The standard set game of a cooperative game

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1 Introduction

The concept set game was introduced by Hoede in [6]. A *set game* is a pair $\langle N_s, v_s \rangle$, where N_s is a nonempty, finite set, called *the player set* and $v_s : 2^{N_s} \to 2^{U_s}$ associates with every coalition S of players a subset of U_s, called the *value (worth)* of coalition S. We assume that $v_s(\emptyset) = \emptyset$.
A *solution* ψ on the set of all set games G associates a so-called *allocation* $\psi(N_s, v_s) = (\psi_i(N_s, v_s))_{i \in N_s} \in (2^{U_s})^{N_s}$ with every set game $\langle N_s, v_s \rangle$. For every $i \in N_s$, ψ_i represents the items that are given , according to the solution ψ, to player i from participating in the game. Several solutions were proposed for set games (see [1,4,5,8]). As stated in [5], these solutions can be included in the class of semi-marginalistic values. A *semi-marginalistic value* ψ on the set game space G has the following form

$$\psi_i(N_s, v_s) = \bigcup_{\substack{S \subseteq N_s \\ S \ni i}} [v_s(S) - \nabla^{v_s}_{S,i}], \text{ for all } \langle N_s, v_s \rangle \in G \text{ and all } i \in N_s, \tag{1}$$

where $\nabla^{v_s}_{S,i}$ is a set determined by the worth of a certain collection of coalitions, somehow determined by S and/or i. For example, by choosing $\nabla^{v_s}_{S,i} := v_s(S - \{i\})$ one obtains the *individually marginalistic value (IM)* introduced in [1], by choosing $\nabla^{v_s}_{S,i} := \bigcup\limits_{T \subset S} v_s(T)$ one obtains the *overall-conditionally marginalistic value (OCM)* introduced in [8], and, finally, by choosing $\nabla^{v_s}_{S,i} := \bigcup\limits_{T \subseteq N_s - i} v_s(T)$ one obtains the *Driessen–Sun value (DS)* introduced in [4].
In many situations, there is a cost associated to each element of the universe U_s and one is interested in how to share the costs between the players. One method of cost sharing is proposed by Hoede in [7]. The allocation of the set game determines for each player i, ψ_i as a set of elements of U_s. If an element $u_S, S \subseteq N$ has cost $c(u_S)$, then this cost is to be shared by all players that

have u_S in their allocation. Our goal is to show the intimate relationship between cooperative games and set games. This paper is structured as follows. In Section 2 we show that with every cooperative game (N_c, v_c) one can associate a set game $\langle N_s, v_s \rangle$, using *basic units*. In Sections 3,4 and 5 we will analyze the relations between several solutions defined for a cooperative game and the solutions defined on the associated standard set game.

2 The standard set game

Consider a cooperative game (N_c, v_c), where N_c is the set of players and $v_c : 2^{N_c} \mapsto \mathbb{R}$ is a mapping which associates with each coalition S the *value* of the coalition. We associate with (N_c, v_c) the following set game, called *the standard set game associated to* (N_c, v_c). The universe U_s is $U_s = \{u_{S'} | S' \subseteq N_c\}$, where the elements u_S are called *basic units*. The set of players N_s is $N_s = N_c$ and $v_s : 2^{N_s} \to 2^{U_s}$ is defined by $v_s(\emptyset) = \emptyset$ and $v_s(S) = \{u_{S'} | S' \subseteq S, S' \neq \emptyset\}$. Every basic unit $u_S, S \subseteq N_s, S \neq \emptyset$ has a cost $c(u_S) = \sum\limits_{\substack{T \subseteq S \\ T \neq \emptyset}} (-1)^{|S|-|T|} v_c(T)$, while $c(u_\emptyset) = 0$. One can easily verify that for each $S, S \subseteq N_s$,

$$v_c(S) = \sum_{T \subseteq S} c(u_T). \tag{2}$$

3 The Shapley value

In this section we will first show two ways of obtaining the Shapley value for a cooperative game (N_c, v_c) via the standard set game associated with it. Let (N, v_c) be a cooperative game and $\langle N, v_s \rangle$ its associated standard set game. Consider the solutions IM, OCM and DS for $\langle N, v_s \rangle$. From the definition of these solutions (see Section 1) and the definition of a standard set game follows that

$$\psi_i = IM(N, v_s) = OCM(N, v_s) = DS(N, v_s) = \bigcup_{S \ni i} \{u_S\}.$$

Consider for this ψ_i the cost sharing method $a = (a_i)_{i \in N}$ defined as described before, leading to

$$a_i = \sum_{\substack{S \subseteq N \\ S \ni i}} \frac{1}{|S|} c(u_S), \tag{3}$$

which is the Shapley value.
Next we will show how, for a restricted class of games, the Shapley value comes forward with the help of the *excess vector (complaint vector)*. Suppose that

$I(v_c) \neq \emptyset$. For every $x \in I(v_c)$, the *excess vector (complaint vector)* $\theta(x)$ has as its coordinates the excesse

$$e(S,x) := v_c(S) - \sum_{i \in S} x_i, \text{ for every coalition } S,$$

and these excesses are written down in decreasing order. Using (2) the excesses can be written as$e(S,x) = \sum_{T \subseteq S} c(u_T) - \sum_{i \in S} x_i$. Suppose now that $c(u_S) \geq 0$, for all $S \subseteq N$, and that we are interested in finding an imputation x that keeps all the excesses negative. A natural way is to proceed as follows. First set $x_i = c(u_i)$ for each $i \in N_s$. Clearly, $e(\{i\},x) \leq 0$, for every $i \in N_s$ but for S with $|S| \geq 2$ it may happen that $e(S,x) > 0$. Next we change x such that

$$e(S,x) \leq 0 \text{ for every } S \subseteq N, |S| \leq 2. \tag{4}$$

Clearly,

$$x_i = c(u_{\{i\}}) + \frac{1}{2} \sum_{\substack{T \subseteq S \\ |T|=2}} c(u_T), \text{ for every } i \in N,$$

satisfies (4). Proceeding in this way, we finally obtain that the imputation

$$x_i = \sum_{\substack{S \subseteq N \\ S \ni i}} \frac{1}{|S|} c(u_S)$$

maintains all the excesses non-positive. Obviously, x is exactly the Shapley value for (N, v_c).

4 The standard set game and the core

¿From (2) follows that the core is the set of all imputations x satisfying $x(S) \geq \sum_{T \subseteq S} c(u_T)$ and $x(N) = \sum_{T \subseteq N} c(u_T)$. First of all remark that if, in the standard set game, $c(u_S) \geq 0$ for every $S \subseteq N$, then the core is nonempty. In Section 3 we already saw that in this case the Shapley value has the property that it makes the excesses non-positive. So it is in the core. In fact, the nonnegativity of the costs of the basic units implies a stronger statement with respect to (N_c, v_c), namely that the game is convex. It is easy to see that a cooperative game is convex if for the standard set game associated with it the following holds

$$\sum_{\substack{S' \subseteq S \cup T \\ S' \cap (S \setminus T) \neq \emptyset \\ S' \cap (T \setminus S) \neq \emptyset}} c(u_{S'}) \geq 0, \text{for every } S \subseteq N, T \subseteq N.$$

Hence, a game for which all basic units have nonnegative costs is convex. Therefore, the core is nonempty and the Shapley value is an element of it.

5 Discussion

The axiomatization of values for set games was first studied by Aarts, Funaki and Hoede [2,3]. The papers of Driessen and Sun [4,5] form a continuation of this work. In first instance it was somewhat unclear how set games compare to cooperative games. By the results given in this paper we now know that set games are intimately related to cooperative games with coalition values in the reals. However, they cover only the combinatorial aspect of normal cooperative games. It was for this reason that direct analogs for the τ-value or the nucleolus could not be found for the standard set game.

References

[1] H. Aarts, *Minimum cost spanning tree games and set games*, Ph.D. thesis, Faculty of Mathematical Sciences, University of Twente, Enschede, The Netherlands, 1994.

[2] H. Aarts, Y. Funaki and C. Hoede, A marginalistic value for monotonic set games, *International Journal of Game Theory* 26, pages 97-111, 1997.

[3] H. Aarts, Y. Funaki and C. Hoede, *Set games*, Homo Oeconomicus XVII (1/2), pages 137-154, ACCEDO Verlagsgesellschaft, Munchen, 2000.

[4] T. S. H. Driessen, H. Sun, *A potential approach to solutions for set games*, Memorandum 1571, Faculty of Mathematical Sciences, University of Twente, Enschede, The Netherlands, 2001.

[5] T. S. H. Driessen, H. Sun, *A uniform approach to semi-marginalistic values for set games*, Memorandum 1572, Faculty of Mathematical Sciences, University of Twente, Enschede, The Netherlands, 2001.

[6] C. Hoede, *Graphs and games,* Memorandum 1065, Faculty of Mathematical Sciences, University of Twente, Enschede, The Netherlands, 1992.

[7] C. Hoede, *A note on set games and cost sharing problems*, Memorandum 1642, Faculty of Mathematical Sciences, University of Twente, Enschede, The Netherlands, 2002.

[8] H. Sun, S. Zhang and X. Li, *A coalitional power value for set games.* Working paper, Department of Applied Mathematics, Northwestern Polytechnical University, Xi'an, Shaanxi, China, 1999.