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EXPLICIT EVALUATION OF CERTAIN DEFINITE INTEGRALS INVOLVING POWERS OF LOGARITHMS

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1. INTRODUCTION

In recent years, considerable progress has been made in evaluating algorithmically, by symbolic computation on a computer, several classes of indefinite integrals. On the other hand, many fewer procedures seem to be available for the algorithmic computation of definite integrals. This is hardly surprising in view of the many different methods which are required to evaluate such integrals, some of them needing special tricks and a certain experience. There are, however, some classes of definite integrals which are well-suited to formal evaluation, if not by hand -- because of the complexity of the resulting (elementary) expressions -- at least by symbolic computation. In particular, certain integrals which can be represented as repeated derivatives of products or quotients of gamma functions are of this kind. We shall present a few of these here. Their derivation can be found in the corresponding references.

2. AN AUXILIARY PROCEDURE

For the evaluation of some of the integrals we shall need the coefficients of the power series of a certain product of gamma functions multiplied by the exponential function. These coefficients can be computed by symbolic algebra. We start from the power series expansion [Gradshteyn, Ryzhik (1980), No. 8.342]

$$h(a;\alpha_1,\alpha_2,\alpha_3) = \ln\left[a^{-x}\,\frac{\Gamma(1+\alpha_1 x)\Gamma(1+\alpha_2 x)}{\Gamma(1+\alpha_3 x)}\right] =$$

$$-\left[\ln a + \gamma(\alpha_1+\alpha_2-\alpha_3)\right] + \sum_{k=2}^{\infty} (-1)^k \frac{1}{k}\,\zeta(k)(\alpha_1^k+\alpha_2^k-\alpha_3^k)\;x^k \tag{2.1}$$

$$(a > 0,\ |x| < 1/\max\,(|\alpha_1|,\ |\alpha_2|,\ |\alpha_3|))\ ,$$

where $\gamma = 0.57721\ldots$ is Euler's constant, and $\zeta(k)$ is the Riemann zeta function.

In order to obtain the coefficients of the power series for

$$H(a;\alpha_1,\alpha_2,\alpha_3) = \exp h(a;\alpha_1,\alpha_2,\alpha_3) = \sum_{k=2}^{\infty} C_k(a;\alpha_1,\alpha_2,\alpha_3)\;x^k \tag{2.2}$$

we apply a recurrence procedure given by Knuth (1969), p. 561 and elsewhere, and obtain

$$C_0(a;\alpha_1,\alpha_2,\alpha_3) = 1$$

$$C_k(a;\alpha_1,\alpha_2,\alpha_3) = \frac{1}{k}\sum_{\kappa=1}^{k} c_\kappa(a;\alpha_1,\alpha_2,\alpha_3)\;C_{k-\kappa}(a;\alpha_1,\alpha_2,\alpha_3) \qquad (\kappa > 1) \tag{2.3}$$

where, from (2.1),

$$c_\kappa(a;\alpha_1,\alpha_2,\alpha_3) = \begin{cases} -\ln a - \gamma(\alpha_1+\alpha_2-\alpha_3) & (\kappa = 1) \\ (-1)^\kappa(\alpha_1^\kappa + \alpha_2^\kappa - \alpha_3^\kappa)\zeta(\kappa) & (\kappa > 1) \ . \end{cases}$$

3. THE INTEGRALS

In what follows, $C_k(a;\alpha_1,\alpha_2,\alpha_3)$ is defined by (2.3); B_j and E_j are the Bernoulli and Euler numbers, respectively, as defined in Abramowitz, Stegun (1966), No. 23.1.1-3. $S_k^{(j)}$ are the Stirling numbers of the first kind, defined by recurrence, or directly by Schlömilch's representation (Abramowitz, Stegun (1966), No. 24.1.3, Comtet (1974), p. 216).

We also introduce the abbreviation

$$\hat{S}(p,q) = \sum_{j=p}^{q} (-1)^j \, S_q^{(j)} \binom{j}{p} \, 2^{p-j} \ .$$

We now list nine integrals which have recently been treated in detail in Kölbig (1982) [formula (3.1)], (1983a) (3.2), (1983b) (3.3) and (3.4), (1985) (3.5) to (3.9):

$$\begin{aligned} &\int_0^1 t^{-1} \ln^n t \ln^p (1-t)\, dt \\ &= -2^{n+p+1} \int_0^{\pi/2} \cot t \ln^n \sin t \ln^p \cos t \, dt \\ &= n!p! \sum_{\nu=0}^{n} C_{n-\nu}(1;1,0,0) \sum_{\rho=0}^{p} \binom{\nu+\rho+1}{\rho} C_{p-\rho}(1;1,0,0) C_{\nu+\rho+1}(1;0,0,1) \ , \end{aligned} \tag{3.1}$$

$$(n = 0,1,2,\ldots,\ p = 1,2,3,\ldots) \ ,$$

$$\begin{aligned} &\int_0^1 t^{-\frac{1}{2}} \ln^n t \ln^p (1-t)\, dt \\ &= 2^{n+p+1} \int_0^{\pi/2} \ln^n \sin t \ln^p \cos t \, dt \\ &= \pi n!p! \sum_{\nu=0}^{n} C_{n-\nu}(4;2,0,1) \sum_{\rho=0}^{p} \binom{\nu+\rho}{\rho} C_{p-\rho}(4;2,0,1) C_{\nu+\rho}(1;0,0,1) \ , \end{aligned} \tag{3.2}$$

$$(n,p = 0,1,2,\ldots) \ ,$$

$$\int_0^\infty e^{-\mu t}\ t^n\ \ln^m t\ dt$$

$$= 2^{m+1}\int_0^\infty e^{-\mu t^2}\ t^{2n+1}\ \ln^m t\ dt \tag{3.3}$$

$$= (-1)^n\ \mu^{-n-1}\ m!\ \sum_{\rho=0}^{\min(m,n)} (-1)^\rho\ S_{n+1}^{(\rho+1)} C_{m-\rho}(\mu;1,0,0)\ ,$$

$$(\mathrm{Re}\ \mu > 0;\ n,m = 0,1,2,\ldots)\ ,$$

$$\int_0^\infty e^{-\mu t}\ t^{n-\frac{1}{2}}\ \ln^m t\ dt$$

$$= 2^{m+1}\int_0^\infty e^{-\mu t^2}\ t^{2n}\ \ln^m t\ dt = \tag{3.4}$$

$$= \sqrt{\pi}(-1)^n\mu^{-n-\frac{1}{2}}m!\ \sum_{\rho=0}^{\min(m,n)} \hat{S}(\rho,n)C_{m-\rho}(4\mu;2,0,1)\ ,$$

$$(\mathrm{Re}\ \mu > 0;\ n,m = 0,1,2,\ldots)\ ,$$

$$\int_0^\infty \frac{x^n\ \ln^m x}{(1+\beta x)^\ell}\ dx$$

$$= \frac{(-1)^{\ell+m}}{\beta^{n+1}}\ \frac{m!}{(\ell-1)!}\ \sum_{m_1=0}^{\min(m,n)} S_{n+1}^{(m_1+1)} \sum_{m_2=0}^{\min(m-m_1,\ \ell-n-2)} (-1)^{m_2}\ S_{\ell-n-1}^{(m_2+1)}$$

$$\sum_{m_3=0}^{m-m_1-m_2} \frac{|2^{m_3}-2|\,|B_{m_3}|}{m_3!(m-m_1-m_2-m_3)!}\ \pi^{m_3}\ \ln^{m-m_1-m_2-m_3}\beta\ , \tag{3.5}$$

$$(|\arg\beta| < \pi,\ n \geq 0,\ \ell \geq n+2,\ m \geq 0)\ ,$$

$$\int_0^\infty \frac{x^n \ln^m x}{(1-x)^\ell}\, dx \tag{3.6}$$

$$= (-1)^{m+n+\ell} \frac{m!}{(\ell-1)!} \sum_{m_1=0}^{n} S_{n+1}^{(m_1+1)} \sum_{m_2=0}^{\ell-n-2} (-1)^{m_2} S_{\ell-n-1}^{(m_2+1)} \frac{(2\pi)^{m-m_1-m_2} |B_{m-m_1-m_2}|}{(m-m_1-m_2)!} ,$$

$$(n \geq 0,\ \ell \geq n+2,\ m \geq \ell-1) ,$$

$$\int_0^\infty \frac{x^{n-\frac{1}{2}} \ln^m x}{(1+\beta x)^\ell}\, dx$$

$$= \frac{(-1)^{\ell+m+1}}{\beta^{n+\frac{1}{2}}} \frac{m!}{(\ell-1)!} \sum_{m_1=0}^{\min(m,n)} (-1)^{m_1} \hat{S}(m_1,n) \sum_{m_2=0}^{\min(m-m_1,\ \ell-n-1)} \hat{S}(m_2,\ell-n-1)$$

$$\sum_{m_3=0}^{m-m_1-m_2} \frac{|E_{m_3}| \pi^{m_3+1}}{m_3!(m-m_1-m_2-m_3)!} \ln^{m-m_1-m_2-m_3} \beta , \tag{3.7}$$

$$(|\arg \beta < \pi,\ n \geq 0,\ \ell \geq n+1,\ m \geq 0) ,$$

$$\int_0^\infty \frac{x^{n-\frac{1}{2}} \ln^m x}{(1-x)^\ell}\, dx$$

$$= (-1)^{\ell+m+n+1} \frac{m!}{(\ell-1)!} \sum_{m_1=0}^{n} (-1)^{m_1} \hat{S}(m_1,n)$$

$$\sum_{m_2=0}^{\ell-n-1} \hat{S}(m_2,\ell-n-1)(2\pi)^{m-m_1-m_2+1} \frac{(2^{m-m_1-m_2+1}-1)|B_{m-m_1-m_2+1}|}{(m-m_1-m_2+1)!} , \tag{3.8}$$

$$(n \geq 0,\ \ell \geq n+1,\ m \geq \ell-1) ,$$

$$\int_0^\infty \frac{x^n \ln^m x}{(1+\beta x)^{\ell-\frac{1}{2}}}\, dx$$

$$= \frac{(-1)^\ell}{\beta^{n+1}} \frac{2^{\ell-1} m!}{(2\ell-3)!!} \sum_{m_1=0}^{\min(m,n)} (-1)^{m_1} S_{n+1}^{(m_1+1)} \tag{3.9}$$

$$\sum_{m_2=0}^{\min(m-m_1,\, \ell-n-2)} (-1)^{m_2} \hat{S}(m_2, \ell-n-2) C_{m-m_1-m_2}\left(\frac{1}{4}\beta; 1, -2, -1\right) ,$$

$$(|\arg \beta| < \pi,\ n \geq 0,\ \ell \geq n+2,\ m \geq 0) .$$

In formulae (3.5) to (3.9), n, m, and ℓ are integers. The integrals in (3.6) and (3.8) are to be understood as Cauchy principal value integrals if $m = \ell - 1$.

Note that

$$\int_0^\infty \frac{x^{n-\frac{1}{2}} \ln^m x}{(1+\beta x)^{\ell-\frac{1}{2}}}\, dx = (-1)^m \beta^{\frac{1}{2}-\ell} \int_0^\infty \frac{x^{\ell-n-2} \ln^m x}{(1+x/\beta)^{\ell-\frac{1}{2}}}\, dx .$$

Hence this integral can be obtained from (3.9). For each of the integrals (3.5) to (3.8), there exist relations connecting integrals with different parameter values. These relations and a short explicit table of these integrals for small values of n, m, and ℓ, obtained by REDUCE [Hearn (1984)] on an IBM 3081 and a Siemens 7880 at CERN, are given in Kölbig (1985).

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