PERGAMON

Annual Reviews in Control 26 (2002) 45–56

Annual Reviews in Control

www.elsevier.com/locate/arcontrol

H_2-OPTIMIZATION — THEORY AND APPLICATIONS TO ROBUST CONTROL DESIGN

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Abstract. H_2-optimization removes the stochastics from LQG optimization (Doyle, Glover, Khargonekar, and Francis, 1989.) It relies on the observation that the customary signal-based mean square criterion of LQG optimization may be re-interpreted as a system norm (in particular, the 2-norm), without direct reference to the signals that are involved. A moment's thought, however, reveals that the H_2-paradigm allows the consideration of design problems that the conventional LQG formulation and solution does not permit. These extended problems include quite naturally frequency dependent weighting functions and colored measurement noise

Although LQG optimization has been generalized to include these "singular" problems a long time ago these results are not widely used for control system design and no standard software appears to be available for their application. Nevertheless, the extra flexibility provided by the general H_2-problem is quite attractive. Moreover, the H_2-paradigm allows treating the stochastic problem parameters such as noise intensities as the design parameters that they really are. Frequency dependent weighting functions permit to design for integrating action in a fashion that is considerably less ad hoc than is usual in the LGQ context. They also provide other loop shaping tools for robust control design such as explicit control of high-frequency roll-off.

After surveying the potential applications of H_2-optimization some of the solution algorithms that are currently available are reviewed. The best-known solution of the standard H_2 problem is described by Doyle, Glover, Khargonekar, and Francis (1989) but applies to a limited class of problems only that does not extend much beyond conventional LQG. Early polynomial matrix solutions (Hunt, Šebek and Kučera, 1994) suffer from complexity. Recent versions of the polynomial matrix solution (Meinsma, 2000; Kwakernaak, this paper) and the descriptor solution (Takaba and Katayama, 1998; Kwakernaak, this paper) offer implementations that are suitable for a wide and flexible class of useful design applications.

Implementations of both algorithms have been tested with the help of the Polynomial Toolbox for MATLAB (www.polyx.com). The paper concludes with several sample applications and a design example.

Keywords: H2-optimization, control system design, polynomial solution, descriptor solution, mixed sensitivity problem, LQG design.

PII: S1367-5788(02)00008-1

1. INTRODUCTION

H_2-optimization removes the stochastics from LQG optimization (Doyle *et al.*, 1989). The block diagram of Fig. 1 shows the LQG paradigm. The plant P is described by the state differential equations

$$\dot{x}(t) = Ax(t) + Bu(t) + Fv(t)$$

where the white noise v models the disturbances. The controlled output z and the measured output y are given by

$$z(t) = Dx(t)$$
$$y(t) = Cx(t) + w(t)$$

where the white noise w represents the sensor noise. Assuming that the controlled output z and the input u are suitably scaled, LQG optimization amounts to finding a feedback compensator K that stabilizes the system and minimizes

$$\lim_{t\to\infty} E\left(z^T(t)z(t) + u^T(t)u(t)\right)$$

This famous and seminal problem has been widely studied and its solution is of course well known. Under suitable assumptions about the controllability and observability of the plant the optimal compensator is the interconnection of a Kalman filter and a state feedback law. Finding the Kalman filter and the feedback law requires solving two algebraic Riccati equations.

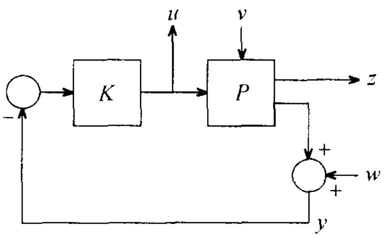
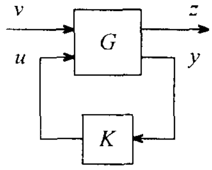


Fig. 1. Feedback system for LQG optimization

The formulation of the LQG problem and its solution require familiarity with the theory of stochastic processes. We recall how, first of all, the LQG problem may be generalized to the so-called "standard problem" and, next, the stochastic interpretation may be eliminated.

It is easy to see that the configuration of Fig. 1 is a special case of the "standard" configuration of Fig. 2. G represents the generalized plant, v comprises the driving signals for the shaping filters for disturbances, measurement noise and reference inputs, z is the control error signal, y is the measured output and u the control input. The system of Fig. 2 reduces to that of Fig. 1 by redefining

$$z := \begin{bmatrix} z \\ u \end{bmatrix}, \quad v := \begin{bmatrix} v \\ w \end{bmatrix},$$



$$G(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix}$$

Fig. 2. Standard configuration

with

$$G_{11}(s) = \begin{bmatrix} D(sI-A)^{-1}F & 0 \\ 0 & 0 \end{bmatrix}, \quad G_{12}(s) = \begin{bmatrix} D(sI-A)^{-1}B \\ I \end{bmatrix}$$

$$G_{21}(s) = \begin{bmatrix} C(sI-A)^{-1}F & I \end{bmatrix}, \quad G_{22}(s) = C(sI-A)^{-1}B$$

The LGQ problem now amounts to the minimization of the steady-state value of $E(z^T(t)z(t))$. In the block diagram of Fig. 2 the output z may be expressed in terms of Laplace transforms as

$$z = H(s)v$$

where H is the closed-loop transfer matrix

$$H(s) = G_{11}(s) + G_{12}(s)[I - K(s)G_{22}(s)]^{-1}K(s)G_{21}(s)$$

If v is white noise with intensity matrix I and the closed-loop system is stable then

$$\lim_{t\to\infty} E\left(z^T(t)z(t)\right) = \frac{1}{2\pi}\operatorname{tr}\int_{-\infty}^{\infty} H^T(-j\omega)H(j\omega)\,d\omega$$

The expression on the right-hand side is the square of the 2-norm

$$\|H\|_2 = \left(\frac{1}{2\pi}\operatorname{tr}\int_{-\infty}^{\infty} H^T(-j\omega)H(j\omega)\,d\omega\right)^{1/2}$$

of the stable transfer matrix H. Hence, LQG optimization is tantamount to minimization of the 2-norm of the closed-loop system. This minimization problem is the celebrated H_2 problem.

2. SCOPE OF THE H_2 PROBLEM

The H_2 formulation replaces the stochastic minimum least squares interpretation of LQG optimization with the minimization of the 2-norm of the closed-loop system. One considerable advantage of this viewpoint is that there is no need to interpret parameters such as the intensity of the various white noise processes that enter into the LQG problem as stochastic data, whose values need to be determined by modelling or identification experiments. They can simply be taken as tuning parameters for the design process. In LQG practice this is of course common procedure. A pedagogical advantage of the H_2 formulation is that there is no need to go into the intricacies of white noise and the pitfalls of the proof of the separation theorem may be detoured.

The insight that conventional LQG optimization merely is a special case of the far more general "standard" H_2 problem has important consequences. The conventional LQG problem, for instance, is immediately seen to be a special case of the generalized LQG problem of Fig. 3. This generalized problem allows for

- colored disturbances and measurement noise, whose frequency contents are determined by the shaping filters V_1 and V_2, and
- frequency weighting of the controlled output and of the input determined by the weighting functions W_1 and W_2

Colored disturbances and frequency dependent weighting of the controlled output may of course easily be handled by conventional LQG optimization. Colored measurement noise and frequency dependent weighting of the input, on the other hand, lead to singular versions of the LQG problem. Although LQG optimization has been generalized to include these singular problems a long time ago (some of them are already considered in Kwakernaak and Sivan, 1972) these results are not widely used for control system design and no standard software appears to be available for their application. The generalized LQG problem actually is the H_2 version of the mixed sensitivity problem (Kwakernaak, 1993) and is further discussed in Section 3.

In later sections of this paper solutions of the H_2 problem are discussed that take singular problems in their stride. Standard software for solving general H_2 problems therefore constitutes a far more powerful design tool than software for standard LQG solutions alone.

3. H_2 CONTROL SYSTEM DESIGN

In this section we discuss the application of H_2 optimization to linear control system design. Control system design aims at achieving

1. closed-loop stability,
2. closed-loop performance, and
3. closed-loop robustness

These goals are not as competitive or mutually exclusive as is sometimes believed. In fact, the three targets may simultaneously be accomplished by

- making the loop gain large at low frequencies,
- making the loop gain small at high frequencies, and
- keeping the loop gain away from the critical point -1 at crossover frequencies

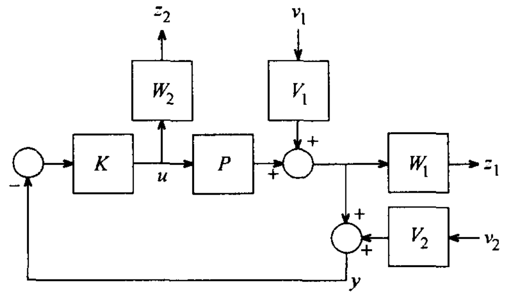


Fig. 3. Generalized LQG problem – the H_2 mixed sensitivity problem

The loop gain is an open-loop quantity. It has a direct effect on important closed-loop transfer functions, which determine the 2-norm, such as the sensitivity S, and the complementary sensitivity T. For the single-loop (but possibly multivariable) configuration of Fig. 3 these closed-loop system functions are

$$S(s) = \left(I + P(s)K(s)\right)^{-1}$$
$$T(s) = P(s)K(s)\left(I + P(s)K(s)\right)^{-1}$$

The sensitivity function S determines the effect of the disturbance on the output of the control system. The complementary sensitivity T satisfies the identity $S + T = I$, and is important for the closed-loop response, the effect of measurement noise and the amount of control effort. In terms of these two functions the design targets may be rephrased as follows:

- make the sensitivity S small at low frequencies,
- make the complementary sensitivity T small at high frequencies, and
- prevent both S and T from peaking at crossover frequencies

We discuss how the configuration of Fig. 3 may be utilized to achieve these targets. If we choose $V_2 = 0$ then we have

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} W_1 S V_1 \\ W_2 U V_1 \end{bmatrix} v_1 \tag{1}$$

U is the input sensitivity function

$$U(s) = K(s)\left(I + P(s)K(s)\right)^{-1}$$

It is directly related to the complementary sensitivity function T as $T = PU$. Shaping U is equivalent to shaping T. Equation (1) shows that the closed-loop transfer function H is given by

$$H = \begin{bmatrix} W_1 S V_1 \\ W_2 U V_1 \end{bmatrix}$$

Accordingly, in the SISO case minimization of the 2-norm amounts to minimization of

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(|W_1(j\omega)S(j\omega)V_1(j\omega)|^2+|W_2(j\omega)U(j\omega)V_1(j\omega)|^2\right)d\omega \tag{2}$$

This is clearly the H_2 version of the well-known mixed sensitivity problem of H_∞ optimization (Kwakernaak, 1993)

We expect to achieve the design goals by suitable choices of the functions V_1, W_1 and W_2, and present a number of considerations for the choice of these functions. The discussion is limited to the single-input single-output case but similar arguments apply to the MIMO case.

Choice of V_1. For the choice of the shaping filter V_1 we let us guide by the LQG problem. For the standard LQG problem (assuming non-inferential control, that is, $D = C$) we have $V_1(s) = C(sI - A)^{-1}F$. The precise form of this function of course depends on the choice of F. Commonly, especially when loop transfer recovery (Saberi, Chen, and Sannuti, 1993) is pursued, F is set equal to B so that

$$V_1(s) = P(s) = C(sI - A)^{-1}B$$

P is the open-loop plant transfer function. This choice of V_1 therefore means that the frequency content of the disturbance is shaped according to the open-loop plant frequency response function.

This choice is not always adequate, however, because often low-frequency disturbances prevail. This may be accounted for by including a supplementary factor $(s+\alpha)/s$ in V_1, so that

$$V_1(s) = P(s)\frac{s+\alpha}{s} \tag{3}$$

The constant α is a design parameter. Effectively, the presence of a pole at 0 in V_1 forces the sensitivity function $S(s)$ to be 0 at $s = 0$. Inspection of (2) shows that if S does not have a zero at 0 then the 2-norm cannot be finite.

Since $S(0) = 0$ can only be achieved by integral action, this choice of the weighting function implies that the resulting design necessarily involves integrating action. If the plant has "natural" integrating action (that is, P has a pole at 0) then it is not necessary to include a pole at 0 in V_1 except if it is desired to design a type k system with $k > 1$.

If the plant transfer function P is strictly proper then V_1 as given by (3) is also strictly proper. For sensible control systems the sensitivity function S has the property $S(\infty) = 1$ and, therefore is proper but not strictly proper. For this reason, whichever way the weighting function V_1 is chosen the product W_1V_1 needs to be strictly proper to allow convergence of the integral in the expression for the 2-norm.

Choice of W_1. Letting the weighting function $W_1(s)$ have a pole at 0 may also enforce integrating action. This has the same effect as letting $V_1(s)$ have such a pole. If we choose $V_1(s)$ to have a pole at 0 then $W_1(s)$ may be used for fine tuning. A safe initial choice is $W_1(s) = 1$.

Choice of W_2. Selecting W_2 is slightly more intricate. Consider the situation that V_1 is chosen to have a pole at 0 to achieve integrating action. Inspection of (2) shows that to make the 2-norm of the closed-loop system finite necessarily W_2 needs to have a zero at 0 that cancels the corresponding pole of V_1 (because U can never have a zero at 0 if $S(0) = 0$.) The reason for this zero is that if the closed-loop system is to reject constant disturbances then we cannot penalize constant control inputs.

The second function of W_2 is to control the high-frequency roll-off of the compensator transfer function K and, hence, that of the input sensitivity U and the complementary sensitivity T. In the SISO case we have

$$U(s) = \frac{K(s)}{1+P(s)K(s)}, \quad T(s) = \frac{P(s)K(s)}{1+P(s)K(s)} \tag{4}$$

Assume that the plant transfer function has pole excess $e > 0$. The pole excess is the difference between the number of poles and the number of zeros, that is, the difference of the degree of the denominator and that of the numerator. The pole excess of a transfer function equals its high-frequency roll-off if the latter is expressed in decades per decade.

Inspection of (4) shows that if the compensator has nonnegative roll-off then the high-frequency roll-off of U equals that of the compensator K, while the high-frequency roll-off of T equals e plus the roll-off of the compensator.

Inspection of (2) reveals that to ensure convergence of the integral the high-frequency roll-off of $W_2(s)U(s)V_1(s)$ needs to be at least 1. If V_1 has roll-off 1, say, then it is enough if W_2U has roll-off 0. We can make sure that U has roll-off 1 or more by letting W_2 have a zero-pole excess, that is, by choosing W_2 to be nonproper. Here are two options for the choice of W_2, both under the assumption that as argued previously it needs a zero at 0 when designing for integral control:

- $W_2(s) = \dfrac{\rho s}{s+\alpha}$

  In this case U is expected to have at least zero roll-off. The constants ρ and α are available for fine-tuning.

- $W_2(s) = \dfrac{\rho s}{s+\alpha}(1+\tau s)$

  U now has a minimal roll-off of 1 dec/dec, which is expected to set in at the frequency $1/\tau$.

In the design example of Section 7 we demonstrate the procedure.

4. SOLUTIONS OF THE H_2 PROBLEM

Since Doyle *et al.* (1989) introduced the standard H_2 problem a number of solutions have become available. In this presentation we mention the original state space solution of Doyle *et al.* (1989), a polynomial matrix solution by Hunt, Šebek, and Kucera (1994), a frequency domain solution by Meinsma (2000), another polynomial matrix solution described in Section 5, and a descriptor solution by Takaba and Katayama (1998), which is extended in Section 6.

The starting point of the state space solution (Doyle, Glover, Khargonekar and Francis, 1989; Zhou, Doyle and Glover, 1995) is the state space representation

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + B_1 v(t) + B_2 u(t) \\
z(t) &= C_1 x(t) + D_{12} u(t) \\
y(t) &= C_2 x(t) + D_{21} v(t) + D_{22} u(t)
\end{aligned}
$$

of the generalized plant of Fig. 2.The H_2 problem may be solved by reducing it to an LQG problem. The derivation necessitates the introduction of the following assumptions:

- The system $\dot{x}(t) = Ax(t) + B_2 u(t)$, $z(t) = C_1 x(t)$ is stabilizable and detectable.
- The system $\dot{x}(t) = Ax(t) + B_1 v(t)$, $y(t) = C_2 x(t)$ is stabilizable and detectable.
- The matrix $D_{12}^T D_{12}$ is nonsingular. This is equivalent to the assumption that D_{12} is tall and has full column rank. It is basically the LQG assumption that the weighting matrix of the control input be nonsingular.
- The matrix $D_{21} D_{21}^T$ is nonsingular. This is equivalent to the assumption that D_{21} is wide and has full row rank. It is basically the LQG assumption that the observation noise is white.

Under these assumptions the optimal output feedback controller is

$$
\begin{aligned}
\dot{\hat{x}}(t) &= A\hat{x}(t) + B_2 u(t) + K\left[y(t) - C_2 \hat{x}(t) - D_{22} u(t)\right] \\
u(t) &= -F\hat{x}(t)
\end{aligned}
$$

The observer and state feedback gain matrices are given by

$$
\begin{aligned}
F &= (D_{12}^T D_{12})^{-1}(B_2^T X + D_{12}^T C_1) \\
K &= (Y C_2^T + B_1 D_{21}^T)(D_{21} D_{21}^T)^{-1}
\end{aligned}
$$

The symmetric matrices X and Y are the unique positive-definite solutions of the algebraic Riccati equations

$$
\begin{aligned}
A^T X + XA + C_1^T C_1 - (XB_2 + C_1^T D_{12})(D_{12}^T D_{12})^{-1}(B_2^T X + D_{12}^T C_1) &= 0 \\
AY + YA^T + B_1 B_1^T - (YC_2^T + B_1 D_{21}^T)(D_{21} D_{21}^T)^{-1}(C_2 Y + D_{21} B_1^T) &= 0
\end{aligned}
$$

These AREs are conveniently solved by application of the ordered Schur transformation to the appropriate Hamiltonian matrices (Laub, 1979).

The solution is simple and elegant but the assumptions effectively restrict H_2 optimization to the LQG framework.

5. POLYNOMIAL SOLUTION

Various polynomial solutions of the standard H_2 problem are available. That by Hunt, Šebek and Kučera (1994) builds on the well known polynomial solution of the LQG problem of Kučera (1979), based on completion of squares and Diophantine equations (see also Kučera, 1996). Algorithmically the solution is rather involved.

In a paper presented during this conference Meinsma (2000) describes an elegant solution of the standard H_2 problem based on factorizations over polynomial matrices and stable matrices. The details of the algorithm are not completely worked out, however, and the assumptions under which the problem is solved are not quite as general as desired.

We consider another polynomial solution, which relies on representing the standard plant G in the left and right coprime polynomial matrix fraction forms

$$
G = \begin{bmatrix} D_{11} & D_{12} \\ 0 & D_{22} \end{bmatrix}^{-1} \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} = \begin{bmatrix} \bar{N}_{11} & \bar{N}_{12} \\ \bar{N}_{21} & \bar{N}_{22} \end{bmatrix} \begin{bmatrix} \bar{D}_{11} & 0 \\ \bar{D}_{21} & \bar{D}_{22} \end{bmatrix}^{-1}
$$

We furthermore define D_o as a greatest left divisor of D_{22} and N_{22} so that $D_{22} = D_o \breve{D}_{22}$, $N_{22} = D_o \breve{N}_{22}$ with $\breve{D}_{22}$ and $\breve{N}_{22}$ left coprime, and $\bar{D}_o$ as a greatest right divisor of $\bar{D}_{22}$ and $\bar{N}_{22}$ so that $\bar{D}_{22} = \hat{D}_{22} \bar{D}_o$, $\bar{N}_{22} = \hat{N}_{22} \bar{D}_o$ with $\hat{D}_{22}$ and $\hat{N}_{22}$ right coprime. As a result we have the left and right coprime fractions $G_{22} = \breve{D}_{22}^{-1} \breve{N}_{22} = \hat{N}_{22} \hat{D}_{22}^{-1}$.

Assumptions: For the solution of the H_2 problem we introduce the following modest assumptions:

1. $G_{12}(s)$ has full column rank for almost all s and has no zeros on the imaginary axis.
2. $G_{21}(s)$ has full row rank for almost all s and has no zeros on the imaginary axis.

If $G_{12}(s)$ does not have full column rank or $G_{21}(s)$ does not have full row rank then the plant input and the measured output may normally be transformed to meet the assumptions.

The roots of the polynomial matrices D_{11} and D_o are "fixed" poles of the closed-loop system, that is, they stay in place no matter what the compensator is. If these two polynomial matrices have any right-half plane roots then the closed-loop system cannot be stabilized. Fixed poles frequently arise from shaping filters. Strictly anti-stable poles of shaping filters may often be reflected into the left-half complex plane by spectral factorization. Shaping filter poles on the imaginary axis are useful for designing for integral control or vibration suppression and, hence, need to be allowed. Thus, our next assumption is

3. The fixed poles, that is, the zeros of D_{11} and D_o, lie in the closed left-half complex plane (including the imaginary axis).

Under these three assumptions the H_2 problem may have no solution because no compensator exists that makes the closed-loop transfer matrix strictly proper and cancels the fixed poles on the imaginary axis in the closed-loop transfer matrix. Both are needed for the 2-norm to be finite. Hence, we need the final assumption

4. There exists a compensator so that the corresponding closed-loop transfer matrix is strictly stable and strictly proper.

The algorithm that is proposed detects whether or not a solution exists.

Youla-Kučera parametrization. The compensators that we consider stabilize the controllable part of the closed-loop system. The Youla-Kučera parametrization of all such compensators is of the form $K = YX^{-1}$, where the polynomial matrices X and Y are given by

$$X = X_o + \hat{N}_{22}Q$$
$$Y = Y_o + \hat{D}_{22}Q$$

$Q = PD_{cl}^{-1}$ is the strictly stable "parameter," where P and D_{cl} are arbitrary polynomial matrices of the correct dimensions such that D_{cl} is strictly Hurwitz. X_o and Y_o form a solution of the Bézout equation

$$I = \breve{D}_{22}X_o - \breve{N}_{22}Y_o$$

With this parametrization, the closed-loop transfer matrix is given by

$$H = H_o + \bar{N}_{12}\bar{D}_o^{-1}QD_o^{-1}N_{21} \tag{5}$$

where H_o is the stable but not necessarily proper transfer matrix

$$H_o = D_{11}^{-1}N_{11} - D_{11}^{-1}(D_{12}X_o - N_{12}Y_o)D_o^{-1}N_{21}$$

Optimality condition. Given this parametrization it may be proved that if the closed-loop transfer matrix H is strictly stable and strictly proper then it is optimal if and only if

$$\Phi^{-1}N_{21}H^{\sim}\bar{N}_{12}\Psi^{-1} \tag{6}$$

is strictly proper and strictly stable. Here we write $H^{\sim}(s) = H^T(-s)$, while Φ and Ψ are a strictly Hurwitz polynomial spectral co-factor and spectral factor, respectively, defined by

$$N_{21}N_{21}^{\sim} = \Phi\Phi^{\sim}, \quad \bar{N}_{12}^{\sim}\bar{N}_{12} = \Psi^{\sim}\Psi$$

Construction of the compensator. The optimal compensator may now be constructed as follows. By substituting the parametrization (5) of the closed-loop transfer matrix H into (6) we see that if H is strictly proper and strictly stable then a sufficient and necessary condition for optimality is that

$$(\Psi^{-1})^{\sim}\bar{N}_{12}^{\sim}H_oN_{21}^{\sim}(\Phi^{-1})^{\sim} + \Psi\bar{D}_o^{-1}QD_o^{-1}\Phi$$

be strictly anti-stable and strictly proper. To determine Q so that this condition is satisfied we decompose

$$(\Psi^{-1})^{\sim}\bar{N}_{12}^{\sim}H_{11}N_{21}^{\sim}(\Phi^{-1})^{\sim} = R_- + R_+$$

with R_+ strictly proper strictly anti-stable and R_- stable but not necessarily strictly stable and typically not proper. We may now solve the optimal parameter Q as

$$Q = -\bar{D}_o\Psi^{-1}R_-\Phi^{-1}D_o \tag{7}$$

If the parameter Q obtained from (7) does not make the closed-loop transfer matrix H strictly proper and strictly stable then the corresponding compensator is not optimal. In this case no optimal solution exists.

The solution may be implemented by standard polynomial matrix operations such as available in the Polynomial Toolbox for MATLAB.

The details of the proof and algorithm and an implementation of the algorithm for the Polynomial Toolbox for MATLAB are available at the website www.polyx.com.

Example. Consider the generalized plant

$$G(s) = \left[\begin{array}{c|c} 1 & s-1 \\ 0 & s-1 \\ \hline 1 & 0 \end{array}\right]$$

The closed-loop transfer matrix is

$$\begin{aligned} H(s) &= G_{11}(s) + G_{12}(s)[I - K(s)G_{22}(s)]^{-1}K(s)G_{21}(s) \\ &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} s-1 \\ s-1 \end{bmatrix} K(s) \end{aligned}$$

Inspection shows that there exists no compensator K that cancels the polynomial part of H, and, hence, there is no compensator that makes the 2-norm finite. Application of the algorithm results in the compensator

$$K(s) = -\frac{0.5}{s+1}$$

For this compensator we have

$$\Phi^{-1}(s)N_{21}(s)H^{\sim}(s)\bar{N}_{12}(s)\Psi^{-1}(s) = -\frac{2\sqrt{2}}{s+1},$$

$$H(s) = 0.5\begin{bmatrix} s+3 \\ -s+1 \end{bmatrix}\frac{1}{s+1}$$

The first expression is strictly stable and strictly proper, but the closed-loop transfer matrix H is nonproper. Hence, the compensator is not optimal.

6. DESCRIPTOR SOLUTION

Takaba and Katayama (1998) present the descriptor solution of the standard H_2 problem. Kučera (1986) solves a scalar version of the H_2 problem for descriptor systems by polynomial methods. We discuss a somewhat different and more general solution than that of Takaba and Katayama, which mixes descriptor and polynomial methods. The starting point is the descriptor representation

$$\begin{aligned} E\dot{x} &= Ax + B_1 v + B_2 u \\ z &= C_1 x + D_{11} v + D_{12} u \\ y &= C_2 x + D_{21} v \end{aligned} \tag{8}$$

of the standard plant G. Like the polynomial formulation the descriptor representation supports nonproper transfer functions, which may arise from the application of nonproper weighting functions.

Assumptions. The assumptions are the same as for the polynomial version of the problem. We require

$$G_{12}(s) = C_1(sE - A)^{-1} B_2 + D_{12}$$

to have full column rank for almost all s and

$$G_{21}(s) = C_2(sE - A)^{-1} B_1 + D_{21}$$

to have full row rank for almost all s. Neither G_{12} nor G_{21} is allowed to have zeros on the imaginary axis. Any fixed poles of the system may lie on the imaginary axis but not in the open right-half plane, and we assume the existence of a compensator that makes the closed-loop system strictly proper and strictly stable.

Preparation. The descriptor representation (8) needs to be arranged so that it has the following properties:

1. There is no term $D_{22}u$ in the output equation for y. This causes no loss of generality because if such a term is present then defining an additional set of state variables $x' = D_{22}u$ eliminates it.
2. The matrix $Q = D_{12}^T D_{12}$ is nonsingular. Again this causes no loss of generality because the condition may be arranged to hold by introducing extra state variables if needed. Suppose for instance that the entire term $D_{12}u$ is missing in the equation for z. In this case we may add the term $D_{12}u + x'$ to this equation, with D_{12} an arbitrary full rank matrix of the correct dimensions, while at the same time including the equation $D_{12}u + x' = 0$ in the descriptor equations.
3. The matrix $R = D_{21}D_{21}^T$ is nonsingular. Again, this causes no loss of generality because the equations may always be rearranged so that the condition holds.

Optimality condition. The starting point for the derivation of the optimal compensator is again the condition that the expression (6) be strictly proper and strictly stable. We rewrite (6) in the form

$$G_{21}^{ci} H^{\sim} G_{12}^{i} \tag{9}$$

The inner function $G_{12}^i = \bar{N}_{12}\Psi^{-1}$ has the property

$$(G_{21}^i)^{\sim} G_{21}^i = I$$

and the co-inner function $G_{21}^{ci} = \Phi^{-1} N_{21}$ satisfies

$$G_{21}^{ci}(G_{21}^{ci})^{\sim} = I$$

Both functions are strictly stable and proper but not strictly proper.

The computation of the inner and co-inner functions and the solution of the H_2 problem rely on the solution of two generalized algebraic Riccati equations. The first of these GAREs is

$$\begin{gathered} X^T A + A^T X + C_1^T C_1 - (X^T B_2 + C_1^T D_{12}) Q^{-1} (B_2^T X + D_{12}^T C_1) = 0 \\ X^T E = E^T X \end{gathered} \tag{10}$$

Given the correct solution X of this equation (see later) the inner function G_{12}^i is given by

$$G_{12}^i(s) Q^{1/2} = (C_1 - D_{12}F)(sE - A + B_2 F)^{-1} B_2 + D_{12}$$

where $F = Q^{-1}(B_2^T X + D_{12}^T C_1)$. The other GARE that needs to be considered is

$$\begin{gathered} YA^T + AY^T + B_1 B_1^T - (YC_2^T + B_1 D_{21}^T) R^{-1} (C_2 Y + D_{21} B_1^T) = 0 \\ YE^T = EY^T \end{gathered} \tag{11}$$

Defining $K = (YC_2^T + B_1 D_{21}^T)R^{-1}$ the co-inner function G_{21}^{ci} follows from

$$R^{1/2} G_{21}^{ci}(s) = C_2(sE - A + KC_2)^{-1}(B_1 - KD_{21}) + D_{21}$$

Construction of the compensator. Takaba and Katayama (1998) consider the compensator given in descriptor form by

$$\begin{aligned} E\dot{\hat{x}} &= A\hat{x} + B_2 u + K(y - C_2\hat{x}) \\ u &= -F\hat{x} - L(y - C_2\hat{x}) \end{aligned} \tag{12}$$

with the gains F and K as given. The additional gain L is yet to be determined. In the paper by Takaba and Katayama L is a constant matrix but we allow it to be a polynomial matrix.

It may be proved that the compensator (12) makes the expression (9) stable for any choice of the gain L but not necessarily strictly stable or strictly proper. The gain L needs to be chosen so that the closed-loop transfer matrix H is strictly proper so that also (9) is strictly proper. The closed-loop transfer matrix H is actually given by

$$H(s) = H_0(s) - G_{12}^i(s)Q^{1/2}LR^{1/2}G_{21}^{ci}(s)$$

where H_0 is the closed-loop transfer matrix obtained by setting $L = 0$. Since both G_{12}^i and G_{21}^{ci} are proper it follows that if H is strictly proper then

$$\left(G_{12}^i(-s)\right)^T H(s)\left(G_{21}^{ci}(-s)\right)^T$$
$$= \left(G_{12}^i(-s)\right)^T H_0(s)\left(G_{21}^{ci}(-s)\right)^T - Q^{1/2}LR^{1/2}$$

is also strictly proper. Therefore, L needs to be chosen such that $Q^{1/2}L(s)R^{1/2} = P(s)$, where P is the polynomial part of

$$\left(G_{12}^i(-s)\right)^T H_0(s)\left(G_{21}^{ci}(-s)\right)^T$$

This determines L uniquely. Given L, the compensator is optimal only if the corresponding closed-loop transfer matrix H is both strictly proper and strictly stable.

Since both G_{12}^i and G_{21}^{ci} are strictly stable H is strictly stable iff H_0 is strictly stable, which may be checked before computing L. If H_0 is not strictly stable then no optimal compensator exists. If after computing L the closed-loop system transfer matrix H is not strictly proper then no solution exists.

The gain matrix L often is constant but it is not difficult to find examples where it is polynomial. If L is polynomial then it needs to be converted to descriptor form to obtain an augmented descriptor representation of the optimal compensator. Alternatively, (12) may be converted to polynomial matrix fraction form by elimination of $\hat{x}$.

Solution of the GAREs. The GARE (10) may be solved by transforming the Hamiltonian-type matrix pencil

$$\begin{bmatrix} B_2Q^{-1}B_2^T & sE - A + B_2Q^{-1}D_{12}^TC_1 \\ -sE^T - A^T + C_1^TD_{12}Q^{-1}B_2^T & -C_1^TC_1 + C_1^TD_{12}Q^{-1}D_{12}^TC_1 \end{bmatrix} \tag{13}$$

to anti-triangular form. By Clements' algorithm (Clements, 1993; see also Kwakernaak, 1998, and Kwakernaak, 2000) we may determine an orthogonal matrix

$$W = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}$$

so that pre-multiplication of (13) by W and post-multiplication by W^T brings (13) in the form

$$\begin{bmatrix} 0 & sE_1 - A_1 \\ -sE_1^T - A_1^T & sE_2 - A_2 \end{bmatrix} \tag{14}$$

with $sE_1 - A_1$ anti-Hurwitz. The desired solution of the GARE now is given by

$$X = W_{11}^T(W_{12}^T)^{-1} = -W_{21}^{-1}W_{22}$$

and $sE - A + B_2F$ is Hurwitz. If the system $E\dot{x} = Ax + B_2u$, $z = C_1x + D_{12}u$ is not stabilizable then the matrices W_{12} and W_{21} are singular and there exists no finite solution X of the GARE as needed. To resolve this difficulty, write

$$FW_{12}^T = Q^{-1}(B_2^TW_{11}^T + D_{12}^TC_1W_{12}^T) \tag{15}$$

It may be shown by decomposing the system into Kalman canonical form (Banaszuk, Kociecki, and Lewis, 1992) that even if W_{12} is singular this equation has a (non-unique) solution for F that stabilizes the controllable modes and results in the correct inner function.

The solution of the second GARE (11) is similarly obtained by transforming the matrix pencil

$$\begin{bmatrix} -B_1B_1^T + B_1D_{21}^TR^{-1}D_{21}B_1^T & sE - A + B_1D_{21}^TR^{-1}C_2 \\ -sE^T - A^T + C_2^TR^{-1}D_{21}B_1^T & C_2^TR^{-1}C_2 \end{bmatrix}$$

to the Clements form (14), where now $sE_1 - A_1$ is Hurwitz. If this transformation is accomplished by

$$V = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}$$

then the desired solution is

$$Y = -V_{22}^{-1}V_{21} = V_{12}^T(V_{11}^T)^{-1}$$

and $sE - A + C_2K$ is Hurwitz. In the event of lack of detectability of the underlying system the computation may be modified as in the case of the GARE (10).

The details of the proof and algorithm and an implementation of the algorithm for version 2 of the Polynomial Toolbox for MATLAB are available at the website www.polyx.com.

Example. Consider the standard H_2 problem defined by the block diagram of Fig. 4.

The plant has transfer function $1/(s+1)$. The block $1/s$ is included to obtain integral control; the factor s in the weighting function cs is needed to allow this.

Taking $c = 1$ the generalized plant has a (non-minimal) descriptor representation given by

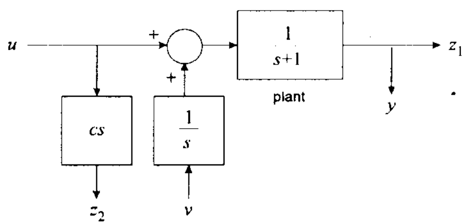


Fig. 4. Sample H_2 optimization problem

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} x + \left[\begin{array}{c|c} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \\ 0 & -1 \\ 0 & 1 \\ 1 & 0 \end{array}\right] \begin{bmatrix} v \\ \hline u \end{bmatrix}$$

$$\begin{bmatrix} z_1 \\ z_2 \\ \hline y \end{bmatrix} = \left[\begin{array}{ccccc} 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ \hline 2 & 0 & 0 & 0 & 1 \end{array}\right] x + \left[\begin{array}{c|c} 0 & 0 \\ 0 & 1 \\ \hline 1 & 0 \end{array}\right] \begin{bmatrix} v \\ \hline u \end{bmatrix}$$

This representation has the required features. Numerical computation shows that the GARE (10) has a finite solution X that results in the required inner factor with gain

$$F = [-0.5359 \quad -1.4641 \quad 1 \quad 2.7321 \quad 0]$$

The GARE (11), however, does not have a finite solution Y but by solving an equation similar to (15) it may be found that the gain

$$K^T = [0 \quad 0.5 \quad 0 \quad -1.1892 \quad 1]$$

realizes the desired co-inner function. With these data we may compute the inner and co-inner functions G^i_{12} and G^{ci}_{21} and the closed-loop transfer matrix H_0 for $L = 0$. We have

$$\begin{aligned} H(s) &= H_0(s) - G^i_{12}(s) Q^{\frac{1}{2}} L(s) R^{\frac{1}{2}} G^{ci}_{21}(s) \\ &= \frac{1}{1 + 1.7s + s^2} \begin{bmatrix} 4.2 + 1.2s \\ -1 + 2.5s + 4.4s^2 + 1.2s^3 \end{bmatrix} \\ &\quad + \frac{1}{1 + 1.7s + s^2} \begin{bmatrix} 1 \\ s(s+2) \end{bmatrix} L(s) \end{aligned}$$

The polynomial part

$$P(s) = \begin{bmatrix} 0 \\ 2.4 + 1.2s \end{bmatrix}$$

of H_0 may be cancelled by taking

$$L(s) = -3.2 - 1.2s$$

From this, the optimal compensator may be found to be given by

$$K(s) = -\frac{1 + 0.73s}{s}$$

The compensator has integrating action as expected. The polynomial algorithm yields the same compensator.

7. APPLICATIONS

The polynomial and descriptor algorithms for the solution of the H_2 problem may be applied to various types of problems.

Wiener filtering problem. A class of Wiener filtering problems may be defined as follows. A message signal x is given by

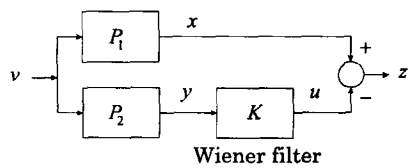


Fig. 5. Wiener filter configuration

$$x = P_1(s) v$$

where v is a standard white noise process. The observed signal y is related to the message process by

$$y = P_2(s) v$$

P_1 and P_2 are stable rational transfer matrices. It is desired to estimate the message signal x by filtering the observed signal y.

Fig. 5 shows the system configuration. Inspection shows that the generalized plant that defines the H_2-problem is given by

$$\begin{bmatrix} z \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} P_1 & -I \\ P_2 & 0 \end{bmatrix}}_{G} \begin{bmatrix} v \\ u \end{bmatrix}$$

By way of example, suppose that x and y are related as

$$y = x + n$$

where the observation noise n is independent of the message signal x. The message signal is generated by the shaping filter

$$x = \frac{1}{(s+1)^2} v_1$$

with v_1 white noise, and the noise is given by

$$n = \frac{\omega_o^2}{s^2 + 2\zeta\omega_o s + \omega_o^2} \sigma v_2$$

where the white noise v_2 is independent of v_1. We let $\omega_o = 1$, $\zeta = 0.01$ and $\sigma = 0.1$ so that the measurement noise is not very large but has a relatively sharp peak at the cut-off frequency of the message signal. This defines

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad P_1(s) = \begin{bmatrix} \frac{1}{(s+1)^2} & 0 \end{bmatrix}$$

$$P_2(s) \begin{bmatrix} \frac{1}{(s+1)^2} & \frac{\omega_o^2 \sigma}{s^2 + 2\zeta\omega_o s + \omega_o^2} \end{bmatrix}$$

so that

$$G(s) = \left[\begin{array}{cc|c} \frac{1}{(s+1)^2} & 0 & -1 \\ \hline \frac{1}{(s+1)^2} & \frac{\omega_o^2 \sigma}{s^2 + 2\zeta\omega_o s + \omega_o^2} & 0 \end{array}\right]$$

Both the polynomial and the descriptor algorithm return the transfer function of the Wiener filter as

$$K(s) = \frac{0.91 + 0.018s + 0.91s^2}{1 + 0.2s + s^2}$$

It is a notch filter that removes the colored measurement noise as best as it can. Fig. 6 shows the Bode magnitude plot of the filter.

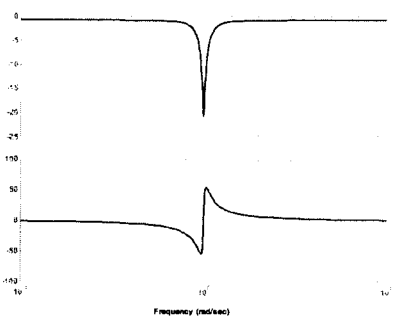


Fig. 6. Bode plot of the Wiener filter

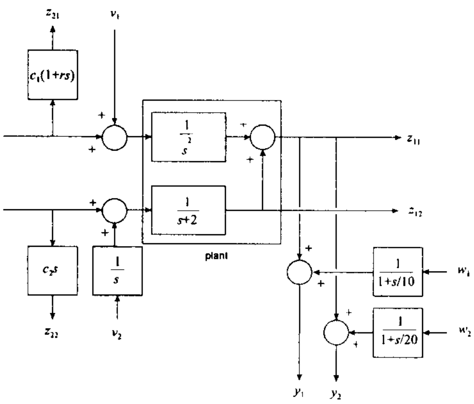


Fig. 7. MIMO problem

MIMO system. Fig. 7 defines a MIMO control problem. It features colored measurement noise on both output channels, a constant disturbance model on the second input channel to ensure integrating action in both channels, and nonproper weighting functions on both inputs. The parameters are chosen as $c_1 = 1$, $r = 5$, and $c_2 = 1$.

The polynomial and descriptor algorithms both produce the same MIMO compensator with strictly proper transfer matrix

$$K(s) = -\begin{bmatrix} \dfrac{0.45 + 2.6s + 2.8s^2 + 1.1s^3 + 0.08s^4}{4.4 + 13s + 17s^2 + 14s^3 + 5.6s^4 + s^5} & \dfrac{-0.033 - 0.81s - 0.56s^2 - 0.14s^3 - 0.0058s^4}{4.4 + 13s + 17s^2 + 14s^3 + 5.6s^4 + s^5} \\ \dfrac{0.94 + 4.4s + 8s^2 + 6.9s^3 + 2.3s^4 + 0.17s^5}{s(4.4 + 13s + 17s^2 + 14s^3 + 5.6s^4 + s^5)} & \dfrac{1.9 + 3.4s + 3s^2 + 1.7s^3 + 0.42s^4 + 0.017s^5}{s(4.4 + 13s + 17s^2 + 14s^3 + 5.6s^4 + s^5)} \end{bmatrix}$$

The compensator provides the second channel with integrating action as planned.

Design example. In this subsection we illustrate the design procedure for SISO systems outlined in Section 3. Consider the plant with transfer function

$$P(s) = \frac{1 + 0.2s}{(s + 0.1)(s + 10)}$$

The design specifications for the closed-loop system consist of a bandwidth of 1 rad/s, integral control and less than 3 dB peaking of the sensitivity and complementary sensitivity functions. To achieve integral control we let

$$V_1(s) = P(s)\frac{s + \alpha}{s} = \frac{(1 + 0.2s)(s + \alpha)}{(s + 0.1)(s + 10)s}$$

with the design parameter α to be chosen. Since the target bandwidth is 1 rad/s we expect that α should be about equal to 1 rad/s or somewhat larger. As argued in Section 3 we let $V_2(s) = 0$ and $W_1(s) = 1$ and consider choosing

$$W_2(s) = \frac{\rho s}{s + \alpha}(1 + \tau s)$$

Setting $\tau = 0$ is expected to produce a proper but not strictly proper compensator while choosing $\tau > 0$ is anticipated to provide the compensator with a roll-off of 1 dec/dec.

The generalized plant is

$$G(s) = \left[\begin{array}{c|c} W_1(s)V_1(s) & W_1(s)P(s) \\ 0 & W_2(s) \\ \hline -V_1(s) & -P(s) \end{array}\right]$$

The polynomial and descriptor algorithms work equally well for this problem. Not much experimenting is needed to find that $\alpha = 2$, $\rho = 0.2$ and $\tau = 0$ yield a quite acceptable design with the proper compensator

$$K(s) = \frac{7.231(s + 0.7077)(s + 9.7697)}{s(s + 5)}$$

which has integrating action as intended. The closed-loop poles are $-0.7751 \pm j0.6528$, -10.04 and -5.

Fig. 8 shows the sensitivity function and complementary sensitivity function. They exhibit no peaking and the plots confirm that the closed-loop bandwidth is 1 rad/s as required. Fig. 9 displays the response of the closed-loop system to a step disturbance and a step reference input. The reference signal r is supplied to the system in the form

$$X(d/dt)u(t) = Y(d/dt)y(t) + r(t)$$

This eliminates the numerator dynamics of the plant and leaves the option of using a prefilter to improve the response further.

By assigning a nonzero value to the design parameter τ additional roll-off of the compensator may be introduced. We expect that setting $\tau = 0.1$ lets the roll-off set in at about 10 rad/s. Application of the polynomial or descriptor algorithm result in the compensator

$$K(s) = \frac{75.62(s+9.8902)(s+0.6685)}{s(s+5)(s+11.41)}$$

The compensator is now strictly proper, and Fig. 10 shows that the complementary sensitivity function T has extra high-frequency roll-off. The peak value of the sensitivity function now is 0.9 dB and that of the complementary sensitivity function is 1.8 dB. The step responses are not shown because they differ very little from those of Fig. 9. The closed-loop poles are $-0.7540 \pm j0.6551$, $-10.00 \pm j0.4247$ and -5.

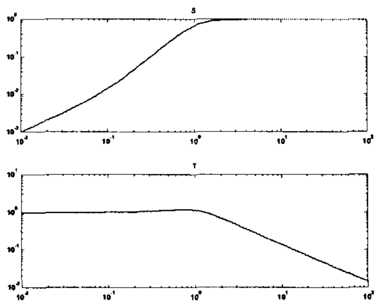


Fig. 8. Sensitivity function (top) and complementary sensitivity function (bottom)

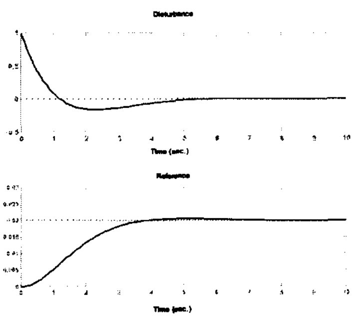


Fig. 9. Response to step disturbance (top) and step reference signal (bottom)

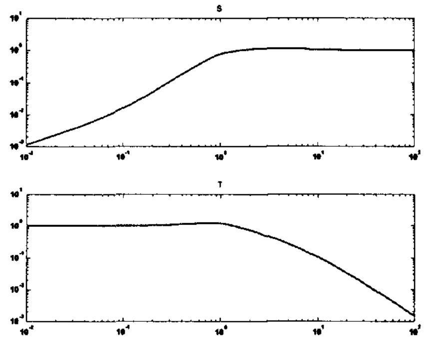


Fig. 10. Sensitivity function (top) and complementary sensitivity function (bottom) with extra roll-off

The design procedure is straightforward and works well. It is easy to predict the effect of the various design parameters.

8. CONCLUSIONS

The history of H_2-optimization dates back to Wiener filtering (Wiener, 1949) and its early applications to control (Newton, Gould and Kaiser, 1957). The optimal control and state space era contributed the LQ and LGQ problems. The H_2-problem itself is a spin-off of the robust control period although it is seldom recognized as a robust control tool. In this paper an attempt is made to show that H_2-optimization may be a valuable instrument to design linear multivariable control systems that have generically good performance and robustness properties. H_2-optimization, on the other hand, is not a good tool to design for protection against specific, non-generic perturbations such as caused by large parameter variations. Neither is H_∞-optimization (Landau, 1995).

A suitable paradigm for the design of generically good control systems is the H_2 mixed sensitivity problem discussed in Section 3. To take full advantage of the power of H_2-optimization the LQG and state space versions of the problem that have made their way into textbooks (Saberi, Sannuti and Chen, 1995; Burl, 1998) are not adequate. The polynomial and descriptor solutions reviewed in Sections 5 and 6 of this paper allow the use of shaping and weighting filters with poles on the imaginary axis, including poles at infinity. Colored measurement noise and other singular problems are handled as a matter of routine. These features greatly enhance the flexibility and applicability of H_2-optimization as a design tool.

The polynomial and descriptor solutions described in this paper have both been implemented with the help of the Polynomial Toolbox for MATLAB. Beta versions of the MATLAB-macros for use with version 2.0

of the Polynomial Toolbox may be collected at www.polyx.cz or www.polyx.com. Detailed proofs and descriptions of the algorithms are available at this website as well. Standard versions of the macros will be included in future distributions of the Polynomial Toolbox.

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