The Ramsey Numbers of Paths Versus Fans

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Abstract

For two given graphs G and H, the Ramsey number $R(G, H)$ is the smallest positive integer p such that for every graph F on p vertices the following holds: either F contains G as a subgraph or the complement of F contains H as a subgraph. In this paper, we study the Ramsey numbers $R(P_n, F_m)$, where P_n is a path on n vertices and F_m is the graph obtained from m disjoint triangles by identifying precisely one vertex of every triangle (F_m is the join of K_1 and mK_2). We determine exact values for $R(P_n, F_m)$ for the following values of n and m: $n = 1, 2$ or 3 and $m \geq 2$; $n \geq 4$ and $2 \leq m \leq (n+1)/2$; $n \geq 7$ and $m = n-1$ or $m = n$; $n \geq 8$ and $(k \cdot n - 2k + 1)/2 \leq m \leq (k \cdot n - k + 2)/2$ with $3 \leq k \leq n-5$; $n = 4, 5$ or 6 and $m \geq n-1$; $n \geq 7$ and $m \geq (n-3)^2/2$.

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1 Introduction

Throughout this paper, all graphs are finite and simple. Let G be such a graph. The graph $\overline{G}$ is the *complement* of G, i.e., the graph obtained from the complete graph on $|V(G)|$ vertices by deleting the edges of G. A *fan* F_m is a graph on $2m + 1$ vertices obtained from m disjoint triangles (K_3s) by identifying precisely one vertex of every triangle (F_m is the *join* of K_1 and mK_2). The vertex corresponding to K_1 is called the *hub* of the fan. Given two graphs G and H, the *Ramsey number* $R(G, H)$ is defined as the smallest positive integer p such that every graph F on p vertices satisfies the following condition: F contains G as a subgraph or $\overline{F}$ contains H as a subgraph.

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In 1967, Geréncser and Gyárfás [3] determined all Ramsey numbers for paths versus paths. After that, Ramsey numbers $R(P_n, H)$ for paths versus other graphs H have been investigated in several papers, for example in [5], [1], [6], [4], [2], [7] and [8]. We study Ramsey numbers for paths versus fans.

2 Main results

We determine the Ramsey numbers $R(P_n, F_m)$ for the following values of n and m: $n = 1, 2$ or 3 and $m \geq 2$; $n \geq 4$ and $2 \leq m \leq (n+1)/2$; $n \geq 7$ and $m = n-1$ or $m = n$; $n \geq 8$ and $(k \cdot n - 2k + 1)/2 \leq m \leq (k \cdot n - k + 2)/2$ with $3 \leq k \leq n-5$; $n = 4, 5$ or 6 and $m \geq n-1$; $n \geq 7$ and $m \geq (n-3)^2/2$.

Proposition 1 *Let $m \geq 2$. Then* $R(P_n, F_m) = \begin{cases} 1 & \textit{for } n = 1 \\ 2m+1 & \textit{for } n = 2 \textit{ or } 3. \end{cases}$

We skip the easy proof of Proposition 1. The next lemma plays a key role in the proofs for the remaining cases.

Lemma 2 *Suppose G is a graph on $n \geq 4$ vertices containing no P_n. Let the paths $P^1, P^2, P^3, \ldots, P^k$ in G be chosen in the following way: $\bigcup_{j=1}^{k} V(P^j) = V(G)$, P^1 is a longest path in G, and, if $k > 1$, P^{i+1} is a longest path in $G - \bigcup_{j=1}^{i} V(P^j)$ for $1 \leq i \leq k-1$. Let z be an end vertex of P^k. Then:*
(i) $|V(P^1)| \geq |V(P^2)| \geq \ldots \geq |V(P^k)|$;
(ii) If $|V(P^k)| \geq \lfloor n/2 \rfloor$, then $|N(z)| \leq |V(P^k)| - 1$;
(iii) If $|V(P^k)| < \lfloor n/2 \rfloor$, then $|N(z)| \leq \lfloor n/2 \rfloor - 1$.

Proof. (i) is obvious.
(ii) follows from the choice of the paths and the observation that all neighbors of z are on P^k.
(iii) Now assume $|V(P^k)| < \lfloor n/2 \rfloor$. If z has no neighbors in $V(G) \setminus V(P^k)$, we are done. If z has some neighbors in $V(G) \setminus V(P^k)$, simple counting arguments yield the result: Denote by $\ell_1, \ldots, \ell_t$ the numbers of vertices on the paths in $P^1, \ldots, P^k$ that contain a neighbor of z, chosen in such a way that $\ell_t \geq \ldots \geq \ell_1$, and denote by $d_1, \ldots, d_t$ the numbers of neighbors of z on the corresponding paths. Then, $\ell_1 = |V(P^k)| \geq d_1 + 1$ and $\ell_2 \geq 2\ell_1 + 2d_1 - 1$. Observing that z connects any two of the considered paths, and using elementary counting techniques, we get (if $t \geq 3$) $\ell_j \geq 2(\frac{\ell_{j-1}-1}{2} + 2) + 2d_j - 1 = \ell_{j-1} + 2d_j + 2$ for $3 \leq j \leq t$. This implies (for $t \geq 2$) that $\ell_t \geq 2(d_1 + \ldots + d_t) + 2(t-2) + 1 \geq |N(z)| + 1$. Since $\ell_t \leq n-1$ and $|N(z)|$ in an integer, this yields the desired result. $\square$

Theorem 3 *Let $n \geq 4$ and $2 \leq m \leq (n+1)/2$. Then $R(P_n, F_m) = 2n-1$.*

Proof. The graph $2K_{n-1}$ shows that $R(P_n, F_m) > 2n-2$. Let G be a graph on $2n-1$ vertices and assume G contains no P_n. We are going to show that $\overline{G}$ contains an F_m. Choose the paths $P^1, \ldots, P^k$ and the vertex z as in Lemma 2. Lemma 2 implies $|N(z)| \leq (2n-1)/3 - 1$. Hence, z is not a neighbor of at least $2m$ vertices. It is not hard to show by case distinction that there is an F_m in $\overline{G}$ with hub z. We leave the details to the reader. $\square$

The following lemma provides upper bounds that yield several exact Ramsey numbers in the sequel.

Lemma 4 *If $n \geq 4$ and $m \geq n-1$, then*

$$R(P_n, F_m) \leq \begin{cases} 2m+n-1 \textit{ for } 2m = 1 \; mod(n-1) \\ 2m+n-2 \textit{ for other values of } m. \end{cases}$$

Proof. Let G be a graph that contains no P_n and has order

$$|V(G)| = \begin{cases} 2m+n-1 \text{ for } 2m = 1 \; mod(n-1) \\ 2m+n-2 \text{ for other values of } m. \end{cases} \tag{1}$$

Choose the paths $P^1, \ldots, P^k$ and the vertex z in G as in Lemma 2. Because of (1), not all P^i can have $n-1$ vertices, so $|V(P^k)| \leq n-2$. By Lemma 2, $|N(z)| \leq n-3$. Hence, z is not a neighbor of (at least) $(2m+n-2)-1-(n-3) = 2m$ vertices. We will use the following result that has been proved in [1]: $R(P_t, C_s) = s + \lfloor t/2 \rfloor - 1$ for $s \geq \lfloor (3t+1)/2 \rfloor$. We distinguish the following cases.

Case 1 $|N(z)| \leq \lfloor n/2 \rfloor - 2$ or n is odd and $|N(z)| = \lfloor n/2 \rfloor - 1$.
Since $|V(G) \setminus N[z]| \geq 2m + \lfloor n/2 \rfloor - 1$, we find that $\overline{G - N[z]}$ contains a C_{2m}. So, there is an F_m in $\overline{G}$ with z as a hub.

Case 2 n is even and $|N(z)| = \lfloor n/2 \rfloor - 1$.
Since $|V(G) \setminus N[z]| = (2m+n-2) - n/2 = 2m + n/2 - 2$, we find that $\overline{G - N[z]}$ contains a C_{2m-1}; denote its vertices by $v_1, v_2, v_3, \ldots, v_{2m-1}$ in the order of appearance on the cycle with a fixed orientation. There are $n/2 - 1$ vertices in $U = V(G) \setminus (V(C_{2m-1}) \cup N[z])$, say $u_1, u_2, \ldots, u_{n/2-1}$. If some vertex v_i $(i = 1, \ldots, 2m-1)$ is no neighbor of some vertex u_j $(j = 1, \ldots, n/2-1)$, w.l.o.g. assume $v_{2m-1}u_1 \notin E(G)$. Then $\overline{G}$ contains an F_m with hub z and additional

edges $v_1v_2, v_3v_4, \ldots, v_{2m-3}v_{2m-2}, v_{2m-1}u_1$. So each of the v_i is adjacent to all u_j in G. For every choice of a subset of $n/2+1$ vertices from $V(C_{2m-1})$, there is a path on $n-1$ vertices in G alternating between the vertices of this subset and the vertices of U, starting and terminating in two arbitrary vertices from the subset. Since G contains no P_n, there are no edges $v_iv_j \in E(G)$ $(i, j \in \{1, \ldots, 2m-1\})$. This implies that $V(C_{2m-1}) \cup \{z\}$ induces a K_{2m} in $\overline{G}$. Since G contains no P_n, no v_i is adjacent to a vertex of $N(z)$. This implies that $\overline{G}$ contains a $K_{2m+1} - e$ for some edge zw with $w \in N(z)$, and hence $\overline{G}$ contains an F_m with one of the v_i as a hub.

Case 3 Suppose that there is no choice for P^k and z such that one of the former cases applies. Then $|N(w)| \geq \lfloor n/2 \rfloor$ for any end vertex w of a path on $|V(P^k)|$ vertices in $G - \bigcup_{j=1}^{k-1} V(P^j)$. This implies all neighbors of such w are in $V(P^k)$ and $|V(P^k)| \geq \lfloor n/2 \rfloor + 1$. So for the two end vertices z_1 and z_2 of P^k we have that $|N(z_i) \cap V(P^k)| \geq \lfloor n/2 \rfloor \geq |V(P^k)|/2$. By standard arguments in hamiltonian graph theory we obtain a cycle on $|V(P^k)|$ vertices in G. This implies that any vertex of $V(P^k)$ could serve as w. By the assumption of this last case, we conclude that there are no edges in G between $V(P^k)$ and the other vertices. This also implies that all vertices of P^k have degree at least $2m$ in $\overline{G}$.

Similar arguments for $P^{k-1}, \ldots, P^1$ can be used to show that all vertices of G have degree at least $2m-1$ in $\overline{G}$. We omit the details. Now let $|V(P^k)| = \ell$. Then in the graph $H = \overline{G} - V(P^k)$ all vertices have degree at least $2m-1-\ell \geq m-1-\ell+n-1 = \frac{1}{2}(2m-4+2n-2\ell) \geq \frac{1}{2}(2m-4+2n-\ell-(n-2)) = \frac{1}{2}(2m+n-\ell-2) \geq \frac{1}{2}(|V(H)|-1)$. This implies there exists a Hamilton path in H. Since $|V(H)| \geq 2m$ and z is a neighbor of all vertices in H, it is clear that $\overline{G}$ contains an F_m with z as a hub. This completes the proof of Lemma 4. $\square$

The next corollaries can be obtained by indicating suitable graphs for providing sharp lower bounds, and combining them with the upper bounds from Lemma 4. We omit the details.

Corollary 5 *If $n \geq 7$ and $m = n-1$ or $m = n$, then $R(P_n, F_m) = 2m+n-2$.*

Corollary 6 *If $n \geq 8$ and $(k \cdot n - 2k + 1)/2 \leq m \leq (k \cdot n - k + 2)/2$ for $3 \leq k \leq n-5$, then* $R(P_n, F_m) = \begin{cases} 2m+n-1 \text{ for } 2m = 1 \bmod (n-1) \\ 2m+n-2 \text{ for other values of } m. \end{cases}$

Corollary 7 *If either $n = 4, 5$ or 6 and $m \geq n-1$ or $n \geq 7$ and $m \geq$*

$(n-3)^2/2$, *then* $R(P_n, F_m) = \begin{cases} 2m+n-1 \text{ for } 2m = 1 \ mod(n-1) \\ 2m+n-2 \text{ for other values of } m. \end{cases}$

References

[1] R.J. Faudree, S.L. Lawrence,T.D. Parsons and R.H. Schelp, Path-cycle Ramsey numbers, *Discrete Mathematics*, **10** (1974), 269–277.

[2] R.J. Faudree, R.H. Schelp and M. Simonovits, On some Ramsey type problems connected with paths, cycles and trees, *Ars Combinatoria*, **29A** (1990), 97–106.

[3] L. Geréncser and A. Gyárfás, On Ramsey-type problems, *Annales Universitatis Scientiarum Budapestinensis, Eötvös Sect. Math.*, **10** (1967), 167–170.

[4] R. Häggkvist, On the path-complete bipartite Ramsey numbers, *Discrete Mathematics*, **75** (1989), 243–245.

[5] T.D. Parsons, The Ramsey numbers $r(P_m, K_n)$, *Discrete Mathematics*, **6** (1973), 159–162.

[6] T.D. Parsons, Path-star Ramsey numbers, *Journal of Combinatorial Theory*, Series B, **17** (1974), 51–58.

[7] Surahmat and E.T. Baskoro, On the Ramsey number of a path or a star versus W_4 or W_5, *Proceedings of the 12th Australasian Workshop on Combinatorial Algorithms*, Bandung, Indonesia (2001), 174–178.

[8] C. Yaojun, Z. Yunqing and Z.K. Min, The Ramsey numbers of paths versus wheels, *preprint* (2002).