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Hougaard, Jens Leth; Tvede, Mich

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Jens Leth Hougaard
Mich Tvede

Studiestræde 6, DK-1455 Copenhagen K., Denmark
Tel. +45 35 32 30 82 - Fax +45 35 32 30 00
http://www.econ.ku.dk

Benchmark Selection: An axiomatic approach

Jens Leth Hougaard
Copenhagen Business School
and
University of Copenhagen

Mich Tvede
University of Copenhagen

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Abstract

Within a production theoretic framework, this paper considers an axiomatic approach to benchmark selection. It is shown that two simple and weak axioms; efficiency and comprehensive monotonicity characterize a natural family of benchmarks which typically becomes unique. Further axioms are added in order to obtain a unique selection.

Keywords: Benchmarking, production economics, selection

JEL-classification: C6, D2, L2, M2

Address: Jens Leth Hougaard, University of Copenhagen, Institute of Economics, Studiestraede 6, DK-1455 Copenhagen K., Denmark.
E-mail: Jens.Leth.Hougaard@econ.ku.dk
Phone: ++ 35 32 30 87, Fax: 35 32 30 00.

1 Introduction

Although, in theory, firms in various markets are supposed to operate efficiently numerous empirical studies have shown the opposite. Often such efficiency studies reveal quite a large potential for performance improvements. For example Berger & Humphrey (1997) find that the average technical efficiency score for banks is below 0.8 (i.e. suggest an average improvement potential of around 20 %) based on a survey of 122 efficiency studies.

As part of a management strategy, benchmarking appears to be a natural and often used technique to improve performance on all levels of organisation. Implicitly the management uses benchmarking when they set up production and target plans. Explicitly, some firms use a specific benchmark unit (either internal or external) in a process where they try to learn (and copy) certain elements of the performance of the peer. Such a benchmarking process consists of several steps ranging from planning to implementation (see e.g. Camp 1989) and an important part is the identification of benchmarks.

Recently, the Xerox corporation has been particularly known for its use of benchmarking techniques but also several other major companies (AT&T, Ford Motor, IBM, etc.) have been involved with benchmarking in the explicit sense: See, for example, Elmuti (1998) and Voss et al. (1997) for empirical studies.

As one of the overall conclusions from these studies Elmuti op cit. finds that;

> "*unclear and inadequately understood objectives and goals of benchmarking projects is ranked first among all the critical factors for benchmarking projects failure*" (p. 9).

The present paper provides an approach to identify benchmarks that deals directly with these problems in the benchmarking process by enforcing an explicit description of the production activity as well as of a set of fundamental principles for benchmark selection. In fact, as noted by Voss et al. op cit, one of the main advantages of benchmarking seems to be that it promotes the performance directly through identification of practices and performance goals.

Production economics (e.g. as in Debreu (1951), Koopmans (1951), Shephard (1970) and Färe (1988)) provides a well suited framework for a theoretical model of benchmark identification. Each firm (or branch, department, team etc.) is considered as a production plan consisting of a finite number of inputs and outputs which will be considered as *net outputs*[1]. The firms are operating under some specified production technology in the net output space. The term "net output" is not to be taken too literally (in a physical sense) since, for example, an index of customer satisfaction may enter the model as an output category. As such most quantifiable aspects of performance can be made to fit the model. Indeed, motivated by applications, recent research has tried to broaden the scope of the production model by incorporating value judgements etc., see e.g. Allen et al. (1997) and Thanassoulis and Allen (1998).

Based on a production economic model the present paper introduces an axiomatic approach to benchmark identification. That is, we specify a set of axioms where each axiom represents a fundamental principle for benchmark selection and show that these axioms characterize one, or possibly a small class, of selections procedures.

To specify a number of general selection principles helps to ensure a *consistent* benchmarking procedure as well as *openness* in the benchmark identification process. Openness is important for the subsequent learning process following the identification of benchmarks. If the management more or less dictatorially perform the benchmark identification some groups of interest within the firm may not have incentives to ease the subsequent improvement and adaptation process. However, if the management succeed in obtaining widespread agreement on a set of overall selection principles the specific result of the benchmark identification procedure is more likely to meet general acceptance. In other words, consistency and openness helps to induce organizational learning.

To take this point a bit further: If benchmarking is viewed as an alternative to performance evaluation in the sense of efficiency (or productivity) analysis the ability to induce organizational learning becomes a central argument in favor of the benchmark approach. Following a stream of literature on organizational learning initiated by Chris Argyris (cf. e.g. Leibenstein and Maital 1994) the main source of technical inefficiency in organizations is 'defensive behaviour' in the sense that people are often reluctant to admit that things can be done better than status quo. To evaluate organizations us-

[1]That is, outputs are indicated by positive numbers and inputs are indicated by negative numbers

ing, for example, productivity analysis seems to intensify defensive behaviour rather than to ease the learning process. Using the benchmark approach on the other hand seems in a more direct way to induce learning and adaptation processes.

Under very weak assumptions on the production technology we show that two simple and weak axioms; efficiency and comprehensive monotonicity, characterize the benchmark selection of an intuitively natural family of selection procedures, i.e. procedures that select benchmarks in proportion to ideal performance in a sense to be made precise in the following. Thus, the result of the axioms 'efficiency' and 'comprehensive monotonicity' may be construed as a generalization of the potential improvements approach introduced in Bogetoft and Hougaard (1999). It is argued that such a family of selection procedures typically result in one and the same benchmark selection. However, since multiplicity of benchmarks may be considered unfortunate for some purposes we add further axioms in order to focus on a unique selection.

Benchmarking and productivity analysis are closely related fields since both areas are concerned with the possible lack of efficiency. Hence by providing a theoretical foundation for benchmark selection our approach has immediate application to the growing field of productivity analysis where Data Envelopment Analysis (DEA) is a popular tool, see e.g. Charnes et al. (1978) and Charnes et al. (1991). It is important to stress however, that the focus is radically different: In productivity analysis the aim is to measure the improvement potential in production whereas in benchmarking the aim is to identify peer performance and secondly to learn from, and adapt to, these performance standards.

Thus, basically there is a clear distinction between productivity analysis and benchmarking with regard to their aims. Moreover, they also differ from a technical perspective since efficiency measures do not necessarily relate to benchmarks (neither implicitly nor explicitly). Indeed an efficiency measure of the relative 'size' of the dominating set does not relates to any specific benchmark unit. Furthermore, the Farrell index used in DEA relate, at least implicitly, to benchmark selections – one for the input space and one for the output space – but these selections are typically not identical.

However, if we restrict attention to either inputs or outputs it is clear that there is some link between axioms for benchmark selection and the axiomatic litterature on efficiency indices cf. e.g. Russell (1998) and Christensen et al.

(1999). That is, axioms for efficiency indices may induce axioms on benchmark selection (and vice versa) as shown in Bogetoft and Hougaard (1999) for the implicit Farrell selection.

There may be several advantages connected with the benchmark selection approach in efficiency analysis. First of all, the criteria for benchmark selection become explicit (via the axioms) and are hence open for discussion as mentioned above. In productivity analysis this is particularly important when one has to convince the management of the inefficient firms that it is 'fair' to make a comparison with the selected benchmark unit. Secondly, relevant efficiency indices can be constructed on the basis of the selected benchmark unit along the lines of Bogetoft & Hougaard op. cit.

To focus on benchmark selection in relation to efficiency measurement is a relatively recent project. So far, there have only been attempts to combine DEA with interactive decision procedures from Multiple Criteria Decision Making theory, see fx. Golany (1988), Belton and Vickers (1993) and Post and Spronk (1999).

The paper is organised as follows: Section 2 presents the model. Section 3 present the two fundamental axioms, efficiency and comprehensive monotonicity which together characterize a natural family of benchmark selection procedures. In Section 4 we characterize a unique benchmark selection procedure by adding two axioms: Affine invariance and Comprehensive independence of irrelevant production plans. Moreover, we assume the existence of a solution (with certain properties) on a small subclass $\mathcal{S}^n(I,t)$ and show that given the axioms there is a unique extension of this solution to entire class $\mathcal{S}$ of benchmark selection problems. Section 5 provides an example. Section 6 closes with final remarks.

2 The model

Consider a firm described by a production plan a which consists of m net outputs, i.e. negative coordinates are inputs and positive coordinates are outputs – that is, $a \in \mathbf{R}^m$.

Now, assume that the firm operates under a production technology $Y \subset \mathbf{R}^m$. Let,

$$D(a,Y) \;=\; \{y \in Y | y \geq a\},$$

be the set of feasible production plans in Y which (weakly) dominates the observed production plan a, i.e. the weak dominance set.

Let $\mathcal{Y}$ denote the class of technologies for which $D(a, Y)$ is compact for all $a \in \mathbf{R}^m$. This class of technologies is very broad and includes for example Cobb-Douglas, Leontief and nonconvex technologies.

A pair (a, Y), where $a \in Y$ and $Y \in \mathcal{Y}$ is called a *benchmark selection problem.* Denote by $\mathcal{S}$ the set of such problems.

Let a map $f : \mathcal{S} \to \mathbf{R}^m$ with $f(a, Y) \in Y$ be a *benchmark selection* for production plan a relative to the technology $Y \in \mathcal{Y}$ and let $E(Y)$ be the set of strictly efficient production plans in Y, i.e. $E(Y) = \{y \in Y | D(y, Y) = \{y\}\}$.

In order to ease the exposition a few additional concepts need to be defined. Consider a given production plan a and its weak dominance set $D(a, Y)$. Let $M_j(a, Y)$ be the the maximal value of the j'th coordinate in $D(a, Y)$, i.e.

$$M_j(a, Y) = \max_{y \in D(a,Y)} \mathrm{pr}_j y,$$

where pr_j is the projection on the j'th coordinate. Denote by $M(a, Y) = (M_1(a, Y), \ldots, M_m(a, Y))$ the *ideal* production plan relative to a. Clearly, if a is strictly efficient, a and $M(a, Y)$ are identical – but typically $M(a, Y) \notin Y$.

Finally, let $d(a, Y)$ be the line through a and $M(a, Y)$, i.e. the diagonal of (a, Y)

$$d(a, Y) = \{x \in \mathbf{R}^m | x = a + t(M(a, Y) - a) \text{ for some } t \in \mathbf{R}\}.$$

Moreover, let the *reference production plan* $s(a, Y) = E((D(a, Y) - \mathbf{R}^m_+) \cap d(a, Y))$ be the maximal element in the intersection of the free disposal hull of the weak dominance set, $D(a, Y)$, and the diagonal, $d(a, Y)$. Figure 1 provides an illustration of the different concepts of the production model in case of two net outputs (one input and one output). The ideal point $M(a, Y)$ is found by maximizing output while keeping the amount of input fixed and by maximizing (negative) input while keeping the amount of output fixed. The diagonal $d(a, Y)$ is the line through a and $M(a, Y)$ and the reference point $s(a, Y)$ is seen to be the intersection between the free disposal hull of Y and the diagonal $d(a, Y)$.

In particular, let $\mathcal{S}^* \subset \mathcal{S}$ denote the class of selection problems for which the reference production plan $s(a, Y)$ is strictly efficient, i.e.

$$\mathcal{S}^* = \{(a, Y) \in \mathcal{S} | s(a, Y) \in E(Y)\}.$$

Figure 1. *Illustration of the various concepts of the production model in case of two net outputs.*

Note that even for convex selection problems with at least three net outputs, $s(a, Y)$ need not be efficient, cf. Roth (1980).

A "well behaved" subclass of $\mathcal{S}^*$ was examined in Bogetoft & Hougaard (1999) who characterize axiomatically a so-called potential improvements selection for convex input correspondences.

3 A set of suitable properties for benchmark selection.

As mentioned in the introduction there may be several advantages connected with the axiomatic approach to benchmark selection. In this section two simple axioms will be introduced and discussed.

Firms ought to strive toward efficient production as this makes additional resources available compared to inefficient production by improved input utilization. Therefore it is natural to demand that a selected benchmark unit must be strictly efficient.

Axiom 1 *(Strict efficiency)*

$$f(a,Y) \in E(Y)$$

for all $(a,Y) \in \mathcal{S}$.

If for some reason inefficient benchmark units are preferred, the production technology seems to be misspecified: Inefficient benchmark units must have some unmodelled qualities compared to efficient units – otherwise they should not be benchmark units – and these qualities should be included in the description of the technology, perhaps as additional net outputs.

Moreover, there is no reason to limit strict efficiency to the comparison between a given production plan and its benchmark, it should also be applied to comparisons between benchmarks (given the production plan): If more productions possibilities become known relative to a given production plan then the new benchmark should (weakly) dominate the old benchmark. That is, some kind of monotonicity condition seems appropriate when selecting benchmarks.

However, some caution is necessary when choosing the specific version of the monotonicity condition. Indeed, let for example $a = (-5, 4)$ and

$$Y = \{y \in \mathbf{R}^2 | y \in \{t(-1,1)\} - \mathbf{R}^2_+ \text{ for } t \in \mathbf{R}^2_+\}.$$

Now, let $Y' = Y \cup \{(-5, 8)\}$ and $Y'' = Y \cup \{(-1, 4)\}$ implying that $D(a,Y) \subset D(a,Y'), D(a,Y'')$ because $Y \subset Y', Y''$. Comparing the three selection problems $(a,Y), (a,Y')$ and (a,Y'') it seems reasonable to suggest that their solutions should satisfy $f_1(a,Y') < f_1(a,Y)$ and $f_2(a,Y') > f_2(a,Y)$ and $f_1(a,Y'') > f_1(a,Y)$ and $f_2(a,Y'') < f_2(a,Y)$ in order to reflect the differences in improvement potential. Hence, monotonicity is problematic in this case because in each of the pairwise comparisions one of the coordinates of the benchmarks $f(a,Y')$ and $f(a,Y'')$ becomes smaller than in the benchmark $f(a,Y)$.

Now, the most striking difference between the three selection problems above is that their diagonals differ reflecting their different relative improvement potentials. One way to incorporate this aspect as well as securing a weak axiom is to restrict monotonicity to selection problems having the same diagonal as in comprehensive monotonicity below.

Axiom 2 *(Comprehensive monotonicity)*

$$f(a, Y') \leq f(a, Y)$$

for all $(a, Y') \in \mathcal{S}^*$ *with* $d(a, Y') = d(a, Y)$ *and* $D(a, Y') \subset D(a, Y) - \mathbf{R}^m_+$.

Comprehensive monotonicity states the following: Consider a selection problem (a, Y). The benchmark of this problem must weakly dominate the benchmark of any selection problem with efficient reference point and common diagonal where the weak dominance set is included in the free disposal hull of the weak dominance set of (a, Y).

Now, by the axioms above it is possible to prove the following result:

Theorem 1 *A benchmark selection* $f : \mathcal{S} \to \mathbf{R}^m$ *satisfies strict efficiency (Axiom 1) and comprehensive monotonicity (Axiom 2) if and only if*

$$f(a, Y) \in \{y \in E(Y) | y \geq s(a, Y)\}$$

for all $(a, Y) \in \mathcal{S}$.

Proof: Let $f : \mathcal{S} \to \mathbf{R}^m$ be such that

$$f(a, Y) \in \{y \in E(Y) | y \geq s(a, Y)\},$$

then Axiom 1 is satisfied by definition. Moreover, if $(a, Y') \in \mathcal{S}^*$ with $D(a, Y') \subset D(a, Y) - R^m_+$ and $d(a, Y') = d(a, Y)$ then $s(a, Y') \leq s(a, Y)$ and $f(a, Y') = s(a, Y')$ because $s(a, Y') = \{y \in E(Y') | y \geq s(a, Y')\}$. Therefore $f(a, Y) \geq f(a, Y')$ because $f(a, Y') = s(a, Y') \leq s(a, Y) \leq f(a, Y)$, i.e. Axiom 2 is satisfied.

Now, suppose that $f : \mathcal{S} \to \mathbf{R}^m$ satisfies Axiom 1 and Axiom 2. Let $Y' = \{a, s(a, Y)\}$ then $f(a, Y') = s(a, Y)$ by Axiom 1 (efficiency) and $D(a, Y') \subset D(a, Y) - \mathbf{R}^m_+$ with $d(a, Y') = d(a, Y)$. Therefore $f(a, Y') \leq f(a, Y)$ by Axiom 2 because $(a, Y') \in \mathcal{S}^*$. Hence, $f(a, Y') = s(a, Y) \leq f(a, Y)$. $\square$

According to Theorem 1, the requirements for benchmark selection represented by efficiency and comprehensive monotonicity implies that benchmark selections are efficient production plans which dominate the reference production plan. Figure 2 illustrates a specific example involving two net outputs. Given the inefficient production plan a and the technology Y, the benchmark

Figure 2. *The possibility of multiple benchmarks.*

selections that satisfy Axiom 1 and 2 are given by the 'corner points' $f(a, Y)$ and $f'(a, Y)$ using Theorem 1.

An important consequence of the above axiomatic system is that if the weak dominance sets, are identical for two benchmark selection problems, (a, Y) and (a, Y') (i.e. $D(a, Y) = D(a, Y')$), then these problems select the same benchmark. In other words, the shape of the production set outside the weak dominance set does not influence the benchmark selection. Note, however, that two incomparable production plans a and b where neither $b \in D(a, Y)$ nor $a \in D(b, Y)$, may be compared indirectly under an assumption about constant returns to scale by scaling them such that they become comparable.

We consider the fact that benchmarks are selected from the weak dominance set only as a force of the axiomatic system since it is in line with production economics stating that without further information (such as for example a set of prices) dominance is the only principle left to rank production plans in the net output space[2]. In the literature on efficiency measurement such an 'axiom' of dominance is also well-known. For example, Hougaard and Keiding (1998) explicitly use a dominance axiom in their axiomatic system characterizing a large family of efficiency indices which includes the familiar Farrell and Färe-Lovell indices.

4 Unique benchmark selection

It might be considered unfortunate that Theorem 1 characterizes a family of benchmarks rather than a unique selection even though intuition suggests that multiplicity of benchmarks occurs very rarely. For example, in Figure 2, it can be noted that a small pertubation moving a away from the diagonal $d(a, Y)$ results in uniqueness. Moreover, consider the following example:

Example Let $m = 2$ and consider the problem $(a, Y) \in \mathcal{S}$ with $a = (-5, 7)$ and

$$\begin{aligned} Y &= \operatorname{conv}\{(-5,7),(-4,8),(-1,7),(-1,10+\varepsilon)\} \\ &\quad \cup \operatorname{conv}\{(-5,7),(-4,8),(-5,11),(-2,11)\} \end{aligned}$$

where $\varepsilon \geq 0$. Then for $\varepsilon = 0$ both $(-1, 10)$ and $(-2, 11)$ are possible benchmarks according to Theorem 1, while $(-1, 10 + \varepsilon)$ is the unique benchmark for $\varepsilon > 0$. □

Though multiplicity of benchmarks is a relatively rare phenomenon (assuming that axioms 1 and 2 are satisfied) it is nevertheless a theoretical challenge to obtain uniqess. Thus, this section will consider additional axioms which lead to a unique benchmark selection. The general idea is to postulate the existence of a unique selection with certain properties on a very small class of selection problems (to be denoted $\mathcal{S}^n(I, t)$) and then show

[2]Clearly dominance does not have to be confined to the cone $\mathbf{R}^m_+$. We may consider any (larger) family of cones or other sets including $\mathbf{R}^m_+$. Such an approach is easily included in our framework. However the notion of dominance then loses its economic relevance. Assume that some decision maker chooses benchmarks according to a utility function u on the set of production plans. If u is unknown (as is usually the case) then it is well-known that if dominance is confined to the positive cone $\mathbf{R}^m_+$ each utility function u obtains its maximum among the set of undominated production plans and each undominated plan is optimum for some utility function u.

that, given certain axioms of consistency, this selection extends uniquely to the entire domain of selection problems $\mathcal{S}$.

To be more specific we add an axiom of affine invariance stating that the choice of benchmark shall not depend on the units of measurement for the net outputs. Moreover, we also add an axiom of independence of irrelevant production plans stating that if a given plan is chosen as benchmark for some selection problem then if this plan is also feasible in a smaller (less productive) selection problem it must be chosen as benchmark for this 'small' problem too.

Formally, we impose consistency over the set of selection problems by,

Axiom 3 *(Affine invariance)*

$$f(h(a), h(Y)) \;=\; h(f(a, Y)),$$

for any $h : \mathbf{R}^m \to \mathbf{R}^m$ *with* $h(y) = (\alpha_1 y_1 + \beta_1, \ldots, \alpha_m y_m + \beta_m)$ *where* $\alpha > 0$.

and

Axiom 4 *(Comprehensive independence of irrelevant production plans)*

$$f(a, Y') \;=\; f(a, Y)$$

for all selection problems with $M(a, Y') = M(a, Y)$, $D(a, Y') \subset D(a, Y) - \mathbf{R}^m_+$ *and* $f(a, Y) \in D(a, Y')$.

Note that independence is only required for the class of selection problems having the same ideal production plan and the difference between the 'small' and the 'large' problem is made w.r.t. the comprehensive hull of the dominance set.

However, axioms of consistency are not enough. There will be selection problems where we are forced to rank the net output dimensions according to their importance in order to obtain uniqueness. If, for example, we end up in a symmetric situation as in the example above (for $\varepsilon = 0$) we are forced to prefer improvements in one net output to improvements in the other.

Clearly, such a choice cannot be founded on any overall theoretical argument but must be *ad hoc*.[3]

As a consequence we are forced to assume the existence of a solution on some class of selection problems and clearly, this class ought to be as small as possible. Moreover, such a selection must be compatible with the other axioms and hence some properties need to be imposed. Imagine, for example, that a decision maker is confronted with a series of particularly simple selection problems (which are normalized by affine invariance and consists of the production plan, the unit vectors and two additional production plans with identical minimal coordinate) and then is asked to choose a benchmark for each problem. This selection cannot be made at random since at least it has to comply with axioms 1 and 2. Moreover, we shall assume that the preference induced by the selection satifies weak transitivity and continuity requirements in order to enable extension of the selection to the entire domain. Note that the lexicographic selection procedure is a straightforward example of a selection which satisfies axiom 1 and 2 and secures a unique selection of benchmark. However, the induced preference is not continuous so a weaker form of continuity is required. Lemma 1 in the appendix establishes such weak conditions under which a preference relation has maximal elements.

Formally, let $\mathcal{S}^n(I) \subset \mathcal{S}$ be defined by

$$\mathcal{S}^n(I) = \{(a, Y) \in \mathcal{S} | a = (0, \ldots, 0) \text{ and } \sum_{i \in I} e_i = M(a, Y)\},$$

where $I \subset \{1, \ldots, m\}$ and e_i is the i'th unit vector. Next, let $L^n(I, t) \subset [0, 1]^m$ be defined by

$$L^n(I, t) = \{y \in [0, 1]^m | \min_{i \in I} y_i = t \text{ and } y_j = 0 \text{ for all } j \notin I\},$$

where $t \in [0, 1]$. Finally, let $\mathcal{S}^n(I, t) \subset \mathcal{S}^n(I)$ be defined by

$$\mathcal{S}^n(I, t) = \{(a, Y) \in \mathcal{S}^n(I) | \exists\, y, y' \in L(I, t) : Y = \{a, (e_i)_{i \in I}, y, y'\}\}.$$

[3]Suppose, for example, that the model consists of one input producing one output and we end up in a symmetric situation like above. In this case one can either select the benchmark which minimizes the use of input or the benchmark which maximizes the production of output. If the overall aim of the benchmarking process is the improve the firms input utilization (cost minization) it seems natural to choose a benchmark guided by the result of an input minimization. On the other hand, if the overall aim is to increase production (revenue maximization) the benchmark ought to be chosen using output maximization.

Now, the following result can be obtained.

Theorem 2 *Suppose that there exists a family of selections, $g(a, Y; I, t)$: $\mathcal{S}^n(I, t) \to \mathbf{R}^m$, which satisfy efficiency and dominate the reference production plan $s(a, Y)$ such that the induced preferences on the $L(I, t)$'s,*

$$\begin{aligned} V(x; I, t) &= \{y \in L(I, t) | \exists (a, Y) \in \mathcal{S}^n(I, t) : \\ & \qquad x, y \in Y, y \neq x \text{ and } y = g(a, Y; I, t)\}, \end{aligned}$$

satisfies the assumptions of Lemma 1 (in the appendix). Then there exists a unique benchmark selection, $f : \mathcal{S} \to \mathbf{R}^m$, which satisfy axioms 1-4 such that $f(a, Y) = g(a, Y; I, t)$ for all $(a, Y) \in \mathcal{S}^n(I, t)$.

Proof: Firstly, we propose a selection, secondly we note that it satisfies axioms 1-4 and thirdly, we show that no other selection satisfies axioms 1-4. Due to Axiom 3 we restrict attention to the $\mathcal{S}^n(I)$'s.

Fix $I \subset \{1, \ldots, m\}$. Consider the correspondence $f : \mathcal{S}^n(I) \to 2^{\mathbf{R}^m}$ defined by

$$f(a, Y) = \{y \in Y \cap L(I, t) | V(y; I, t) \cap (Y \cap L(I, t)) = \emptyset\},$$

where $s(a, Y) = tM(a, Y)$. For a problem $(a, Y) \in \mathcal{S}^n(I)$ let

$$Y' = (Y \cap L(I, t)) \cup \{a, (e_i)_{i \in I}\}.$$

By Lemma 1 there exists $y \in Y'$ such that $V(y; I, t) \cap (Y' \cap L(I, t)) = \emptyset$ therefore $V(y; I, t) \cap (Y \cap L(I, t)) = \emptyset$ because $V(y; I, t) \subset L(I, t)$. Thus, $f(a, Y)$ is non-empty for all $(a, Y) \in \mathcal{S}^n(I)$. Now, suppose that $V(y'; I, t) = V(y; I, t) = \emptyset$ for $y, y' \in Y \cap L(I, t)$, and consider $Y' = \{a, e_1, \ldots, e_m, y, y'\}$, then $(a, Y') \in \mathcal{S}^n(I, t)$. By assumption (a, Y') has a unique selection, $g(a, Y'; I, t)$. Therefore $y' = y = g(a, Y'; I, t)$ is the unique selection from Y. Thus, $f(a, Y)$ contains at most one element. Consequently, $f : \mathcal{S}^n(I) \to 2^{\mathbf{R}^m}$ is a benchmark selection.

Clearly, by construction axioms 1,2 and 4 are satisfied by the selection defined above.

Now, suppose that a selection $h : \mathcal{S}^n(I) \to \mathbf{R}^m$ satisfies axioms 1-4 such that $h(a, Y) = g(a, Y; I, t)$ for all $(a, Y) \in S^n(I, t)$. Consider (a, Y') where

$Y' = \{a, e_1, \ldots, e_m, f(a,Y), h(a,Y)\}$ then $(a,Y') \in \mathcal{S}^n(I,t)$ for $s(a,Y) = tM(a,Y)$. Therefore, $f(a,Y') = h(a,Y') = g(a,Y';I,t)$ by construction and

$$f(a,Y) = f(a,Y') = h(a,Y') = h(a,Y),$$

by Axiom 4. Hence, $f : S^n(I) \to 2^{\mathbf{R}^m}$ is unique. $\square$

5 An example

This section will provide a short illustration of how the result of Theorem 1 can be used.

Imagine a set of n observed production plans, $\{a^1, \ldots, a^j, \ldots, a^n\}$, where each dominated plan has to find its benchmark within a technology estimated on the basis of the data points themselves. Assume, for example, that the production technology is estimated as the comprehensive hull of the data points (the FDH-technology) or the convex cone of the data points (the CRS-technology)[4].

Fix the production plan to be benchmarked, a^0. To find the ideal production plan $M(a^0, Y)$ one has to solve m LP-problems (one for each net output) of the form:

$$\max_{\lambda, \theta_i} \quad \theta_i$$

$$\text{s.t.} \quad \begin{array}{rcl} \sum_{j=1}^{n} \lambda^j a_i^j & \geq & \theta_i \\ \sum_{j=1}^{n} \lambda^j a_k^j & \geq & a_k^0 \text{ for all } k \neq i \\ \lambda & \geq & 0 \text{ (CRS)} \\ \sum_{j=1}^{n} \lambda^j & = & 1 \text{ and } \lambda^j \in \{0,1\} \text{ for all } j \text{ (FDH)} \end{array}$$

yielding $M(a^0, Y) = (\theta_1, \ldots, \theta_m)$.

Secondly, in order to find the reference production plan $s(a^0, Y)$ one has to solve the following LP-problem:

[4]FDH and CRS is short for Free Disposal Hull and Constant Returns to Scale respectively, see e.g. Färe et al. (1994).

$$
\begin{array}{llcl}
\max_{\lambda,\beta} & \beta & & \\
& \sum_{j=1}^{n} \lambda^j a^j & \geq & a^0 + \beta(M(a^0, Y) - a^0) \\
\text{s.t.} & \lambda & \geq & 0 \text{ (CRS)} \\
& \sum_{j=1}^{n} \lambda^j & = & 1 \text{ and } \lambda^j \in \{0,1\} \text{ for all } j \text{ (FDH)}
\end{array}
$$

If $s(a^0, Y) = a^0 + \beta(M(a^0, Y) - a^0)$ is efficient we have a unique benchmark. Otherwise, all efficient points which dominate $s(a^0, Y)$ are benchmarks. This can easily be checked given the technological assumptions.

6 Final remarks

In the present paper the choice of benchmark has been axiomatized. The proposed axiomatization was carried out in two steps: First two natural axioms were imposed and it was shown that these axioms restricted the set of feasible benchmarks considerably: Only efficient production plans dominating the reference production plan are feasible. Secondly two additional axioms were imposed and the existence of a solution with certain properties on a very small domain of selection problems was assumed. It was then shown that such a solution uniquely extends to the entire domain given the additional axioms. Lexicographic maxmin (on the set of normalized selection problems) is an example of a selection procedure which complies with all axioms.

As mentioned in the introduction there is a link between the axiomatic approach to benchmarking and the axiomatic approach to (some) efficiency indices although the two approaches are in no way dual. For example, it is clear that the axiom of efficiency (Axiom 1) of the present approach in some sense resembles the axiom of 'indication' from the axiomatic literature on efficiency indices (see e.g. Russell 1998). Moreover, to consider monotonicity properties w.r.t. the technology has also been used to characterize both the Farrell and the Färe-Lovell efficiency index in Christensen et al. (1999). However, the precise relation between the axiomatic approach to benchmark selection and the axiomatic approach to efficiency indices is left for future research.

Appendix

In order to establish Theorem 2 the following lemma is needed.

Lemma 1 *Suppose that X is a compact set and that $U : X \to 2^X$ where $x \notin U(x)$ and $U(x) = \cup_{n \in \mathbf{N}} G_n(x)$ ($\mathbf{N}$ being the set of natural numbers) for all $x \in X$ satisfies*

- *if $x_2 \in G_n(x_1), \ldots, x_m \in G_n(x_{m-1})$ then $x_1 \notin G_n(x_m)$ for all $x_1, \ldots, x_m$*
- *$G_{n+1}(x) \subset \{y | G_n(y) \subset G_n(x)\}$*
- *$G_1^{-1}(x)$ is open and $G_{n+1}^{-1}(x)$ is open in $\{y | G_n(y) = G_n(x)\}$*

Then $\{x \in X | U(x) = \emptyset\}$ is nonempty and compact.

Proof: Let

$$M_n \;=\; \{x \in X | \cup_{j=1}^{n} G_j(x) = \emptyset\}$$

for all $n \in \mathbf{N}$ then $M_{n+1} \subset M_n$ for all $n \in \mathbf{N}$. Moreover, M_1 is nonempty and compact according to Bergstrom (1975) and Walker (1977) and M_{n+1} is nonempty and compact provided that M_n is nonempty and compact because M_n has the same properties as X and G_{n+1} has the same properties on M_n as G_1 has on X. Therefore

$$\{x \in X | U(x) = \emptyset\} \;=\; \bigcap_{n \in \mathbf{N}} M_n$$

is nonempty and compact because M_n is nonempty and compact and $M_{n+1} \subset M_n$ for all $n \in \mathbf{N}$ and X has the finite intersection property. $\square$

The first assumption of Lemma 1 ensures that the preference relation is acyclic while the two subsequent assumptions ensure some weak form of continuity. Indeed, the usual continuity assumption, $V^{-1}(x)$ is open, is too strong to ensure that there exists exactly one point such that $V(x) \cap X' = \emptyset$ for all $X' \subset X$ where X' is compact. Note that the lexicographic preference satisfies the assumptions of Lemma 1: The lexicographic preference is defined as $y \in V(x)$ if and only if $y_1 > x_1$, $y_1 = x_1$ and $y_2 > x_2$, ... or $y_1 = x_1$, ..., $y_{n-1} = x_{n-1}$ and $y_n > x_n$. Hence, let $V(x) = \cup_{n \in \mathbf{N}} G_n(x)$ where

$$G_n(x) \;=\; \{y \in \mathbf{R}^m | y_1 = x_1, \ldots, y_{n-1} = x_{n-1} \text{ and } y_n > x_n\}$$

for all $n \leq m$ and $G_n(x) = \emptyset$ for all $n \geq m+1$, then $V(\cdot)$ is the lexicographic preference on $\mathbf{R}^m$. Hence, it is clear that the lexicographic preference relation is acyclic and

$$\begin{aligned} G_{n+1}(x) &\subset \{y \in \mathbf{R}^m | G_n(y) \subset G_n(x)\} \\ &= \{y \in \mathbf{R}^m | y_1 = x_1, \ldots, y_{n-1} = x_{n-1} \text{ and } y_n \geq x_n\} \end{aligned}$$

for $n \leq m$. Also, $G_1^{-1}(x)$ is open and $G_{n+1}^{-1}(x)$ is open in $\{y \in X | G_n(y) = G_n(x)\}$.

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