Formulæ for the Number of Partitions of n into at most m parts (Using the Quasi-Polynomial Ansatz)

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Abstract

The purpose of this short article is to announce, and briefly describe, a Maple package, `PARTITIONS`, that (inter alia) *completely automatically* discovers, and then proves, explicit expressions (as sums of quasi-polynomials) for $p_m(n)$ for any desired m. We do this to demonstrate the power of "rigorous guessing" as facilitated by the quasi-polynomial ansatz.

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1. Introduction

Recall that a *partition* of a non-negative integer n is a non-increasing sequence of positive integers $\lambda_1 \dots \lambda_m$ that sum to n. For example the integer 5 has the following seven partitions: $\{5, 41, 32, 311, 221, 2111, 11111\}$. The **bible** on partitions is George Andrews' *magnum opus* [1].

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We denote by $p_m(n)$ the number of partitions of n into at most m parts. By a classic theorem [1, p. 8, Thm. 1.4], $p_m(n)$ also equals the number of partitions of n into parts that are at most m. There is an extensive literature concerning formulæ for $p_m(n)$, including contributions by Cayley, Sylvester, Glaisher, and Gupta. For additional references and historical notes, see George Andrews' fascinating article [2, §3] and Gupta's *Tables* [8, pp. i–xxxix]. For an exhaustive history through 1920, see Dickson [4, Ch. 3].

More recently, George Andrews' student, Augustine O. Munagi, developed a beautiful theory of so-called q-partial fractions [11], where the denominators in the decomposition are always expressions of the form $(1-q^r)^s$, rather than powers of cyclotomic polynomials as is the case with the ordinary partial fraction decomposition. Accordingly, formulæ for $p_m(n)$ derived from the q-partial fraction decomposition of the generating function are most naturally expressed in terms of binomial coefficients.

It is well-known and easy to see that for any m, $p_m(n)$ is a sum of *quasi-polynomials of periods* $1, 2, 3, \ldots, m$. A *quasi-polynomial of period* r is a function $f(n)$ on the integers such that there exist r polynomials $P_1(n), P_2(n), \ldots, P_r(n)$ such that $f(n) = P_i(n)$ if $n \equiv i \pmod{r}$. We represent such a quasi-polynomial as a list $[P_1(n), \ldots, P_r(n)]$.

Thus, e.g., we have, for $n \geq 0$,

$$p_1(n) = 1, \tag{1}$$

$$p_2(n) = \left[\frac{n}{2} + \frac{3}{4}\right] + \left[-\frac{1}{4}, \frac{1}{4}\right], \tag{2}$$

$$p_3(n) = \left[\frac{n^2}{12} + \frac{n}{2} + \frac{47}{72}\right] + \left[-\frac{1}{8}, \frac{1}{8}\right] + \left[-\frac{1}{9}, -\frac{1}{9}, \frac{2}{9}\right] \tag{3}$$

$$\begin{aligned} p_4(n) = &\left[\frac{n^3}{144} + \frac{5n^2}{48} + \frac{15n}{32} + \frac{175}{288}\right] + \left[-\frac{n+5}{32}, \frac{n+5}{32}\right] + \left[0, -\frac{1}{9}, \frac{1}{9}\right] \\ &+ \left[0, -\frac{1}{8}, 0, \frac{1}{8}\right] \end{aligned} \tag{4}$$

$$\begin{aligned} p_5(n) = &\left[\frac{n^4}{2880} + \frac{n^3}{96} + \frac{31}{288}n^2 + \frac{85}{192}n + \frac{50651}{86400}\right] + \left[-\frac{n}{64} - \frac{15}{128}, \frac{n}{64} + \frac{15}{128}\right] \\ &+ \left[-\frac{1}{27}, -\frac{1}{27}, \frac{2}{27}\right] + \left[\frac{1}{16}, -\frac{1}{16}, -\frac{1}{16}, \frac{1}{16}\right] \end{aligned}$$

$$+\left[-\frac{1}{25}, -\frac{1}{25}, -\frac{1}{25}, -\frac{1}{25}, \frac{4}{25}\right]. \qquad (5)$$

Eqs. (1)–(5) were given in 1856 by Cayley [3, p. 132] in a somewhat different form. In 1909, Glaisher [6] presented formulæ for $p_m(n)$ for $m = 1, 2, \ldots, 10$. In 1958, Gupta [8] extended Glaisher's results to the cases $m = 11, 12$. In his 2005 Ph.D. thesis [10], Munagi gave formulæ for the cases $m = 1, 2, \ldots, 15$. Munagi's formulæ were derived with the aid of a Maple package he developed, and are of a somewhat different character than earlier contributions, as they follow from his theory of q-partial fractions [11].

2. The `PARTITIONS` Maple package

2.1. Overview

The purpose of this short article is to announce and briefly describe a Maple package, `PARTITIONS`, that *completely automatically* discovers and proves explicit expressions (as sums of quasi-polynomials) for $p_m(n)$ for any desired m. So far we only bothered to derive the formulæ for $1 \le m \le 70$, but one can easily go far beyond.

Not only that, we can, more generally, derive (and prove!), completely automatically, expressions, as sums of quasi-polynomials, for the number of ways of making change for n cents in a country whose coins have denominations of any given set of positive integers.

Not only that, we can derive (and prove!), completely automatically, expressions (as sums of quasi-polynomials) for $D_k(n)$, the number of partitions of n whose *Durfee square* has size k, for any desired, (numeric) positive integer k. (Recall that the size of the Durfee square of a partition $\lambda_1 \ldots ... \lambda_m$ is the largest k such that $\lambda_k \ge k$.)

Not only that, we (or rather our computers (and of course yours, if it has Maple and is loaded with our package)) can derive **asymptotic expressions**, to *any desired order*, for both $p_m(n)$ and $D_k(n)$. As far we we know the formula for $D_k(n)$ is brand-new, and the previous attempts for the asymptotic formula for $p_m(n)$ by humans G.J. Rieger [14] and E.M. Wright [16] (of Hardy-and-Wright fame) only went as far as $O(n^{-2})$ and $O(n^{-4})$ respectively. We go all the way to $O(n^{-100})$! (and of course can easily go far beyond).

Not only that, we implement George Andrews' ingenious way [2, sec. 3] to convert any quasi-polynomial to a polynomial expression where one is also allowed to use the integer-part function $\lfloor n \rfloor$. This enabled our computers to find Andrews-style expressions for $p_m(n)$ for $1 \le m \le 70$.

All these feats (and more!) are achieved by the **Maple package** `PARTITIONS`.

2.2. Using the PARTITIONS package

In order to use PARTITIONS, you must have `Maple`^{TM} installed on your computer. Then download the file:

`http://www.math.rutgers.edu/~zeilberg/tokhniot/PARTITIONS` and save it as `PARTITIONS`. Then launch Maple, and at the prompt, enter:

`read PARTITIONS`:

and follow the on-line instructions. Let's just highlight the most important procedures.

`AS100(m,n)`: shows the *pre-computed* first 100 terms of the asymptotic expression, in n, of $p_m(n)$ for *symbolic* m.

`ASD80(k,n)`: shows the *pre-computed* first 80 terms of the asymptotic expression, in n, of $D_k(n)$ for *symbolic* k.

`BuildDBpmn(n,M)`: inputs a symbol n and a positive integer M and outputs a list of size M whose i-th entry is an expression for $p_i(n)$ as a sum of i quasi-polynomials

`DiscoverAS(m,n,k)`: discovers the asymptotic expansion to order k of $p_m(n)$ (the number of partitions of n into at most m parts) for large n and fixed, but *symbolic*, m.

`DiscoverDAS(k,n,r)`: discovers the asymptotic expansion to order r of $D_k(n)$ (the number of partitions of n whose Durfee square has size k) for large n and fixed, but *symbolic* k.

`Durfee(k,n)`: discovers (rigorously!) the quasi-polynomial expression, in n, for $D_k(n)$, for any desired positive integer k. It is extremely fast for small k, but of course gets slower as k gets larger.

`DurfeePC(k,n)`: does the same thing (much faster, of course!) using the pre-computed expressions of `Durfee(k,n);` for $k \le 40$.

`EvalQPS(L,n,n0)`: evaluates the sum of the quasi-polynomials in the variable n given in the list L at $n = n_0$.

`HRR(n,T)`: evaluates in floating point the sum of the first T terms of the Hardy-Ramanujan-Rademacher formula for $p(n)$, the number of unrestricted partitions of n:

$$p(n)=\frac{1}{\pi\sqrt{2}}\sum_{k\geq 1}\sqrt{k}\sum_{\substack{0\leq h<k\\ \gcd(h,k)=1}} e^{\pi i(s(h,k)-2nh/k)}\frac{d}{dn}\left(\frac{\sinh\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n-\frac{1}{24}\right)}\right)}{\sqrt{n-\frac{1}{24}}}\right),$$

where $s(h,k)=\sum_{j=1}^{k-1}\left(\frac{j}{k}-\lfloor\frac{j}{k}\rfloor-\frac{1}{2}\right)\left(\frac{hj}{k}-\lfloor\frac{hj}{k}\rfloor-\frac{1}{2}\right)$is the Dedekind sum.

Please be warned that for larger n you need to increase `Digits`. In order to get reliable results you may want to use procedure `HRRr(n,T,k)`.

`pmn(m,n)`: discovers (rigorously!) the quasi-polynomial expression, in n, for $p_m(n)$, for any desired positive integer m. It is extremely fast for small m, but of course gets slower as m gets larger.

`pmnPC(m,n)`: does the same thing (much faster, of course!) using the pre-computed expressions of `pmn(m,n);` for $m\leq 70$.

`pmnAndrews(m,n)`: discovers (rigorously!) the Andrews-style expression, in n, for $p_m(n)$ for any desired positive integer m. Instead of using quasi-polynomials explicitly (that some humans find awkward), it uses the integer-part function $\lfloor n\rfloor$, denoted by `trunc(n)` in Maple.

`pn(n)`: the number of partitions of n, $p(n)$, using Euler's recurrence. It is useful for checking, since $p_n(n)=p(n)$.

`pnSeq(N)`: the list of the first N values of $p(n)$. The output of `pnSeq(50000):` can be gotten from

`http://www.math.rutgers.edu/~zeilberg/tokhniot/oPARTITIONS9` where this list of 50000 terms is called `pnTable`.

`pSn(S,n,K)`: the more general problem where the parts are drawn from the list S of positive integers. It outputs an explicit expression, as a sum of quasi-polynomials, for $p_S(n)$, the number of integer partitions of n whose parts are drawn from the finite list of positive integers S. K is a guessing parameter, that should be made higher if the procedure returns `FAIL`.

`pmnNum(m,n0)`: like `pmn(m,n);` but for both *numeric* m and $n0$. The output is a number. For $m\leq 70$ it is extremely fast, since it uses the *pre-computed* values of $p_m(n)$ gotten from `pmnPC(m,n);`. For example to get the number of integer partitions of a googol (10^{100}) into at most 60 parts, you would get, in 0.02 seconds, the 5783-digit integer, by simply typing

`pmnNum(60,10**100);` .

One of us (DZ) posed this is a 100-dollar challenge to the users of the very useful `Mathoverflow` forum. This was taken-up, successfully, by user *joro*[5], whose computer did it correctly in about 2 hours, using `PARI`. User `joro` generously suggested that instead of sending him a check, DZ should donate it in `joro`'s honor, to a charity of DZ's choice, and the latter decided on the Wikipedia Foundation.

Sample input and output can be gotten from the "front" of this article:
`http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/pmn.html`
.

3. Methodology: Rigorous Guessing

The idea of deriving formulæ for $p_m(n)$ and $p_S(n)$ with the aid of the partial fraction decomposition of the generating function dates back at least to Cayley [3]. We ask Maple to convert the generating function

$$\sum_{n\geq 0} p_m(n)q^n = \frac{1}{(1-q)(1-q^2)\cdots(1-q^m)}$$

or in the case of $p_S(n)$, where $S = \{s_1, s_2, \ldots, s_j\}$,

$$\sum_{n\geq 0} p_S(n)q^n = \frac{1}{(1-q^{s_1})(1-q^{s_2})\cdots(1-q^{s_j})}$$

into partial fractions. Then for each piece, Maple finds the first few terms of the Maclaurin expansion, and then fits the data with an appropriate quasi-polynomial using *undetermined* coefficients. The output is the list of these quasi-polynomials whose sum is the desired expression for $p_m(n)$ or $p_S(n)$. See the source-code for more details.

Example. Consider the case $m = 4$. We have Maple calculate that

$$\begin{aligned}\sum_{n\geq 0} p_4(n)q^n &= \frac{1}{(1-q)(1-q^2)(1-q^3)(1-q^4)}\\ &= \frac{17/72}{1-q} + \frac{59/288}{(1-q)^2} + \frac{1/8}{(1-q)^3} + \frac{1/24}{(1-q)^4} + \frac{1/8}{1+q} + \frac{1/32}{(1+q)^2} + \frac{(1+q)/9}{1+q+q^2}.\end{aligned} \tag{6}$$

At this point we could, as Cayley did, expand each term as a series in q, collect like terms, and then the coefficient of q^n will be a formula for $p_4(n)$, but why bother? From Sylvester [15] and Glaisher [7], we know that

$$p_4(n) = \sum_{j=1}^{4} W_j(n),$$

where each $W_j(n)$ is a quasi-polynomial $[P_{j1}(n), P_{j2}(n), \dots, P_{jj}(n)]$ of period j. Further, $W_j(n)$ is of degree $\lfloor \frac{m-j}{j} \rfloor$, and arises from those terms of (6) with denominator a power of the j-th cyclotomic polynomial. Instead, let us allow Maple to guess the correct quasi-polynomials: We know *a priori* that $W_1(n)$ is of the form $[a_0 + a_1 n + a_2 n^2 + a_3 n^3]$ and let Maple calculate the (beginning of the) Maclaurin series for the terms of (6) that contribute to $W_1(n)$:

$$\frac{17/72}{1-q} + \frac{59/288}{(1-q)^2} + \frac{1/8}{(1-q)^3} + \frac{1/24}{(1-q)^4} = \frac{175}{288} + \frac{19}{16}q + \frac{581}{288}q^2 + \frac{113}{36}q^3 + O(q^4).$$

Thus,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1^0 & 1^1 & 1^2 & 1^3 \\ 2^0 & 2^1 & 2^2 & 2^3 \\ 3^0 & 3^1 & 3^2 & 3^3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 175/288 \\ 19/16 \\ 581/288 \\ 113/36 \end{bmatrix},$$

which immediately implies that

$$W_1(n) = \left[\frac{1}{144}n^3 + \frac{5}{48}n^2 + \frac{15}{32}n + \frac{175}{288} \right].$$

Similarly, for $W_2(n)$, which must be of the form

$$[a_1 + a_3 n, a_0 + a_2 n],$$

we find

$$\frac{1/8}{1+q} + \frac{1/32}{(1+q)^2} = \frac{5}{32} - \frac{3}{16}q + \frac{7}{32}q^2 - \frac{1}{4}q^3 + O(q^4),$$

so that

$$\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_2 \end{bmatrix} = \begin{bmatrix} 5/32 \\ 7/32 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_3 \end{bmatrix} = \begin{bmatrix} -3/16 \\ -1/4 \end{bmatrix},$$

and thus

$$W_2(n) = \left[-\frac{5}{32} - \frac{n}{32}, \frac{5}{32} + \frac{n}{32} \right].$$

Analogous reasoning yields $W_3(n) = \left[0, -\frac{1}{9}, \frac{1}{9}\right]$ and $W_4(n) = \left[0, -\frac{1}{8}, 0, \frac{1}{8}\right]$.

4. Conclusion

The present approach uses *very naïve* guessing to discover, and *prove* (rigorously!), formulas (or as Cayley and Sylvester would say, *formulæ*) for the number of partitions of the integer n into at most parts m parts for $m \leq 70$, and of course, one can easily go far beyond. The core of the idea goes back to Arthur Cayley, and is familiar to any second-semester calculus student: partial fractions! But dear Arthur could only go so far, so his good buddy, James Joseph Sylvester, designated a sophisticated theory of "waves" [15] that facilitated hand calculations, which were later dutifully carried out by J. W. L. Glaisher in [7]. But, with modern computer algebra systems (Maple in our case), one can go much further just using Cayley's original ideas.

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References

[1] G. E. Andrews, The Theory of Partitions, Addison-Welsley, 1976. Reprinted by Cambridge University Press, 1984. First paperback edition, 1998.

[2] G. E. Andrews, Partitions: at the interface of q-series and modular forms, Ramanujan J. 7 (2003) 384-400.

[3] A. Cayley, Researches on the partition of numbers, Phil. Trans. Royal Soc. of London, 146 (1856) 127–140.

[4] L. E. Dickson, History of the Theory of Numbers, vol. 2: Diophantine Analysis, Carnegie Institute, Washington, 1920. Reissued by AMS Chelsea and Dover.

[5] "`joro`", answer to Doron Zeilberger's 100-dollar challenge, `http://mathoverflow.net/questions/71092/how-many-integer-partitions-of-a-googol-10100-into-at-most-60-parts`

[6] J. W. L. Glaiser, On the number of partitions of a number into a given number of parts, Quart. J. Pure Appl. Math. 40 (1909) 57–143.

[7] J. W. L. Glaisher, Formulæ for partitions into given elements, derived from Sylvester's theorem, Quart. J. Pure Appl. Math. 40 (1909) 275–348.

[8] H. Gupta, Tables of Partitions, Royal Society Mathematical Tables, vol. 4, Cambridge University Press, 1958.

[9] G. H. Hardy and S. Ramanujan, Asymptotic formulæ in combinatory analysis, Proc. London Math. Soc. (2) 17 (1918), 75–115.

[10] A. O. Munagi, Restricted Partition Formulas, q-Partial Fractions, and the Rademacher Conjecture, Ph.D. thesis, University of Lagos, Nigeria, 2005.

[11] A. O. Munagi, Computation of q-partial fractions, INTEGERS 7 (2007) #A25.

[12] H. Rademacher, On the partition function $p(n)$, Proc. London Math. Soc. (2) 43 (1938) 241–254.

[13] H. Rademacher, On the expansion of the partition function in a series, Ann. Math. (2) 44 (1943) 416–422.

[14] G. J. Rieger, Über Partitionen, Math. Annalen 138 (1959), 356-362.

[15] J. J. Sylvester, On subinvariants, i.e. semi-invariants to binary quantics of an unlimited order. With an excursus on rational fractions and partitions, Quart. J. Math. (Oxford Series (2)) 2 (1882) 85–108.

[16] E.M. Wright, Partitions into k parts, Math. Annalen 142 (1961), 311-316.