following property: if p is a $\delta_{\text{open},\partial}$–name for a proper simply connected region $U \subset \mathbb{C}$ and q is a ρ^2-name for a point $z_0 \in U$, then $F\langle p, q\rangle$ exists and is a (ρ^2, ρ^2)–tracking name for a conformal mapping $f : U \to \mathbb{C}$ so that $f(U)$ is a proper subset of the unit disk D, that $f(z_0) = 0$ and that $f'(z_0) > 0$.

2. We show that there is computable functional F_2 with the following property: if p is a $\delta_{\text{open},\partial}$–name for a simply connected region U which contains the point 0 and is a proper subset of the unit disk D, then $F_2(p)$ exists and is a (ρ^2, ρ^2)–tracking name for a conformal mapping $f : U \to \mathbb{C}$ with $f(U) = D$, with $f(0) = 0$ and with $f'(0) > 0$.

Before we prove the existence of the two functionals F_1 and F_2 we show how to obtain the final computable functional F of the first part of Theorem 5.1 from them. Let H be a computable functional whose existence was proved in Theorem 4.7, let Comp denote the computable functional from Corollary 2.10, and let Inverse denote the computable functional from Theorem 4.5. Assuming that we have F_1 and F_2 with the properties above, we define the functional F by

$$F\langle p, q\rangle := \text{Inverse} \circ \text{Comp}\langle F_2 \circ H\langle p, F_1\langle p, q\rangle\rangle, F_1\langle p, q\rangle\rangle .$$

It is clear that F is computable. Given names p and q as in the first part of Theorem 5.1 we first compute $F_1\langle p, q\rangle$, which is a name for a conformal mapping f_1 on $U = \delta_{\text{open},\partial}(p)$, then $F_2 \circ H\langle p, F_1\langle p, q\rangle\rangle$, which is a name for a conformal bijection f_2 from $f_1(U)$ onto D, then a name for their composition $f_2 \circ f_1$ and finally a name for the inverse mapping $f := (f_2 \circ f_1)^{-1}$, which is the desired conformal bijection from D onto U with $f(0) = z_0$ and $f'(0) > 0$.

We come to the first step, to the construction of the computable functional F_1. It is sufficient to describe how F_1 works for valid input. Let p be a $\delta_{\text{open},\partial}$–name for a proper, simply connected region U and let q be a ρ^2–name for a point $z_0 \in U$. The geometric idea is the following:

we compute a point not in U (actually a point in the boundary ∂U), shift it to 0, take an appropriate square root. Then the complement of the image contains an open set. By shifting again appropriately and inverting we obtain a conformal image of U which is contained in some bounded disk. Shifting a third time and multiplying with a suitable factor gives a conformal image of U which is a proper subset of the unit disk. Furthermore we can achieve that the given point z_0 is mapped to zero and the overall mapping has positive derivative in z_0.

We show that the sketched construction above can be performed effectively. Using $\pi_2^2 p$ (the "∂–component" of the $\delta_{\text{open},\partial}$–name for U) we can compute a ρ^2–name $r^{(1)}$ of a point in the boundary ∂U of U. Let $z_1 := \rho^2(r^{(1)})$ be this point. Using q we can compute a ρ^2–name $r^{(2)}$ for $\alpha := |z_0 - z_1|$ and a ρ^2–name $r^{(3)}$ for a real number β with $z_0 - z_1 = \alpha \cdot e^{\mathrm{i}\beta}$. The mapping

$$h_1 : \mathbb{C} \to \mathbb{C} \quad \text{with} \quad h_1(z) := e^{-\mathrm{i}\beta} \cdot (z - z_1)$$

is a conformal mapping. The image $V := h_1(U)$ is a simply connected region with $0 = h_1(z_1) \notin V$ and $\alpha = h_1(z_0) \in V$. Therefore the analytic branch $\sqrt{}^{V,\alpha}$ according to Proposition 4.2 of the square root function on V is well-defined. The function $\sqrt{}^{V,\alpha}$ is injective on V because applying $\sqrt{}^{V,\alpha}$ and then squaring gives the identity on V. Furthermore,

$$\text{if } z \in \sqrt{V}^{V,\alpha}, \text{ then the symmetric point } -z \text{ is not contained in } \sqrt{V}^{V,\alpha}. \tag{1}$$

Assume on the contrary that there is complex number $z \in \sqrt{V}^{V,\alpha}$ with $-z \in \sqrt{V}^{V,\alpha}$. Then by the injectivity of $\sqrt{}^{V,\alpha}$ there are different numbers $z_2 \neq z_3$ in V with $\sqrt{z_2}^{V,\alpha} = z = -\sqrt{z_3}^{V,\alpha}$. But squaring gives $z_2 = z_3$, a contradiction.

We have $\sqrt{\alpha}^{V,\alpha} = \sqrt{\alpha} \in \sqrt{V}^{V,\alpha}$. Therefore, (1) tells us $-\sqrt{\alpha} \notin \sqrt{V}^{V,\alpha}$. Hence, the function $h_2 : V \to \mathbb{C}$ with

$$h_2(z) := 1/(\sqrt{z}^{V,\alpha} + \sqrt{\alpha})$$

is well-defined. It is a conformal mapping. Therefore the set $h_2(V)$ is a conformal image of V, hence a simply connected region. We claim that it is contained in a bounded disk and that we can compute a radius of such a disk (using the names p, q and the already computed names $r^{(1)}$, $r^{(2)}$, and $r^{(3)}$).

First let us go back to the function h_1. Using the ρ^2–names $r^{(1)}$ for z_1 and $r^{(3)}$ for β we can compute a (ρ^2, ρ^2)–tracking name $r^{(4)}$ for the function h_1 (by the smn Theorem for η). Applying the computable functional of Theorem 4.7 to the $\delta_{\text{open},\partial}$–name p for U and to $r^{(4)}$ gives us a $\delta_{\text{open},\partial}$–name $r^{(5)}$ for V. The computable functional of Proposition 4.2, applied to $r^{(5)}$ and $r^{(2)}$ yields a (ρ^2, ρ^2)–tracking name $r^{(6)}$ for $\sqrt{}^{V,\alpha}$. Applying the functional of Theorem 4.7 to $r^{(5)}$ and $r^{(6)}$ yields a $\delta_{\text{open},\partial}$–name $r^{(7)}$ for the set $\sqrt{V}^{V,\alpha}$. From $r^{(2)}$, the already computed ρ^2–name for α we can also compute a ρ^2–name $r^{(8)}$ for $\sqrt{\alpha}$. Using $r^{(8)}$ and the first component of $r^{(7)}$ we can compute a dyadic number $\delta > 0$ with $\mathrm{S}(\sqrt{\alpha}, \delta) \subseteq \sqrt{V}^{V,\alpha}$. Now we can apply (1) and see that $\mathrm{S}(-\sqrt{\alpha}, \delta) \cap \sqrt{V}^{V,\alpha} = \emptyset$. This implies $h_2(V) \subseteq \mathrm{S}(0, \delta^{-1})$.

Now we only have to shift the set and to multiply it by a scalar so that the image of z_0 lands in 0, the derivative of the overall function f_1 is positive in z_0, and the whole set is mapped to a proper subset of $\mathrm{S}(0,1)$. We define $f_1 : U \to \mathbb{C}$ by

$$f_1(z) := \frac{\delta}{2} \cdot e^{\mathrm{i}\beta} \cdot \frac{|h_2'(\alpha)|}{h_2'(\alpha)} \cdot (h_2 h_1(z) - h_2(\alpha)) \, .$$

It is clear that this mapping is conformal and satisfies $f_1(z_0) = 0 \in f_1(U)$. We see

$$f_1(U) \subseteq \mathrm{S}(-\frac{\delta}{2} \cdot e^{\mathrm{i}\beta} \cdot \frac{|h_2'(\alpha)|}{h_2'(\alpha)} \cdot h_2(\alpha), \frac{1}{2}) \subset D$$

and

$$f_1'(z_0) = \frac{\delta}{2} \cdot e^{\mathrm{i}\beta} \cdot \frac{|h_2'(\alpha)|}{h_2'(\alpha)} \cdot h_2'(\alpha) \cdot h_1'(z_0) = \frac{\delta}{2} \cdot |h_2'(\alpha)| > 0\,.$$

We claim that, using the input p and q, we can compute a (ρ^2, ρ^2)–tracking name for f_1.

Using the ρ^2–name $r^{(8)}$ for $\sqrt{\alpha}$ and the (ρ^2, ρ^2)–tracking name $r^{(6)}$ for $\sqrt{\ }^{V,\alpha}$, we can also compute a (ρ^2, ρ^2)-tracking name $r^{(9)}$ for h_2. Using this name and the ρ^2–name $r^{(2)}$ for α we can compute a ρ^2–name $r^{(10)}$ for $h_2'(\alpha)$ according to Proposition 4.1. We summarize: from the input p and q we have we have computed the rational number δ, we have computed ρ^2-names for β, for α, and for $h_2'(\alpha)$, and we have computed (ρ^2, ρ^2)–tracking names for h_1 and h_2. By Corollary 2.10 and because the elementary functions addition, subtraction, multiplication, division, absolute value and exponentiation are computable, we can compute a (ρ^2, ρ^2)–tracking name for f_1. This finishes the first step of the construction.

We come to the second step of the construction, to the construction of the computable functional F_2. We shall show that every set in the following class of sets

$$\mathcal{S} := \{U \subseteq \mathbb{C} \mid 0 \in U,\ U \subseteq D,\ U \neq D,\ U \text{ is open, connected, and simply connected}\}\,.$$

can be mapped effectively by a conformal mapping $f : U \to D$ with $f(0) = 0$ and $f'(0) > 0$ onto the unit disk D, if a $\delta_{\mathrm{open},\partial}$–name for the set is given. The conformal mapping f will be obtained as the limit of a sequence of functions which are compositions of certain dilating maps, called *Koebe maps.*

For the definition of the Koebe maps we need automorphisms of the unit disk D. For $z_0 \in D$ the mapping

$$\mu_{z_0} : D \to D \quad \text{with} \quad \mu_{z_0}(z) := \frac{z - z_0}{\overline{z_0} z - 1} \quad \text{for } z \in D$$

is a conformal automorphism of D which interchanges the points 0 and z_0, that is, $\mu_{z_0}(0) = z_0$ and $\mu_{z_0}(z_0) = 0$. The following lemma is obvious.

Lemma 6.1 *There exists a computable functional G_1 with the following property: if p is a ρ^2–name of a point $z_0 \in D$, then $G_1(p)$ is defined and a (ρ^2, ρ^2)–tracking name for μ_{z_0}.*

For the definition of the Koebe map and the first three statements of the following lemma we fix a set $U \in \mathcal{S}$ and a point $z_0 \in D \setminus U$. Let $\alpha := |z_0|$ and β be a real number with $z_0 = \alpha \cdot e^{-\mathrm{i}\beta}$. We define the *Koebe map* $\kappa_{U,z_0} : U \to \mathbb{C}$ by

$$\kappa_{U,z_0}(z) := e^{-\mathrm{i}\beta} \cdot \left(\mu_{\sqrt{\alpha}} \circ \sqrt{\ }^{\mu_\alpha(e^{\mathrm{i}\beta}U),\alpha} \circ \mu_\alpha\right) \left(e^{\mathrm{i}\beta} z\right)\,.$$

Lemma 6.2 *1. The map κ_{U,z_0} is a well-defined conformal function with $\kappa_{U,z_0}(0) = 0$ and $\kappa_{U,z_0}(U) \in \mathcal{S}$.*

2. $\kappa_{U,z_0}'(0) = \frac{1+\alpha}{2\sqrt{\alpha}}$ is a real number greater than 1.

3. $d(\mathbb{C} \setminus U, 0) < d(\mathbb{C} \setminus \kappa_{U,z_0}(U), 0)$.

4. There exists a computable functional G_2 with the following property: if p is a δ_{open}–name for a set $U \in \mathcal{S}$ and q is a ρ^2–name for a point $z_0 \in D \setminus U$, then $G_2\langle p, q\rangle$ is defined and a (ρ^2, ρ^2)–tracking name for κ_{U,z_0}.

Proof. The first three statements are pure complex analysis and the proofs can be found in various sources. We give the proofs nevertheless for completeness sake.

1. By $z \mapsto \mu_\alpha(e^{i\beta} \cdot z)$ a conformal automorphism of D is defined. Hence, the set $\mu_\alpha(e^{i\beta}U)$ is an open connected, simply connected subset of D with $\alpha = \mu_\alpha(e^{i\beta} \cdot 0) \in \mu_\alpha(e^{i\beta}U)$ and $0 = \mu_\alpha(e^{i\beta} \cdot z_0) \notin \mu_\alpha(e^{i\beta}U)$. Therefore the analytic branch $\sqrt{}^{\mu_\alpha(e^{i\beta}U),\alpha}$ of the square root function introduced in Proposition 4.2 is well defined on $\mu_\alpha(e^{i\beta}U)$. It maps the set $\mu_\alpha(e^{i\beta}U)$ into D. Since the mapping $z \mapsto e^{-i\beta} \cdot \mu_{\sqrt{\alpha}}(z)$ is a conformal isomorphism of D, we conclude that κ_{U,z_0} is a conformal mapping on U with $\kappa_{U,z_0}(U) \subseteq D$. The image κ_{U,z_0} is open, connected and simply connected, it contains the point $0 = \kappa_{U,z_0}(0)$ but not the point $e^{-i\beta} \cdot \sqrt{\alpha}$. Hence, $\kappa_{U,z_0}(U) \in \mathcal{S}$.

2. Straightforward computation yields:

$$\kappa'_{U,z_0}(0) = e^{-i\beta} \cdot \mu'_{\sqrt{\alpha}}(\sqrt{\alpha}) \cdot \frac{1}{2 \cdot \sqrt{\alpha}} \cdot \mu'_\alpha(0) \cdot e^{i\beta} = \frac{1+\alpha}{2 \cdot \sqrt{\alpha}}\,.$$

From $\alpha \in (0,1)$ one deduces immediately $\frac{1+\alpha}{2\cdot\sqrt{\alpha}} > 1$.

3. Let square : $\mathbb{C} \to \mathbb{C}$ denote the squaring function square$(z) := z^2$. The function

$$\lambda : D \to D \quad \text{with} \quad \lambda(z) := e^{-i\beta} \cdot (\mu_\alpha \circ \text{square} \circ \mu_{\sqrt{\alpha}})(e^{i\beta} \cdot z)$$

satisfies $\lambda(0) = 0$ and is not a rotation. Therefore, by the Schwarz Lemma (see e.g. Ahlfors [1])

$$|\lambda(z)| < |z| \quad \text{for all} \quad z \in D \setminus \{0\}\,. \tag{2}$$

Since for any $\tilde{z} \in D$ the function $\mu_{\tilde{z}}$ is its own inverse, we see

$$\lambda \circ \kappa_{U,z_0}(z) = z$$

for all $z \in U$. Let us fix a point $z_1 \in \partial\kappa_{U,z_0}(U)$ with $|z_1| = d(\mathbb{C} \setminus \kappa_{U,z_0}(U), 0)$. Then $z_1 \in D$. We choose a sequence $(\tilde{z}_n)_n$ of points in U with $\lim_n \kappa_{U,z_0}(\tilde{z}_n) = z_1$. We can assume that the sequence $(\tilde{z}_n)_n$ converges itself since it is bounded and we can switch to a converging subsequence. The limit $z_2 := \lim_n \tilde{z}_n$ does not lie in U since otherwise by continuity of κ_{U,z_0} we had $z_1 = \kappa_{U,z_0}(z_2) \in \kappa_{U,z_0}(U)$. On the other hand we have $\lambda \circ \kappa_{U,z_0}(\tilde{z}_n) = \tilde{z}_n$ for all n. Hence, by continuity of λ we obtain $\lambda(z_1) = z_2$. With (2) we obtain $|z_2| < |z_1|$. We conclude

$$d(\mathbb{C} \setminus U, 0) \le |z_2| < |z_1| = d(\mathbb{C} \setminus \kappa_{U,z_0}(U), 0)\,.$$

4. Assume that a δ_{open}–name p for a set $U \in \mathcal{S}$ and a ρ^2–name q for a point $z_0 \in D \setminus U$ are given. Using the functional G_1 from Lemma 6.1 we can compute a (ρ^2, ρ^2)–tracking name for the function which assigns to each number $z \in D$ the value $\mu_\alpha(e^{i\beta} \cdot z)$. According to the Effective Open Mapping Theorem we can use this name and p in order to compute a δ_{open}–name for the set $\mu_\alpha(e^{i\beta}U)$. Now we can apply the functional of Proposition 4.2 to this name and to a ρ^2–name for

α, computed from q, and compute a (ρ^2, ρ^2)–tracking name for the analytic branch $\sqrt{}^{\mu_\alpha(e^{i\beta}U),\alpha}$ of the square root function on $\mu_\alpha(e^{i\beta}U)$. Finally, using the functional G_1 from Lemma 6.1 we can also compute a (ρ^2, ρ^2)–tracking name for $\mu_{\sqrt{\alpha}}$. We have now (ρ^2, ρ^2)–tracking names for the three functions in the middle of the definition of the Koebe function and, by using q, also for the rotations $z \mapsto e^{i\beta}z$ and $z \mapsto e^{-i\beta}z$. We can compute a (ρ^2, ρ^2)–tracking name for the composition of these functions by Corollary 2.10. This proves the assertion. □

The third property of Koebe maps is the reason why they are useful for our purpose: they are dilating maps. By choosing the point $z_0 \in D \setminus U$ as close as possible to the center 0 of the disk D one can achieve that the dilation of the Koebe map κ_{U,z_0} is large, i.e. the ratio between the *inner radius* $d(\mathbb{C} \setminus \kappa_{U,z_0}(U), 0)$ and $d(\mathbb{C} \setminus U, 0)$ of U is large enough so that by iterating this process the images of U form a sequence of sets in $\mathcal{S}$ whose inner radii tend to 1. We shall see that the composition of all of these maps tends to a conformal mapping of U onto D.

In general, it is impossible to compute a point $z_0 \in D \setminus U$ which is really as close as possible to 0, that is, satisfies $|z_0| = d(\mathbb{C} \setminus U, 0)$. But it is sufficient to have a point $z_0 \in D \setminus U$ with $|z_0| < \frac{1}{2} \cdot (1 + d(\mathbb{C} \setminus U, 0))$.

Lemma 6.3 *There exists a computable functional G_3 with the following property: if p is a $\delta_{\mathrm{open},\partial}$–name for a set $U \in \mathcal{S}$, then $G_3(p)$ is defined and a (ρ^2, ρ^2)–tracking map for a Koebe map κ_{U,z_U}, where z_U is a point in $D \setminus U$ with $|z_U| < \frac{1}{2} \cdot (1 + d(\mathbb{C} \setminus U, 0))$.*

Proof. Assume that a $\delta_{\mathrm{open},\partial}$–name p for a set $U \in \mathcal{S}$ is given. By Theorem 3.9 we can compute a $\delta_{\mathrm{dist,open},\partial}$–name for U. Using this we can compute the inner radius $d(\mathbb{C} \setminus U, 0)$ of U with arbitrary precision. On the other hand, using the ∂–part $\pi_2^2 p$ of p we can compute a sequence of points which form a dense subset of the boundary ∂U of U. Since we know the inner radius of U we can especially compute a ρ^2–name q for a point $z_U \in \partial U$ with $|z_U| < \frac{1}{2} \cdot (1 + d(\mathbb{C} \setminus U, 0))$. Applying the functional G_2 of Lemma 6.2.4 to $\pi_1^2 p$ and q yields a (ρ^2, ρ^2)–tracking name for a Koebe map with the desired properties. □

We come to the iteration. For $p \in \Sigma^\omega$ we define two sequences $(p^{(n)})_n$ and $(q^{(n)})_n$ of sequences in Σ^ω by

$$\begin{aligned} p^{(0)} &:= p, \\ q^{(n)} &:= G_3(p^{(n)}), \\ p^{(n+1)} &:= H\langle p^{(n)}, q^{(n)} \rangle \end{aligned}$$

where G_3 is the computable functional from Lemma 6.3 and H is the computable functional from Theorem 4.7.

Furthermore, using the functional Comp from Corollary 2.10 we define

$$r^{(0)} := q^{(0)}, \quad r^{(n+1)} := \mathrm{Comp}(q^{(n+1)}, r^{(n)}).$$

Then the functional G_4 with

$$G_4(p) := \langle r^{(0)}, r^{(1)}, r^{(2)}, \ldots \rangle$$

is computable.

From now on we assume that p is a $\delta_{\mathrm{open},\partial}$–name for a set $U \in \mathcal{S}$. Then we can define for each $n \in \mathbb{N}$:

$$\begin{aligned} U_n &:= \delta_{\mathrm{open},\partial}(p^{(n)}), \\ \kappa_n &:= \text{the Koebe map which is } (\rho^2,\rho^2)\text{–tracked by } q^{(n)}, \\ f_n &:= \kappa_n \circ \ldots \circ \kappa_0 \\ &= \text{the function which is } (\rho^2,\rho^2)\text{–tracked by } r^{(n)}. \end{aligned}$$

By the lemmata shown in this section the sets U_n are elements of $\mathcal{S}$ and the functions f_n are conformal bijections from U onto U_{n+1} with $f_n(0) = 0$ and $f_n'(0) > 0$. We wish to show that the functions f_n converge towards a conformal bijection f from U onto D and that from p we can compute a (ρ^2,ρ^2)–tracking name for f. First we show

Lemma 6.4 *The inner radii $d(\mathbb{C} \setminus U_n, 0)$ of the sets U_n tend to 1 for n tending to infinity.*

Proof. For each $n \in \mathbb{N}$ the function $z \mapsto f_n(z \cdot d(\mathbb{C} \setminus U, 0))$ maps D into D and 0 to 0. By the Schwarz Lemma we conclude

$$f_n'(0) \leq 1/d(\mathbb{C} \setminus U, 0)\,.$$

Let z_n be the point with $\kappa_n = \kappa_{U_n,z_n}$. Then by Lemma 6.2.2

$$f_n'(0) = \kappa_n'(0) \cdot \ldots \cdot \kappa_0'(0) = \frac{1+|z_n|}{2\sqrt{|z_n|}} \cdot \ldots \cdot \frac{1+|z_0|}{2\sqrt{|z_0|}}\,.$$

Each of the factors $\frac{1+|z_n|}{2\sqrt{|z_n|}}$ is greater than 1 by Lemma 6.2.2. Hence, due to the last inequality the factors $\frac{1+|z_n|}{2\sqrt{|z_n|}}$ must tend to 1 for n tending to infinity. Since the function $h : (0,1] \to \mathbb{R}$ with $h(x) := \frac{1+x}{2\sqrt{x}}$ is strictly decreasing on the interval $(0,1]$ and takes the value $h(1) = 1$ at the point 1, also $\lim_{n\to\infty} |z_n| = 1$. Our choice $|z_n| < \frac{1}{2}(1 + d(\mathbb{C} \setminus U_n, 0))$ finally implies that also $\lim_{n\to\infty} d(\mathbb{C} \setminus U_n, 0) = 1$. □

Thus, according to this lemma and Lemma 6.2.3 the sequence $(d(\mathbb{C} \setminus U_n, 0))_n$ of inner radii is an increasing sequence of positive real numbers with limit 1. We claim that the functions f_n converge locally uniformly and that, given p, we can compute a modulus of convergence in the sense of Proposition 2.15.

Lemma 6.5 *There exists a computable functional G_5 with the following property: if p is a $\delta_{\mathrm{open},\partial}$–name for a set $U \in \mathcal{S}$, then $G_5(p)$ exists and describes a modulus of convergence in the sense of Proposition 2.15 for the sequence $(f_n)_n$ of functions f_n on U, defined as above.*

For the proof we follow closely Bishop and Bridges [2]. We shall need Proposition 4.11 and the following lemma. For the proof the reader is referred to Bishop and Bridges [2].

Lemma 6.6 (Bishop and Bridges [2, Chapter 5, Corollary 7.5]) *Let $b \in (0,1)$ and let $U \subseteq D$ be an open set with $\mathrm{S}(0,b) \subseteq U$. Furthermore let $g : U \to \mathbb{C}$ be a conformal mapping with $g(0) = 0$, with $g'(0) > 0$, and with $\mathrm{S}(0,b) \subseteq g(U) \subseteq D$. Then*

$$|g(z) - z| \leq 3 \cdot \frac{\sqrt{1-b}}{b^2 - |z|}$$

for all $z \in \mathrm{S}(0,b^2)$.

Proof of Lemma 6.5. Let p be a $\delta_{\text{open},\partial}$–name for a set $U \in \mathcal{S}$, let $p^{(n)}$, U_n, and f_n be derived from p as above. We show that we can create a list of triples (w, m, l) of strings w and numbers m, l such that all pairs (w, m) with $\nu_{Sc^2}(w) \subseteq U$ and $m \in \mathbb{N}$ occur as the first two components of triples in this list and for each triple (w, m, l) in the list

$$|f_j(z) - f_k(z)| < 2^{-m}$$

for all $j > k \geq l$ and all $z \in \nu_{Sc^2}(w)$. Indeed, using p and Theorem 3.2 we can create a list of all w with $\nu_{Sc^2}(w) \subseteq U$. We must show that for any such w and any m we can find a suitable number l.

First, using w, we can compute a $\delta_{\mathcal{K}}^{>}$–name for the compact set $K := \{0\} \cup \nu_{Sc^2}(w)$, which is contained in U. From this name and from p we can compute a number $n \in \mathbb{N}$ with $|f_i(z)| \leq \frac{3^n - 1}{3^n}$ for all $i \in \mathbb{N}$ and $z \in K$, by Proposition 4.11 (remember that $f_i(0) = 0$ for all i). We set $a := \frac{3^n - 1}{3^n}$ and choose a rational number b with

$$a < b^2 < 1 \quad \text{and} \quad 3 \cdot \frac{\sqrt{1-b}}{b^2 - a} < 2^{-m}\,.$$

Using the $\delta_{\text{open},\partial}$–names $p^{(n)}$ for the sets U_n we can by Theorem 3.9 compute the inner radii $d(\mathbb{C} \setminus U_n, 0)$ with arbitrary precision. Since they tend to 1 for n tending to infinity we can find a number l with $d(\mathbb{C} \setminus U_l, 0) \geq b$. We claim that this number has the demanded property.

Since the sequence of inner radii is increasing we have $d(\mathbb{C} \setminus U_k, 0) \geq b$ for all $k \geq l$. Therefore, if $j > k \geq l$, then we can apply Lemma 6.6 to the set U_k and the function $\kappa_j \circ \ldots \circ \kappa_{k+1}$ and obtain for any $z \in K$

$$|f_j(z) - f_k(z)| = |\kappa_j \circ \ldots \circ \kappa_{k+1} \circ f_k(z) - f_k(z)| \leq 3 \cdot \frac{\sqrt{1-b}}{b^2 - |f_k(z)|} \leq 3 \cdot \frac{\sqrt{1-b}}{b^2 - a} < 2^{-m}\,.$$

□

The non-effective content of Lemma 6.5 already tells us that the sequence $(f_n)_n$ of functions converges locally uniformly on U. By theorems of Weierstrass and Hurwitz the limit of a locally uniformly converging sequence of conformal functions exists and is a conformal function again. Let f be the conformal limit of the functions f_n. Since $f_n(U) = U_{n+1} \subseteq D$ for all n, also $f(U) \subseteq D$. Since $f_n(0) = 0$ and $f_n'(0) > 0$ we conclude $f(0) = 0$ and $f'(0) > 0$. Furthermore, since the inner radii $d(\mathbb{C} \setminus U_n, 0)$ tend to 1, we have $f(U) = D$. Thus, f is a conformal bijection from U onto D. Finally, let H be the computable functional of Proposition 2.15 for $n = m = 2$. We define the functional F_2 by

$$F_2(p) := H\langle G_4(p), G_5(p)\rangle\,.$$

Then F_2 is computable and $F_2(p)$ is a (ρ^2, ρ^2)–tracking name for f. This ends the construction of the functional F_2 of the second step of the total construction. We have proved Theorem 5.1.

We conclude this section with two remarks on the proof. In the construction of the conformal mapping we have used only Koebe maps. There are many other possible families of suitable maps, compare Henrici [8] and Remmert [17]. Others might be better for complexity theoretic reasons.

Theorem 4.7 was essential for the classically simple direction of the Riemann Mapping Theorem: for going from information about the conformal mapping to topological information about the set. But we have used it also in the main construction and have applied it to Koebe functions and the square root function. This application was not really necessary since for the square root function and the Koebe maps one can obtain direct estimates, see Bishop and Bridges [2].

7 Conclusion

We have formulated several effective versions of the Riemann Mapping Theorem. The strongest version showed that certain topological information — essentially enumerating a covering set of open spheres in the set and a dense set of points in the boundary — about a proper simply connected region is equivalent to geometric information about a conformal bijection between this set and the unit circle in the complex plane. This also gave a characterization of those proper simply connected regions for which there exists a computable conformal bijection onto the unit disk.

We conclude with several open problems. In preparation for the results about the Riemann Mapping Theorem we analyzed various types of information about open subsets of Euclidean spaces and derived computability classes of open sets. These types of information and these classes should be analyzed more thoroughly. Also, effective complex analysis could be developed more systematically and broader along the lines of the results of Section 4. For the Effective Riemann Mapping Theorem itself, one can aim for generalizations to multiply connected regions or Riemann surfaces, thus, going further along the suggestions by Pour–El and Richards [15, Problem 5]. Another interesting problem is to analyze the computational complexity of the transformations in the Effective Riemann Mapping Theorem.

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