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The analytic hierarchy process: can wash criteria be ignored?

J.S. Finan[a], W.J. Hurley[b,*]

[a]Department of Political and Economic Science, Royal Military College of Canada, CP 17000, Succursale Forces, Kingston, Ont., Canada K7K 7B4

[b]Department of Business Administration, Royal Military College of Canada, CP 17000, Succursale Forces, Kingston, Ont., Canada K7K 7B4

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Abstract

We define a *wash criterion* as one where the decision-maker is indifferent among the alternatives when they are compared on that criterion. In view of the Belton–Gear example and other such anomalies associated with the analytic hierarchy process (AHP), we ask whether eliminating a wash criterion will affect the overall ranking of objects. In the case where there is only one level of criteria, the rank-order of objects is unaffected by leaving out a wash criterion. However, in the case where the wash criterion is a subcriterion, the rank order may be affected by leaving it out.

Scope and purpose

A *wash criterion* is defined as a criterion where the decision-maker is indifferent among the alternatives when they are compared on that criterion. We would like to think that the overall rank-order of objects would be unaffected in the case where the wash criterion is excluded. We give an example of an AHP hierarchy where this is not the case. In our view this presents another challenge to the AHP methodology.

Keywords: Multicriteria decision analysis; AHP; Belton–Gear

1. Introduction

In the case where the analytic hierarchy process (AHP, see Saaty [1–3]) is applied to a multicriteria decision, we define a *wash criterion* as one where the decision-maker (DM) is indifferent

* Corresponding author. Tel.: + 1-613-541-6000 (Ext: 6468); fax: + 1-613-541-6315.
E-mail address: hurley_w@rmc.ca (W.J. Hurley).

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among the alternatives when they are compared on that criterion. This type of criterion is sometimes termed a *non-discriminating* criterion. In view of the Belton–Gear [4] example and other such anomalies associated with the AHP, we examine whether eliminating a wash criterion will affect the overall ranking of objects.

In our experience, this problem has come up in a variety of contexts. One example concerned the choice of strategic direction for an integrated oil and gas firm in Western Canada. Senior management felt that it was important that the firm grow in order to remain competitive. There were three options: buy a large competitor; buy a small competitor; and the status quo (grow as fast as internally generated funds would allow). There were four criteria, and of these, one of the most important was earnings per share. But when the numbers were run, these three options produced an identical earnings per share. The analyst concluded that earnings per share was a wash criterion and eliminated it. Subsequently, the three options were assessed on the three remaining criteria.

Here is another example. Some years ago the Canadian forces were interested in purchasing an unmanned battlefield surveillance system. One of the criteria was mission survivability — the probability the vehicle would survive a well-defined average scenario. The manufacturers' glossies all estimated this survival probability to be 0.90 give or take a couple of percent depending on the manufacturer. Given that the decision exercise was an initial screening (the top three to four moved on), and there was no way to differentiate the manufacturers on this criterion, our base assumption was that mission survivability was a wash criterion.

We consider a general AHP hierarchy in the case where a DM is trying to rank-order the alternatives. We denote an AHP hierarchy where there are t levels of criteria as $H(t)$. Hence $H(1)$ is a hierarchy with only one level of criteria; $H(2)$ is a hierarchy with subcriteria. In view of the popularity of the multiplicative AHP (see Barzilai and Golany [5], Barzilai [6], and Barzilai et al. [7]), we consider two schemes for collapsing the hierarchy into an overall set of weights: one is the additive or Saaty method (SAHP); the other is the multiplicative procedure (MAHP).

We show the following results. In the case where there is only one level of criteria and the DM is perfectly consistent, the rank-order of objects is unaffected by leaving out a wash criterion regardless of which evaluation procedure is used. However, in the case where the wash criterion is a subcriterion, the rank-order may be affected by leaving it out.

2. Proof that $H(1)$ wash criteria are irrelevant

Suppose the DM begins with $n+1$ criteria indexed by the set $J = \{0, 1, \ldots, n\}$ and m choice alternatives indexed by $I = \{1, 2, \ldots, m\}$. The DM's problem is to determine a rank-order of the m alternatives. The wash criterion is indexed by 0. We index the reduced set of criteria by $\bar{J} = \{1, \ldots, n\}$. Note that, as defined, this is an $H(1)$ hierarchy.

We assume the DM is perfectly consistent. Denote the set of weights for the full criteria set J by c_j, and for the reduced criteria set by $\bar{c}_j$. Then we have that

$$c_j = (1 - c_0)\bar{c}_j \quad \text{for } j = 1, 2, \ldots, n. \tag{2.1}$$

To see this, suppose the elements of the pairwise comparison matrix for the full criteria set has elements a_{ij} and note that

$$\frac{c_i}{c_j} = a_{ij} = \frac{\bar{c}_i}{\bar{c}_j} \quad \text{for all } i, j \geqslant 1. \tag{2.2}$$

Note that, under the assumption in (2.1), the set of weights for the full criteria set sums to 1:

$$\begin{aligned}\sum_{i=0}^{m} c_i &= c_0 + \sum_{i=1}^{m} (1 - c_0)\bar{c}_i \\ &= c_0 + (1 - c_0)\sum_{i=1}^{m} \bar{c}_i \\ &= 1 \quad \text{since } \sum_{i=1}^{m} \bar{c}_i = 1.\end{aligned} \tag{2.3}$$

Let u_{ij} be the weight of alternative i measured on criterion j assuming that the SAHP evaluation procedure is used. Then $\sum_i u_{ij} = 1$ for all j. In particular, we have that

$$u_{i0} = \frac{1}{m} \quad \text{for all } i. \tag{2.4}$$

Let the SAHP overall weights of the alternatives for the full criteria set be denoted w_i^+, and for the reduced criteria set, $\bar{w}_i^+$. We now show that rank-order of alternatives is unaffected by eliminating the wash criterion.

Proposition 1. $\bar{w}_i^+ \gtreqless \bar{w}_j^+ \Leftrightarrow w_i^+ \gtreqless w_j^+$ *for all* $i, j \in I$.

Proof. The overall weight for alternative i over the reduced set is

$$\bar{w}_i^+ = \sum_k \bar{c}_k u_{ik}, \tag{2.5}$$

and for alternative j

$$\bar{w}_j^+ = \sum_k \bar{c}_k u_{jk}. \tag{2.6}$$

Taking the difference, we have

$$\bar{w}_i^+ - \bar{w}_j^+ = \sum_k \bar{c}_k u_{ik} - \sum_k \bar{c}_k u_{jk}. \tag{2.7}$$

Now examine $w_i^+ - w_j^+$:

$$w_i^+ - w_j^+ = \sum_k c_k u_{ik} - \sum_k c_k u_{jk}$$

$$
\begin{aligned}
&= \frac{1}{m}c_0 + \sum_k (1-c_0)\bar{c}_k u_{ik} - \frac{1}{m}c_0 - \sum_k (1-c_0)\bar{c}_k u_{jk} \\
&= (1-c_0)\left\{\sum_k \bar{c}_k u_{ik} - \sum_k \bar{c}_k u_{jk}\right\} \\
&= (1-c_0)\{\bar{w}_i^+ - \bar{w}_j^+\}.
\end{aligned}
\tag{2.8}
$$

Hence, the sign of $w_i^+ - w_j^+$ is the same as the sign of $\bar{w}_i^+ - \bar{w}_j^+$ and the proof is complete. □

The conclusion is that the rank-order of alternatives in an $H(1)$ hierarchy is unaffected by eliminating a wash criterion in the case where the SAHP evaluation is used. This result is easily extended to the case where there are a number of wash criteria. It also extends to the case where the MAHP evaluation procedure is used, as we now show.

Let v_{ij} be the weight of alternative i measured on criterion j assuming that the MAHP evaluation procedure is used. Then $\prod_i v_{ij} = 1$ for all j. In particular, we have

$$v_{i0} = 1 \quad \text{for all } i. \tag{2.9}$$

Let the MAHP overall weights for the full criteria set be denoted $w_i^\times$, and for the reduced criteria set, $\bar{w}_i^\times$.

Proposition 2. $\bar{w}_i^\times \gtreqless \bar{w}_j^\times \Leftrightarrow w_i^\times \gtreqless w_j^\times$ *for all* $i, j \in I$.

Proof. The overall weight for alternative i over the reduced set is

$$\bar{w}_i^\times = v_{i1}^{\bar{c}_1} v_{i2}^{\bar{c}_2} \dots v_{im}^{\bar{c}_m}. \tag{2.10}$$

Over the full set, it is

$$
\begin{aligned}
w_i^\times &= 1^{c_0} v_{i1}^{c_1} v_{i2}^{c_2} \dots v_{im}^{c_m} \\
&= v_{i1}^{(1-c_0)\bar{c}_1} v_{i2}^{(1-c_0)\bar{c}_2} \dots v_{im}^{(1-c_0)\bar{c}_m} \\
&= (v_{i1}^{\bar{c}_1} v_{i2}^{\bar{c}_2} \dots v_{im}^{\bar{c}_m})^{1-c_0} \\
&= (\bar{w}_i^\times)^{1-c_0}.
\end{aligned}
\tag{2.11}
$$

Therefore, we have

$$w_i^\times = (\bar{w}_i^\times)^{1-c_0} \tag{2.12}$$

and the result of the proposition follows directly. □

Hence, Propositions 1 and 2 demonstrate that, regardless of the evaluation scheme, the rank-order of objects in an $H(1)$ hierarchy is unaffected by ignoring a wash criterion. It is important to note that our general result that wash criteria can be ignored in a hierarchy with one level of criteria depends critically on the DM being perfectly consistent. We cannot prove the same result in the case of an imperfectly consistent DM.

3. What about wash subcriteria?

Consider the following $H(2)$ hierarchy:

Goal	G				
Main criteria	J			J'	
Main criteria weights	0.55			0.45	
Subcriteria	J_0	J_1	J_2	J'_1	J'_2
Subcriteria weights	0.6	0.2	0.2	0.5	0.5
Option A_1	0.5	0.8	0.4	0.2	0.6
Option A_2	0.5	0.2	0.6	0.8	0.4

where J_0 is the wash subcriterion. The following table gives the overall SAHP weights of A_1 and A_2 in two cases: one where J_0 is included and the other where it is not:

	With J_0	Without J_0
Weight of A_1	0.477	0.51
Weight of A_2	0.523	0.49

(3.1)

Note that, with J_0, A_2 is preferred to A_1, and in the case where J_0 is left out A_1 is preferred to A_2. Hence this simple example demonstrates that wash subcriteria cannot be ignored when the SAHP evaluation procedure is used.

But the MAHP is no better. If the MAHP is applied to this same hierarchy, we get the following weights:

	With J_0	Without J_0
Weight of A_1	0.967	1.050
Weight of A_2	1.034	0.952

(3.2)

And again note the reversal with and without the wash criterion.

4. Conclusion

Our results have the flavour of the Belton–Gear example. We would like to think that the overall rank-order of objects should be unaffected by including or excluding wash criteria. But this is not the case. While it is true that, for a hierarchy with a single level of criteria, that rank-order is unaffected, the same does not hold for hierarchies with multiple levels of criteria. Even the MAHP technique for computing the overall weights does not work in this latter case. In view of the fact that every hierarchy with multiple levels of criteria can, in principle, be modelled as a hierarchy with a single level of criteria, it must be that our methods for collapsing a hierarchy with multiple levels of criteria are incorrect. In sum, we view our results as a serious challenge to the AHP methodology.

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W.J. Hurley is a Professor in the Department of Business Administration at the Royal Military College of Canada. His research interests are decision analysis, game theory, and transportation modeling.

J.S. Finan is a Professor in the Department of Politics and Economics at the Royal Military College of Canada. His primary research interest is decision analysis and the application of decision analytic techniques to problems in international politics.