Correspondences between Classical, Intuitionistic and Uniform Provability

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Abstract

Based on an analysis of the inference rules used, we provide a characterization of the situations in which classical provability entails intuitionistic provability. We then examine the relationship of these derivability notions to uniform provability, a restriction of intuitionistic provability that embodies a special form of goal-directedness. We determine, first, the circumstances in which the former relations imply the latter. Using this result, we identify the richest versions of the so-called abstract logic programming languages in classical and intuitionistic logic. We then study the reduction of classical and, derivatively, intuitionistic provability to uniform provability via the addition to the assumption set of the negation of the formula to be proved. Our focus here is on understanding the situations in which this reduction is achieved. However, our discussions indicate the structure of a proof procedure based on the reduction, a matter also considered explicitly elsewhere.

Key Words: classical logic, intuitionistic logic, proof theory, uniform provability, proof search, logic programming.

1 Introduction

We address three questions pertaining to derivability relations over sequents in this paper. The first of these concerns the correspondence between classical and intuitionistic provability. It is well known that the former is a stronger relation than the latter: while every intuitionistic proof is also a classical one, there are some sequents that are derivable only in classical logic. However, it is possible in principle to obtain the reverse correspondence by restricting the syntax of formulas considered or the kinds of inference rules used in a classical proof. We examine this possibility here. In particular, we provide a characterization at the level of inference rule usage of the situations in which classical provability implies intuitionistic provability. Our analysis is "coarse-grained" in that it pays attention only to the inference rules used, and not to their interaction in particular proofs as is done in a restricted setting in [PRW96], but it is complete at this level of granularity. While our study is one that has been independently conducted, results similar to ours have previously been obtained by Orevkov [Ore68] as we discuss in Section 4. The results that we present have uses in proof search. One possible application is that it permits intuitionistic proof procedures to be employed in settling questions of classical validity in special situations. This approach has benefits and has also been employed in the past: for example, it underlies the procedure commonly used relative to Horn clause logic with the virtue that proof search at any point is driven by a *single* goal formula. Another application of our observations is that it supports the use of classical principles in intuitionistic proof search. Thus, the treatment of quantifier dependencies can, in special circumstances, be achieved by a static (dual) Skolemization process instead of a costly dynamic accounting mechanism.

The second question we consider concerns the correspondence between classical and intuitionistic provability on the one hand and uniform provability on the other. Uniform proofs as identified in [MNPS91] are intuitionistic proofs restricted so as to capture a goal-directedness in proof search. One reason for interest in this category of proofs is that it provides a framework for interpreting the logical symbols in the formulas being proved as primitives for directing search and the inference rules pertaining to these symbols as specifications of their search semantics. This viewpoint has been exploited in [MNPS91] in describing a proof-theoretic foundation for logic programming. By its very definition, uniform provability is a less inclusive relation than either classical or intuitionistic provability. However, by a suitable restriction of the context, it is possible to obtain a correspondence between these three relations. We provide, once again, a complete characterization at the level of inference rule usage of the situations in which intuitionistic provability entails and uniform provability. When combined with the earlier result, this analysis yields a similar characterization relative to classical logic. As one application of these observations, they enable us to identify the richest possible logic programming languages within classical and intuitionistic logic; our remarks relative to intuitionistic logic are similar to those in [Har94].

The final question we consider concerns the reduction of classical and intuitionistic provability to uniform provability. Efficient procedures can be designed for searching for uniform proofs. Towards exploiting this possibility, it is worth considering a modification of the given formula or sequent in a way that does not alter the original derivability question but, nevertheless, succeeds in reducing it to one of uniform provability. One such modification that has been studied in the past is the addition of the negation of the formula that is to be proved to the assumptions [NL95, Nad96]. This transformation is sound with respect to classical logic. We characterize the situations in which it also achieves the desired reduction. Since the transformation can be applied to intuitionistic provability without loss of soundness whenever this notion coincides with classical provability, we obtain information indirectly about the reducibility in this case as well.

2 Logical preliminaries

We will work within the framework of a first-order logic in this paper. The logical symbols that we assume as primitive are $\top$, $\bot$, $\wedge$, $\vee$, $\supset$, $\exists$, and $\forall$. The first two symbols in this collection denote the tautologous and the contradictory propositions, respectively. Note that we consider these logical constants to be distinct from atomic formulas. Negation is a defined notion in our language, $\neg A$ being an abbreviation for $(A \supset \bot)$.

Notions of derivation that are of interest to us are formalized by sequent calculi. A sequent in our context is a pair of *multisets* of formulas. Assuming that Γ and Δ are its elements, the pair is written as $\Gamma \longrightarrow \Delta$ and Γ and Δ are referred to as its antecedent and succedent, respectively. Such a sequent is an axiom if either $\top \in \Delta$ or for some A that is either $\bot$ or an atomic formula, it is the case that $A \in \Gamma$ and $A \in \Delta$. The rules that may be used in constructing sequent proofs are those that can be obtained from the schemata shown in Figure 1. In these schemata, Γ, Δ and Θ stand for multisets of formulas, B and D stand for formulas, c stands for a constant, x stands for a variable and t stands for a term. The notation B, Γ (Δ, B) is used here for a multiset containing the formula B whose remaining elements form the multiset Γ (respectively, Δ). Further, expressions of the form $[t/x]B$ are used to denote the result of replacing all free occurrences of x in B by t, with bound variables being renamed as needed to ensure the logical correctness of these replacements. There is the usual proviso with respect to the rules produced from the schemata $\exists$-L and $\forall$-R: the constant that replaces c should not appear in the formulas that form the lower sequent. A *contraction* rule is one that is obtained from either the contr-L or the contr-R schema. All other rules are referred to as *operational* rules and the formula in the lower sequent that is explicitly affected by such a rule is called its *principal* formula.

$$\frac{B, B, \Gamma \longrightarrow \Delta}{B, \Gamma \longrightarrow \Delta}\ \text{contr-L} \qquad \frac{\Gamma \longrightarrow \Delta, B, B}{\Gamma \longrightarrow \Delta, B}\ \text{contr-R}$$

$$\frac{\Gamma \longrightarrow \Delta, \bot}{\Gamma \longrightarrow \Delta, D}\ \bot\text{-R}$$

$$\frac{B, \Gamma \longrightarrow \Delta}{B \wedge D, \Gamma \longrightarrow \Delta}\ \wedge\text{-L} \qquad \frac{D, \Gamma \longrightarrow \Delta}{B \wedge D, \Gamma \longrightarrow \Delta}\ \wedge\text{-L}$$

$$\frac{B, \Gamma \longrightarrow \Delta \qquad D, \Gamma \longrightarrow \Delta}{B \vee D, \Gamma \longrightarrow \Delta}\ \vee\text{-L}$$

$$\frac{\Gamma \longrightarrow \Delta, B \qquad \Gamma \longrightarrow \Delta, D}{\Gamma \longrightarrow \Delta, B \wedge D}\ \wedge\text{-R}$$

$$\frac{\Gamma \longrightarrow \Delta, B}{\Gamma \longrightarrow \Delta, B \vee D}\ \vee\text{-R} \qquad \frac{\Gamma \longrightarrow \Delta, D}{\Gamma \longrightarrow \Delta, B \vee D}\ \vee\text{-R}$$

$$\frac{\Gamma \longrightarrow \Delta, B \qquad D, \Gamma \longrightarrow \Theta}{B \supset D, \Gamma \longrightarrow \Delta, \Theta}\ \supset\text{-L} \qquad \frac{B, \Gamma \longrightarrow \Delta, D}{\Gamma \longrightarrow \Delta, B \supset D}\ \supset\text{-R}$$

$$\frac{[t/x]B, \Gamma \longrightarrow \Delta}{\forall x\, B, \Gamma \longrightarrow \Delta}\ \forall\text{-L} \qquad \frac{\Gamma \longrightarrow \Delta, [t/x]B}{\Gamma \longrightarrow \Delta, \exists x\, B}\ \exists\text{-R}$$

$$\frac{[c/x]B, \Gamma \longrightarrow \Delta}{\exists x\, B, \Gamma \longrightarrow \Delta}\ \exists\text{-L} \qquad \frac{\Gamma \longrightarrow \Delta, [c/x]B}{\Gamma \longrightarrow \Delta, \forall x\, B}\ \forall\text{-R}$$

Figure 1: Rules for deriving sequents

Finally, we refer to contr-L and the operational rules whose principal formulas are in the antecedent of the lower sequent as *left* rules and to the remaining rules as *right* rules.

We are interested in three notions of derivability for sequents of the form $\Gamma \longrightarrow \Delta$. A **C**-proof for such a sequent is a derivation obtained by making arbitrary uses of the inference rules. **I**-proofs are **C**-proofs in which every sequent has exactly one formula in its succedent. Notice that, by this stipulation, Δ must itself consist of a single formula. Finally, a *uniform proof* or **O**-proof is an **I**-proof in which any sequent that has a non-atomic formula distinct from $\perp$ in its succedent occurs only as the lower sequent of an inference rule that introduces the top-level logical symbol of that formula.

In the case that Δ is a single formula, we shall write $\Gamma \vdash_C \Delta$, $\Gamma \vdash_I \Delta$ and $\Gamma \vdash_O \Delta$ to indicate the existence of, respectively, a **C**-proof, an **I**-proof and an **O**-proof for $\Gamma \longrightarrow \Delta$. The first two notions correspond to classical and intuitionistic provability respectively. The sequent calculi that we have used here to characterize these derivability relations are transparently related to those in [Pra65]: we have treated antecedents and succedents as multisets rather than sets but have added the contraction rules to realize arbitrary multiplicity of formulas and, in the intuitionistic setting, we do not permit sequents of the form $\Gamma \longrightarrow$ that are not derivable in the system of [Pra65]. Uniform provability corresponds to the existence of an **O**-proof. This notion indicates the possibility for a goal-directedness in the search for a derivation, with the top-level structure of the formula in the succedent controlling the next step in the search at each stage.

We observe certain properties of our derivation calculi for classical and intuitionistic logic that will be used in later sections. First, any sequent in which the antecedent and succedent have a common formula has a **C**-proof and, if the succedent has a single element, an **I**-proof. Thus, a modification to our calculi that considers all such sequents to be axioms does not change the set of provable sequents. The second observation concerns the so-called *Cut* inference rules that are obtained from the schemata

$$\frac{\Gamma_1 \longrightarrow \Delta_1, B \qquad\qquad B, \Gamma_2 \longrightarrow \Delta_2}{\Gamma_1, \Gamma_2 \longrightarrow \Delta_1, \Delta_2}$$

Notice that in generating *Cut* rules in the intuitionistic context, Δ_1 must be instantiated by an empty multiset and Δ_2 by a singleton multiset. Now, these *Cut* rules are admissible with respect to classical and intuitionistic provability as formulated here, *i.e.*, the same set of sequents have **C**-proofs and **I**-proofs even if we allow these additional rules to be used in derivations. This property can be demonstrated by describing a procedure for eliminating occurrences of the *Cut* rules from any given derivation. An examination of a typical such procedure—for example, the procedure contained in [Gen69]—actually allows a stronger conclusion to be drawn: a derivation that uses *Cut* rules, contraction rules and operational inference rules obtained from a restricted subset of the schemata in Figure 1 can be transformed into one in which only contraction rules and operational rules obtainable from the restricted schemata set appear. Furthermore, this property holds even when the notion of an axiom is strengthened as described earlier in this paragraph.

3 Building contraction into other inference rules

The contraction rules allow for a profligate multiplicity of formulas. The necessary multiplicity can be characterized more precisely by identifying derived forms of some of the operational rules that incorporate contraction into their structure and thereby permit the contraction rules themselves to be omitted from the calculus. We describe below a convenient form of these derived rules that is presented, for instance, in [Dra79].

We consider first the case for classical provability. The new rules that are of interest are those obtained from the following schemata:

$$\frac{A, B, \Gamma \longrightarrow \Delta}{A \wedge B, \Gamma \longrightarrow \Delta}\ \wedge\text{-L}^* \qquad \frac{\Gamma \longrightarrow \Delta, A, B}{\Gamma \longrightarrow \Delta, A \vee B}\ \vee\text{-R}^*$$

$$\frac{\Gamma \longrightarrow B, \Delta \qquad\qquad D, \Gamma \longrightarrow \Delta}{B \supset D, \Gamma \longrightarrow \Delta}\ \supset\text{-L}^*$$

$$\frac{\forall x\, P, [t/x]P, \Gamma \longrightarrow \Delta}{\forall x\, P, \Gamma \longrightarrow \Delta}\ \forall\text{-L}^* \qquad \frac{\Gamma \longrightarrow \Delta, \exists x P, [t/x]P}{\Gamma \longrightarrow \Delta, \exists x P}\ \exists\text{-R}^*$$

These rules are obviously derived ones: a **C**-proof of the lower sequent of each rule can be obtained from **C**-proof(s) of the upper sequent(s) by using an instance of the 'asterisk-less' version of the schema followed by some number of contraction rules. By a $\mathbf{C}^+$-proof let us mean a derivation constructed in a calculus obtained from the one for **C**-proofs by replacing the rules $\wedge$-L, $\vee$-R, $\supset$-L, $\forall$-L and $\exists$-R with the ones obtained from the schemata above. It is then easily seen that a sequent has a **C**-proof if and only if it has a $\mathbf{C}^+$-proof.

Let an $\mathbf{C}^*$-proof be a $\mathbf{C}^+$-proof in which contraction rules are not used. Our objective is to show that a sequent has a $\mathbf{C}^*$-proof whenever it has a $\mathbf{C}^+$-proof, *i.e.*, contraction can be eliminated from **C**-proofs under the described strengthening of the $\wedge$-L, $\vee$-R, $\supset$-L, $\forall$-L and $\exists$-R rules. This can be done through the following sequence of steps that culminate in Theorem 3.

Lemma 1 *Let* $\Gamma \longrightarrow \Delta$ *have a* $\mathbf{C}^*$*-proof of height* h *and let all references to rules be to ones for constructing* $\mathbf{C}^*$*-proofs.*

1. *For any single upper sequent rule of the form*

$$\frac{\Gamma' \longrightarrow \Delta'}{\Gamma \longrightarrow \Delta}$$

it is the case that $\Gamma' \longrightarrow \Delta'$ *has a* $\mathbf{C}^*$*-proof of height at most* h*.*

2. *For any rule with two upper sequents of the form*

$$\frac{\Gamma' \longrightarrow \Delta' \qquad\qquad \Gamma'' \longrightarrow \Delta''}{\Gamma \longrightarrow \Delta}$$

it is the case that both $\Gamma' \longrightarrow \Delta'$ *and* $\Gamma'' \longrightarrow \Delta''$ *have* $\mathbf{C}^*$*-proofs of height at most* h*.*

Proof. The cases for $\forall$-L* and $\exists$-R* follow from observing that if a $\mathbf{C}^*$-proof exists for a certain sequent, then there is a derivation of similar structure and identical height for any sequent obtained from it by adding formulas to the antecedent or succedent. The remaining cases, that show the invertibility of all the other rules, are covered by the Inversion Lemma of [TS96].

□

Let Σ be a multiset of formulas. We use the notation $\hat{\Sigma}$ to denote the *set* of formulas appearing in Σ.

Lemma 2 *Let* $\Gamma \longrightarrow \Delta$ *have a* $\mathbf{C}^*$*-proof of height* h *and let* Γ' *and* Δ' *be such that* $\hat{\Gamma} \subseteq \hat{\Gamma}'$ *and* $\hat{\Delta} \subseteq \hat{\Delta}'$*. Then* $\Gamma' \longrightarrow \Delta'$ *has a* $\mathbf{C}^*$*-proof of height at most* h*.*

Proof. By an induction on the height of the derivation for $\Gamma \longrightarrow \Delta$. We leave the reader to fill out the details, perhaps by consulting [TS96].

□

Theorem 3 *A sequent* $\Gamma \longrightarrow \Delta$ *has a* $\mathbf{C}$*-proof if and only if it has a* $\mathbf{C}^*$*-proof.*

Proof. It suffices to show that $\Gamma \longrightarrow \Delta$ has a $\mathbf{C}^+$-proof if and only if it has a $\mathbf{C}^*$-proof. The 'if' direction is obvious. For the other direction we use an induction on the number of contractions in the $\mathbf{C}^+$-proof. If there are none, then we already have a $\mathbf{C}^*$-proof. Otherwise, we find a contraction that is the first one in the derivation in the path from an axiom to the final sequent. By Lemma 2, this contraction can be dispensed with, yielding a $\mathbf{C}^+$-proof with one less contraction.

□

Theorem 3 and Lemma 1 provide the basis for a proof procedure for classical logic that is worth noting. All the rules that may be used in $\mathbf{C}^*$-proofs that are distinct from the $\forall$-L and $\exists$-R rules have an upper sequent or two upper sequents with fewer logical symbols than those in the lower sequent. We may therefore use these rules repeatedly in a terminating process to reduce a given sequent to a new set of sequents for which derivations must be constructed. If every sequent in the set so produced is an axiom, then we will have established classical provability. If at least one of the new sequents is not an axiom and cannot be the lower sequent of either a $\forall$-L* or a $\exists$-R*, then the original sequent can have no **C**-proof. Otherwise each non-axiom sequent is reduced by a simultaneous use of all the $\forall$-L* and $\exists$-R* rules that are applicable to it and the process is repeated. The procedure as presently stated is not quite practical since the use of a $\forall$-L* or a $\exists$-R* rule also involves picking the 'right' instantiation term. However, this choice can be delayed by introducing instead a variable that may be instantiated later and determining bindings for such variables by using unification when checking if a sequent is an axiom. Unification must, of course, not lead to the constraint on constants in the $\exists$-L and $\forall$-R rules to be violated. The best way to achieve this effect in the classical setting is to transform the original sequent by a process referred to as Herbrandization in [Sha92] that eliminates at the outset all quantifiers that might require a $\exists$-L and $\forall$-R to be used in proof search.

Contraction is applicable only to antecedent formulas in the intuitionistic setting. The essential uses of these rules occur in conjunction with the $\wedge$-L, $\forall$-L, and $\supset$-L rules. To realize the effects of these uses, we may replace the $\wedge$-L and the $\forall$-L schemata by $\wedge$-L* and $\forall$-L* respectively, and $\supset$-L by the following:

$$\frac{B \supset D, \Gamma \longrightarrow B \qquad\qquad D, \Gamma \longrightarrow \Delta}{B \supset D, \Gamma \longrightarrow \Delta} \supset\text{-L}_I^*$$

Notice that, in contrast to the situation in classical logic, the present modification to the $\supset$-L rules incorporates a contraction in the *antecedent*.

We refer to a derivation constructed in the calculus for **I**-proofs with the indicated replacements for the $\wedge$-L, $\forall$-L and $\supset$-L rules as an $\mathbf{I}^+$-proof. If contr-L is not used in such a derivation, we shall call it an $\mathbf{I}^*$-proof. The adequacy of $\mathbf{I}^*$-proofs in settling questions of intuitionistic provability is stated in the following theorem whose proof may be modelled on the arguments in [TS96].

Theorem 4 *A sequent* $\Gamma \longrightarrow F$ *has a* **I***-proof if and only if it has a* $\mathbf{I}^*$*-proof.*

Once again, Theorem 4 has content that can be utilized in structuring proof search in intuitionistic logic. However, there are important differences from the classical case.

First, the static Herbrandization step is not sound in the new setting [Nad93, Sha92]. An alternative approach that can be used in this case is to treat the $\exists$-L and $\forall$-R rules explicitly in proof search and to employ a dynamic form of Herbrandization to ensure that the required constraints are satisfied by quantifier instantiation terms [Fit90, Nad93, Sha92]. Second, a detailed analysis reveals that the process for reducing sequents must delay consideration of the rules $\supset$-L_I^*, $\vee$-R and $\exists$-R in addition to $\forall$-L^*.[1] Further, the order in which these rules are eventually considered may be important and it may be necessary to backtrack over particular orders of reduction.

4 Correspondence between classical and intuitionistic provability

It is clear from the definitions that, if Γ is a (multi)set of formulas and Δ is a single formula, then $\Gamma \vdash_I \Delta$ implies $\Gamma \vdash_C \Delta$. The converse is not always true. A 'canonical' demonstration of this fact is obtained by taking Γ to be the empty (multi)set and letting Δ consist of the formula $((q \supset s) \supset q) \supset q$. However, the truth of the converse and, hence, the equivalence of classical and intuitionistic provability, can be assured when the syntax of the assumption and conclusion formulas is restricted in certain ways. We describe these syntactic restrictions in this section. Our characterization is based, first of all, on the inference rules used in a **C**-proof and is a complete one at this level: we identify four classes of **C**-proofs determined by the non-use of certain inference rules and show that (a) an **I**-proof exists for the final sequent of a **C**-proof belonging to any of these classes and (b) for each possible way for violating all the restrictions on inference rule usage, there is a **C**-proof with a corresponding final sequent for which no **I**-proof exists. Now, the syntactic structure of the formulas in a given sequent determines the inference rules that can appear in a (cut-free) **C**-proof of that sequent. This observation enables us to translate the restriction on inference rules into the desired syntactic constraints on formulas.

The following theorem identifies one of the classes of **C**-proofs that are of interest.

Theorem 5 *Let $\Gamma \longrightarrow \Delta$ have a* **C***-proof in which no $\supset$-R or $\vee$-L rule is used. Then, for some G in Δ, it is the case that $\Gamma \longrightarrow G$ has an* **I***-proof. In particular, if Δ consists of a*

[1] There is actually another problem with regard to the $\supset$-L_I^* rules: the principal formula of this rule appears again in the left upper sequent and so it is not certain that a use of the rule will produce less 'complex' sequents. Dyckhoff [Dyc92] and Hudelmaier [Hud90] have proposed alternative sequent calculi for propositional logic that overcomes this problem but, to our knowledge, no similar calculus has been described for the situation where quantifiers are included.

single formula, then $\Gamma \longrightarrow \Delta$ *itself has an* **I***-proof.*

Proof. We use an induction on the heights of **C**-proofs. If $\Gamma \longrightarrow \Delta$ is an axiom, then, clearly, there is a G in Δ such that $\Gamma \longrightarrow G$ is also an axiom. Thus, the theorem is true for **C**-proofs of height 1. If the height of the derivation is greater than 1, we consider each possibility for the last rule used. If this is a rule with a single upper sequent, then, by assumption, it must be distinct from an $\supset$-R. In all the remaining cases, the induction hypothesis combined possibly with a rule obtained from the same schema yields the desired conclusion. If the last rule has two upper sequents, then, since it is distinct from an $\vee$-L, it must be either an $\supset$-L or an $\wedge$-R. Suppose it is the first. Then the derivation at the end has the structure

$$\frac{\Gamma' \longrightarrow \Delta_1, B \qquad\qquad D, \Gamma' \longrightarrow \Delta_2}{B \supset D, \Gamma' \longrightarrow \Delta_1, \Delta_2}$$

where Γ is $B \supset D, \Gamma'$ and Δ_1 and Δ_2 constitute a (multiset) partition of Δ. By hypothesis, either $\Gamma' \longrightarrow G$ has an **I**-proof for some G in Δ_1 or $\Gamma' \longrightarrow B$ has an **I**-proof. In the first case, it is easily seen that $B \supset D, \Gamma' \longrightarrow G$ also has an **I**-proof. In the second case, we use the hypothesis again to observe that for some G in Δ_2 it is the case that $D, \Gamma' \longrightarrow G$ has an **I**-proof. These observations used together with an $\supset$-L rule yields the theorem in this case. A similar argument can be provided when the last rule is an $\wedge$-R.

□

We translate the restriction on proof rules in Theorem 5 into restrictions on the syntax of formulas. Consider the classes of formulas defined by the following mutually recursive syntax rules, assuming A represents atomic formulas:

$$\begin{array}{lcl} G & ::= & \top \mid \bot \mid A \mid G \wedge G \mid G \vee G \mid \forall x\, G \mid \exists x\, G \\ D & ::= & \top \mid \bot \mid A \mid G \supset D \mid D \wedge D \mid \exists x\, D \mid \forall x\, D \end{array}$$

A sequent in which the succedent consists of a G-formula and the antecedent contains only D-formulas is classically provable just in case it is intuitionistically provable. We observe that the G- and D-formulas defined here subsume the so-called goal formulas and program clauses of Horn clause logic [MNPS91].

There is an auxiliary utility to Theorem 5: it has content relevant to defining a multi-formula succedent sequent calculus for intuitionistic logic. Such a calculus is of interest because it permits a postponement in proof search of decisions about which disjunct of a disjunctive formula in the succedent is to be chosen. Consider the calculus for constructing $\mathbf{C}^*$-proofs with the $\vee$-L and $\supset$-R rules replaced, respectively, with ones obtained from the following schemata:

$$\frac{B, \Gamma \longrightarrow F \qquad D, \Gamma \longrightarrow F}{B \vee D, \Gamma \longrightarrow \Delta, F}$$

$$\frac{B, \Gamma \longrightarrow D}{\Gamma \longrightarrow \Delta, B \supset D}$$

A sequent of the form $\Gamma \longrightarrow \Delta$ in which Δ is a singleton multiset has a derivation in this calculus if and only if it has an **I**-proof; the 'if' direction is obvious and the 'only if' direction follows from Theorem 5 and an easy induction on the number of occurrences of the 'new' $\vee$-L and $\supset$-R rules in the given $\mathbf{C}^*$-proof. We may also allow Δ to contain more than one formula by interpreting it as the disjunction of these formulas. We note, however, that the modification to the $\vee$-L schema is essential even under such an interpretation for quantificational logic: without this modification, the sequent $\forall x\,(p(x) \vee q) \longrightarrow (\forall x\, p(x)) \vee q$ would, for example, have a derivation even though it has no **I**-proof.

The following theorem identifies a second interesting class of **C**-proofs.

Theorem 6 *Let Γ be a multiset of formulas and let $B_1, \ldots, B_n$ be formulas such that $\Gamma \longrightarrow B_1, \ldots, B_n$ has a **C**-proof in which no $\supset$-R or $\forall$-R rule is used. Then the sequent $\Gamma \longrightarrow B_1 \vee \ldots \vee B_n$ has an **I**-proof. In the case that $n = 1$, $\Gamma \longrightarrow B_1$ has an **I**-proof.*

Proof. This theorem can be proved, once again, by an induction on the heights of **C**-proofs. We do not provide an explicit proof here, noting only that an argument that is similar to, but simpler than, that for Theorem 7 below suffices. In particular, a complication arises in the (inductive) proof of Theorem 7 from having to consider a $\forall$-R as the last rule in the **C**-proof of $\Gamma \longrightarrow B_1, \ldots, B_n$. The premise of the present theorem rules out this possibility. There is an additional case that has to be considered here in that a $\forall$-L rule could be the last one used. However, the argument for this case is a relatively simple one. $\square$

Following earlier lines, we can rephrase Theorem 6 in terms of a restriction on the syntax of formulas. Consider the following classes of formulas, assuming, again, that A represents atomic formulas:

$$\begin{array}{lll} G & ::= & \top \mid \bot \mid A \mid G \wedge G \mid G \vee G \mid \exists x\, G \\ D & ::= & \top \mid \bot \mid A \mid G \supset D \mid D \wedge D \mid D \vee D \mid \exists x\, D \mid \forall x\, D. \end{array}$$

If Γ is a (multi)set of D-formulas and F is a G-formula, then $\Gamma \vdash_C F$ only if $\Gamma \vdash_I F$. The classes of G- and D-formulas described by the present rules constitute a generalization of similarly named classes in [NL95] and have been studied there as the basis for disjunctive logic programming [LMR92].

Analogously to Theorem 5, Theorem 6 can be used to justify a multi-formula succedent sequent calculus for intuitionistic logic. Consider the calculus for constructing $\mathbf{C}^*$-proofs with the $\forall$-R and $\supset$-R rules replaced, respectively, with ones obtained from the following schemata:

$$\frac{\Gamma \longrightarrow [c/x]B}{\Gamma \longrightarrow \Delta, \forall x\, B}$$
$$\frac{B, \Gamma \longrightarrow D}{\Gamma \longrightarrow \Delta, B \supset D}$$

A sequent of the form $\Gamma \longrightarrow \Delta$ in which Δ is a singleton multiset has a derivation in this calculus if and only if it has an $\mathbf{I}$-proof. As before, we may also allow Δ to contain more than one formula by interpreting it as the disjunction of these formulas. We note that the calculus that is so described for intuitionistic logic differs superficially—in particular, only in the manner in which the logical constant $\perp$ is treated—from the GHPC calculus of Dragalin [Dra79].

A third category of $\mathbf{C}$-proofs is identified in the following theorem.

Theorem 7 *Let Γ be a multiset of formulas and let $B_1, \ldots, B_n$ be formulas such that $\Gamma \longrightarrow B_1, \ldots, B_n$ has a $\mathbf{C}$-proof in which no $\supset$-R or $\forall$-L rule is used. Then the sequent $\Gamma \longrightarrow B_1 \vee \ldots \vee B_n$ has an $\mathbf{I}$-proof. In the case that $n = 1$, $\Gamma \longrightarrow B_1$ has an $\mathbf{I}$-proof.*

Proof. It is convenient to prove the theorem assuming derivation calculi with the stronger notion of axioms described in Section 2, *i.e.*, ones in which any sequent whose antecedent and succedent have a common formula is considered an axiom. Further, we show a stronger property than that required: If $\Gamma \longrightarrow B_1, \ldots, B_n$ has a $\mathbf{C}$-proof in which no $\supset$-R or $\forall$-L rule is used, then $\Gamma \longrightarrow B_1 \vee \ldots \vee B_n$ has an $\mathbf{I}$-proof in which no $\forall$-L rule is used. We prove this property by means of an induction on the height of the $\mathbf{C}$-proof for $\Gamma \longrightarrow B_1, \ldots, B_n$.

The base case corresponds to $\Gamma \longrightarrow B_1, \ldots, B_n$ being an axiom. In this case, we can construct an $\mathbf{I}$-proof of $\Gamma \longrightarrow B_1 \vee \ldots \vee B_n$ by using a sequence of $\vee$-R rules below a suitably chosen axiom. Note that no $\forall$-L rule appears in this derivation.

For the inductive step, we consider the various possibilities for the last rule used in the derivation. The argument is straightforward for all permitted left rules, *i.e.*, ones that are distinct from $\forall$-L, in which the succedent is identical in the upper and lower sequents—we invoke the induction hypothesis and use an instance of the same rule schema to get an $\mathbf{I}$-proof for $\Gamma \longrightarrow B_1 \vee \ldots \vee B_n$, and we note that this derivation must not contain a $\forall$-L rule occurrence.

The only remaining possibility for a left rule is that it is an $\supset$-L. In this case, the derivation at the end has the structure

$$\frac{\Gamma' \longrightarrow \Delta_1, F \qquad D, \Gamma' \longrightarrow \Delta_2}{F \supset D, \Gamma' \longrightarrow \Delta_1, \Delta_2}$$

where Γ is $F \supset D, \Gamma'$ and Δ_1 and Δ_2 constitute some partition of $B_1, \ldots, B_n$. The argument follows the pattern of that for the other left rules in the case that Δ_1 is empty. We therefore assume that it is nonempty. We also assume that the final sequent in the derivation has exactly two formulas in the succedent and that Δ_1 is B_1 and Δ_2 is B_2; these assumptions are not critical, and may be dispensed with in a more detailed argument.

Now, using the induction hypothesis, we see that $\Gamma' \longrightarrow B_1 \vee F$ and $D, \Gamma' \longrightarrow B_2$ have **I**-proofs in which no $\forall$-L rule is used. From this it follows that $F \supset D, \Gamma' \longrightarrow B_1 \vee F$ and $F, D, \Gamma' \longrightarrow B_2$ also have such **I**-proofs. Using the latter, we can construct an **I**-proof for $B_1 \vee F, F \supset D, \Gamma' \longrightarrow B_1 \vee B_2$ as follows:

$$\dfrac{\dfrac{B_1, F \supset D, \Gamma' \longrightarrow B_1}{B_1, F \supset D, \Gamma' \longrightarrow B_1 \vee B_2}\ \vee\text{-R} \qquad \dfrac{F, \Gamma' \longrightarrow F \qquad \dfrac{F, D, \Gamma' \longrightarrow B_2}{F, D, \Gamma' \longrightarrow B_1 \vee B_2}\ \vee\text{-R}}{F, F \supset D, \Gamma' \longrightarrow B_1 \vee B_2}\ \supset\text{-L}}{B_1 \vee F, F \supset D, \Gamma' \longrightarrow B_1 \vee B_2}\ \vee\text{-L}$$

Notice that no $\forall$-L rule appears in this derivation. We can combine this derivation with the one for $F \supset D, \Gamma' \longrightarrow B_1 \vee F$ by means of a *Cut* rule and some contr-L rules to get a derivation for $F \supset D, \Gamma' \longrightarrow B_1 \vee B_2$. Finally, by the observation in Section 2, the *Cut* rule can be eliminated from this derivation to obtain an **I**-proof for $F \supset D, \Gamma' \longrightarrow B_1 \vee B_2$ in which no $\forall$-L rules appear.

To complete the proof, we have to consider the possibility that the last rule in the **C**-proof is a right rule. Suppose that it is, in fact, a $\exists$-R. Then the derivation at the end has the following form:

$$\frac{\Gamma \longrightarrow B_1, \ldots, B_{i-1}, [t/x]B_i', \ldots, B_n}{\Gamma \longrightarrow B_1, \ldots, B_{i-1}, \exists x\, B_i', \ldots, B_n.}$$

We have assumed here that B_i is actually a formula of the form $\exists x\, B_i'$. By the induction hypothesis, $\Gamma \longrightarrow B_1 \vee \ldots \vee B_{i-1} \vee [t/x]B_i' \vee \ldots \vee B_n$ has an **I**-proof. Now, it is easily seen that

$$B_1 \vee \ldots \vee B_{i-1} \vee [t/x]B_i \vee \ldots \vee B_n \longrightarrow B_1 \vee \ldots \vee B_{i-1} \vee \exists x\, B_i' \vee \ldots \vee B_n$$

has an **I**-proof in which no $\forall$-L rule is used. The desired conclusion follows in this case first from using a *Cut* rule and then noting that this rule can be eliminated from the derivation without introducing any occurrences of the $\forall$-L rule.

An argument similar to that for $\exists$-R can be provided for all other permitted right rules except $\forall$-R. For the case of $\forall$-R, we need a further observation: If a sequent of the form $\Gamma \longrightarrow P_1 \vee \ldots \vee [c/x]P'_i \vee \ldots \vee P_n$ has an **I**-proof in which no $\forall$-L rule is used and if Γ and $P_1, \ldots, P'_i, \ldots P_n$ are such that the constant c does not appear in them, then $\Gamma \longrightarrow P_1 \vee \ldots \vee \forall x\, P'_i \vee \ldots \vee P_n$ has an **I**-proof in which no $\forall$-L rule is used. This observation can be established by a routine induction on the height of the given **I**-proof. Further, it can be used together with the present induction hypothesis to yield an argument for the only remaining case in the proof of the main claim.

$\square$

We can, as usual, rephrase Theorem 7 in terms of a restriction on the syntax of formulas. Once again, consider the following classes of formulas, assuming that A represents atomic formulas:

$$\begin{array}{lcl} G & ::= & \top \mid \bot \mid A \mid G \wedge G \mid G \vee G \mid \exists x\, G \mid \forall x\, G \\ D & ::= & \top \mid \bot \mid A \mid G \supset D \mid D \wedge D \mid D \vee D \mid \exists x\, D. \end{array}$$

If Γ is a (multi)set of D-formulas and F is a G-formula, then $\Gamma \vdash_C F$ only if $\Gamma \vdash_I F$.

The following theorem identifies a fourth, and final, class of **C**-proofs that are of interest from the perspective of this section.

Theorem 8 *Let Γ be a multiset of formulas and let B be a formula such that $\Gamma \longrightarrow B$ has a **C**-proof in which no $\supset$-L, $\vee$-R or $\exists$-R rule is used. Then $\Gamma \longrightarrow B$ has an **I**-proof.*

Proof. We claim that if $\Gamma \longrightarrow B$ has a **C**-proof in which no $\supset$-L, $\vee$-R or $\exists$-R rule is used, then this sequent also has a **C***-proof in which no $\supset$-L*, $\vee$-R* or $\exists$-R* rule is used. Towards seeing this, we first make the easy observation that, under the given assumption, $\Gamma \longrightarrow B$ must have a **C**$^+$-proof in which the latter rules do not appear. Now, it is easily determined that the **C***-proofs mentioned in the Lemmas 1 and 2 may be qualified to be ones in which the $\supset$-L*, $\vee$-R* and $\exists$-R* rules do not appear. Finally, an argument similar to that provided for Theorem 3 allows us to conclude that $\Gamma \longrightarrow B$ has a **C***-proof that does not contain any $\supset$-L*, $\vee$-R* or $\exists$-R* rules.

The claim easily yields the theorem: Every sequent in the **C***-proof of restricted form must have exactly one formula in the succedent. Each occurrence of an $\wedge$-L* and $\forall$-L* rule in this derivation can be eliminated in favor of a contr-L rule paired with some number of $\wedge$-L and $\forall$-L rules, respectively, to produce a **C**-proof. The number of formulas in the succedent of each sequent remains unchanged by this transformation and so the **C**-proof that is produced is also an **I**-proof.

$\square$

Towards rephrasing Theorem 8 in terms of a restriction on formulas, we define the following classes of formulas, assuming, as usual, that A represents atomic formulas:

$$\begin{array}{rcl} G & ::= & \top \mid \bot \mid A \mid G \wedge G \mid D \supset G \mid \forall x\, G \\ D & ::= & \top \mid \bot \mid A \mid D \wedge D \mid D \vee D \mid \exists x\, D \mid \forall x\, D. \end{array}$$

It follows from the theorem that if Γ is a (multi)set of D-formulas and F is a G-formula, then $\Gamma \vdash_C F$ only if $\Gamma \vdash_I F$.

Theorem 9 *Theorems 5-8 provide a characterization at the level of proof rules of the conditions under which classical provability implies intuitionistic provability that is complete in the following sense: for each way of violating all the restrictions on inference rule usage described in the mentioned theorems, there is a sequent with a singleton succedent that has a violating* **C***-proof but no* **I***-proof.*

Proof. **C**-proofs may be categorized into those that do and those that do not contain occurrences of the $\supset$-R rules.

We consider first the collection of **C**-proofs in which the $\supset$-R rules are *not* used. To violate the restrictions on proof rule usage contained in Theorems 5-8, a derivation of this kind must contain at least one occurrence of an $\vee$-L, a $\forall$-R and a $\forall$-L rule and of either an $\supset$-L, an $\vee$-R or a $\exists$-R rule. We list sequents below that have **C**-proofs satisfying each of these requirements and note that none of these has an **I**-proof:

$$\begin{array}{l} (\forall x\, p(x)) \supset q, \forall x\, (p(x) \vee q) \;\longrightarrow\; q \\ \forall x\, (p(x) \vee q) \;\longrightarrow\; (\forall x\, p(x)) \vee q \\ \forall x\, \forall y\, (r(x,a) \vee r(y,b)) \;\longrightarrow\; \exists y\, \forall x\, r(x,y). \end{array}$$

We assume in these sequents that q is a proposition symbol, p is a unary predicate symbol, r is a binary predicate symbol and a and b are constants.

A **C**-proof in which an $\supset$-R rule is used must also contain an occurrence of one of the $\supset$-L, $\vee$-R and $\exists$-R rules in order to violate the restrictions described in Theorems 5-8. The following sequents have **C**-proofs satisfying each of these requirements:

$$\begin{array}{l} (q \supset s) \supset q \;\longrightarrow\; q \\ \longrightarrow\; q \vee (q \supset s) \\ \longrightarrow\; \exists x\, (p(x) \supset p(f(x))) \end{array}$$

In these sequents, we assume additionally that s is a proposition symbol and that f is a unary function symbol. It is easily seen that none of these sequents has an **I**-proof, thus verifying the theorem even in this case.

□

We stress, once again, the observation made in Section 1 that our analysis of the correspondence between classical and intuitionistic provability is coarse-grained in that it pays attention only to the rules used in a derivation and not to the particular interactions between rules in it. Thus, there are sequents whose only **C**-proofs violate all the conditions in Theorems 5-8 but which, nevertheless, have **I**-proofs. For example, consider the sequent

$$\longrightarrow ((\forall x\, (r(x,a) \vee r(x,b))) \supset ((((\forall x\, \exists y\, r(x,y)) \supset q) \supset q) \vee s))$$

in which we have used the non-logical vocabulary described in the proof of Theorem 9. Any **C**-proof of this sequent must use an $\vee$-L, an $\supset$-R, a $\forall$-R, a $\forall$-L, an $\vee$-R, a $\exists$-R and an $\supset$-L rule. However, this sequent has an **I**-proof. We note that it is possible to conduct an alternative analysis of the correspondence between classical and intuitionistic provability that focuses specifically on the *interactions* between the rules that appear in a derivation. Such a study has, for instance, been carried out in [PRW96] for a propositional logic that has $\supset$, $\wedge$ and $\neg$ as its only logical symbols. An analysis of this sort indicates when a given classical derivation may be interpreted as having intuitionistic force and may be used in driving a search for a **C**-proof with such a force given one without it. The results of this section are relevant to such a study in that they provide insight into the rules between which interactions should be considered carefully.

After the completion of this paper, it has come to our attention that a study similar to the one presented in this section has previously been conducted by Orevkov [Ore68]. In this work, the notion of a σ-class is identified as a list of logical symbols with positive or negative markings. A sequent is said to belong to a given σ-class if a logical symbol occurs positively (negatively) in the sequent only if it does not occur with a corresponding positive (negative) marking in the listing denoting the σ-class. Viewed differently, a σ-class describes a restriction to the syntax of formulas that are permitted to appear in sequents. A *completely Glivenko class* is now defined to be a σ-class such that any sequent with a singleton succedent belonging to that class is derivable in classical predicate logic only if it is also derivable in intuitionistic predicate logic.[2] Analogous to our Theorem 9, Orevkov provides a complete description of all completely Glivenko classes. The two characterizations are not exactly identical because negation is treated in [Ore68] as a primitive symbol. However, a comparison of the results can still be made. Ignoring the negation symbol, the two characterizations coincide. Treating the negation symbol explicitly allows for distinctions in [Ore68] that, in our context, would translate not into restrictions in rule usage,

[2] In reality, it is predicate logic with *equality* that is considered in [Ore68], but this appears not to be significant to the analysis.

but into distinguishing different roles for implication and paying attention to the polarity of occurrences of $\perp$.

5 Relationship to uniform provability

We consider now the relationship between classical and intuitionistic provability on the one hand and uniform provability on the other. Our analysis covers two kinds of questions. First, we examine restrictions in the syntax of formulas that ensure a coextensiveness between these different proof relations. Following this, we consider the reduction of classical provability to uniform provability in situations where these relations are *not* coextensive.

5.1 Correspondence with uniform provability

Our first goal is to describe the sequents for which the existence of an **I**-proof implies the existence of an **O**-proof. Since an **O**-proof is a special case of an **I**-proof, we can combine this characterization with the results of the previous section to obtain a similar relationship between classical and uniform provability. The following theorem provides the desired characterization in terms of the inference rules used in the **I**-proof.

Theorem 10 *Let Γ be a multiset of formulas and let G be a formula. If the sequent $\Gamma \longrightarrow G$ has an* **I***-proof in which*

1. *either no $\vee$-L rule is used or no $\vee$-R and no $\exists$-R rules are used, and*
2. *either no $\exists$-L rule or no $\exists$-R rule is used,*

then it also has a uniform proof. Moreover, this characterization is tight in that, for each possible way of violating these restrictions, there is a sequent with an **I***-proof but no uniform proof.*

Proof. The first part of the theorem is an immediate consequence of the permutability properties of inference rules in intuitionistic sequent calculi established, for instance, in [Kle52]. To complete the proof of the theorem, we list a suitable set of sequents:

$p(a) \vee p(b) \;\longrightarrow\; \exists x\, p(x)$,
$q \vee s \;\longrightarrow\; s \vee q$, and
$\exists x\, (p(x) \wedge q) \;\longrightarrow\; \exists x\, p(x)$.

We assume here that q and s are proposition symbols, p is a predicate symbol and a and b are constants. None of these sequents has a uniform proof. However all of them have **I**-proofs: the first has one in which an $\vee$-L and a $\exists$-R rule are used, the second has one in which an $\vee$-L and an $\vee$-R rule are used and the last has one in which a $\exists$-L and a $\exists$-R rule are used.

$\square$

The notion of uniform provability is useful in identifying logical languages that provide a basis for programming [MNPS91]. In particular, letting $\mathcal{D}$ and $\mathcal{G}$ denote collections of formulas and $\vdash$ denote a chosen proof relation, an *abstract logic programming language* is defined to be a triple $\langle \mathcal{D}, \mathcal{G}, \vdash \rangle$ such that, for all finite subsets $\mathcal{P}$ of $\mathcal{D}$ and all $G \in \mathcal{G}$, $\mathcal{P} \vdash G$ if and only if $\mathcal{P} \vdash_O G$. In the programming interpretation of such a triple, elements of $\mathcal{D}$ function as program clauses and elements of $\mathcal{G}$ serve as queries or goals. The virtue of this definition is that it supports a broad interpretation of logic programming based on a duality in the meaning of logical symbols: on the one hand, these symbols have a declarative reading given by the proof relation $\vdash$ and, on the other, they are accorded a search-related interpretation given by the rules for introducing each of them on the right in sequent proofs.

An interesting question is that of how rich the syntax of program clauses and goals can be in the cases where $\vdash$ is interpreted as classical or intuitionistic provability. Before answering this question, we note that these formulas must contain certain syntactic components in order to be useful for programming: the procedural interpretation of program clauses relies on universal quantification and implications being permitted at the top-level in these formulas and outermost existential quantification is important in goals in making sense of the result of finding a derivation. In light of Theorem 10, the second requirement precludes outermost occurrences of disjunction and existential quantification in program clauses. Thus, if $\vdash$ is interpreted as intuitionistic provability, the collection of G- and D-formulas given by the following syntax rules represent the largest possible classes for goals and program clauses:

$$
\begin{array}{lcl}
G & ::= & \top \mid \bot \mid A \mid G \wedge G \mid G \vee G \mid D \supset G \mid \forall x\, G \mid \exists x\, G \\
D & ::= & \top \mid \bot \mid A \mid G \supset D \mid D \wedge D \mid \; \mid \forall x\, D
\end{array}
$$

We assume, as before, that A represents atomic formulas in these rules. The only essential difference between the abstract logic programming language given by these classes of formulas and intuitionistic provability and the language of hereditary Harrop formulas studied in [MNPS91] is that the logical constant $\bot$ is permitted to appear here in goals and program clauses.

In the case that classical provability is used instead to clarify the declarative semantics, further restrictions have to be placed on formulas to ensure coextensiveness with intuitionistic provability, a prelude to coextensiveness with uniform provability. By virtue of Theorem 5, one way to achieve this effect is to exclude the case involving implication from the syntax rule for G-formulas above. The language that results from this restriction is closely related to the Horn clause logic that underlies the language Prolog: in particular, it extends Horn clause logic as presented in [MNPS91] by including universal quantification in goals and allowing $\bot$ to appear in goals and program clauses.

However, it is not necessary to exclude implications in goals even when the chosen proof relation is classical provability.[3] What the examples used in the proof of Theorem 9 show is that implications must not appear negatively in program clauses or embedded within disjunctions or existential quantifications in goals. We can modify the definition of G- and D-formulas as follows to satisfy these requirements:

$$\begin{array}{rcl} G & ::= & G' \mid D \supset G \mid G \wedge G \mid \forall x\, G \\ G' & ::= & \top \mid \bot \mid A \mid G' \wedge G' \mid G' \vee G' \mid \forall x\, G' \mid \exists x\, G' \\ D & ::= & \top \mid \bot \mid A \mid G' \supset D \mid D \wedge D \mid \ \mid \forall x\, D \end{array}$$

Using Theorems 5 and 10 and the easy observations that (a) $\Gamma \longrightarrow F_1 \wedge F_2$ has a **C**-proof only if $\Gamma \longrightarrow F_1$ and $\Gamma \longrightarrow F_2$ also have **C**-proofs, (b) $\Gamma \longrightarrow F_1 \supset F_2$ has a **C**-proof only if $F_1, \Gamma \longrightarrow F_2$ also has one, and (c) $\Gamma \longrightarrow \forall x\, F$ has a **C**-proof only if, for some constant c not appearing in Γ or F, $\Gamma \longrightarrow [c/x]F$ also has one, it can be seen that these definitions in fact yield an abstract logic programming language. Moreover, this is the largest such language based on classical logic that also meets the mentioned requirements for programming.

5.2 Reduction to uniform provability

The succedent formula can be used to direct the search for a uniform proof for a sequent in a fairly deterministic fashion. By exploiting this fact, it is possible to define efficient proof procedures for logical languages that have a derivability relation that is coextensive with uniform provability. This idea has been used previously relative to abstract logic programming languages; see, for instance, [Mil91, Nad93]. Now, even in situations where the proof relation of interest deviates from uniform provability, it may still be possible to

[3]This is not in contradiction to Theorem 9. As noted already, the analysis in the theorem does not pay attention to the order in which rules are used and so is not fine-grained enough to provide a tight constraint on the syntax of formulas.

utilize the latter notion in structuring proof search. For instance, it may be possible to modify the sequent whose derivability status is to be verified in some predetermined and sound way to produce a new sequent that has a derivation in the relevant sense just in case it has an **O**-proof. One approach of this kind that has been considered in the past in conjunction with classical logic [NL95, Nad96]. In this approach, the attempt to prove a sequent of the form $\Gamma \longrightarrow F$ is transformed into an attempt to prove $F \supset \bot, \Gamma \longrightarrow F$ instead. As we see below, the indicated augmentation of the antecedent can be made implicit by being built into new inference rules. The virtue of the resulting derivation system is that it provides the basis for a goal-directed proof procedure with the characteristic that the attempt to prove the original goal is *restarted* with a modified set of premises at certain points in the search [Gab85, GR93, LR91, Nad96].

A crucial requirement in using this method is that the described augmentation of the sequent reduce the question of classical provability to that of uniform provability. Towards understanding the applicability of this method, we wish to circumscribe the sequents for which this reduction is actually achieved. We begin by observing that the overall approach is actually sound:

Lemma 11 *Let Γ be a multiset of formulas and let F be a formula. Then $F \supset \bot, \Gamma \vdash_C F$ if and only if $\Gamma \vdash_C F$.*

Proof. This follows easily from the admissibility of the *Cut* inference rules and the fact that $\longrightarrow (F \supset \bot) \vee F$ has a **C**-proof.

$\square$

Since **O**-proofs are **I**-proofs of a special form, a useful first step towards the desired characterization is to understand when the augmentation of a sequent succeeds in reducing classical provability to intuitionistic provability.

Theorem 12 *Let Γ be a multiset of formulas and let F be a formula such that $\Gamma \longrightarrow F$ has a* **C***-proof. Then there is an* **I***-proof for $F \supset \bot, \Gamma \longrightarrow F$ if any one of the following conditions holds relative to the* **C***-proof of $\Gamma \longrightarrow F$:*

1. *no $\forall$-R rule is used,*

2. *no $\supset$-R and no $\vee$-L rule is used,*

3. *no $\supset$-R and no $\forall$-L rule is used, and*

4. *no $\supset$-L, $\vee$-R and $\exists$-R rule is used.*

Further, for each way of violating all these conditions, there is a sequent $\Gamma \longrightarrow F$ *with a violating* **C***-proof such that* $F \supset \bot, \Gamma \longrightarrow F$ *does not have an* **I***-proof.*

Proof. Using the results in [MO63], it can be established that $\Gamma \longrightarrow F$ has a **C**-proof in which no $\forall$-R rule is used, then $F \supset \bot, \Gamma \longrightarrow F$ has an **I**-proof. This fact is also independently and explicitly established in [Nad96]. Further, it is obvious that $F \supset \bot, \Gamma \longrightarrow F$ has an **I**-proof if $\Gamma \longrightarrow F$ has one. If any one of conditions 2-4 is true, then, by virtue of Theorems 5, 7 and 8, $\Gamma \longrightarrow F$ has an **I**-proof. Thus, if any one of the listed conditions is true, then $F \supset \bot, \Gamma \longrightarrow F$ must have an **I**-proof.

It only remains to be shown that, corresponding to each way of violating all the conditions, there is a sequent that has a **C**-proof but whose augmented version does not have an **I**-proof. Clearly, we need to consider only those situations in which a $\forall$-R rule is used in the **C**-proof. Now, our analysis breaks up into two parts, depending on whether or not a $\supset$-R rule appears in the **C**-proof. Suppose, first, that it does not. Then the **C**-proof must contain occurrences of both an $\vee$-L and a $\forall$-L rule and of one of the $\supset$-L, $\vee$-R and $\exists$-R rules. The following sequents have **C**-proofs respectively meeting each of these requirements:

$$\forall x \, \forall y \, (p(x) \vee q(y)), (\forall x \, p(x)) \supset (\forall y \, q(y)) \longrightarrow \forall y \, q(y),$$
$$\forall x \, \forall y \, (p(x) \vee q(y)) \longrightarrow (\forall x \, p(x)) \vee (\forall y \, q(y)),$$
$$\forall x \, \forall y \, (r(x,a) \vee r(y,b)) \longrightarrow \exists y \, \forall x \, r(x,y).$$

We assume that p and q are unary predicate symbols, that r is a binary predicate symbol and that a and b are constants in these sequents. Now, denoting the antecedent by Γ and the formula in the succedent by F in each case, it can be seen that in none of these cases does $F \supset \bot, \Gamma \longrightarrow F$ have an **I**-proof.

To complete the argument, we consider the situation in which an $\supset$-R rule appears in the **C**-proof. In this case, one of the $\supset$-L, $\vee$-R and $\exists$-R must also appear in the **C**-proof. But then consider the following sequents:

$$\forall x \, ((p(x) \supset \bot) \supset \bot) \longrightarrow \forall x \, p(x),$$
$$\longrightarrow \forall x \, (p(x) \vee (p(x) \supset s)), \text{ and}$$
$$\longrightarrow \exists y \, \forall x \, (p(y) \supset p(x)).$$

In these sequents we assume additionally that s is a proposition symbol. Now, these sequents have **C**-proofs respectively meeting each of the requirements. However, it can be easily seen that none of the augmented versions of these sequents have an **I**-proof.

□

It remains only to characterize the situations in which the augmentation suffices to reduce intuitionistic provability to uniform provability. Part of this task has already been performed in [Nad96]. In particular, it has been shown there that if $G \supset \bot, \Gamma \longrightarrow G$ has an **I**-proof in which no $\forall$-R rule is used, then this sequent also has an **O**-proof. In determining the other situations in which a similar property holds, we find it convenient to use a modified version of our calculus for constructing **I**-proofs. Towards this end, we consider the following inference rules that are parameterized by a specific formula G:

$$\frac{B, \Delta \longrightarrow F \qquad\qquad D, \Delta \longrightarrow G}{B \vee D, \Delta \longrightarrow F}\ \vee\text{-L}_G$$

$$\frac{\Delta \longrightarrow G}{\Delta \longrightarrow F}\ \mathrm{res}_G$$

We assume that B, D and F are schema variables for formulas in these rules and that Δ denotes a multiset of formulas. These rules are obviously derived ones relative to the calculus for constructing **I**-proofs in the case that Δ contains the formula $G \supset \bot$. Moreover, every use that is made of the "additional" formula $G \supset \bot$ in an **I**-proof of $G \supset \bot, \Gamma \longrightarrow G$ can actually be transformed into a use of a res_G rule. Thus, in constructing an **I**-proof of a sequent of the form $G \supset \bot, \Gamma \longrightarrow G$, we may use these rules and also make the augmentation of the antecedent implicit by strengthening the proviso on the $\exists$-L and $\forall$-R rules to disallow the use of constants appearing in G. We are actually interested in a calculus that results from the above modifications and the *removal* of the $\vee$-L rule. Let us refer to derivations constructed within this calculus as $\mathbf{I}_G$-proofs. We then have the following observation.

Lemma 13 *Let the sequent $G \supset \bot, \Gamma \longrightarrow G$ have an* **I***-proof in which no $\forall$-L or $\supset$-R rules are used. Then there is an $\mathbf{I}_G$-proof for $\Gamma \longrightarrow G$ in which no $\forall$-L and $\supset$-R rules are used.*

Proof. By an $\mathbf{I}'_G$-proof let us mean a derivation that does not contain any $\forall$-L or $\supset$-R rules and that would be an $\mathbf{I}_G$-proof except for the fact that some number of $\vee$-L rules appear in it. From the premises of the lemma, it follows that $\Gamma \longrightarrow G$ has an $\mathbf{I}'_G$-proof. Thus, it suffices to show that an $\mathbf{I}'_G$-proof of $\Gamma \longrightarrow G$ with some $\vee$-L rules in it can be transformed into one that does not contain any $\vee$-L rules. We do this by an inductive argument based on the number of $\vee$-L rules in the given $\mathbf{I}'_G$-proof. We shall assume in this argument that this derivation satisfies two additional properties: (a) the antecedent(s) of the upper sequent(s) of each left operational rule contains (contain) an occurrence of the principal formula of that rule and (b) each $\exists$-L and $\forall$-R rule uses a distinct constant all of whose occurrences are restricted to the part of the derivation appearing above that rule. We may have to introduce some contr-L rules into the original derivation to make sure that the first requirement is satisfied and a consistent renaming of some constants suffices to ensure the second property. These 'preprocessing' steps may be applied with impunity since they do not increase the number of $\vee$-L rules in the derivation and they also produce something that is itself an $\mathbf{I}'_G$-proof of the same final sequent.

An explicit argument is needed only in the case that at least one $\vee$-L rule appears somewhere in the derivation. Suppose this happens to be of the form

$$\frac{B, \Sigma \longrightarrow F \qquad\qquad D, \Sigma \longrightarrow F}{B \vee D, \Sigma \longrightarrow F}$$

We may replace this with an $\vee$-L_G rule, thereby reducing the number of occurrences of $\vee$-L rules, provided we can produce an $\mathbf{I}_G$-proof for $D, \Sigma \longrightarrow G$. It suffices, for this purpose, to exhibit an $\mathbf{I}'_G$-proof for $D, \Sigma \longrightarrow G$ with fewer $\vee$-L rules in it than in the given derivation for $\Gamma \longrightarrow G$. Such a derivation can be constructed based on the one for $\Gamma \longrightarrow G$ by retaining unchanged the portion of the latter derivation above the sequent $D, \Sigma \longrightarrow F$ and by transforming the portion below this sequent as follows:

1. Replacing all left rules that are not $\vee$-L_G rules above whose right upper sequent $D, \Sigma \longrightarrow F$ appears by the sequent $D, \Sigma \longrightarrow F'$ where F' is the succedent of the upper and lower sequents of this rule; applications of this transformation to a sequence of such rules will result in replacement by a single sequent.

2. Erasing the portion of the derivation up to and including the left upper sequent of all remaining $\vee$-L_G rules and renaming these to res_G rules.

3. Replacing each right rule with an instance of the same schema but with D, Σ as the antecedent of the upper and lower sequents.

4. Replacing the derivation above the upper sequent of an $\wedge$-R rule that is different from the one above which the sequent $D, \Sigma \longrightarrow F$ appears by one that uses the same rule schemata but with suitably modified antecedents.

Clearly, this construction eliminates at least one $\vee$-L rule from the given $\mathbf{I}'_G$-proof. However, some care is needed in ascertaining that it yields something that is indeed an $\mathbf{I}'_G$-proof. First, each $\forall$-R rule below $D, \Sigma \longrightarrow F$ in the new 'derivation' uses the same constant as is used in the derivation of $\Gamma \longrightarrow G$ and we must verify that this is acceptable. We see this to be the case by observing that this constant cannot appear in D, Σ since the given derivation does not contain occurrences of either the $\forall$-L or the $\supset$-R rules. Second, it must be possible to construct the derivation above the other upper sequent of an $\wedge$-R rule as described; in particular, all the $\forall$-R and left rules needed in this construction must be legitimate ones. Our assumptions concerning the constants used in $\exists$-L and $\forall$-R rules and the relationship between the antecedents of the upper and lower sequents of each left operational rule in the given $\mathbf{I}'_G$-proof ensure that this is the case.

□

We now relativize the notion of a uniform proof to our modified calculus. In particular, let an $\mathbf{O}_G$-proof be an $\mathbf{I}_G$-proof with the following characteristic: if there is a sequent in this proof whose succedent contains a non-atomic formula, then that sequent occurs as the lower sequent of an inference rule that introduces the top-level logical symbol of that formula. The following may then be observed:

Lemma 14 *If $\Gamma \longrightarrow G$ has a $\mathbf{I}_G$-proof in which no $\forall$-L or $\supset$-R rules appear, then it has an $\mathbf{O}_G$-proof.*

Proof. By the *nonuniformity measure* of a left rule in an $\mathbf{I}_G$-proof let us mean the count of right rules pertaining to logical symbols in the succedent of the lower sequent of the left rule that appear *above* the left rule in the derivation. Further, let the nonuniformity measure of the $\mathbf{I}_G$-proof itself be defined to be the sum of the nonuniformity measures of the left operational rules contained in it. Now, let us refer to an $\mathbf{I}_G$-proof in which no $\forall$-L or $\supset$-R rules appear as an $\mathbf{I}'_G$-proof. We claim then that if $\Gamma \longrightarrow G$ has an $\mathbf{I}'_G$-proof, then it has one whose nonuniformity measure is 0. We prove this claim by induction on the measure. We shall assume in our argument that the given derivation satisfies two additional properties: (a) the antecedent(s) of the upper sequent(s) of each left operational rule contains (contain) an occurrence of the principal formula of that rule and (b) each $\exists$-L and $\forall$-R rule uses a distinct constant all of whose occurrences are restricted to the part of the derivation appearing above that rule. We may have to apply the preprocessing steps discussed in the proof of Lemma 13 to ensure that these requirements are met, but we can do this without changing the nonuniformity measure of the derivation.

In order to establish the claim, it is sufficient to show that if $\Gamma \longrightarrow G$ has an $\mathbf{I}'_G$-proof with nonzero nonuniformity measure, then it has one with a smaller such measure. From the assumption it follows that the given derivation contains a left operational rule with right operational rules pertaining to the succedent of its lower sequent appearing above it. We focus on a left rule that is the *first* along some path in the derivation to have this characteristic. It is easily seen that a contr-L rule can be moved above any right rule in an $\mathbf{I}_G$-proof. Thus, we may assume that the left operational rule of interest appears *immediately after* the relevant right rule in the given $\mathbf{I}'_G$-proof. Our objective, now, is to show that these two rules can be reordered in a way that decreases the nonuniformity measure of the overall derivation.

A simple transformation can be used to achieve this effect when the left rule is not a $\exists$-L or the right rule is not a $\exists$-R. We illustrate this by considering one particular case: that when the left rule is an $\supset$-L and the right rule is an $\wedge$-R. In this case, the subderivation at the end has the following structure:

$$\dfrac{\Delta \longrightarrow B \qquad \dfrac{D,\Delta \longrightarrow F_1 \qquad D,\Delta \longrightarrow F_2}{D,\Delta \longrightarrow F_1 \wedge F_2}\,\wedge\text{-R}}{B \supset D,\Delta \longrightarrow F_1 \wedge F_2}\,\supset\text{-L}$$

By assumption, the nonuniformity measure of the derivation of $\Delta \longrightarrow B$ is 0. We may reuse the $\mathbf{I}'_G$-proofs of $\Delta \longrightarrow B$, $D,\Delta \longrightarrow F_1$ and $D,\Delta \longrightarrow F_2$ to produce an alternative subderivation of $B \supset D,\Delta \longrightarrow F_1 \wedge F_2$ that has the structure

$$\dfrac{\dfrac{\Delta \longrightarrow B \qquad D,\Delta \longrightarrow F_1}{B \supset D,\Delta \longrightarrow F_1}\,\supset\text{-L} \qquad \dfrac{\Delta \longrightarrow B \qquad D,\Delta \longrightarrow F_2}{B \supset D,\Delta \longrightarrow F_2}\,\supset\text{-L}}{B \supset D,\Delta \longrightarrow F_1 \wedge F_2}\,\wedge\text{-R}$$

at the end. The nonuniformity measure of the new subderivation is obviously less than that of the earlier one, and it also does not have any new occurrences of right rules that could increase the nonuniformity measure of left operational rules appearing later in the derivation. Thus, the desired effect is achieved by this transformation.

For the only remaining case, let us suppose that it occurs in a subderivation that has the structure

$$\dfrac{\dfrac{[c/x]B,\Delta \longrightarrow [t/y]D}{[c/x]B,\Delta \longrightarrow \exists y\, D}\,\exists\text{-R}}{\exists x\, B,\Delta \longrightarrow \exists y\, D}\,\exists\text{-L}$$

at the end. Now, it can be shown that $[c/x]B,\Delta \longrightarrow G$ has an $\mathbf{I}'_G$-proof of smaller nonuniformity measure than that of the one for $\Gamma \longrightarrow G$; as in the proof of Lemma 13, we construct such a derivation essentially by mimicking the structure of the given $\mathbf{I}'_G$-proof of $\Gamma \longrightarrow G$ and note that at least one occurrence of a $\exists$-L rule—the one shown above—that makes a nonzero contribution to the nonuniformity measure is eliminated in the process. From the induction hypothesis, it follows then that $[c/x]B,\Delta \longrightarrow G$ has an $\mathbf{I}'_G$-proof of zero nonuniformity measure. The proviso on a $\exists$-L rule ensures that c does not occur in B, Δ or G. Given this, we may also assume that c does not also appear in t, for, if it does, we simply rename it to a new constant c' that satisfies this additional requirement and use $[c'/x]B, \Delta \longrightarrow G$ and its corresponding derivation in the rest of the argument. Using the known derivation for $[c/x]B, \Delta \longrightarrow G$, we may restructure the $\mathbf{I}'_G$-proof for $\exists x\, B, \Delta \longrightarrow \exists y\, D$ so that it has the form

$$\dfrac{\dfrac{\dfrac{[c/x]B, \Delta \longrightarrow G}{[c/x]B, \Delta \longrightarrow [t/y]D}\,\mathrm{res}_G}{\exists x\, B, \Delta \longrightarrow [t/y]D}\,\exists\text{-L}}{\exists x\, B, \Delta \longrightarrow \exists y\, D}\,\exists\text{-R}$$

at the end. This derivation obviously has a nonuniformity measure less than that of the earlier one and using it instead also decreases the nonuniformity measure of the overall derivation.

We have thus shown that $\Gamma \longrightarrow G$ has an $\mathbf{I}'_G$-proof, and, hence, an $\mathbf{I}_G$-proof, of zero nonuniformity measure. By moving contr-L rules above any immediately preceding right rules in this derivation, we obtain a structure that would be an $\mathbf{O}_G$-proof if an additional property holds: the succedent of the lower sequent of every $\perp$-R and res_G rule is an atomic formula. This may not be true at the outset, but a simple transformation process ensures that it eventually is. To illustrate this process, suppose that there is a res_G rule in the derivation whose lower sequent has the formula $F_1 \wedge F_2$ as its succedent. Now, there must be a last sequent following this one in the derivation that has the same formula as its succedent. Suppose this sequent is $\Delta \longrightarrow F_1 \wedge F_2$. By imitating the derivation of this sequent, we obtain $\mathbf{I}_G$-proofs for $\Delta \longrightarrow F_1$ and $\Delta \longrightarrow F_2$. Further, using these $\mathbf{I}_G$-proofs, we may replace the derivation of $\Delta \longrightarrow F_1 \wedge F_2$ by one that has the structure

$$\dfrac{\Delta \longrightarrow F_1 \qquad\qquad \Delta \longrightarrow F_2}{\Delta \longrightarrow F_1 \wedge F_2}\,\wedge\text{-R}$$

at the end without changing the nonuniformity measure of the overall $\mathbf{I}_G$-proof. The virtue of this transformation is that the res_G rule in the original derivation is replaced by ones whose lower sequent have formulas with fewer logical symbols in their succedents. In a more detailed presentation, we associate with each $\mathbf{I}_G$-proof of zero nonuniformity measure a multiset of numbers that count the logical symbols in the formulas that appear as the succedents of the lower sequents of $\perp$-R and res_G rules used in the derivation. We then use the above form of argument in an induction over the multiset ordering induced by the usual ordering on natural numbers [Der82] to show that $\Gamma \longrightarrow G$ has an $\mathbf{I}_G$-proof of zero nonuniformity measure and in which the succedent of the lower sequent of every $\perp$-R and

res_G rule is atomic.

□

The following theorem states the desired relationship between intuitionistic and uniform provability.

Theorem 15 *Suppose that there is an* **I***-proof for a sequent of the form* $G \supset \bot, \Gamma \longrightarrow G$ *satisfying one of the following restrictions on rule usage:*

1. *No* $\forall$*-R rule is used.*

2. *No* $\vee$*-R or no* $\vee$*-L rule is used and, in addition, either no* $\exists$*-R rule is used or no* $\vee$*-L and* $\exists$*-L rules are used.*

3. *no* $\forall$*-L and* $\supset$*-R rules are used.*

Then there is an **O***-proof for the same sequent. Furthermore, this characterization is complete in the following sense: there is a sequent of the required form that has an* **I***-proof but no* **O***-proof corresponding to each way of violating all the restrictions on inference rule usage.*

Proof. The sufficiency of the first restriction on inference rule usage is shown in [Nad96] and that of the second restriction follows immediately from Theorem 10. The sufficiency of the third restriction is a consequence of Lemmas 13 and 14 and the observation that an $\mathbf{O}_G$-proof for $\Gamma \longrightarrow G$ can be translated into a uniform proof for $G \supset \bot, \Gamma \longrightarrow G$.

We now show the completeness of the characterization in the sense claimed. To begin with, the only situations we need to consider are those in which a $\forall$-R rule is used in the **I**-proof. Now, we may partition these situations based on whether an $\vee$-R or a $\exists$-R rule has been used. Considering the former possibility first, we note that in these situations an $\vee$-L rule and one of the $\forall$-L and $\supset$-R rules must also have been used. The following sequents have **I**-proofs respectively satisfying these requirements on rule usage:

$$(\forall y\,(r(b,y) \vee r(a,y))) \supset \bot, \forall y\,(r(a,y) \vee r(b,y)) \longrightarrow \forall y\,(r(b,y) \vee r(a,y)), \text{ and}$$
$$(\forall y\,((r(b,y) \vee r(a,y)) \supset (r(a,y) \vee r(b,y)))) \supset \bot \longrightarrow$$
$$\forall y\,((r(b,y) \vee r(a,y)) \supset (r(a,y) \vee r(b,y)));$$

we assume that r is a binary predicate symbol and a and b are constants in these sequents. It is easily seen that neither of these sequents has an **O**-proof, as is required.

To finish the proof, we have to consider those situations in which the violation of the restrictions arises from the use of a $\exists$-R rule. In these cases, one of the $\vee$-L and $\exists$-L rules and also one of the $\forall$-L and $\supset$-R rules must also have been used. We list four sequents of the required form that have **I**-proofs respectively satisfying these requirements:

$$(\forall y\, \exists x\, r(x,y)) \supset \bot, \forall y\, (r(a,y) \vee r(b,y)) \;\longrightarrow\; \forall y\, \exists x\, r(x,y),$$
$$(\forall y\, ((r(a,y) \vee r(b,y)) \supset \exists x\, r(x,y))) \supset \bot \;\longrightarrow\; \forall y\, ((r(a,y) \vee r(b,y)) \supset \exists x\, r(x,y)),$$
$$(\forall y\, \exists x\, r(x,y)) \supset \bot, \forall y\, \exists x\, r(x,y) \;\longrightarrow\; \forall y\, \exists x\, r(x,y), \text{ and}$$
$$(\forall y\, ((\exists x\, r(x,y)) \supset \exists x\, r(x,y))) \supset \bot \;\longrightarrow\; \forall y\, ((\exists x\, r(x,y)) \supset \exists x\, r(x,y)).$$

Once again, it can be verified that none of these sequents has an **O**-proof.

□

Combining Theorems 12 and 15, we see that if $\Gamma \longrightarrow G$ has a **C**-proof satisfying one of the following restrictions on rule usage, then there must also be an **O**-proof for $G \supset \bot, \Gamma \longrightarrow G$: (i) no $\forall$-R rule is used, (ii) no $\supset$-L, $\vee$-R and $\exists$-R rule is used, and (iii) no $\supset$-R and $\forall$-L rule is used. These restrictions can be recast in an obvious manner into ones on the syntax of formulas in the sequent for which a derivation is to be constructed and are, in fact, more useful in this form. Of all these conditions, the one that is most easily ensured in practice is that there be no universal quantifiers occurring negatively in the antecedent and positively in the succedent—any sequent can be transformed in one that is equivalent from the perspective of classical provability and that satisfies this additional property through the use of Herbrand functions [Sha92]. A proof procedure based on these observations is described in [Nad96] and connections with other previously presented procedures is also discussed there.

We observe, finally, that the results of this section are also relevant from the perspective of structuring proof search in intuitionistic logic. In particular, the augmentation of sequents is sound with respect to intuitionistic provability whenever the structure of the sequent ensures a coincidence with classical provability. Such an augmentation may then be used to obtain a reduction to uniform provability. One interesting situation in which this approach may be utilized is that when implications and universal quantifications do not appear positively in the succedent and negatively in the antecedent of a sequent. This situation epitomizes disjunctive logic programming and is discussed in more detail in [NL95].

6 Conclusion

We have explored the interrelationships between the notions of classical, intuitionistic and uniform provability in this paper. We have also examined the relevance of our results to proof search in classical and intuitionistic logic and to identifying logic programming languages. We believe there are other applications to our observations as well, especially to our characterization of the correspondence between classical and intuitionistic provability.

Another matter that is only partially studied here and that is worthy of further consideration is the usefulness of uniform provability in designing proof procedures for intuitionistic logic.

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