A coding problem for pairs of subsets

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Abstract: Let X be an n–element finite set, $0 < k \le n/2$ an integer. Suppose that $\{A_1, A_2\}$ and $\{B_1, B_2\}$ are pairs of disjoint k-element subsets of X (that is, $|A_1| = |A_2| = |B_1| = |B_2| = k$, $A_1 \cap A_2 = \emptyset$, $B_1 \cap B_2 = \emptyset$). Define the distance of these pairs by $d(\{A_1, A_2\}, \{B_1, B_2\}) = \min\{|A_1 - B_1| + |A_2 - B_2|, |A_1 - B_2| + |A_2 - B_1|\}$. This is the minimum number of elements of $A_1 \cup A_2$ one has to move to obtain the other pair $\{B_1, B_2\}$. Let $C(n, k, d)$ be the maximum size of a family of pairs of disjoint k-subsets, such that the distance of any two pairs is at least d.

Here we establish a conjecture of Brightwell and Katona concerning an asymptotic formula for $C(n, k, d)$ for k, d are fixed and $n \to \infty$. Also, we find the exact value of $C(n, k, d)$ in an infinite number of cases, by using special difference sets of integers. Finally, the questions discussed above are put into a more general context and a number of coding theory type problems are proposed.

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1 The transportation distance

Let X be a finite set of n elements. When it is convenient we identify it with the set $[n] := \{1, 2, \dots, n\}$. The family of the k-sets of an underlying set X is denoted by $\binom{X}{k}$. For $0 < k \le n/2$ let $\mathcal{Y}$ be the family of unordered disjoint pairs $\{A_1, A_2\}$ of k-element subsets of X (that is, $|A_1| = |A_2| = k, A_1 \cap A_2 = \emptyset$). The *transportation distance* or *Enomoto-Katona distance* d on $\mathcal{Y}$ is defined by

$$d(\{A_1, A_2\}, \{B_1, B_2\}) = \min\{|A_1 - B_1| + |A_2 - B_2|, |A_1 - B_2| + |A_2 - B_1|\}. \tag{1}$$

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In fact, this is an instance of a more general notion. Whenever (Z, ρ) is a metric space, we can define a metric $\rho^{(s)}$ on $Z^{(s)}$, the set of unordered s-tuples from Z, by

$$\rho^{(s)}(\{x_1, \dots, x_s\}, \{y_1, \dots, y_s\}) = \min_{\pi \in S_s} \sum_{i=1}^{s} \rho(x_i, y_{\pi(i)}). \tag{2}$$

It is not hard to verify that $\rho^{(s)}$ satisfies the triangle inequality, i.e., it really is a metric. The transportation distance defined above is obtained by taking $s = 2$, Z to be the set of k-elements subsets of X and ρ is half of their symmetric difference.

The minimization problem (2) (where ρ can be an arbitrary metric) is one of the fundamental combinatorial optimization problems, a so called *assignment problem*, a special case of a more general *Monge-Kantorovich transportation problem* (see, e.g., the monograph [18]).

The transportation distance between finite sets of the same cardinalities is one of the interesting measurements among many different ways to define how two sets differ from each other. In [1], Ajtai, Komlós and Tusnády considered the assignment problem from a different perspective, and determined with high probability the transportation distance between two sets of points randomly chosen in a unit square.

Since the transportation distance is an important notion, especially from the algorithmic point of view, there are monographs and graduate texts about this topic, see, e.g., [18]. It is also mentioned in the *Encyclopedia of Distances* [5] as the "KMMW metric" (p. 245 in Chapter 14) or as the "c-transportation distance". Nevertheless, many combinatorial problems are still unsolved. The packing of sets in spherical spaces with large transportation distance will be discussed in [8].

2 Packings and codes

Given a metric space (Z, ρ) and a distance $h > 0$, the *packing number* $\delta(Z, \geq h)$ is the maximum number of elements in Z with pairwise distance at least h.

A (v, k, t) packing $\mathcal{P} \subseteq \binom{[v]}{k}$ is a family of k-sets with pairwise intersections at most $t - 1$ (here $v \geq k \geq t \geq 1$). In other words, every t-subset is covered at most once. Its maximum size is denoted by $P(v, k, t)$. Obviously,

$$P(v, k, t) \leq \binom{v}{t} / \binom{k}{t}. \tag{3}$$

If here equality holds then $\mathcal{P}$ is called a Steiner system $S(v, k, t)$, or a *t-design* of parameters v, k, t and $\lambda = 1$ (for more definitions concerning symmetric combinatorial structures esp., difference sets, etc. see, e.g., the monograph by Hall [10]). More generally, for a set K of integers, a family $\mathcal{P}$ on v elements is called a (v, K, t)-design (packing) if every t-subset of $[v]$ is contained in exactly one (at most one) member of $\mathcal{P}$ and $|P| \in K$ for every $P \in \mathcal{P}$.

Determining the packing number is a central problem of Coding Theory, it is essentially the same problem as finding the rate of a large-distance error-correcting code.

If equality holds in (3) then every i-subset of $[v]$ is contained in $\binom{v-i}{t-i} / \binom{k-i}{t-i}$ members of $\mathcal{P}$ for $i = 0, 1, \dots, t-1$. We say that v, k, and t satisfy the *divisibility conditions* if these t fractions are integers. It was recently proved by Keevash [13] that for any given k and t there exists a bound $v_0(k,t)$ such that these trivial necessary conditions are also sufficient for the existence of a t-design.

$$\text{An } S(v,k,t) \text{ exists if } v, k, \text{ and } t \text{ satisfy the divisibility conditions and } v > v_0(k,t). \tag{4}$$

This implies Rödl's theorem[17], that for given k and t as $v \to \infty$

$$P(v,k,t) = (1+o(1))\binom{v}{t}/\binom{k}{t}. \tag{5}$$

Even more, (4) implies that here the error term is only $O(v^{t-1})$. The case $t=2$ was proved much earlier by Wilson [19]. For this case he also proved the following more general version. For a finite K there exists a bound $v_0(K,2)$ such that for $v > v_0(K,2)$

$$\text{a } (v,K,2) \text{ design exists if } v \text{ and } K \text{ satisfy the generalized divisibility conditions,} \tag{6}$$

namely, g.c.d.$(\binom{k}{2} : k \in K)$ divides $\binom{v}{2}$ and g.c.d.$(k-1 : k \in K)$ divides $v-1$.

3 Packing pairs of subsets

In this paper, we concentrate on the space $\mathcal{Y}$ of pairs of *disjoint* k-subsets. We say that a set $\mathcal{C} \subset \mathcal{Y}$ of such pairs is a 2-(n,k,d)–*code* if the distance of any two elements is at least d. Let $C(n,k,d)$ be the maximum size of a 2-(n,k,d)-code. Enomoto and Katona in [6] proposed the problem of determining $C(n,k,d)$. For the origin of the problem see [4]. Connections to Hamilton cycles in the Kneser graph $K(n,k)$ are discussed in [12]. The problem makes sense only when $d \le 2k \le n$. It is obvious, that a maximal 2-$(n,k,1)$ code consists of all the pairs, $C(n,k,1) = |\mathcal{Y}| = \frac{1}{2}\binom{n}{k}\binom{n-k}{k}$. A 2-$(n,k,2k)$ code consists of mutually disjoint k-sets, hance $C(n,k,2k) = \lfloor n/2k \rfloor$.

In Section 5 we present a method for the determination the exact value of $C(n,k,2k-1)$ for infinitely many n. However, we were able to complete the cases $k=2,3$ only, the cases of pairs and triple systems.

Theorem 1. *If $n \equiv 1 \bmod 8$ and $n > n_0$ then $C(n,2,3) = \frac{n(n-1)}{8}$.
If $n \equiv 1, 19 \bmod 342$ and $n > n_0$ then $C(n,3,5) = \frac{n(n-1)}{18}$.*

The following theorem was proved in [2]. Let $d \le 2k \le n$ be integers. Then

$$C(n,k,d) \le \frac{1}{2}\frac{n(n-1)\cdots(n-2k+d)}{k(k-1)\cdots\lceil\frac{d+1}{2}\rceil \cdot k(k-1)\cdots\lfloor\frac{d+1}{2}\rfloor}. \tag{7}$$

Quisdorff [16] gave a new proof and using ideas from classical coding theory he significantly improved the upper bound for small values of n (for $n \le 4k$). For completeness, in Section 6 we reprove (7) in an even more streamlined way.

Concerning larger values of n one can build a 2-(n,k,d) code from smaller ones using the following observation. If $|(A_1 \cup A_2) \cap (B_1 \cup B_2)| \le 2k-d$ holds for the disjoint pairs $\{A_1, A_2\} \in \mathcal{Y}$, $\{B_1, B_2\} \in \mathcal{Y}$ then $d(\{A_1,A_2\},\{B_1,B_2\}) \ge d$. Take a $(2k-d+1)$-packing $\mathcal{P}$ on n elements and choose a 2-$(|P|,k,d)$-code on each members $P \in \mathcal{P}$. We obtain

$$\sum_{P \in \mathcal{P}} C(|P|,k,d) \le C(n,k,d). \tag{8}$$

This gives

$$P(n,p,2k-d+1)C(p,k,d) \le C(n,k,d). \tag{9}$$

Fix p (and k, t and d) then Rödl's theorem (5) gives $(1+o(1))\binom{n}{2k-d+1}\binom{p}{2k-d+1}^{-1}C(p,k,d) \le C(n,k,d)$. Rearranging we get, that the sequence $C(n,k,d)/\binom{n}{2k-d+1}$ is essentially nondecreasing in n, for any fixed p (and k, t and d)

$$C(p,k,d)/\binom{p}{2k-d+1} \le (1+o(1))C(n,k,d)/\binom{n}{2k-d+1}.$$

Since, obviously, $C(2k,k,d) \ge 1$ we obtain that $\lim_{n\to\infty} C(n,k,d)/\binom{n}{2k-d+1}$ exists, it is positive, it equals to its supremum, and finite by (7).

It was conjectured ([2], Conjecture 8) that the upper estimate (7) is asymptotically sharp. We prove this conjecture in Section 7.

Theorem 2.

$$\lim_{n\to\infty} \frac{C(n,k,d)}{n^{2k-d+1}} = \frac{1}{2}\frac{1}{k(k-1)\cdots\lceil\frac{d+1}{2}\rceil \cdot k(k-1)\cdots\lfloor\frac{d+1}{2}\rfloor}.$$

4 The case $d=2$, the exact values of $C(n,k,2)$

Besides the cases mentioned in the previous Section (the cases $d=1$, $d=2k$ and $(k,d) \in \{(2,3),(3,5)\}$) we can solve one more case easily, namely if $d=2$. Since $C(2k,k,2)] = |\mathcal{Y}| = \frac{1}{2}\binom{2k}{k}$ the construction (9) gives $P(n,2k,2k-1)\frac{1}{2}\binom{2k}{k} \le C(n,k,2)$. Then the recent result of Keevash (4) gives the lower bound in the following Proposition. The upper bound follows from (7).

Proposition 3. $C(n,k,2) = \binom{n}{2k-1}\frac{1}{4k}\binom{2k}{k}$ *for all* $n > n_0(k)$ *whenever the divisibility conditions of* (4) *hold.* □

5 The case $d=2k-1$, the exact values of $C(n,k,2k-1)$

The distance $\delta(a,b)$ of two integers mod m $(1 \le a,b \le m)$ is defined by

$$\delta(a,b) = \min\{|b-a|, |b-a+m|\}.$$

(Imagine that the integers $1,2,\dots,m$ are listed around the cirle clockwise uniformly. Then $\delta(a,b)$ is the smaller distance around the circle from a to b.) $\delta(a,b) \le \frac{m}{2}$ is trivial. Observe that $b-a \equiv d-c \bmod m$ implies $\delta(a,b)=\delta(c,d)$.

We say that the pair $S=\{s_1,\dots,s_k\}$, $T=\{t_1,\dots,t_k\} \subset \{1,\dots,m\}$ of disjoint sets is *antagonistic* mod m if

(i) all the $k(k-1)$ integers $\delta(s_i,s_j)$ $(i\ne j)$ and $\delta(t_i,t_j)$ $(i \ne j)$ are different,
(ii) the k^2 integers $\delta(s_i,t_j)$ $(1\le i,j\le k)$ are all different and
(iii) $\delta(s_i,t_j) \ne \frac{m}{2}$ $(1 \le i,j \le k)$.

If there is a pair of disjoint antagonistic k-element subsets mod m then $2k^2+1 \le m$ must hold by (ii) and (iii).

Problem 4. *Is there a pair of disjoint, antagonistic k-element sets* mod $2k^2+1$*?*

We have an affirmative answer only in three cases.

Proposition 5. *There is a pair of disjoint, antagonistic k-element sets* mod $2k^2+1$ *when* $k = 1, 2, 3$.

PROOF: We simply give such k-element sets in these cases. It is easy to check that they satisfy the conditions.

$k = 1$: $S = \{1\}, T = \{2\}$.

$k = 2$: $S = \{1, 8\}, T = \{2, 3\}$.

$k = 3$: $S = \{1, 5, 19\}, T = \{2, 13, 15\}$. $\square$

Lemma 6. *If there is a pair of disjoint, antagonistic k-element sets* mod m *then* $C(m, k, 2k-1) \geq m$.

PROOF: Let (S, T) be the antagonistic pair. The shifts $S(u) = \{a + u \bmod m : s \in S\}, T(u) = \{s + u \bmod m : s \in T\} (0 \leq u < m)$ will serve as pairs of disjoint subsets of X.

Suppose that $S(u)$ and $S(v)$ $(u \neq v)$ have two elements in common: $s_1 + u = s_2 + v \neq s_3 + u = s_4 + v$ where $s_1, s_2, s_3, s_4 \in S, (s_1, s_2) \neq (s_3, s_4)$. The difference is $s_1 - s_2 = s_3 - s_4$ contradicting (i). One can prove in the same way that $T(u)$ and $T(v)$ $(u \neq v)$ and $S(u)$ and $T(v)$, respectively, have at most one element in common. In other words the intersection of any pair from the sets $S(u), T(u), S(v), T(v)$ has at most one element.

Suppose now that both $S(u) \cap S(v)$ and $T(u) \cap T(v)$ are non-empty for some $u \neq v$. Then $s_1 + u = s_2 + v, t_1 + u = t_2 + v$ holds for some $s_1, s_2 \in S, t_1, t_2 \in T$. This leads to $v - u = s_1 - s_2 = t_1 - t_2$, contradicting (i), again.

Finally, suppose that both $S(u) \cap T(v)$ and $T(u) \cap S(v)$ are non-empty for some $u \neq v$. Then $s_1 + u = t_1 + v, t_2 + u = s_2 + v$ is true for some $s_1, s_2 \in S, t_1, t_2 \in T$. Here $v - u = s_1 - t_1 = t_2 - s_2$ is obtained, contradicting either (ii) or (iii) (the latter one, if $s_1 - t_1 = t_1 - s_1$ is obtained).

This proves that the distance of the pairs $(S(u), T(u))$ and $(S(v), T(v))$ $(u \neq v)$ is at least $2k - 1$. $\square$

Corollary 7. *Suppose that there is Steiner family $\mathcal{S}(n, 2k^2+1, 2)$ and a disjoint, antagonistic pair of k-element subsets* mod $2k^2+1$ *then*

$$C(n, k, 2k-1) = \frac{n(n-1)}{2k^2}.$$

PROOF: The upper bound $C(n, k, 2k-1) \leq n(n-1)/2k^2$ is a corollary of (7).

The lower estimate is obtained from (9). By Lemma 6 one can choose $2k^2+1$ pairs of disjoint k-subsets with distance $2k-1$ in a set of $2k^2+1$ elements. This can be done in each of the members of $\mathcal{S}(n, 2k^2+1, 2)$. Since the members have at most one common element, the distance of two pairs in distinct members of $\mathcal{S}(n, 2k^2+1, 2)$ will have distance at least $2k-1$. Therefore all the

$$|\mathcal{S}(n, 2k^2+1, 2)|(2k^2+1) = \frac{\binom{n}{2}}{\binom{2k^2+1}{2}}(2k^2+1) = \frac{n(n-1)}{2k^2}$$

pairs have distance at least 1. $\square$

Proof of Theorem 1. We only need lower bounds, i.e., constructions. The case $k = 3$ follows from Wilson's theorem (4) of the existence of $S(n, 19, 2)$, Proposition 5 and Corollary 7.

Similarly, the case $k = 2$ for $n \equiv 1, 9 \bmod 72$ follows in the same way using Steiner systems $S(n, 9, 2)$ and the fact $C(9, 2, 3) = 9$ from Corollary 7. However, one can see that $C(17, 3, 2) = 34$ and then the results follows from Wilson's theorem (6) of the existence of $S(n, \{9, 17\}, 2)$ for all large $n \equiv 1 \bmod 8$ and construction (8).

The construction for $C(17, 2, 3)$ is similar to the proof of Lemma 6. The 9 pairs there are defined as $\{\{x + 1, x + 8\}, \{x + 2, x + 3\}\} : x \in Z_9\}$. These correspond to a perfect edge decomposition of K_9 into C_4's with side lengths $1, 3, 4$, and 2. For $n = 17$ we take the pairs $\{\{x, x+7\}, \{x+2, x+6\}\} : x \in Z_{17}\}$ and $\{\{y, y+11\}, \{y+7, y+8\}\} : y \in Z_{17}\}$ which correspond to C_4's of side lengths $(2, 5, 1, 6)$ and $(7, 4, 3, 8)$, respectively. $\square$

Note that the method gives that $C(n, 1, 1) = \frac{n(n-1)}{2}$ when $n \equiv 1, 3 \bmod 6$. This, however, is trivial for all n.

6 A new proof of the upper estimate

The upper estimate in (7) was proved in [2]. We give a new, more illuminating proof here.

Given a pair $\{A, B\}$ of disjoint k-element sets let $\mathcal{P}(\{A, B\}, u, v)$ denote the family of pairs $\{U, V\}$ where $|U| = u, |V| = v$ and $U \subseteq A, V \subseteq B$ or vice versa. We have

$$|\mathcal{P}(\{A, B\}, u, v)| = 2\binom{k}{u}\binom{k}{v}.$$

Suppose first $u < v$. Then the total number of pairs $\{U, V\}, U \cap V = \emptyset, |U| = u, |V| = v$ in an n-element set is

$$\binom{n}{u}\binom{n-u}{v}.$$

Let $\{A_1, B_1\}, \{A_2, B_2\}$ be two pairs with distance at least d, and $u < v$ be two nonnegative integers such that $u + v = 2k - d + 1$. By definition (1), $\mathcal{P}(\{A_1, B_1\}, u, v)$ and $\mathcal{P}(\{A_2, B_2\}, u, v)$ are disjoint. We have

$$C(n, k, d) \leq \frac{\binom{n}{u}\binom{n-u}{v}}{2\binom{k}{u}\binom{k}{v}} = \frac{n(n-1)\dots(n-2k+d)}{2k(k-1)\dots(k-u+1)k(k-1)\dots(k-v+1)} \tag{10}$$

for every pair u, v that satisfies the above requirements. If $u = v$, then equality (10) holds by similar arguments.

The numerator does not depend on u, and the denominator is maximized when u and v are as close as possible, i.e., for $u = 2k - \lceil \frac{d-1}{2} \rceil$ and $v = 2k - \lfloor \frac{d-1}{2} \rfloor$. Substituting these values, we obtain the upper estimate in (7). $\square$

7 Nearly perfect selection

Let $\mathcal{W}$ be the family of pairs $\{U, V\}$ such that $U, V \subseteq [n]$, $U \cap V = \emptyset$, and $|U| + |V| = 2k - d + 1$ holds. Note that $|\mathcal{W}| = \frac{1}{2}\sum_{0 \leq u \leq 2k-d+1} \binom{n}{u}\binom{n-u}{(2k-d+1)-u}$. For a pair $\{A, B\}$ of disjoint k-element sets, let $\mathcal{P}(\{A, B\})$ denote the family of pairs $\{U, V\} \in \mathcal{W}$ for which $U \subseteq A$ and $V \subseteq B$, or vice versa.

Lemma 8. *$d(\{A_1, B_1\}, \{A_2, B_2\}) \le d-1$ holds if and only if $\mathcal{P}(\{A_1, B_1\}) \cap \mathcal{P}(\{A_2, B_2\}) \neq \emptyset$.*

Proof: Suppose that $\{U, V\} \in \mathcal{P}(\{A_1, B_1\}) \cap \mathcal{P}(\{A_2, B_2\})$, say $U \subset A_1 \cap A_2$ and $V \subset B_1 \cap B_2$. Then $|A_1 - A_2| \le k - |U|, |B_1 - B_2| \le k - |V|$ imply $|A_1 - A_2| + |B_1 - B_2| \le 2k - |U| - |V| = d - 1$ proving the statement. The other case is analogous.

Conversely, if the distance is at most $d-1$ then either $|A_1 - A_2| + |B_1 - B_2| \le d-1$ or $|A_1 - B_2| + |B_1 - A_2| \le d-1$ must hold. Suppose that the first one is true. Then $|A_1 \cap A_2| + |B_1 \cap B_2| \ge 2k - d + 1$ follows. Take $U = A_1 \cap A_2$ and a $V \subseteq B_1 \cap B_2$ such that $|V| = 2k - d + 1 - |U|$. Then $\mathcal{P}(\{A_1, B_1\}) \cap \mathcal{P}(\{A_2, B_2\}) \neq \emptyset$ holds, as claimed. $\square$

We can view the sets $\mathcal{P}(\{A, B\})$ as the edges of a hypergraph on the vertex set $\mathcal{W}$. Let us call this hypergraph $\mathcal{H}$. Then a 2-(n, k, d)-code corresponds to a *matching* in $\mathcal{H}$.

In his celebrated paper [17], Rödl established (5) in the following way. He viewed the t-element sets as vertices of a $\binom{k}{t}$-uniform hypergraph $\mathcal{H}_n$ whose edges correspond to the k-element subsets of $[n]$. Equality (5) is in fact a statement about the existence of an almost perfect matching in $\mathcal{H}_n$. Using the same key proof idea, a powerful generalization by Frankl and Rödl [7] guarantees the existence of almost perfect matchings in hypergraphs satisfying certain more general conditions. Various generalizations and stronger versions versions were later proved, e.g., by Pippenger and Spencer [15].

A function $t : E(\mathcal{H}) \to \mathbb{R}$ is a *fractional matching* of the hypergraph $\mathcal{H}$ if $\sum_{e \in E(\mathcal{H}); x \in e} t(e) \le 1$ holds for every vertex $x \in V(\mathcal{H})$. The *fractional matching number*, denoted $\nu^*(\mathcal{H})$ is the maximum of $\sum_{e \in E(\mathcal{H})} t(e)$ over all fractional matchings. If $\nu(\mathcal{H})$ denotes the maximum size of a matching in $\mathcal{H}$, then clearly

$$\nu(\mathcal{H}) \le \nu^*(\mathcal{H}).$$

Kahn [11] proved that under certain conditions, asymptotic equality holds. Both the hypotheses and the conclusion are in the spirit of the Frankl–Rödl theorem.

Given a hypergraph $\mathcal{H}$ with vertex set $[n]$, a fractional matching t and a subset $W \subseteq [n]$, define $\bar{t}(W) = \sum_{W \subseteq e \in E(\mathcal{H})} t(e)$ and $\alpha(t) = \max\{\bar{t}(\{x, y\}) : x, y \in V(\mathcal{H}), x \neq y\}$. In other words, $\alpha(t)$ is a fractional generalization of the codegree. Let $t(\mathcal{H})$ denote $\sum_{e \in E(\mathcal{H})} t(e)$. We say that $\mathcal{H}$ is *s-bounded* if each of its edges has size at most s.

Theorem 9 ([11])**.** *For every s and every $\varepsilon > 0$ there is a δ such that whenever $\mathcal{H}$ is an s-bounded hypergraph and t a fractional matching with $\alpha(t) < \delta$, then*

$$\nu(\mathcal{H}) > (1 - \varepsilon) t(\mathcal{H}).$$

Proof of Theorem 2. In the light of Lemma 8 it suffices to verify the conditions of Theorem 9 and to produce a fractional matching t of the hypergraph $\mathcal{H}$ of the desired size.

Define a constant weight function $t : E(\mathcal{H}) \to \mathbb{R}$ by

$$t(e) = \frac{\lceil \frac{d-1}{2} \rceil ! \lfloor \frac{d-1}{2} \rfloor !}{n^{d-1}}.$$

For a vertex $x = \{U, V\} \in \mathcal{W}$ with $|U| = u$ and $|V| = v$ we have

$$\deg(\{U, V\}) = \binom{n-u-v}{k-u} \binom{n-k-v}{k-v} \le \frac{n^{d-1}}{(k-u)!(k-v)!} \le \frac{n^{d-1}}{\lceil \frac{d-1}{2} \rceil ! \lfloor \frac{d-1}{2} \rfloor !}$$

hence t is indeed a fractional matching. Note that $t(\mathcal{H})$ is is asymptotically equal to the quantity in the statement of the Theorem 2.

The hypergraph $\mathcal{H}$ is not regular but s-bounded with $s = \frac{1}{2}\sum_u \binom{k}{u}\binom{k}{(2k-d+1)-u}$. Here s does not depend on n. For $x, y \in V(\mathcal{H}) = \mathcal{W}$ let $\deg(x, y)$ denote the codegree of $x = \{U, V\}$ and $y = \{U', V'\}$, i.e., the number of hyperedges $\mathcal{P}(\{A, B\})$ that contain both x and y. If $U \cup V = U' \cup V'$ (they partition the same $(2k-d+1)$-element set) then the codegre $\deg(x, y) = 0$. Otherwise, $|U \cup U' \cup V \cup V'| \geq 2k - d + 2$ and $(U \cup U' \cup V \cup V') \subset (A \cup B)$ imply that

$$\deg(\{U, V\}, \{U', V\}) = O(n^{d-2}).$$

Hence $\alpha(t) = \deg(\{U, V\}, \{U', V\}) \cdot t(e) = o(1)$ and Kahn's theorem completes the proof. $\square$

8 s-tuples of sets, q-ary codes

Let $\mathcal{Y}^{(s)}$ be the family of s-tuples of pairwise disjoint k-element subsets of $[n]$. A natural definition of a metric on $\mathcal{Y}^{(s)}$ was already mentioned in the introduction, in equation (2). With ρ being half the symmetric difference, the distance is defined as

$$\rho^{(s)}(\{A_1, \dots, A_s\}, \{B_1, \dots, B_s\}) = \min_{\pi \in S_s} \sum_{i=1}^{s} |A_i \setminus B_{\pi(i)}|.$$

Let $C_s(n, k, d)$ denote the maximum size of a subfamily $\mathcal{S}$ of $\mathcal{Y}^{(s)}$ such that any two elements in $\mathcal{S}$ have distance at least d. The proofs presented in Sections 7 and 6 can be easily adapted to determining $C_s(n, k, d)$, as well. The proof of the lower and the upper bounds in Theorem 10 is completely analogous to the proofs of inequality (7) and Theorem 2.

Theorem 10.

$$\lim_{n\to\infty} \frac{C_s(n, k, d)}{n^{sk-d+1}} = \frac{1}{s!} \frac{\lceil \frac{d-1}{s} \rceil ! \lceil \frac{d-2}{s} \rceil ! \dots \lceil \frac{d-s}{s} \rceil !}{(k!)^s}. \qquad \square$$

Let $\mathcal{Y}_q$ be the set of q-ary vectors of length n and weight k (weight is the number of nonzero entries). Let $A_q(n, d, k)$ be the maximum size of a subset $\mathcal{C} \subseteq \mathcal{Y}_q$ such that $\rho'(u, v) \geq d$ whenever $u, v \in \mathcal{C}$. Here ρ' is the Hamming distance.

With a slightly more technical proof along the same lines, the following can be proven.

Theorem 11. *Fix $q \geq 2$, k and d. If d is odd, then, as $n \to \infty$,*

$$A_q(n, d, k) \sim \frac{n^{k-\frac{d-1}{2}} (q-1)^{k-\frac{d-1}{2}} \left(\frac{d-1}{2}\right)!}{k!}.$$

If $d \geq 2$ is even, then, as $n \to \infty$,

$$A_q(n, d, k) \sim \frac{n^{k-\frac{d}{2}+1} (q-1)^{k-\frac{d}{2}+1} \left(\frac{d}{2}-1\right)!}{k!}. \qquad \square$$

To use random methods constructing codes is not a new idea. The best known general bounds for the *covering radius* problems are obtained in this way, see, e.g., [9, 14].

We can also consider pairs (or more generally s-tuples) of q-ary vectors of weight k. For simplicity, we will only state the results for pairs here. Define the set $\mathcal{Y}_q^{(2)}$ of pairs $\{u, v\}$ of vectors such that

- $u, v \in \{0, 1, \ldots, q-1\}^n$
- each of u and v has exactly k nonzero entries
- the supports of u and v are disjoint (i.e. $u_i = 0$ for all i such that $v_i \neq 0$, and $v_i = 0$ for all i such that $u_i \neq 0$).

Define the distance between these pairs by

$$\delta(\{u,v\},\{w,z\}) = \min\{\rho'(u,w) + \rho'(v,z), \rho'(u,z) + \rho'(v,w)\}$$

where ρ' is again the Hamming distance.

In the following, $A_q^2(n,d,k)$ will denote the maximum size of a subset $\mathcal{C} \subseteq \mathcal{Y}_q^{(2)}$ such that $\delta(\{u,v\},\{w,z\}) \geq d$ for any pair $\{u,v\},\{w,z\}$ of members of $\mathcal{C}$.

Theorem 12. *Fix q, d and k. If d is odd and $q \geq 3$, then, as $n \to \infty$,*

$$A_q^2(n,d,k) \sim \frac{1}{2} \cdot \frac{n^{2k-\frac{d-1}{2}} \cdot (q-1)^{2k-\frac{d-1}{2}} \cdot \lfloor \frac{d-1}{4} \rfloor ! \lceil \frac{d-1}{4} \rceil !}{(k!)^2}.$$

If $d \geq 2$ is even and $q \geq 2$, then, as $n \to \infty$,

$$A_q^2(n,d,k) \sim \frac{1}{2} \cdot \frac{n^{2k-\frac{d}{2}+1} \cdot (q-1)^{2k-\frac{d}{2}} \cdot \lfloor \frac{d}{4} \rfloor ! \left(\lceil \frac{d}{4} \rceil - 1 \right)!}{(k!)^2}. \qquad \square$$

The distance δ used here is twice the distance defined in Section 1, hence the apparent inconsistency of this result for $q = 2$ with Theorem 2.

For $q = 2$ and d odd we have $A_q(n,d,k) = A_q(n,d+1,k)$.

9 Open problems

We believe that for an arbitrary pair of k and d, there are infinitely many n's with equality in inequality (7).

10 Further developments

Let us note that since announcing the first version of the present paper Theorem 1 has been greatly extended by Chee, Kiah, Zhang and Zhang [3]. They determined the exact value of $C(n,2,d)$ completely, and for any fixed k the exact value of $C(n,k,2k-1)$ for all $n > n_0(k)$ satisfying either $n = 0 \mod k$ or $n = 1 \mod k$ and $n(n-1) = 0 \mod 2k^2$. Their proofs are different: they use more design theory. However, our Section 5 is still interesting for its own sake and Problem 4 is still open.

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