ON MULTIPLE PATTERN AVOIDING SET PARTITIONS

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Abstract

We study classes of set partitions determined by the avoidance of multiple patterns, applying a natural notion of partition containment that has been introduced by Sagan. We say that two sets S and T of patterns are equivalent if for each n the number of partitions of size n avoiding all the members of S is the same as the number of those that avoid all the members of T.

Our goal is to classify the equivalence classes among two-element pattern sets of several general types. First, we focus on pairs of patterns $\{\sigma, \tau\}$, where σ is a pattern of size three with at least two distinct symbols and τ is an arbitrary pattern of size k that avoids σ. We show that pattern-pairs of this type determine a small number of equivalence classes; in particular, the classes have on average exponential size in k. We provide a (sub-exponential) upper bound for the number of equivalence classes, and provide an explicit formula for the generating function of all such avoidance classes, showing that in all cases this generating function is rational.

Next, we study partitions avoiding a pair of patterns of the form $(1212, \tau)$, where τ is an arbitrary pattern. Note that partitions avoiding 1212 are exactly the non-crossing partitions. We provide several general equivalence criteria for pattern pairs of this type, and show that these criteria account for all the equivalences observed when τ has size at most six.

In the last part of the paper, we perform a full classification of the equivalence classes of all the pairs $\{\sigma, \tau\}$, where σ and τ have size four.

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1. Introduction

If $n \geq 1$, then a *partition* of $[n] = \{1, 2, \ldots, n\}$ is any collection of nonempty, pairwise disjoint subsets, called *blocks*, whose union is $[n]$. (If $n = 0$, then there is a single empty partition of $[0] = \varnothing$ which has no blocks.) A partition Π having exactly k blocks is called a *k-partition*. We will denote the set of all k-partitions of $[n]$ by $P_{n,k}$ and the set of all partitions of $[n]$ by P_n. The number n is referred to as

the size of a partition. A partition Π is said to be in *standard form* if it is written as $\Pi = B_1/B_2/\cdots$, where $\min(B_1) < \min(B_2) < \cdots$. One may also represent the partition $\Pi = B_1/B_2/\cdots/B_k \in P_{n,k}$, equivalently, by the *canonical sequential form* $\pi = \pi_1\pi_2\cdots\pi_n$, wherein $j \in B_{\pi_j}$, $1 \le j \le n$ (see, e.g., [25]). Throughout this paper, we will represent set partitions by their canonical forms and consider the problem of avoidance on these words.

For instance, the partition $\Pi = 1, 3, 8/2, 5/4, 7/6 \in P_{8,4}$ has the canonical sequential form $\pi = 12132431$. Note that $\pi = \pi_1\pi_2\cdots\pi_n \in P_{n,k}$ is a *restricted growth function* from $[n]$ to $[k]$ (see, e.g., [18] for details), meaning that it satisfies the following three properties: (i) $\pi_1 = 1$, (ii) π is onto $[k]$, and (iii) $\pi_{i+1} \le \max\{\pi_1, \pi_2, \ldots, \pi_i\} + 1$ for all i, $1 \le i \le n-1$. We remark that restricted growth functions are often encountered in the study of set partitions [23, 26] as well as other related topics, such as Davenport-Schinzel sequences [4, 19].

Let $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ and $\tau = \tau_1\tau_2\cdots\tau_m$ be two partitions, represented by their canonical sequences. We say that σ *contains* τ if σ contains a subsequence that is order-isomorphic to τ; that is, σ has a subsequence $\sigma_{f(1)}, \sigma_{f(2)}, \ldots, \sigma_{f(m)}$, where $1 \le f(1) < f(2) < \cdots < f(m) \le n$, such that for each $i, j \in [m]$, we have $\sigma_{f(i)} < \sigma_{f(j)}$ if and only if $\tau_i < \tau_j$ and $\sigma_{f(i)} > \sigma_{f(j)}$ if and only if $\tau_i > \tau_j$. Otherwise, we say that σ *avoids* τ. In this context, τ is usually called a *pattern*. For example, the partition σ avoids the pattern 1212 if there exist no indices $i < j < k < \ell$ with $\sigma_i = \sigma_k < \sigma_j = \sigma_\ell$ and avoids 1232 if there exist no such indices with $\sigma_i < \sigma_j = \sigma_\ell < \sigma_k$.

The concept of pattern-avoidance described above was introduced by Sagan [23], who considered, among other topics, the enumeration of partitions avoiding patterns of size three. Several other notions of pattern-avoidance of set partitions have been studied, see, e.g., the works of Klazar [8], Chen et al. [3], or Goyt [5].

We will use the following notation. If $\{\tau_1, \tau_2, \ldots\}$ is a set of patterns, then let $P_n(\tau_1, \tau_2, \ldots)$ and $P_{n,k}(\tau_1, \tau_2, \ldots)$ denote the subsets of P_n and $P_{n,k}$, respectively, which avoid all of the patterns. We will denote the cardinalities of $P_n(\tau_1, \tau_2, \ldots)$ and $P_{n,k}(\tau_1, \tau_2, \ldots)$ by $p_n(\tau_1, \tau_2, \ldots)$ and $p_{n,k}(\tau_1, \tau_2, \ldots)$, respectively. From the definitions, note that $p_n(\tau_1, \tau_2, \ldots) = \sum_{k\ge 0} p_{n,k}(\tau_1, \tau_2, \ldots)$. In accordance with the terminology first used for permutations (see, e.g., [10]), we will say that two sets of partition patterns $T = \{\tau_1, \tau_2, \ldots, \}$ and $R = \{\rho_1, \rho_2, \ldots\}$ are (Wilf) *equivalent*, denoted by $T \sim R$, if $p_n(\tau_1, \tau_2, \ldots) = p_n(\rho_1, \rho_2, \ldots)$ for all $n \ge 0$.

The pattern avoidance question is a rather broad one in enumerative combinatorics and has been the topic of much research, starting with Knuth [9] and Simion and Schmidt [24] on permutations. See also, for example, [20, 22, 16]. More recently, the problem has been considered on further structures such as k-ary words and compositions.

Jelínek and Mansour [6] have determined all the equivalences among singleton sets of patterns of size at most seven. In this paper, we focus on classes of partitions determined by two forbidden patterns. We address three main problems. First, in Section 2 we consider set partitions avoiding a pair of patterns $\{\sigma, \tau\}$, where σ is a pattern of size three and τ is an arbitrary pattern not containing σ. The situation when $\sigma = 111$ corresponds to single-pattern avoidance in partial matchings, which has been previously addressed [7]. We therefore restrict our attention to the cases when $\sigma \ne 111$.

We derive general criteria for Wilf-equivalence between pairs of patterns $\{\sigma, \tau\}$ of this form. In particular, we show that when τ has size k, these pairs form at most $\xi_k + 1$ equivalence classes, where ξ_k is the number of integer partitions having no summand equal to 2 [21, sequence A027336]. This implies that on average the equivalence classes have exponential size. For small values of k (up to $k = 20$), we are able to verify that the estimate $\xi_k + 1$ is sharp and all the equivalence classes may be described explicitly.

We also derive explicit formulas for the generating functions $\sum_{n\geq 0} p_n(\sigma, \tau) x^n$, where $\sigma \neq 111$ is of size three and τ is any pattern not containing σ. In particular, we show that all these generating functions are rational.

Next, in Section 3, we study the equivalences among pairs of patterns of the form $\{1212, \tau\}$, where τ is a pattern that avoids 1212. Note that the partitions avoiding 1212 correspond to the classical non-crossing partitions. We may therefore regard this section as the study of single-pattern avoidance among non-crossing partitions. We derive several general criteria for equivalences of pairs of patterns of this form. It turns out that some of the equivalence classes have size that is exponential in the size of τ. We verify, with the help of computer enumeration, that our criteria are sufficient to fully describe the equivalences among the pairs $\{1212, \tau\}$ for τ of size at most six.

Finally, in Section 4, we perform a systematic classification of the equivalences among the pairs $\{\sigma, \tau\}$, where σ and τ are distinct patterns of size four. Partial results in this direction have already been provided by previous research [11, 12, 15]. We provide several new results concerning the avoidance of two or more patterns, including ones involving infinite families of patterns. By combining these results with some specific cases which are worked out, we are able to provide a complete solution to the problem of identifying all of the equivalence classes corresponding to two patterns of size four.

We shall employ the following notation: if $\tau = \tau_1, \tau_2, \ldots, \tau_n$ is a sequence of numbers, then $\tau + 1$ refers to the sequence $\tau_1 + 1, \tau_2 + 1, \ldots, \tau_n + 1$. Also, if a is a symbol and $q \geq 0$ an integer, then a^q refers to the constant sequence $a, a, \ldots, a$ of length q.

2. Avoiding a pattern of size three and another pattern

Our first main goal is to study classes of partitions that avoid a pair of patterns (σ, τ), where σ is a pattern of size three.

Note first that a set partition avoids 111 if and only if each of its blocks has size at most two. Such a partition is known as a *partial matching*. Pattern avoidance in partial matchings has already been addressed in a previous paper [7]. We therefore focus on the remaining patterns of size three, that is, we assume $\sigma \in \{112, 121, 122, 123\}$. We may also assume that τ does not contain σ, otherwise $P_n(\sigma, \tau) = P_n(\sigma)$.

Let us remark that Sagan [23] has shown that for any pattern σ from the set $\{112, 121, 122, 123\}$, we have $p_n(\sigma) = 2^{n-1}$.

Let us say that a pair of patterns (σ, τ) is a *(3, k)-pair* if $\sigma \in \{112, 121, 122, 123\}$ and τ is a pattern of size k that avoids σ. Our first results deal with general criteria for equivalences among $(3, k)$-pairs. These criteria will apply to $(3, k)$-pairs for any value of k. For values of k up to $k = 20$, we have verified that our criteria account

for all equivalences among the $(3, k)$-pairs. We conjecture that this is the case for larger k as well.

We also give an explicit formula for the generating function of partitions avoiding an arbitrary given $(3, k)$-pair.

2.1. **The patterns** 112 **and** 121. Let us first consider the $(3, k)$-pairs (σ, τ), where σ is equal to either 121 or 112. We will show that the two avoidance classes $P_n(121)$ and $P_n(112)$ are closely related. More precisely, these two classes form isomorphic posets under the containment relation.

Notice that a partition τ avoids 121 if and only if τ is a weakly increasing pattern of the form $1^{a_1} 2^{a_2} \cdots m^{a_m}$ for some $m \geq 1$ and some sequence $a = (a_1, \ldots, a_m)$ of positive integers. In particular, there is a bijection between 121-avoiding partitions of size n and sequences of positive integers whose sum is n.

Similarly, a partition τ avoids 112 if and only if τ has the form $12 \cdots m m^{a_m - 1} (m - 1)^{a_{m-1} - 1} \cdots 1^{a_1 - 1}$, for some $m \geq 1$ and some sequence $a = (a_1, \ldots, a_m)$ of positive integers. We use the term *composition* to refer to any finite sequence of positive integers. The *size* of a composition is the sum of its components, and the *length* of a composition is the number of its components.

For a composition $a = (a_1, \ldots, a_m)$, let $\tau_{121}(a)$ denote the 121-avoiding pattern $1^{a_1} 2^{a_2} \cdots m^{a_m}$ and let $\tau_{112}(a)$ denote the 112-avoiding pattern $12 \cdots m m^{a_m - 1} (m - 1)^{a_{m-1} - 1} \cdots 1^{a_1 - 1}$. Note that $\tau_{112}(a)$ is the unique 112-avoiding partition with m blocks whose i-th block has size a_i, and similarly for $\tau_{121}(a)$.

Let $a = (a_1, \ldots, a_m)$ and $b = (b_1, \ldots, b_k)$ be two compositions. We say that b *dominates* a, if there is an m-tuple of indices $i(1), i(2), \ldots, i(m)$ such that $1 \leq i(1) < i(2) < \cdots < i(m) \leq k$, and $a_j \leq b_{i(j)}$ for each $j \in [m]$. In other words, b dominates a if b contains a subsequence of length m whose every component is greater than or equal to the corresponding component of a.

We present the following simple fact without proof.

Observation 2.1. *For any two compositions a and b, the following are equivalent:*

- b *dominates a,*
- $\tau_{112}(b)$ *contains $\tau_{112}(a)$,*
- $\tau_{121}(b)$ *contains $\tau_{121}(a)$.*

Observation 2.1 shows that the classes $P_n(112)$ and $P_n(121)$ ordered by containment and the set of all integer compositions ordered by domination are three isomorphic posets, with size-preserving isomorphisms identifying a composition a with $\tau_{112}(a)$ and $\tau_{121}(a)$.

Corollary 2.2. *For any integer composition a, the $(3, k)$-pairs $(112, \tau_{112}(a))$ and $(121, \tau_{121}(a))$ are equivalent.*

For two compositions a and a', let us write $a \stackrel{d}{\sim} a'$ if for every n, the number of compositions of size n dominating a is equal to the number of compositions of size n dominating a'. Observation 2.1 implies that $a \stackrel{d}{\sim} a'$ if and only if $(112, \tau_{112}(a)) \sim (112, \tau_{112}(a'))$ which is if and only if $(121, \tau_{121}(a)) \sim (121, \tau_{121}(a'))$.

For a composition $a = (a_1, \ldots, a_m)$, let $M(a)$ denote the multiset $\{a_1, \ldots, a_m\}$.

Lemma 2.3. *Let a and a' be two compositions such that $M(a) = M(a')$. Then $a \stackrel{d}{\sim} a'$.*

Proof. It is enough to prove the lemma in the case when a' is obtained from a by exchanging the order of two consecutive elements. Let m be the length of a and of a'. Suppose a equals $(a_1, \ldots, a_m)$, and that a' is obtained from a by exchanging the components a_r and a_{r+1} for some $r < m$, so that we have

$$a' = (a_1, \ldots, a_{r-1}, a_{r+1}, a_r, a_{r+2}, \ldots, a_m).$$

We prove the lemma bijectively. Let $b = (b_1, \ldots, b_k)$ be a composition of size n that dominates a. Let $i \in [k]$ be the smallest index such that $(b_1, \ldots, b_i)$ dominates $(a_1, \ldots, a_{r-1})$. Let $j \in [k]$ be the largest index such that $(b_{j+1}, \ldots, b_k)$ dominates $(a_{r+2}, \ldots, a_m)$. Since b dominates a, we know that $i + 2 \leq j$ and that $(b_{i+1}, b_{i+2}, \ldots, b_j)$ dominates (a_r, a_{r+1}). Consider now a composition b' obtained from b by reversing the order of the elements $b_{i+1}, b_{i+2}, \ldots, b_j$, that is,

$$b' = (b_1, \ldots, b_i, b_j, b_{j-1}, \ldots, b_{i+1}, b_{j+1}, \ldots, b_k).$$

Clearly, b' dominates a', and the mapping $b \mapsto b'$ is a size-preserving bijection between compositions that dominate a and those that dominate a'. $\qquad\square$

Lemma 2.4. *Let $a = (a_1, \ldots, a_m)$ be a composition, with $a_m = 2$. Define another composition $a' = (a_1, \ldots, a_{m-1}, 1, 1)$. Then $a \overset{d}{\sim} a'$.*

Proof. Fix a size n. We provide a bijection f between compositions of size n that do not dominate a and compositions of size n that do not dominate a'. Suppose that $b = (b_1, \ldots, b_k)$ is a composition of size n that does not dominate a. If b does not even dominate $(a_1, \ldots, a_{m-1})$, then b does not dominate a', and we put $f(b) = b$.

Suppose now that b dominates $(a_1, \ldots, a_{m-1})$. Let $i \in [k]$ be the smallest index such that $(b_1, \ldots, b_i)$ dominates $(a_1, \ldots, a_{m-1})$. Since b does not dominate a, we know that all of the components $(b_{i+1}, b_{i+2}, \ldots, b_k)$ must be equal to 1. Define a new composition $b' = f(b)$ obtained from b by replacing all the components $(b_{i+1}, b_{i+2}, \ldots, b_k)$ with a single component equal to $b_{i+1} + b_{i+2} + \cdots + b_k$ (if $i = k$, then we put $b' = b$). Clearly, the new composition b' has size n and does not dominate a', and the mapping f is the required bijection. $\qquad\square$

From Lemmas 2.3 and 2.4, we see that every composition a is $\overset{d}{\sim}$-equivalent to a composition a' that has the property that its components are weakly decreasing and none of them are equal to 2. Let us call such a composition a' a *2-free integer partition*. Let ξ_k be the number of 2-free integer partitions of size n. Note that the sequence $(\xi_k)_{k \geq 0}$ is listed as A027336 in the OEIS [21]. Basic estimates on the number of integer partitions (see, e.g., [1]) imply the bound $\xi_k = 2^{O(\sqrt{k})}$.

Let $a = (a_1, a_2, \ldots, a_\ell)$ be an integer composition, and let $F_a(x, y)$ be the generating function for the number of partitions of $[n]$ having exactly k blocks and avoiding $\{112, \tau_{112}(a)\}$, i.e.,

$$F_a(x, y) = \sum_{n, k \geq 0} p_{n,k}(112, \tau_{112}(a)) x^n y^k.$$

We may give an explicit formula for $F_a(x, y)$ as follows.

Theorem 2.5. *We have*

$$(2.1) \qquad F_a(x, y) = \sum_{j=0}^{\ell-1} \frac{x^{a_1 + \cdots + a_j} y^j (1 - x)}{\prod_{i=1}^{j+1} (1 - x(1 + y) + x^{a_i} y)}.$$

Proof. Let b be the integer composition $(a_2, a_3, \ldots, a_\ell)$. Let π be a nonempty partition from the set $P_{n,k}(112, \tau_{112}(a))$, and let r be the size of the first block of π. We consider the following two cases:

(1) $1 \le r \le a_1 - 1$,
(2) $r \ge a_1$.

In the first case, π must be of the form $1\pi'1^{r-1}$, where π' is some partition on the letters $\{2, 3, \ldots\}$ avoiding $\{112, \tau_{112}(a)\}$, which implies that the generating function counting these partitions is given by

$$xyF_a(x, y) + x^2yF_a(x, y) + \cdots + x^{a_1-1}yF_a(x, y) = \frac{x - x^{a_1}}{1 - x}yF_a(x, y).$$

In the second case, π must be of the form $1\pi'1^{r-1}$, where π' is now a partition on the letters $\{2, 3, \ldots\}$ avoiding $\{112, \tau_{112}(b)\}$ since $r \ge a_1$. Thus, the generating function counting the partitions in this case is given by

$$x^{a_1}yF_b(x, y) + x^{a_1+1}yF_b(x, y) + \cdots = \frac{x^{a_1}y}{1 - x}F_b(x, y),$$

where we put $F_b(x, y) = 0$ in the case $\ell = 1$. Adding the contributions from the two cases above gives

$$F_a(x, y) = 1 + \frac{x - x^{a_1}}{1 - x}yF_a(x, y) + \frac{x^{a_1}y}{1 - x}F_b(x, y),$$

which may be rewritten as

$$(2.2) \qquad F_a(x, y) = \frac{1 - x}{1 - x(1 + y) + x^{a_1}y} + \frac{x^{a_1}y}{1 - x(1 + y) + x^{a_1}y}F_b(x, y).$$

Iterating recurrence (2.2) yields (2.1), as desired. $\qquad\qquad\square$

2.2. **The pattern** 123. Consider now the $(3, k)$-pairs $(123, \tau)$, for a 123-avoiding partition τ. Of course, a partition avoids 123 if and only if it has at most two blocks. We will distinguish two cases, depending on whether τ has a single block or whether it has two blocks.

The first case is trivial:

Observation 2.6. *A partition avoids the pair of patterns $(123, 1^k)$ if and only it has at most two blocks and each block has size at most $k - 1$. In particular, the generating function of the class $P_n(123, 1^k)$ is given by the formula*

$$\sum_{n \ge 0} p_n(123, 1^k)x^n = 1 + \sum_{a=1}^{k-1}\sum_{b=0}^{k-1}\binom{a + b - 1}{b}x^{a+b}.$$

To deal with the pairs $(123, \tau)$, where τ has two blocks, we first prove a more general theorem.

Theorem 2.7. *Let $m \ge 2$ be an integer. Let $\tau = \tau_1\tau_2\cdots\tau_k$ be a partition with exactly m blocks and with the property that $\tau_i = i$ for each $i \in [m - 1]$. Then the generating function $\sum_{n \ge 0} p_n(12\cdots(m + 1), \tau)x^n$ is equal to*

$$\left(\sum_{n \ge 0} p_n(12\cdots(m + 1))x^n\right) - \left(\frac{x}{1 - (m - 1)x}\right)^{k-m}\frac{x}{1 - mx}\prod_{j=1}^{m-1}\frac{x}{1 - jx}.$$

In particular, the generating function depends on the size of τ but not on τ itself.

Proof. Let Q_n be the set of partitions of size n that avoid $12 \cdots (m+1)$ but contain τ; in other words, $Q_n = P_n(12 \cdots (m+1)) \setminus P_n(12 \cdots (m+1), \tau)$. Define the generating function $G(x) = \sum_{n \geq 0} |Q_n| x^n$. To prove the theorem, we need to prove the formula

$$(2.3) \qquad G(x) = \left(\frac{x}{1 - (m-1)x} \right)^{k-m} \frac{x}{1 - mx} \prod_{j=1}^{m-1} \frac{x}{1 - jx}.$$

Consider a partition $\pi \in Q_n$. Clearly π must have exactly m blocks. Since π contains τ as a pattern, and since τ has m blocks as well, we see that π even contains τ as a subsequence. Recall that $\tau_i = i$ for $i \in [m-1]$, that is, τ has the form $12 \cdots (m-1)\tau_m \tau_{m+1} \cdots \tau_k$. By fixing the leftmost occurrence of the subsequence τ in π, we see that π can be decomposed as

$$\pi = 1w_1 2w_2 3w_3 \cdots (m-1)w_{m-1} \tau_m w_m \tau_{m+1} w_{m+1} \cdots w_{k-1} \tau_k w_k,$$

where the w_j's are determined as follows:

- for $1 \leq j < m-1$, w_j is an arbitrary word over the alphabet $[j]$,
- for $m-1 \leq j < k$, w_j is an arbitrary word over $[m] \setminus \{\tau_{j+1}\}$, and
- w_k is an arbitrary word over $[m]$.

Conversely, any sequence with such a decomposition is an element of Q_n. This directly implies formula (2.3). $\qquad \square$

Applying Theorem 2.7 to the case $m = 2$, and noting that $\sum_{n \geq 0} p_n(123)x^n = (1-x)/(1-2x)$, we get the next result.

Corollary 2.8. *For every k, the $(3,k)$-pairs of the form $(123, \tau)$ where τ has two blocks are all equivalent, and the generating function of any such pair is*

$$\sum_{n \geq 0} p_n(123, \tau)x^n = \frac{1-x}{1-2x} - \left(\frac{x}{1-x} \right)^{k-1} \frac{x}{1-2x} = \sum_{i=0}^{k-1} \left(\frac{x}{1-x} \right)^i.$$

Comparing the generating function of the previous corollary with the formula of Theorem 2.5, we can say even more.

Corollary 2.9. *For every k and every partition $\tau \in P_k$ with two blocks, the $(3,k)$-pair $(123, \tau)$ is equivalent to the $(3,k)$-pair $(112, 12 \cdots k)$.*

2.3. **The pattern 122.** Note that a partition τ avoids 122 if and only if each block of τ except possibly the first one has size one, or equivalently, any number greater than 1 appears at most once in τ.

We will show that for every k, all the $(3,k)$-pairs of the form $(122, \tau)$ are equivalent to $(112, 1^k)$. To this end, we first describe a bijection between 122-avoiding and 123-avoiding partitions which, under suitable assumptions, preserves containment. Let $\tau = \tau_1 \tau_2 \cdots \tau_k$ be a 122-avoiding partition. Define a new partition $f(\tau) = \tau_1' \tau_2' \cdots \tau_k'$ by putting $\tau_i' = 1$ if $\tau_i = 1$ and $\tau_i' = 2$ if $\tau_i > 1$. For example, if $\pi = 1123145$, then $f(\pi) = 1122122$. Note that the mapping f defined by these properties is a bijection from the set of 122-avoiding partitions to the set of 123-avoiding partitions.

Lemma 2.10. *Let $\tau = \tau_1 \cdots \tau_k$ be a 122-avoiding partition with at least two blocks, and let $\pi = \pi_1 \cdots \pi_n$ be any 122-avoiding partition. Then π contains τ if and only if $f(\pi)$ contains $f(\tau)$.*

Proof. Let us write $f(\pi) = \pi'_1 \cdots \pi'_n$ and $f(\tau) = \tau'_1 \cdots \tau'_k$.

Assume that π contains τ, and let $i(1) < i(2) < \cdots < i(k)$ be indices such that the sequence $\pi_{i(1)}\pi_{i(2)}\cdots\pi_{i(k)}$ is order isomorphic to τ. We may assume without loss of generality that $i(1) = 1$. It then follows that $\pi'_{i(1)}\pi'_{i(2)}\cdots\pi'_{i(k)}$ is order-isomorphic to $f(\tau)$ and hence $f(\pi)$ contains $f(\tau)$.

Conversely, assume that $f(\pi)$ contains $f(\tau)$. Since τ has at least two blocks, $f(\tau)$ has exactly two blocks, and therefore $f(\pi)$ contains $f(\tau)$ even as a subsequence, not just as a pattern. This implies that π contains τ. $\qquad\square$

Proposition 2.11. *For any partition $\tau \in P_k(122)$ with at least two blocks, the $(3,k)$-pair $(122, \tau)$ is equivalent to the $(3,k)$-pair $(123, f(\tau))$.*

Proof. Lemma 2.10 shows that f maps $P_n(122, \tau)$ bijectively to $P_n(123, f(\tau))$. $\quad\square$

Combining Proposition 2.11 with Corollary 2.9, we get the following result.

Corollary 2.12. *For any k and any partition $\tau \in P_k(122)$ with at least two blocks, the $(3,k)$-pair $(122, \tau)$ is equivalent to the $(3,k)$-pair $(112, 12 \cdots k)$.*

It remains to deal with $(3,k)$-pairs of the form $(122, 1^k)$. It turns out that these pairs are also equivalent to all the other $(3,k)$-pairs of the form $(122, \tau)$.

Proposition 2.13. *The $(3,k)$-pairs $(122, 1^k)$ and $(122, 12 \cdots k)$ are equivalent.*

Proof. Note that a 122-avoiding partition of $[n]$ is uniquely determined by specifying which of the elements of the set $\{2, 3, \ldots, n\}$ belong to the first block.

Thus, a 122-avoiding partition avoids 1^k if and only if its first block has at most $k-2$ elements from $\{2, 3, \ldots, n\}$, and it avoids $12 \cdots k$ if and only if the complement of the first block has at most $k-2$ elements from $\{2, 3, \ldots, n\}$. Clearly, in both cases there are exactly $\sum_{i=0}^{k-2} \binom{n-1}{i}$ possibilities to specify the first block, and therefore the whole partition. $\qquad\square$

Combining Corollary 2.12 and Proposition 2.13, we obtain the main result of this subsection.

Corollary 2.14. *For any k, the $(3,k)$-pairs of the form $(122, \tau)$ are all equivalent, and they are equivalent to the pair $(112, 12 \cdots k)$.*

2.4. **Summary of equivalences among $(3,k)$-pairs.** Let us summarize the equivalences among $(3,k)$-pairs that follow from the results established so far (see Table 1 for an example with $k = 4$).

- There is an equivalence class containing all the patterns $(112, \tau_{112}(a))$ and $(121, \tau_{121}(a))$ for all compositions a of size k all of whose components are equal to 1 or 2 (Observation 2.1, Lemma 2.3, and Lemma 2.4). That same class also contains all the $(3,k)$-pairs of the form $(122, \tau)$ (Corollary 2.14). By Corollary 2.9, the same class also contains all the pairs of the form $(123, \tau)$, where τ is different from 1^k.
- The pair $(123, 1^k)$ is not equivalent to any other $(3,k)$-pair. There are only finitely many partitions avoiding both 123 and 1^k, whereas any other $(3,k)$-pair is avoided by infinitely many partitions.
- For every 2-free integer partition a of size k, there is an equivalence class containing all the pairs from the set

$$\{(121, \tau_{121}(a')), a' \overset{d}{\sim} a\} \cup \{(112, \tau_{112}(a')), a' \overset{d}{\sim} a\}.$$

(ρ, τ)	Generating function for the sequence $\{p_n(\rho, \tau)\}_{n \geq 0}$
$(112, 1231) \sim (112, 1232) \sim (112, 1233)$ $\sim (121, 1122) \sim (121, 1123) \sim (121, 1223)$ $\sim (121, 1233) \sim (121, 1234) \sim (122, 1111)$ $\sim (122, 1112) \sim (122, 1121) \sim (122, 1123)$ $\sim (122, 1211) \sim (122, 1213) \sim (122, 1231)$ $\sim (122, 1234) \sim (112, 1221) \sim (123, 1112)$ $\sim (123, 1121) \sim (123, 1122) \sim (123, 1211)$ $\sim (123, 1212) \sim (123, 1221) \sim (123, 1222)$ $\sim (112, 1234)$	$\sum_{i=0}^{3} \frac{x^i}{(1-x)^i}$
$(123, 1111)$	$\sum_{i=1}^{3} \sum_{j=0}^{3} \binom{i+j-1}{j} x^{i+j}$
$(121, 1222) \sim (121, 1112)$ $\sim (112, 1211) \sim (112, 1222)$	$\frac{1-x+x^3}{(1-x)(1-x-x^2)}$
$(121, 1111) \sim (112, 1111)$	$\frac{1}{1-x-x^2-x^3}$

TABLE 1. The equivalence classes of $(3, 4)$-pairs

Note that if $a = (1, 1, \ldots, 1)$, then this class corresponds to the equivalence class mentioned in the first item of this list. By Lemmas 2.3 and 2.4, any composition is $\overset{d}{\sim}$-equivalent to a 2-free integer partition, therefore, the classes mentioned so far contain all the $(3, k)$-pairs.

Corollary 2.15. *For each k and each $(3, k)$-pair (σ, τ), the generating function of (σ, τ)-avoiders is rational, and can be computed explicitly.*

Corollary 2.16. *For each $k \geq 3$, the $(3, k)$-pairs form at most $1 + \xi_k$ equivalence classes, where ξ_k is the number of 2-free integer partitions (A027336).*

We do not know whether the bound of the previous corollary is tight or whether there actually exist some more equivalences among the $(3, k)$-pairs. Note that if such 'hidden' equivalences exist, they must involve τ of size at least 21, because for size 20 and less, we can check (with the aid of a computer) that the classes listed above are all non-equivalent. Also the additional equivalences must involve $\sigma = 112$ (or equivalently $\sigma = 121$) because all the pairs of the form $(122, \tau)$ are equivalent, and all the patterns of the form $(123, \tau)$ are equivalent to them as well, except for $(123, 1^k)$, which is not equivalent to any other $(3, k)$-pair.

Problem 2.17. *Are there any more equivalences among the $(3, k)$-pairs of the form $(112, \tau)$ other than those that we know about? Equivalently, are there any two distinct 2-free integer partitions that are $\overset{d}{\sim}$-equivalent?*

3. PATTERN AVOIDANCE IN NON-CROSSING PARTITIONS

Our goal is to study partitions that avoid the pattern 1212 and another pattern. Note that a partition avoids 1212 if and only if it is non-crossing.

We write $\sigma \overset{nc}{\sim} \tau$ if $(1212, \sigma)$ is equivalent to $(1212, \tau)$. If $\sigma \overset{nc}{\sim} \tau$, we say that σ and τ are *nc-equivalent* ('nc' stands for 'non-crossing').

To simplify our notation, we will employ the following convention: whenever we write $[\sigma]$ as a subsequence of a longer pattern π, we assume that $[\sigma]$ refers to the sequence $\sigma + k$, where k is the number of distinct symbols of π that appear before the first symbol of σ in π. Thus, for example, $1[112]1$ refers to the sequence 12231, and $\pi = 11[121][112]$ should be understood as $\pi = 11232445$.

Let us say that a set partition π is *connected*, if it cannot be written as $\pi = \sigma[\tau]$ where σ and τ are nonempty partitions. Note that a non-crossing partition is connected if and only if its last element belongs to the first block. For any set partition π, there is a unique sequence of nonempty connected partitions $\sigma_1, \ldots, \sigma_m$ such that $\pi = \sigma_1[\sigma_2][\sigma_3] \cdots [\sigma_m]$. We call the partitions σ_i *the components* of π.

We say that two non-crossing partition patterns σ and τ are *cc-equivalent*, denoted by $\sigma \overset{cc}{\sim} \tau$, if there is a bijection f from the set of $(1212, \sigma)$-avoiding partitions to the set of $(1212, \tau)$-avoiding partitions, such that for every non-crossing σ-avoider π, the partition $f(\pi)$ has the same size and the same number of components as π. In particular, cc-equivalence is a refinement of nc-equivalence.

Suppose that $\sigma = \sigma_1 \cdots \sigma_k$ and $\tau = \tau_1 \cdots \tau_n$ are two partitions. We say that a sequence $I = (i(1), i(2), \ldots, i(k))$ is an *occurrence* of σ in τ, if $1 \leq i(1) < i(2) < \cdots < i(k) \leq n$ and $\tau_{i(1)}, \tau_{i(2)}, \ldots, \tau_{i(k)}$ is order-isomorphic to σ. We say that an occurrence $(i(1), \ldots, i(k))$ of σ in τ is a *leftmost occurrence* if $i(k)$ has the smallest possible value among all occurrences of σ in τ, or equivalently, σ has no occurrence in $\tau_1, \ldots, \tau_{i(k)-1}$. We say that an occurrence $(i(1), \ldots, i(k))$ of σ in τ is a *topmost occurrence* if $\tau_{i(1)}$ has the largest possible value among all the occurrences of σ in τ, or equivalently, if the subsequence of τ formed by all the elements greater than $\tau_{i(1)}$ is order-isomorphic to a σ-avoiding partition. If σ is the empty partition, we assume that the empty sequence is the unique occurrence of σ in τ, and that this occurrence is both leftmost and topmost.

For example, taking $\sigma = 122$ and $\tau = 11233245466233$, we see that $1, 4, 5$ and $2, 4, 5$ are the two leftmost occurrences of σ in τ, both corresponding to a subsequence 133 of τ, while $8, 10, 11$ is the (in this case unique) topmost occurrence of σ in τ, representing the subsequence 566.

Let $I = (i(1), \ldots, i(k))$ be a topmost occurrence of σ in τ, with $b = \tau_{i(1)}$. Suppose that i' is the smallest index such that $\tau_{i'} = b$. Observe that replacing $i(1)$ with i' yields another topmost occurrence of σ in τ.

Lemma 3.1. *Let ρ, σ and τ be non-crossing partitions of size k, ℓ and m, respectively. Let $I = (i(1), \ldots, i(k))$ be a leftmost occurrence of ρ in τ, let $J = (j(1), \ldots, j(\ell))$ be a topmost occurrence of σ in τ, and let $H = (h(1), \ldots, h(k+\ell))$ be any occurrence of $\rho[\sigma]$ in τ. Then $(i(1), \ldots, i(k), h(k+1), \ldots, h(k+\ell))$, as well as $(h(1), \ldots, h(k), j(1), \ldots, j(\ell))$ are both occurrences of $\rho[\sigma]$ in τ. In particular, τ contains $\rho[\sigma]$ if and only if $(i(1), \ldots, i(k), j(1), \ldots, j(\ell))$ is an occurrence of $\rho[\sigma]$.*

Proof. Let us assume that ρ, σ and τ are nonempty, otherwise the lemma is trivial.

Let us prove that $(i(1), \ldots, i(k), h(k+1), \ldots, h(k+\ell))$ is an occurrence of $\rho[\sigma]$. Since we already know that I is an occurrence of ρ and that $h(k+1), \ldots, h(k+\ell)$ is an occurrence of σ, we only need to prove that $i(k) < h(k+1)$, and that every element of $\tau_{i(1)}, \tau_{i(2)} \ldots, \tau_{i(k)}$ is smaller than any element of $\tau_{h(k+1)}, \tau_{h(k+2)}, \ldots, \tau_{h(k+\ell)}$.

Since I is a leftmost occurrence of ρ, we know that $i(k) \leq h(k)$ and therefore $i(k) < h(k+1)$. We now show that $\tau_{i(a)} < \tau_{h(k+b)}$ for every $a \in [k]$ and $b \in [\ell]$. Suppose that we have $\tau_{i(a)} \geq \tau_{h(k+b)}$ for some $a \in [k]$ and $b \in [\ell]$. We know that $\tau_{h(k)} < \tau_{h(k+b)}$. Let us write $x = \tau_{h(k)}$ and $y = \tau_{h(k+b)}$. Let i' and j' be

the indices of the first occurrences of x and y in τ, respectively. Since we know that $x < y \le \tau_{i(a)}$, we know that $i' < j' \le i(a) < h(k)$. Thus, the four indices $i', j', h(k), h(k+b)$ are an occurrence of 1212 in τ, contradicting the assumption that τ is non-crossing.

Let us now show that $h(1), \ldots, h(k), j(1), \ldots, j(\ell)$ is an occurrence of $\rho[\sigma]$. Since J is a topmost occurrence of σ, we know that $\tau_{h(k+1)} \le \tau_{j(1)}$, and therefore $\tau_{h(a)} < \tau_{j(b)}$ for any $a \in [k]$ and $b \in [\ell]$. To show that $h(1), \ldots, h(k), j(1), \ldots, j(\ell)$ is an occurrence of $\rho[\sigma]$, we thus only need to prove that $h(k) < j(1)$. Suppose that this is not the case. Let us write $x = \tau_{h(k)}$ and $y = \tau_{h(k+1)}$, and let i' and j' be the indices of first occurrences of x and y in τ, respectively. Since $x < y \le \tau_{j(1)}$, we know that $i' < j' \le j(1)$, showing that $i', j', h(k), h(k+1)$ is an occurrence of 1212 in τ, a contradiction. $\qquad\square$

Theorem 3.2. *If σ and τ are (possibly empty) non-crossing partitions, and if ρ and ρ' are two cc-equivalent non-crossing partitions, then $\sigma[\rho][\tau] \stackrel{cc}{\sim} \sigma[\rho'][\tau]$.*

Note that in the previous theorem, cc-equivalence cannot be replaced by nc-equivalence. For example, 11 and 12 are nc-equivalent partitions, but $1[11] = 122$ and $1[12] = 123$ are not nc-equivalent.

Proof of Theorem 3.2. Let us write $\alpha = \sigma[\rho][\tau]$ and $\alpha' = \sigma[\rho'][\tau]$. We will define a bijection f that maps α-avoiding non-crossing partitions of size n to α'-avoiding non-crossing partitions of the same size, while preserving the number of connected components.

Let $\pi = \pi_1 \cdots \pi_n$ be a non-crossing partition on n vertices. If π avoids $\sigma[\tau]$, then we may define $f(\pi) = \pi$. Suppose that π contains $\sigma[\tau]$. Let $I = (i(1), \ldots, i(k))$ be the leftmost occurrence of σ in π, and $J = (j(1), \ldots, j(\ell))$ the top-most occurrence of τ. Furthermore, assume that $j(1)$ is chosen as small as possible, that is, $\tau_{j(1)}$ is the first element of its π-block. By Lemma 3.1, we know that $i(1), \ldots, i(k), j(1), \ldots, j(\ell)$ is an occurrence of $\sigma[\tau]$.

Let us define $a = i(k)$ and $b = j(1)$. We will refer to the elements $\pi_1, \pi_2, \ldots, \pi_a$ as the *left part* of π, while $\pi_{a+1}, \pi_{a+2}, \ldots, \pi_{b-1}$ are the *middle part*, and $\pi_b, \ldots, \pi_n$ are the *right part*. A block of π is a *left block* (or *middle block* or *right block*) if its first element appears in the left part of π (middle part, right part, respectively). We say that an element π_i is an *outlier* if it belongs to a different part of π than the first element of its block. In other words, an outlier is an element of a left block belonging to the middle part or right part, or an element of a middle block belonging to the right part.

Let π_M denote the partition obtained from π by deleting all the left blocks and right blocks, and then by deleting all the outliers from the middle blocks. In other words, π_M consists of the elements of π belonging to middle blocks and to the middle part of π. We will abuse the terminology by identifying an element of π_M with the corresponding element of π.

It is clear that if π_M contains ρ, then π contains α. We claim that the converse is true as well, i.e., if π contains α then π_M contains ρ. To see this, fix an occurrence H of α in π, and write H as a concatenation $I'KJ'$, where I', K and J' are occurrences of σ, ρ and τ, respectively. By Lemma 3.1, IKJ is also an occurrence of α, which shows that all the indices in K refer to the middle part of π. To see that for each $k \in K$, π_k belongs to a middle block, notice that $\pi_k > \pi_a$, and if π_k belonged to a

left block, then the block containing π_k would cross the block containing π_a. Thus, K induces an occurrence of ρ in π_M.

Suppose now that π is an α-avoiding partition, and therefore π_M is a ρ-avoiding partition. Let $\kappa_1, \ldots, \kappa_t$ be the connected components of π_M. Note that any outlier appearing in the middle part of π must appear in the 'gap' between two components κ_i and κ_{i+1}, otherwise we would have a crossing between a left block and a middle block. In particular, each κ_i corresponds to a consecutive sequence of elements of π. Note also that if the right part of π contains an element from a middle block, then this middle block must correspond to the first block of one of the components κ_i.

Suppose that the cc-equivalence of ρ and ρ' is witnessed by a bijection g. Assume that the ρ'-avoiding partition $g(\pi_M)$ has components $\kappa_1', \ldots, \kappa_t'$. We now define an α'-avoiding partition π' having the same size and the same number of components as π. The left part of π' is identical to the left part of π. In the middle part, we replace the elements corresponding to κ_i with the elements corresponding to κ_i', for each $i = 1, \ldots, t$. The elements belonging to left blocks and appearing in the gap between κ_i and κ_{i+1} will remain in the same block and will appear in the gap between κ_i' and κ_{i+1}'. Since we do not assume that each κ_i' has the same size as κ_i, it may happen that the position of the gap changes. We also do not assume that $g(\pi_M)$ has the same number of blocks as π_M, so the numbering of right blocks may change as well. Finally, if in π the first block of κ_i contains some elements in the right part of π, then in π' these elements will be inserted into the first block of κ_i'. We now define $f(\pi) = \pi'$. It is clear that f has the required properties. $\square$

Theorem 3.3. *If σ and ρ are non-crossing partitions and σ is connected, then $\sigma[\rho] \overset{cc}{\sim} \rho[\sigma]$.*

Proof. Let us define $\alpha = \sigma[\rho]$ and $\alpha' = \rho[\sigma]$. Of course, a partition that avoids σ must avoid both α and α'. To prove the theorem, we characterize the structure of a partition π that contains σ but not α, as well as the structure of a partition π' containing σ but not α'. From the two characterizations, it will be clear that the two classes are equinumerous and there is a bijection between them preserving the number of components.

We will first describe the structure of an α-avoiding non-crossing partition π that contains σ. Let $I = (i(1), \ldots, i(k))$ be a leftmost occurrence of σ in π. Define $a = i(1)$ and $b = i(k)$. Note that since σ is connected, we know that π_a and π_b correspond to the same block of π. Choose I in such a way that a is the first element of the block containing b.

Define the left part of π to be the elements strictly to the left of π_a, the middle part to be the elements $\pi_a, \ldots, \pi_b$, and the right part to be the rest of π. Define left blocks, middle blocks, right blocks and outliers in the same way as in the previous proof. Note that there are no outliers in the middle part of π.

Let π_L be the left part of π, let π_M be the partition order-isomorphic to the middle part of π, and let π_R be the partition formed by the elements in the right blocks of π. By the choice of I, we know that π_L is σ-avoiding. It is not hard to see that π is $\sigma[\rho]$-avoiding if and only if π_R is ρ-avoiding. Let $\kappa_1, \ldots, \kappa_t$ be the connected components of π_L ordered right-to-left, and let $\zeta_1, \ldots, \zeta_u$ be the connected components of π_R ordered left-to-right. Every outlier in π is in the right part of π and its block is either the first block of π_M or the first block of one of the κ_i. Each outlier must be placed between the last vertex of ζ_i and the first

vertex of ζ_{i+1} for some $i \leq u$, or between π_b and the first vertex of ζ_1. Note that the outliers form a weakly decreasing subsequence in π.

Let x_0 be the number of outliers belonging to the first block of π_M, and for $i \in [t]$, let x_i be the number of outliers from the first block of κ_i. Thus, $\sum_{i=0}^{t} x_i$ is the number of all outliers in π. Let y_0 be the number of outliers appearing between the last element of π_M and the first element of ζ_1, and for $i \in [u]$, let y_i be the number of outliers between the last element of ζ_i and the first element of ζ_{i+1}, with y_u being the number of outliers to the right of ζ_u. The two sequences $x_0, \ldots, x_t$ and $y_0, \ldots, y_u$ determine uniquely the position and value of the outliers in π. Thus, π is uniquely determined by specifying $(\pi_\mathrm{L}, \pi_\mathrm{M}, \pi_\mathrm{R}, (x_i)_{i=0}^{t}, (y_i)_{i=0}^{u})$. We may easily check that this gives a bijection between $\sigma[\rho]$-avoiding partitions containing σ, and five-tuples of the form $(\pi_\mathrm{L}, \pi_\mathrm{M}, \pi_\mathrm{R}, (x_i)_{i=0}^{t}, (y_i)_{i=0}^{u})$, where π_L is a σ-avoiding partition, π_M is a connected partition that contains σ and every occurrence of σ in π_M intersects the last element of π_M, π_R is a ρ-avoiding partition, and $(x_i)_{i=0}^{t}$ and $(y_i)_{i=0}^{u}$ are nonnegative integer sequences of the same sum, where t is the number of components in π_L and u is the number of components of π_R. Note that we rely on the fact that σ is connected, which implies that if π_L avoids σ, then the leftmost occurrence of σ in $\pi_\mathrm{L}[\pi_\mathrm{M}]$ is contained in the component π_M.

From the sequences $(x_i)_{i=0}^{t}$ and $(y_i)_{i=0}^{u}$, we may deduce the number of components of π — each component of π is either equal to κ_i for some i, or equal to ζ_j for some j, or it contains π_M. Moreover, κ_i is a component of π if and only if $x_i = x_{i+1} = \cdots = x_t = 0$, and ζ_j is a component of π if and only if $y_j = y_{j+1} = \cdots = y_u = 0$.

Let us now provide an analogous analysis of the $\rho[\sigma]$-avoiding partitions containing σ. Let π' be such a partition, and let $I' = (i'(1), \ldots, i'(k))$ be a topmost occurrence of σ in π', chosen in such a way that $i'(1)$ is the first element of its block and $i'(k)$ is as small as possible. Put $a = i'(1)$ and $b = i'(k)$, and define the left part, middle part and right part of π' in the same way as in the first part of the proof. Let π'_L be the left part of π', let π'_M be the middle part of π', and let π'_R be the partition induced by the right blocks of π'. Then π'_L is a ρ-avoiding partition and π'_R is a σ-avoiding partition. Suppose that $\kappa'_1, \ldots, \kappa'_{t'}$ are the components of π'_L numbered right-to-left and $\zeta'_1, \ldots, \zeta'_{u'}$ are the components of π'_R numbered left-to-right. Let x'_i be the number of outliers belonging to the same π-block as the first vertex of κ'_i, with x'_0 being the outliers belonging to the block of π_a. Let y'_i be the number of outliers between ζ'_i and ζ'_{i+1}, with y'_0 being the number of outliers between π'_M and ζ_1. Then π' is uniquely determined by $(\pi'_\mathrm{L}, \pi'_\mathrm{M}, \pi'_\mathrm{R}, (x'_i)_{i=0}^{t'}, (y'_i)_{i=0}^{u'})$, and the x'_is and y'_is determine the number of components of π', in the same way as in the case of π. We see that by mapping π_L to π'_R, π_M to π'_M, π_R to π'_L, t to u', u to t', x_i to y'_i and y_i to x'_i, we get the required bijection. $\qquad\square$

Theorem 3.4. *Let* $\sigma_1, \ldots, \sigma_k$ *be a k-tuple of non-crossing partitions, and let p be a permutation of the set $\{1, 2, \ldots, k\}$. Then the partitions $\sigma_1[\sigma_2] \cdots [\sigma_k]$ and $\sigma_{p(1)}[\sigma_{p(2)}] \cdots [\sigma_{p(k)}]$ are cc-equivalent.*

Proof. We may assume, without loss of generality, that all the σ_i are connected and that p is a transposition of adjacent elements. Suppose that for some $i < k$ we have $p(i) = i + 1$, $p(i + 1) = i$, and $p(j) = j$ otherwise. By Theorem 3.3, we know that $\sigma_i[\sigma_{i+1}]$ is cc-equivalent to $\sigma_{i+1}[\sigma_i]$, and then from Theorem 3.2 we obtain

the desired result, by putting $\sigma = \sigma_1[\sigma_2] \cdots [\sigma_{i-1}]$, $\rho = \sigma_i[\sigma_{i+1}]$, $\rho' = \sigma_{i+1}[\sigma_i]$, and $\tau = \sigma_{i+2}[\sigma_{i+3}] \cdots [\sigma_k]$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Theorem 3.5. *Let $\sigma_1, \ldots, \sigma_k$ be a k-tuple of non-crossing partitions, let $q < k$ be an index such that the partition σ_q is empty, or connected, or contains only singleton blocks. Then the partition*

$$\sigma = 1[\sigma_1]1[\sigma_2]1 \cdots 1[\sigma_{q-1}]1[\sigma_q]1[\sigma_{q+1}]1[\sigma_{q+2}]1 \cdots 1[\sigma_k]1$$

is cc-equivalent to

$$\sigma' = 1[\sigma_1]1[\sigma_2]1 \cdots 1[\sigma_{q-1}]1[\sigma_{q+1}]1[\sigma_q]1[\sigma_{q+2}]1 \cdots 1[\sigma_k]1.$$

Proof. We proceed by induction. Fix an integer n, and suppose that for every $n' < n$ and for every p, the number of σ-avoiding partitions of size n' with p components is equal to the number of σ'-avoiding such partitions. Let g be a bijection between σ-avoiders and σ'-avoiders of size less than n, preserving size and number of components. We may assume, without loss of generality, that g has the property that $g(\pi) = \pi$ for any partition π that avoids both σ and σ'. We will define a bijection f mapping σ-avoiders of size n to σ'-avoiders of the same size and number of components.

Let π be a σ-avoiding partition of size n. If π is disconnected, it can be written as $\pi = \pi_1[\pi_2] \cdots [\pi_m]$ for $m > 1$ and π_i connected. We then define $f(\pi) = g(\pi_1)[g(\pi_2)] \cdots [g(\pi_m)]$. This clearly satisfies all the claimed properties.

Assume now that π is connected. Thus, π can be uniquely written as $\pi = 1[\pi_1]1[\pi_2]1 \cdots 1[\pi_m]1$ for some σ-avoiding partitions π_i. We use the following terminology: for a partition ρ, an occurrence $I = (i(1), \ldots, i(\ell))$ of ρ in π is a *top-level occurrence* if it maps the elements of the first block of ρ to the elements of the first block of π; in other words, if $\pi_{i(1)} = 1$. If I is not a top-level occurrence, we say that it is a *deep occurrence*. Note that if ρ is connected, then any deep occurrence of ρ in π must correspond to an occurrence of ρ in one of the partitions $\pi_1, \ldots, \pi_m$.

For $i \leq j \in [m+1]$, let $\pi(i, j)$ denote the partition $1[\pi_i]1[\pi_{i+1}]1 \cdots 1[\pi_{j-1}]1$, i.e., $\pi(i, j)$ is the subpartition of π between the i-th and j-th element of the first block. For an integer $i \leq m + 1$, let $\pi(\leq i)$ denote the partition $\pi(1, i)$ and $\pi(\geq i)$ be the partition $\pi(i, m + 1)$. We apply analogous notation for other connected partitions as well.

Let $\overline{\pi}_i$ denote the partition $g(\pi_i)$, and let $\overline{\pi}$ be the partition $1[\overline{\pi}_1]1[\overline{\pi}_2]1 \cdots 1[\overline{\pi}_m]1$. By induction, we know that for any $i \in [m]$, $\overline{\pi}_i$ is σ'-avoiding and $\overline{\pi}_i$ contains σ if and only if π_i contains σ'. Consequently, $\overline{\pi}$ has no deep occurrence of σ', and $\overline{\pi}$ has a deep occurrence of σ if and only if π has a deep occurrence of σ'. Using the fact that $\overline{\pi}_i = \pi_i$ whenever π_i avoids both σ and σ', we also see that for any $j \in [k]$, $\overline{\pi}_i$ contains σ_j if and only if π_i does, and more generally, for any $h, i \in [m+1]$, $\pi(h, i)$ has a top-level occurrence of $1[\sigma_j]1$ if and only if $\overline{\pi}(h, i)$ does. Consequently, $\overline{\pi}$ has no top-level occurrence of σ, and $\overline{\pi}$ has a top-level occurrence of σ' if and only if π does.

Let $a \in [m+1]$ be the smallest index such that $\overline{\pi}(\leq a)$ has a top-level occurrence of $\sigma(\leq q)$, and let $b \in [m+1]$ be the largest index such that $\overline{\pi}(\geq b)$ has a top-level occurrence of $\sigma(\geq q+2)$. If such a or b do not exist, or if $a + 2 > b$, then $\overline{\pi}$ has no top-level occurrence of either σ or σ', and we define $f(\pi) = \overline{\pi}$.

Suppose that $a + 2 \leq b$, and let c be the smallest integer from $\{a+1, a+2, \ldots, b\}$ such that $\overline{\pi}(a, c)$ has a top-level occurrence of $1[\sigma_q]1$. If no such c exists, we again

put $f(\pi) = \overline{\pi}$. Otherwise, define a partition $\widehat{\pi} = 1[\widehat{\pi}_1]1[\widehat{\pi}_2]1 \cdots 1[\widehat{\pi}_m]1$, by putting $\widehat{\pi}(\leq a) = \overline{\pi}(\leq a)$, $\widehat{\pi}(\geq b) = \overline{\pi}(\geq b)$, $\widehat{\pi}(a, a + b - c) = \overline{\pi}(c, b)$, and $\widehat{\pi}(a + b - c, b)$ being equal to $1[\overline{\pi}_{c-1}]1[\overline{\pi}_{c-2}]1 \cdots 1[\overline{\pi}_{a+1}]1[\overline{\pi}_a]1$. Notice that $\widehat{\pi}(a + b - c, b)$ has a top-level occurrence of $1[\sigma_q]1$, while $\widehat{\pi}(a + b - c + 1, b)$ does not (here we use the assumption that σ_q is empty, or connected, or only contains singleton blocks). We also know that $\overline{\pi}(c, b)$ has no top-level occurrence of $1[\sigma_{q+1}]1$, because $\overline{\pi}$ has no top-level occurrence of σ. This implies that $\widehat{\pi}$ has no top-level occurrence of σ', and therefore $\widehat{\pi}$ is a σ'-avoiding partition. We then define $f(\pi) = \widehat{\pi}$. It is easy to check that f is a bijection between σ-avoiding and σ'-avoiding partitions of size n which preserves the number of components. $\square$

Theorem 3.6. *Let $\sigma_1, \ldots, \sigma_k$ be a k-tuple of non-crossing partitions, let $q < k$ be an index, and let σ'_q be a partition cc-equivalent to σ_q. Then the pattern*

$$\sigma = 1[\sigma_1]1[\sigma_2]1 \cdots 1[\sigma_{q-1}]1[\sigma_q]1[\sigma_{q+1}]1 \cdots 1[\sigma_k]1$$

is cc-equivalent to

$$\sigma' = 1[\sigma_1]1[\sigma_2]1 \cdots 1[\sigma_{q-1}]1[\sigma'_q]1[\sigma_{q+1}]1 \cdots 1[\sigma_k]1.$$

Proof. As in the proof of Theorem 3.5, we proceed by induction. Suppose again that n is given, and that there is a bijection g mapping the σ-avoiders of size less than n to σ'-avoiders of the same size and same number of components. Suppose also that $g(\pi) = \pi$ for any partition that avoids both σ and σ'. We define a bijection f mapping σ-avoiders of size n to σ'-avoiders of the same size and number of components. Let h be a mapping from σ_q-avoiding partitions to σ'_q-avoiding partitions which witnesses the cc-equivalence of σ_q and σ'_q.

Let π be a σ-avoiding partition of size n. If π is disconnected with components $\pi_1, \ldots, \pi_m$, we define $f(\pi)$ to be the partition with components $g(\pi_1), \ldots, g(\pi_m)$. Suppose now that π is connected, and has the form $1[\pi_1]1 \cdots 1[\pi_m]1$.

We will define a new partition $\overline{\pi}$ that has no top-level occurrence of σ'. Let a be the smallest integer such that $\pi(\leq a)$ has a top-level occurrence of $\sigma(\leq q)$, and let b be the largest integer such that $\pi(\geq b)$ has a top-level occurrence of $\sigma(\geq q + 1)$. If such a or b does not exist, or if $a \geq b$, we define $\overline{\pi} = \pi$. Otherwise, let ρ be the partition $\pi_a[\pi_{a+1}] \cdots [\pi_{b-1}]$, and let p_i be the number of connected components of π_i, so that ρ has $p_a + p_{a+1} + \cdots + p_{b-1}$ components. Note that ρ avoids σ_q. Define $\rho' = h(\rho)$, and write ρ' as $\rho' = \overline{\pi}_a[\overline{\pi}_{a+1}] \cdots [\overline{\pi}_{b-1}]$, where each $\overline{\pi}_i$ is chosen so that it has exactly p_i components. We now define the partition $\overline{\pi} = 1[\overline{\pi}_1]1 \cdots 1[\overline{\pi}_m]1$ by putting $\overline{\pi}(\leq a) = \pi(\leq a)$, $\overline{\pi}(\geq b) = \pi(\geq b)$, and $\overline{\pi}(a, b)$ is determined by the partitions $\overline{\pi}_i$ obtained from ρ'.

Note that $\overline{\pi}$ has no top-level occurrence of σ'. Also, $\overline{\pi}(a, b)$ has no deep occurrence of σ', because it does not even have a deep occurrence of σ'_q. Define now a partition $\widehat{\pi} = 1[\widehat{\pi}_1]1 \cdots 1[\widehat{\pi}_m]1$ by putting $\widehat{\pi}_i = \overline{\pi}_i$ for each $a \leq i < b$, and $\widehat{\pi}_i = g(\overline{\pi}_i)$ for each $i < a$ and $i \geq b$. Then $\widehat{\pi}$ has no deep occurrence of σ'. Using the fact that $g(\overline{\pi}_i) = \overline{\pi}_i$ whenever $\overline{\pi}_i$ avoids both σ and σ', we can also see that a is the smallest index such that $\widehat{\pi}(\leq a)$ has a top-level occurrence of $\sigma'(\leq q)$, and b is the largest index such that $\widehat{\pi}(\geq b)$ has a top-level occurrence of $\sigma'(\geq q + 1)$. We put $f(\pi) = \widehat{\pi}$, and easily see that f is the required bijection. $\square$

In the rest of this section, we will often employ generating functions as tools in our proofs. Let us therefore fix the following notation. For a partition π, we let $NC(x; \pi)$ denote the generating function of the set of non-crossing π-avoiding

partitions, and we let $C(x; \pi)$ denote the generating function of the set of nonempty connected non-crossing π-avoiding partitions.

Theorem 3.7. *Let σ and τ be two possibly empty connected partitions. Then $1[\sigma]1[\tau]$ and $1[\tau]1[\sigma]$ are nc-equivalent.*

In the previous theorem, the assumption that σ and τ are connected is necessary, as shown, e.g., by the two patterns $1[1]1[12] = 12134$ and $1[12]1[1] = 12314$, which are not nc-equivalent. Also, nc-equivalence in the conclusion cannot in general be replaced with cc-equivalence. For example, taking σ empty and $\tau = 1$, we see that $1[\sigma]1[\tau] = 112$, while $1[\tau]1[\sigma] = 121$. Since 112 and 121 do not have the same number of components, it is easy to see that they cannot be cc-equivalent.

Proof of Theorem 3.7. Let us first deal with the situation where both σ and τ are nonempty. Let $G(x; \sigma, \tau)$ denote the generating function of non-crossing partitions that avoid $1[\sigma]1[\tau]$ but contain $\sigma[\tau]$, in other words,

$$G(x; \sigma, \tau) = NC(x; 1[\sigma]1[\tau]) - NC(x; \sigma[\tau]).$$

We know from Theorem 3.3 that $NC(x, \sigma[\tau]) = NC(x, \tau[\sigma])$. Therefore, to show that $1[\sigma]1[\tau]$ is nc-equivalent to $1[\tau]1[\sigma]$, it is enough to prove that $G(x; \sigma, \tau) = G(x; \tau, \sigma)$. We will derive a formula for $G(x; \sigma, \tau)$ from which the previous identity will easily follow.

Note that if ρ is a connected partition, a non-crossing partition π avoids ρ if and only if each component of π avoids ρ. In particular, we have the identity $NC(x; \rho) = 1/(1 - C(x; \rho))$. We say that a partition $\pi = \pi_1, \ldots, \pi_n$ is ρ-*minimal* if it is connected, non-crossing, contains ρ, but avoids $1[\rho]1$. Let $M(x; \rho)$ be the generating function of the set of ρ-minimal partitions.

Suppose that $\pi = \pi_1, \ldots, \pi_n$ is a non-crossing partition that avoids $1[\sigma]1[\tau]$ and contains $\sigma[\tau]$. Let $I = (i(1), \ldots, i(k))$ be a leftmost occurrence of σ in π, chosen in such a way that $i(1)$ is as small as possible. This implies that $\pi_{i(1)}$ is the leftmost element of its π-block. Let us write $a = i(1)$.

Let $J = (j(1), ..., j(\ell))$ be a topmost occurrence of τ. Choose J in such a way that $j(1)$ is as small as possible, and write $b = j(1)$. Then π_b is the leftmost element of its block. Let π_c be the rightmost element of the block containing π_b. Then $\pi_b, \pi_{b+1}, \ldots, \pi_c$ is order-isomorphic to a τ-minimal partition, because if it contained a copy of $1[\tau]1$, it would contradict the topmost choice of J.

Let π_{L} denote the partition $\pi_1, \pi_2, \ldots, \pi_{b-1}$. Note that π_{L} avoids $1[\sigma]1$. Let $\pi_{\mathrm{L}}^1, \pi_{\mathrm{L}}^2, \ldots, \pi_{\mathrm{L}}^m$ be the connected components of π_{L}. Let π_{L}^q be the component of π_{L} containing the vertex π_a. Note that π_a must be the leftmost vertex of π_{L}^q, otherwise π_{L}^q would contain $1[\sigma]1$. We see that π_{L}^q is a σ-minimal partition. Note also that all the components preceding π_{L}^q must avoid σ, since π_{L}^q contains the leftmost occurrence of σ.

We say that an element π_i of π is an *outlier*, if $i > c$ and $\pi_i < \pi_c$. In other words, an outlier is an element that does not belong to π_{L}, but belongs to a π-block whose leftmost element belongs to π_{L}. Note that if π_i is an outlier, then the π-block containing π_i intersects a unique component π_{L}^j of π_{L}, and it is the first block of of π_{L}^j; we then say that π_i is an *outlier from π_{L}^j.*

For a component π_{L}^j of π_{L}, define *the zone* of π_{L}^j, denoted by Z_j, inductively as follows. If π_{L}^j has no outlier, then Z_j is empty, otherwise Z_j is the sequence $\pi_g, \pi_{g+1}, \ldots, \pi_h$, where π_g is the leftmost outlier of π_{L}^j and π_h is the rightmost

vertex not belonging to $Z_1 \cup Z_2 \cup \cdots \cup Z_{j-1}$. Let π_d be the rightmost vertex of π not belonging to any zone. The zones $Z_1, \ldots, Z_m$ form a disjoint collection of subsequences whose union is $\pi_{d+1}, \ldots, \pi_n$. Each zone Z_j is order-isomorphic to a partition of the form $1[\rho_1]1[\rho_2]1 \cdots 1[\rho_r]$ in which each occurrence of 1 corresponds to an outlier from π_{L}^j, and each ρ_i is a τ-avoiding partition formed by non-outliers. The generating function of such partitions is

$$Z(x) = \frac{1}{1 - xNC(x;\tau)} = \frac{1}{1 - \frac{x}{1-C(x;\tau)}}.$$

Note also that the elements $\pi_{c+1}, \pi_{c+2}, \ldots, \pi_d$ (which do not belong to any zone and do not contain any outliers) are order-isomorphic to a τ-avoiding partition.

We claim that the generating function of all the non-crossing partitions π avoiding $1[\sigma]1[\tau]$, containing $\sigma[\tau]$, having m components in π_{L}, and with the component π_{L}^q containing the leftmost occurrence of σ is equal to

$$(C(x;\sigma)Z(x))^{q-1} M(x;\sigma)Z(x) (C(x;1[\sigma]1)Z(x))^{m-q} M(x;\tau)\frac{1}{1 - C(x;\tau)}.$$

To see this, note first that each factor $C(x;\sigma)Z(x)$ corresponds to one of the first $q - 1$ components of π_{L}, together with its zone. Next, the factor $M(x;\sigma)Z(x)$ corresponds to the possible choices for the component π_{L}^q and its zone. The factor $C(x;1[\sigma]1)Z(x)$ corresponds to a component π_{L}^i for $i > q$, together with its zone. The factor $M(x;\tau)$ corresponds to the elements from π_b to π_c, and the next factor $(1 - C(x;\tau))^{-1}$ corresponds to the elements $\pi_{c+1}, \ldots, \pi_d$.

Summing the above expression for all possible $m \geq 1$ and $q \in [m]$, we obtain
(3.1)
$$G(x;\sigma,\tau) = \frac{1}{1 - C(x;\sigma)Z(x)}M(x;\sigma)Z(x)\frac{1}{1 - C(x;1[\sigma]1)Z(x)}M(x;\tau)\frac{1}{1 - C(x;\tau)}.$$

Using the identity

$$C(x;1[\rho]1) = x + \frac{x^2}{1 - x - C(x;\rho)} = \frac{x}{1 - \frac{x}{1-C(x;\rho)}},$$

which is valid for any connected non-crossing partition ρ, we define two auxiliary expressions, both of which are symmetric in σ and τ:

$$F_1(x;\sigma,\tau) = \frac{1}{1 - C(x;1[\sigma]1)Z(x)} = \frac{1}{1 - \frac{x(1-C(x;\sigma))(1-C(x;\tau))}{(1-x-C(x;\sigma))(1-x-C(x;\tau))}}$$

and

$$F_2(x;\sigma,\tau) = \frac{Z(x)}{(1 - C(x;\sigma)Z(x))(1 - C(x;\tau))}$$

$$= \frac{1}{1 - x - C(x;\sigma) - C(x;\tau) + C(x;\sigma)C(x;\tau)}.$$

With this notation, (3.1) simplifies into

$$G(x;\sigma,\tau) = M(x;\sigma)M(x;\tau)F_1(x;\sigma,\tau)F_2(x;\sigma,\tau),$$

This makes it clear that $G(x;\sigma,\tau) = G(x;\tau,\sigma)$, completing the proof for the case when both σ and τ are nonempty.

It remains to deal with the case when σ or τ is empty, i.e., to show that $1[\tau]1 \overset{nc}{\sim} 11[\tau]$ for any connected τ. The generating function of $1[\tau]1$-avoiding non-crossing partitions is equal to

$$\frac{1}{1 - C(x; 1[\tau]1)} = \frac{1 - x - C(x; \tau)}{1 - 2x - C(x; \tau) + xC(x; \tau)}.$$

Let us now sketch the argument for the pattern $11[\tau]$. Partitions avoiding 11 have generating function $1/(1-x)$. Let π be a partition containing 11 and avoiding $11[\tau]$. Let $I = (a, b)$ be the leftmost occurrence of 11 in π. All the elements $\pi_1, \pi_2, \ldots, \pi_{b-1}$ belong to distinct blocks of π. An element π_i is an *outlier* if $i > b$ and $\pi_i \leq \pi_a$. The elements $\pi_{b+1}, \ldots, \pi_n$ that are not outliers form a τ-avoiding partition. We may define zones $Z_1, \ldots, Z_a$ in analogy to the previous case. All elements of π to the right of the leftmost outlier (inclusive) belong to a unique zone. This yields a generating function

$$\frac{1}{1 - xZ(x)} \frac{x^2 Z(x)}{1 - x} \frac{1}{1 - C(x; \tau)},$$

where the factor $1/(1 - x)$ counts the elements between π_a and π_b, while the factor $(1 - C(x; \tau))^{-1}$ counts subpartitions formed by the elements to the right of π_b but to the left of the leftmost zone.

Adding $1/(1 - x)$ to the above expression and simplifying shows that $11[\tau]$-avoiding partitions have the same generating function as $1[\tau]1$-avoiding partitions. $\square$

Theorem 3.8. *Let σ be a connected partition, and let τ be a partition of the form $1[\rho]$ for some partition ρ. If σ and τ are nc-equivalent, then $1[\sigma]$ and $\tau1 = 1[\rho]1$ are nc-equivalent as well.*

Proof. Let $NC(x; \pi)$ denote the generating function of the set of π-avoiding non-crossing partitions. By assumption, we have $NC(x; \sigma) = NC(x; \tau)$.

Note that a partition π avoids $1[\sigma]$ if and only if it can be written as

$$\pi = 1[\pi^1]1[\pi^2]1 \cdots 1[\pi^k]$$

for some k, where each π^i is a σ-avoiding partition (here we use the fact that σ is connected). Therefore, we have the identity

$$NC(x; 1[\sigma]) = \frac{1}{1 - xNC(x; \sigma)}.$$

Consider now the partition $\tau1 = 1[\rho]1$. Since this partition is connected, we see that π avoids $\tau1$ if and only if each component of π avoids $\tau1$. Moreover, a connected partition $\pi = \pi_1, \ldots, \pi_n$ avoids $\tau1$ if and only if $\pi_1, \ldots, \pi_{n-1}$ avoids τ (here we use the fact that τ has only one occurrence of the symbol 1, and that $\pi_n = 1$ because π is connected). This implies the identity

$$NC(x; \tau1) = \frac{1}{1 - xNC(x; \tau)}.$$

Since $NC(x; \sigma) = NC(x; \tau)$, we get that $NC(x; 1[\sigma]) = NC(x; \tau1)$. $\square$

As an example of an application of Theorem 3.8, consider the partitions $\sigma = 11$ and $\tau = 12$. Since these two partitions are nc-equivalent, the theorem implies that $1[\sigma] = 122$ and $\tau1 = 121$ are nc-equivalent as well. We may in fact apply the theorem again to this new pair of patterns, and obtain that $1221 \overset{nc}{\sim} 1232$, and a

third application reveals that $12332 \stackrel{nc}{\sim} 12321$. Generalizing this example into a straightforward induction argument, we get the next corollary.

Corollary 3.9. *For any k, the pattern $12 \cdots (k-1)kk(k-1) \cdots 32$ is nc-equivalent to the pattern $12 \cdots (k-1)k(k-1) \cdots 21$, and the pattern $12 \cdots (k-1)kk(k-1) \cdots 21$ is nc-equivalent to $12 \cdots k(k+1)k \cdots 32$.*

Note that the partitions avoiding $12 \cdots (k-1)k(k-1) \cdots 21$ are precisely those that do not have a k-tuple of pairwise nested blocks.

Theorem 3.10. *The partitions 12333 and 12321 are nc-equivalent. In other words, the non-crossing partitions whose every block has size at most two except possibly the first two blocks are equinumerous with the non-crossing partitions that have no 3-nesting.*

Proof. We again let $NC(x; \pi)$ denote the generating function of non-crossing π-avoiding partitions, and let $C(x; \pi)$ be the generating function of nonempty connected non-crossing π-avoiding partitions. As we have already pointed out before, for a connected partition π, we have the identity

$$NC(x; \pi) = \frac{1}{1 - C(x; \pi)},$$

and for arbitrary τ, we have the identity

$$C(x; 1[\tau]1) = \frac{x}{1 - xNC(x; \tau)}.$$

Combining these two identities and simplifying, we deduce that

$$NC(x; 12321) = \frac{1 - 3x + x^2}{(1-x)(1-3x)} = 1 + \sum_{n \geq 1} \frac{3^{n-1} + 1}{2} x^n.$$

Let us now turn to the pattern $\tau = 12333$. The generating function for the empty partition together with those that have a single block is of course $1/(1-x)$. On the other hand, a non-crossing partition with at least two blocks avoids τ if and only if it has a decomposition of the form $11 \cdots 12[\rho^1]2[\rho^2]2 \cdots 2[\rho^k]1[\sigma^1]1[\sigma^2]1 \cdots 1[\sigma^\ell]$ for some $k \geq 1$ and $\ell \geq 0$, where the ρ^i and σ^j are 111-avoiding non-crossing partitions. It is known that non-crossing 111-avoiding partitions are counted by the Motzkin numbers ([21, sequence A001006]), and their generating function satisfies the identity

$$NC(x; 111) = 1 + xNC(x; 111) + (xNC(x; 111))^2.$$

We deduce that

$$NC(x; 12333) = \frac{1}{1-x} + \frac{x^2 NC(x; 111)}{(1-x)(1 - xNC(x; 111))^2},$$

from which the result easily follows. $\square$

We remark that the counting function of 12333-avoiding non-crossing partitions (and therefore also 12321-avoiding non-crossing partitions) has been encountered before in different contexts (see [21, sequence A124302]).

We may again use Theorem 3.8 iteratively to extend the equivalence $12333 \stackrel{nc}{\sim} 12321$ to an infinite sequence of equivalences.

Corollary 3.11. *For every $k \geq 3$, the pattern $12 \cdots k(k+1)k(k-1) \cdots 2$ is nc-equivalent to $12 \cdots (k-1)kkk(k-2) \cdots 1$, and the pattern $12 \cdots (k-1)k(k-1) \cdots 1$ is nc-equivalent to $12 \cdots kkk(k-2) \cdots 2$.*

As an application of the above theorems, one may completely identify the Wilf-equivalence classes corresponding to $\{1212, \tau\}$, where τ is of size at most six. For example, we have the following equivalences in the cases when τ is of size four or five, while the corresponding table for size six can be found on the second author's webpage [17].

- [15] $1232 \overset{nc}{\sim} 1213 \overset{nc}{\sim} 1221 \overset{nc}{\sim} 1122$ by Theorem 3.4, Theorem 3.7 and Corollary 3.9,
- $1123 \overset{nc}{\sim} 1223 \overset{nc}{\sim} 1233$ by Theorem 3.4,
- [12] $1211 \overset{nc}{\sim} 1121$ by Theorem 3.5,
- [11] $1112 \overset{nc}{\sim} 1222$ by Theorem 3.4,
- 1234,
- 1231,
- 1111,
- $12332 \overset{nc}{\sim} 12333 \overset{nc}{\sim} 12133 \overset{nc}{\sim} 11123 \overset{nc}{\sim} 12213 \overset{nc}{\sim} 12321 \overset{nc}{\sim} 12223 \overset{nc}{\sim} 11232$ by Theorem 3.4, Theorem 3.7, Corollary 3.9 and Corollary 3.11,
- $12234 \overset{nc}{\sim} 12334 \overset{nc}{\sim} 12344 \overset{nc}{\sim} 11234$ by Theorem 3.4,
- $12113 \overset{nc}{\sim} 12322 \overset{nc}{\sim} 12232 \overset{nc}{\sim} 11213$ by Theorem 3.4 and Theorem 3.5,
- $12233 \overset{nc}{\sim} 11233 \overset{nc}{\sim} 11223$ by Theorem 3.4,
- $12343 \overset{nc}{\sim} 12134 \overset{nc}{\sim} 12324$ by Theorem 3.4,
- $11122 \overset{nc}{\sim} 12221 \overset{nc}{\sim} 11222$ by Theorem 3.4 and Theorem 3.7,
- $11121 \overset{nc}{\sim} 11211 \overset{nc}{\sim} 12111$ by Theorem 3.5,
- $12331 \overset{nc}{\sim} 12231$ by Theorem 3.6 using Theorem 3.4,
- $12342 \overset{nc}{\sim} 12314$ by Theorem 3.4,
- $11231 \overset{nc}{\sim} 12311$ by Theorem 3.5,
- $12211 \overset{nc}{\sim} 11221$ by Theorem 3.5,
- $12222 \overset{nc}{\sim} 11112$ by Theorem 3.4,
- 12131,
- 12341,
- 12345,
- 11111.

4. Avoiding two patterns of size four

Let us say that a pair of patterns (σ, τ) is a $(4,4)$-*pair*, if σ and τ are two distinct partitions of size four. In this section, we will provide the full classification of equivalences among all $(4,4)$-pairs.

4.1. **Previously known equivalences.** Equivalences among sets of patterns have been previously studied in a series of papers by Mansour and Shattuck [11, 12, 13, 14, 15], and some of the equivalence classes of $(4,4)$-pairs have been identified. Specifically, the following results are known.

Theorem 4.1 (Theorem 1.1 in [12]). *If $n \geq 0$, then $p_n(u,v) = w_n$ for the following pairs (u,v):*

$$(1211, 1212), (1121, 1212), (1121, 1221), (1112, 1123), (1122, 1123).$$

The generating function for the sequence w_n is given by

$$\sum_{n \geq 0} w_n x^n = \frac{(1-x)^2 - \sqrt{1 - 4x + 2x^2 + x^4}}{2x^2}.$$

Theorem 4.2 (Theorem 1.1 in [11])**.** *If $n \geq 0$, then $p_n(u,v) = L_n$ for the following pairs (u,v):*

$$(1222, 1212), (1112, 1212), (1211, 1221), (1222, 1221).$$

The generating function for the sequence L_n is given by

$$\sum_{n \geq 0} L_n x^n = \frac{1 - 3x + \sqrt{1 - 2x - 3x^2}}{2(1 - 3x)}.$$

Furthermore, results from [15] imply the following fact.

Fact 4.3. *These pairs are all equivalent:* $(1112, 1213)$, $(1122, 1212)$, $(1123, 1213)$, $(1123, 1223)$, $(1211, 1231)$, $(1212, 1213)$, $(1221, 1231)$, $(1222, 1223)$, $(1222, 1232)$, $(1212, 1232)$, *and* $(1212, 1221)$*. Moreover, for any such pair (u,v), $p_n(u,v)$ is equal to F_{2n-2}, where F_i is the i-th Fibonacci number.*

4.2. **Known results on pattern equivalences.** In the paper on partial patterns in matchings [7], the authors introduce the notion of strong partition equivalence. We say that two patterns σ and τ are *strongly partition equivalent*, if there exists a bijection f between the sets of σ-avoiding and τ-avoiding partitions with the property that for any σ-avoiding partition ρ, the number of blocks of ρ is equal to the number of blocks of $f(\rho)$, and moreover for any i, the i-th block of ρ has the same size as the i-th block of $f(\rho)$. Intuitively, strong partition equivalence means that we can bijectively map σ-avoiders to τ-avoiders by just permuting the letters of their standard representation.

The concept of strong partition equivalence is first explicitly used in [7], although most pairs of strongly partition equivalent patterns follow from the bijections constructed in an earlier paper [6]. Let us list the known facts about strong partition equivalence (references point to the corresponding statement in [6]).

Fact 4.4 (Lemma 9, Theorem 18, Corollary 18)**.** *For every k and every partition τ, the two partitions*

$$12 \cdots k(\tau + k)12 \cdots k \ \text{ and } \ 12 \cdots k(\tau + k)k(k-1) \cdots 1$$

are strongly partition equivalent.

Fact 4.5 (Theorem 31)**.** *Let $p_n(\rho; a_1, a_2, \ldots, a_m)$ be the number of partitions that avoid ρ such that the i-th block has a_i elements. Then for any partition τ,*

$$p_n(1(\tau + 1); a_1, a_2, \ldots, a_m) = \binom{n-1}{a_1 - 1} p_{n-a_1}(\tau; a_2, a_3, \ldots, a_m).$$

Consequently, if τ and σ are strongly partition-equivalent, then so are $1(\tau + 1)$ and $1(\sigma + 1)$.

Fact 4.6 (Theorem 34)**.** *For every $j \geq 1$ and $k \geq 0$, the partition $1^j 2 1^k$ is strongly partition-equivalent to $1^{j+k} 2$.*

Fact 4.7 (Theorem 42). *For every sequence s over the alphabet $[m]$, for every $p \geq 1$ and $q \geq 0$, the partitions*

$$12 \cdots m(m+1)^p(m+2)(m+1)^q s \text{ and } 12 \cdots m(m+1)^{p+q}(m+2)s$$

are strongly partition-equivalent.

Fact 4.8 (Theorem 48). *For every k, all the partitions of size k that start with 12 and that contain two occurrences of the symbol 1, one occurrence of the symbol 3, and all their remaining symbols are equal to 2, are mutually strongly partition-equivalent.*

In the study of multi-avoidance, the concept of strong partition equivalence becomes relevant through the following simple result.

Theorem 4.9. *Let ρ be a pattern of the form $12 \cdots (k-1)k^a$ for some $a \geq 1$ and $k \geq 1$. That is, ρ is formed by a strictly increasing sequence of length k, followed by another $a - 1$ occurrences of the symbol k. Suppose that the two patterns σ and τ are strongly partition equivalent. Then the two pattern pairs (ρ, σ) and (ρ, τ) are equivalent.*

Proof. Note that a partition avoids ρ if and only if it either has fewer than k blocks, or for every $i \geq k$, its i-th block has size less than a. In other words, avoidance of ρ can be characterized as a property of block sizes.

Now assume that σ and τ are strongly partition equivalent via a bijection f. Since f preserves the sizes of each block, we know that a partition π avoids ρ if and only if $f(\pi)$ avoids ρ. In particular, f maps the set of (ρ, σ)-avoiding partitions bijectively to the set of (ρ, τ)-avoiding partitions.   □

4.3. **General arguments.** Before we deal with individual $(4, 4)$-pairs, we first provide several general results applicable to infinite families of pattern-avoiding classes. Our first argument involves patterns containing one symbol equal to 2 and the remaining symbols equal to 1. Fix such a pattern $\sigma = 1^a 2 1^b$ with $a \geq 1$ and $b \geq 0$. Let $k = a + b + 1$ be the size of σ.

For a set of patterns T, let T' denote the set $\{1(\tau + 1), \tau \in T\}$.

For a set of patterns R, let $P_n(\sigma, R)$ denote the set of partitions of size n that avoid the pattern σ as well as all the patterns in R, and let $P_n(\sigma, R; i)$ denote the set of partitions in $P_n(\sigma, R)$ whose first block has size i. Let $f_n(\sigma, R)$ and $f_n(\sigma, R; i)$ denote the cardinality of $P_n(\sigma, R)$ and $P_n(\sigma, R; i)$, respectively.

Lemma 4.10. *For any set of patterns T, and for σ and T' as above, we have*

$$f_n(\sigma, T') = \sum_{i=1}^{k-2} f_{n-i}(\sigma, T) \binom{n-1}{i-1} + \sum_{i=k-1}^{n} f_{n-i}(\sigma, T) \binom{n-i+k-3}{k-3}.$$

Proof. We will compute the size of $P_n(\sigma, T'; i)$ for $i \in [n]$. Suppose first that $i < k - 1$. Then a partition π of size n belongs to $P_n(\sigma, T'; i)$ if and only if its first block has size i and the remaining blocks induce a partition that belongs to $P_{n-i}(\sigma, T)$. Therefore, we have $f_n(\sigma, T'; i) = f_{n-i}(\sigma, T) \binom{n-1}{i-1}$.

Now suppose that $i \geq k - 1$. Then a partition π of size n belongs to $P_n(\sigma, T'; i)$ if and only if its first block has size i, the remaining blocks induce a partition that belongs to $P_{n-i}(\sigma, T)$, and moreover, no symbol greater than 1 may have a occurrences of 1 before it and b occurrences of 1 after it. That means that every

symbol greater than 1 appears either before the a-th occurrence of 1 or after the $(i - b + 1)$-th occurrence of 1. That gives $f_n(\sigma, T'; i) = f_{n-i}(\sigma, T)\binom{n-i+k-3}{k-3}$.

Summing over all $i \in [n]$ gives the result. $\qquad\square$

Note that the formula in the previous lemma does not depend on a and b. The next statement is a direct consequence of Lemma 4.10.

Corollary 4.11. *Let σ and ρ be two patterns of size k, with $\sigma = 1^a 2 1^b$ and $\rho = 1^c 2 1^d$. Let T and U be two sets of patterns. Let $T' = \{1(\tau+1), \tau \in T\}$ and similarly for U' and U. If $\{\sigma\} \cup T \sim \{\rho\} \cup U$, then $\{\sigma\} \cup T' \sim \{\rho\} \cup U'$.*

Proposition 4.12. *If T is a set of patterns and T' is the set of patterns $1(\tau + 1)$ where $\tau \in T$, then for every $n \geq 1$, we have $p_n(T') = \sum_{k=0}^{n-1} \binom{n-1}{k} p_k(T)$. Consequently, if T and R are sets of patterns such that $T \sim R$, then $T' \sim R'$.*

Proof. It is enough to observe that a partition π avoids T' if and only if the sub-partition of π obtained by removing the first block of π avoids T. Therefore, there are exactly $\binom{n-1}{k} p_k(T)$ partitions in $P_n(T')$ whose first block has size $n - k$. $\qquad\square$

Corollary 4.13. *The $(4, 4)$-pairs $(1222, 1234)$ and $(1211, 1234)$ are equivalent.*

Proof. We express $p_n(1222, 1234)$ using Proposition 4.12 and $p_n(1211, 1234)$ using Lemma 4.10. We have, for $n \geq 1$,

$$p_n(1222, 1234) = \sum_{k=0}^{n-1} \binom{n-1}{k} p_k(111, 123)$$

$$= 1 + (n - 1) + 2\binom{n-1}{2} + 3\binom{n-1}{3} + 3\binom{n-1}{4}$$

and

$$p_n(1211, 1234) = p_{n-1}(123, 1211) + (n-1)p_{n-2}(123, 1211)$$

$$+ 1 + \sum_{k=1}^{n-3} p_k(123, 1211)(k+1).$$

From Theorem 2.7, we deduce that $p_n(123, 1211) = n + \binom{n-1}{2}$. Substituting into the above expression and simplifying shows that both $p_n(1211, 1234)$ and $p_n(1222, 1234)$ are equal to $(n^4 - 6n^3 + 19n^2 - 22n + 16)/8$ for $n \geq 1$. $\qquad\square$

Reasoning similar to that used in the proof of Lemma 4.10 above yields the following result, whose proof we omit.

Proposition 4.14. *Let T be a set of patterns and T' be the set of patterns $1(\tau+1)$, where $\tau \in T$. Let $a_n(R)$ (respectively, $b_n(R)$) denote the number of partitions of $[n]$ that avoid all the patterns in R as well as both 1112 and 1121 (respectively, 1121 and 1211). Then, for all $n \geq 4$, we have*

$$a_n(T') = a_{n-1}(T) + (n-1)a_{n-2}(T) + a_{n-3}(T) + a_{n-4}(T) + \cdots + a_0(T) \text{ and}$$

$$b_n(T') = b_{n-1}(T) + (n-1)b_{n-2}(T) + b_{n-3}(T) + b_{n-4}(T) + \cdots + b_0(T).$$

4.4. **Specific patterns of size four.** In the next theorem, we consider the case of avoiding 1213 and another pattern of the form $1(\tau + 1)$.

Theorem 4.15. *Let $\rho = 1(\tau + 1)$ be any pattern of size at least two such that the rightmost letter of τ is greater than 1. Then the generating function $H_\rho(x)$ is given by*

$$1 + \frac{x}{1 - x} H_\tau(x) + \frac{x^2}{(1 - x)(1 - 2x)} (H_\tau(x) - 1),$$

where $H_\sigma(x)$ denotes the generating function for the number of partitions of $[n]$ that avoid 1213 and a pattern σ. Moreover, if $(1213, \tau) \sim (1213, \tau')$, then $(1213, 1(\tau + 1)) \sim (1213, 1(\tau' + 1))$.

Proof. Let us write an equation for the generating function $H_\rho(x)$. For each nonempty partition π of $[n]$ that avoids 1213 and ρ, either the first block of π contains only 1 or it contains 1 and 2 or it contains 1 and at least one other element, not 2. The contributions from the first two cases are $xH_\tau(x)$ and $x(H_\rho(x) - 1)$, respectively. Each partition π in the last case must have the form $12\pi'1\alpha$, where π' does not contain 1 and α is a (possibly empty) word on $\{1, 2\}$, which implies a contribution of $\frac{x^2}{1-2x}(H_\tau(x) - 1)$. Hence,

$$H_\rho(x) = 1 - x + xH_\tau(x) + xH_\rho(x) + \frac{x^2}{1 - 2x}(H_\tau(x) - 1),$$

which yields the required result. $\qquad\square$

Example 4.16. *We consider some specific examples. Theorem 4.15, together with the fact that $H_{112}(x) = \frac{1-x}{1-2x}$, implies*

$$H_{1223}(x) = 1 + \frac{x(1 - 3x + 3x^2)}{(1 - 2x)^2(1 - x)}.$$

Since $112 \sim 123$, we then have $(1213, 1223) \sim (1213, 1234)$, by Theorem 4.15.
    Using the same reasoning as in the proof of Theorem 4.15 gives

- $H_{1231}(x) = 1 + xH_{1231}(x) + x(H_{1231}(x) - 1) + \frac{x^3}{(1-x)(1-2x)}$, *which implies* $H_{1231}(x) = H_{1223}(x)$, *whence $(1213, 1223) \sim (1213, 1231)$;*
- $H_{1221}(x) = 1 + xH_{1221}(x) + x(H_{1221}(x) - 1) + \frac{x^3}{(1-x)^3}$, *which implies*

$$H_{1221}(x) = \frac{1 - 4x + 6x^2 - 3x^3 + x^4}{(1 - x)^3(1 - 2x)};$$

- $H_{1232}(x) = 1 + xH_{121}(x) + x(H_{1232}(x) - 1) + \frac{x^3}{(1-x)^2(1-2x)}$, *whence $H_{1232}(x) = H_{1221}(x)$.*

Similarly, we obtain

$$H_{1233}(x) = H_{1221}(x) \text{ and } H_{1121}(x) = \frac{(1 - x - x^2)(1 - x)^2}{1 - 4x + 4x^2 - 2x^4}.$$

Example 4.17. *Let $L_\sigma(x)$ be the generating function for the number of partitions of $[n]$ that avoid 1231 and σ. In this example, we study the generating function $L_\sigma(x)$ in a couple of cases.*
    First, we consider the case $\sigma = 1121$. Note that each nonempty partition π that avoids 1231 and 1121 can be expressed as $\pi = 11 \cdots 1\pi'$, $\pi = 122 \cdots 211 \cdots 1\pi''$, or $\pi = 122 \cdots 211 \cdots 122 \cdots 2\pi'''$, where π' does not contain the letter 1, π'' starts with

3 if nonempty (and is such that $22 \cdots 2\pi''$ is a nonempty partition on the letters $\{2, 3, \ldots\}$), and π''' does not contain the letters 1 or 2. Thus the generating function $L_{1121}(x)$ satisfies

$$L_{1121}(x) = 1 + \frac{x}{1-x}L_{1121}(x) + \frac{x^2}{1-x}(L_{1121}(x) - 1) + \frac{x^4}{(1-x)^3}L_{1121}(x),$$

whence $L_{1121}(x) = \frac{(1-x-x^2)(1-x)^2}{1-4x+4x^2-2x^4}$.

Next, we consider the case $\sigma = 1232$. From the structure, each nonempty partition π that avoids 1231 and 1232 can be written as $\pi = 1\pi'$, where π' does not contain the letter 1, or as $\pi = 1\pi''$, where π'' is nonempty, or as $\pi = 122 \cdots 21\alpha\pi'''$, where α is a (possibly empty) word in $\{1, 2\}$ and π''' does not contain the letters 1 or 2. Thus the generating function $L_{1232}(x)$ satisfies

$$L_{1232}(x) = 1 + xL_{121}(x) + x(L_{1232}(x) - 1) + \frac{x^3}{(1-x)(1-2x)}L_{121}(x),$$

whence $L_{1232}(x) = 1 + \frac{x(1-3x+3x^2)}{(1-2x)^2(1-x)}$.

The remaining results in this section are of a more specific nature and cover most of the equivalences in the table below left to be shown concerning the avoidance of two patterns of size four. If m and n are positive integers, then $[m, n] = \{m, m+1, \ldots, n\}$ if $m \leq n$ and $[m, n] = \varnothing$ if $m > n$.

Proposition 4.18. *If* $n \geq 1$*, then*

$$p_n(1221, 1232) = p_n(1221, 1223) = 1 + (n-1)2^{n-2}.$$

Proof. Suppose $\pi \in P_{n,k}(1221, 1232)$, where $k \geq 2$. We consider whether or not π contains a repeated letter greater than one. Thus, we see that π must be of one of the following two forms:

(1) $\pi = 11^{a_1}21^{a_2} \cdots j1^{a_j}j^a(j+1)^{a_{j+1}} \cdots k^{a_k}$, where $2 \leq j \leq k$, $a \geq 1$, $a_i \geq 0$ if $i \in [j]$, and $a_i \geq 1$ if $i \in [j+1, k]$,

(2) $\pi = 11^{b_1}21^{b_2} \cdots k1^{b_k}$, where $b_i \geq 0$ if $i \in [k]$.

This implies

$$p_n(1221, 1232) = 1 + \sum_{k=2}^{n}\sum_{j=2}^{k}\binom{n-k-1+k}{k} + \sum_{k=2}^{n}\binom{n-k+k-1}{k-1}$$

$$= 1 + \sum_{k=2}^{n}(k-1)\binom{n-1}{k} + (2^{n-1} - 1) = 1 + (n-1)2^{n-2}.$$

For the second case, note that if $\pi \in P_{n,k}(1221, 1223)$, where $k \geq 2$, then it must be of one of the following two forms:

(1) $\pi = 11^{a_1}21^{a_2} \cdots k1^{a_k}j^a$, where $2 \leq j \leq k$, $a \geq 1$, and $a_i \geq 0$ if $i \in [k]$,

(2) $\pi = 11^{b_1}21^{b_2} \cdots k1^{b_k}$, where $b_i \geq 0$ if $i \in [k]$.

This implies

$$p_n(1221, 1223) = 1 + \sum_{k=2}^{n}\sum_{a=1}^{n-k}(k-1)\binom{n-a-1}{k-1} + \sum_{k=2}^{n}\binom{n-1}{k-1}$$

$$= 1 + \sum_{k=2}^{n}(k-1)\binom{n-1}{k} + (2^{n-1} - 1) = 1 + (n-1)2^{n-2},$$

which completes the proof. □

Proposition 4.19. *For $n \geq 1$,*

$$p_n(1212, 1123) = p_n(1212, 1233) = p_n(1212, 1223) = 1 + (n-1)2^{n-2}.$$

Proof. From Theorem 3.4, we know that the three pairs of patterns are equivalent. It is therefore enough to show that $p_n(1212, 1223) = 1 + (n-1)2^{n-2}$. Suppose that $\pi \in P_{n,k}(1212, 1223)$, where $k \geq 2$. Then we either have $\pi = 12 \cdots (k-1)k^a \alpha$, where $a \geq 1$ and α is a non-increasing word (i.e., it never strictly increases) on $[k-1]$, or

$$\pi = 11^{b_1} 21^{b_2} \cdots \ell 1^{b_\ell} (\ell+1)(\ell+2) \cdots (k-1)k^a \beta,$$

where $a \geq 1$, $1 \leq \ell \leq k-1$, $b_\ell \geq 1$, $b_i \geq 0$ if $1 \leq i < \ell$, and β is a non-increasing word on $[\ell+1, k-1] \cup \{1\}$. Summing over a and ℓ implies that the members of $P_{n,k}(1212, 1223)$ number

$$\sum_{a=1}^{n-k+1} \binom{n-a-1}{k-2} + \sum_{a=1}^{n-k} \sum_{\ell=1}^{k-1} \binom{n-a-1}{k-1} = \binom{n-1}{k-1} + (k-1)\binom{n-1}{k}.$$

Summing the last expression over $k \geq 2$ then gives $(n-1)2^{n-2}$ and completes the proof. □

Proposition 4.20. *We have*

$$\sum_{n \geq 0} p_n(1122, 1221)x^n = \frac{1 - 4x + 5x^2 - x^3}{(1-x)(1-2x)^2}.$$

Proof. If τ is a word, then let $f_\tau(x)$ denote the generating function for the number of partitions (of size at least $|\tau|$) avoiding $\{1122, 1221\}$ of the form $\tau\pi'$. We first compute $f_\tau(x)$ in the case when $\tau = 12 \cdots m1$, where $m \geq 2$. Suppose π is a member of $P_n(1122, 1221)$ whose first $m+1$ letters are $12 \cdots m1$. Then π must be of the form

$$\pi = 1\alpha_0 1\alpha_1 2\alpha_2 3\alpha_3 \cdots m\alpha_m,$$

where $\alpha_0 = 23 \cdots m$, α_1 is a (possibly empty) word in which all the symbols different from 1 form a strictly increasing sequence $m+1, m+2, \ldots$, and $i\alpha_i$, $2 \leq i \leq m$, is a sequence which is either empty or is nonempty and starts with i, with the letters of α_i comprising a (possibly empty) sequence of consecutive integers, the smallest of which is one more than the largest letter occurring to the left of α_i. For example, if $m = 4$ and $\pi = 1234115116712894$, then $\alpha_0 = 234$, $\alpha_1 = 1511671$, $2\alpha_2 = 289$, $3\alpha_3 = \varnothing$ and $4\alpha_4 = 4$. From the above decomposition, we see that

$$(4.1) \qquad f_{12\cdots m1}(x) = \frac{x^{m+1}}{(1-x)^{m-1}(1-2x)}, \qquad m \geq 1.$$

(Note a simpler argument applies to the $m = 1$ case of (4.1).)

If $m \geq 1$, then let $\widetilde{f}_{12\cdots m}(x)$ denote the generating function for the number of members $\pi = \pi_1 \pi_2 \cdots$ of $P_n(1122, 1221)$ of size at least m such that $\pi_1 \pi_2 \cdots \pi_m =$

$12 \cdots m$, with $\pi_{m+1} \le m$ (if it occurs). From the definitions and (4.1), we have

$$\widetilde{f}_{12\cdots m}(x) = x^m + \sum_{j=1}^{m} f_{12\cdots mj}(x) = x^m + \sum_{j=1}^{m} x^{j-1} f_{12\cdots(m-j+1)1}(x)$$

$$= x^m + \sum_{j=1}^{m} \frac{x^{m+1}}{(1-x)^{m-j}(1-2x)} = \frac{x^m(1 - x(1-x)^{m-1})}{(1-x)^{m-1}(1-2x)}.$$

Thus, we have

$$\sum_{n \ge 0} p_n(1122, 1221) x^n = 1 + \sum_{m \ge 1} \widetilde{f}_{12\cdots m}(x) = 1 + \sum_{m \ge 1} \frac{x^m(1 - x(1-x)^{m-1})}{(1-x)^{m-1}(1-2x)}$$

$$= \frac{1 - 4x + 5x^2 - x^3}{(1-x)(1-2x)^2},$$

which completes the proof. $\hspace{2cm} \square$

From Proposition 4.20, we see that $p_n(1122, 1221) = 1 + (n-1)2^{n-2}$ for all $n \ge 1$.

We now consider the case of avoiding $\{1122, 1223\}$. For this, let us introduce the following notation. Let $f_{a_1 a_2 \cdots a_m} = f_{a_1 a_2 \cdots a_m}(x)$ denote the generating function for the number of members $\pi = \pi_1 \pi_2 \cdots \pi_n \in P_n(1122, 1223)$ where $n \ge m$ such that $\pi_1 \pi_2 \cdots \pi_m = a_1 a_2 \cdots a_m$, and let $f^*_{a_1 a_2 \cdots a_m} = f^*_{a_1 a_2 \cdots a_m}(x)$ denote the generating function counting the same partitions with the further restriction that $\max_{1 \le i \le n}(\pi_i) = \max_{1 \le i \le m}(a_i)$. To establish the case $\{1122, 1223\}$, we will need the following lemma.

Lemma 4.21. *We have*

$$(4.2) \qquad F^*(x) := 1 + \sum_{k \ge 1} f^*_{12\cdots k}(x) = \frac{1 - 4x + 5x^2 - 2x^3 + x^4}{(1-x)(1-2x)^2}.$$

Proof. First note that if $k \ge 2$, then partitions π enumerated by $f^*_{12\cdots k1}$ are of the form

$$\pi = 12 \cdots k1 \alpha_1 \alpha_k \alpha_{k-1} \cdots \alpha_2,$$

where α_i, $1 \le i \le k$, is either a nonempty word of the form $i1^a$ for some $a \ge 0$ or is empty, whence $f^*_{12\cdots k1} = \frac{x^{k+1}}{(1-x)^k}$. If $2 \le j \le k$, then within partitions enumerated by $f^*_{12\cdots kj}$, the letters $j+1, j+2, \ldots, k$ can appear only once (so as to avoid 1223), whence

$$f^*_{12\cdots kj} = x^{k-j} f^*_{12\cdots jj} = x^{k-j+1} f^*_{12\cdots j}.$$

From the definitions, we then have

$$f^*_{12\cdots k} = x^k + f^*_{12\cdots k1} + \sum_{j=2}^{k} f^*_{12\cdots kj}$$

$$(4.3) \qquad = x^k + \frac{x^{k+1}}{(1-x)^k} + \sum_{j=2}^{k} x^{k-j+1} f^*_{12\cdots j}, \qquad k \ge 1.$$

Summing (4.3) over $k \geq 1$, and noting $f_1^* = \frac{x}{1-x}$, then yields

$$F^*(x) - 1 = \frac{x}{1-x} + \frac{x^2}{1-2x} + \sum_{j \geq 2} f_{12\cdots j}^*(x) \sum_{k \geq j} x^{k-j+1}$$

$$= \frac{x}{1-x} + \frac{x^2}{1-2x} + \frac{x}{1-x}\left(F^*(x) - \frac{x}{1-x} - 1\right),$$

which gives (4.2). $\square$

We now prove the case $\{1122, 1223\}$.

Proposition 4.22. *We have*

$$\sum_{n \geq 0} p_n(1122, 1223)x^n = \frac{1 - 4x + 5x^2 - x^3}{(1-x)(1-2x)^2}.$$

Proof. We first consider the generating function $\widetilde{f}_{12\cdots k} = \widetilde{f}_{12\cdots k}(x)$, $k \geq 1$, for the number of partitions $\pi = \pi_1\pi_2\cdots\pi_n \in P_n(1122, 1223)$ having length at least k such that $\pi_1\pi_2\cdots\pi_k = 12\cdots k$ and $\pi_{k+1} \leq k$ (if it exists). From the definitions, we have

$$\widetilde{f}_{12\cdots k} = x^k + f_{12\cdots k1} + \sum_{j=2}^{k} f_{12\cdots kj}$$

$$(4.4) \qquad = x^k + f_{12\cdots k1} + \sum_{j=2}^{k} x^{k-j+1} f_{12\cdots j}^*, \qquad k \geq 1,$$

for if $2 \leq j \leq k$, then $f_{12\cdots kj} = x^{k-j+1} f_{12\cdots j}^*$, since the letters $j+1, j+2, \ldots, k$ can only appear once in a partition enumerated by $f_{12\cdots kj}$, with no letters greater than k occurring. Furthermore, we have

$$(4.5) \qquad f_{12\cdots k1} = \frac{x^{k+1}}{(1-x)^{k-1}(1-2x)}, \qquad k \geq 1,$$

the enumerated partition π having the form

$$\pi = 1\alpha_0 1\alpha_1 \alpha_k \alpha_{k-1} \cdots \alpha_2,$$

where $\alpha_0 = 23\cdots k$, α_1 is a (possibly empty) word obtained by replacing the 2's occurring in a word in $\{1, 2\}$, successively, with the letters $k+1, k+2, \ldots$, and α_i, $2 \leq i \leq k$, is a sequence which is either empty or is nonempty and of the form $i1^a$ for some $a \geq 0$.

Summing (4.4) over $k \geq 1$, and using (4.5), then gives

$$\sum_{n \geq 0} p_n(1122, 1223)x^n = 1 + \sum_{k \geq 1} \widetilde{f}_{12\cdots k}(x)$$

$$= \frac{1}{1-x} + \sum_{k \geq 1} \frac{x^{k+1}}{(1-x)^{k-1}(1-2x)} + \sum_{j \geq 2} f_{12\cdots j}^*(x) \sum_{k \geq j} x^{k-j+1}$$

$$= \frac{1}{1-x} + \frac{x^2(1-x)}{(1-2x)^2} + \frac{x}{1-x}\left(F^*(x) - \frac{x}{1-x} - 1\right),$$

which yields the requested result, by (4.2). $\square$

Proposition 4.23. *For $n \geq 0$, $p_n(1123, 1233) = p_n(1231, 1233) = p_n(1123, 1232)$.*
In addition, we have

$$\sum_{n \geq 0} p_n(1122, 1232)x^n = \frac{1 - 4x + 6x^2 - 3x^3 + x^4}{(1-x)^3(1-2x)}.$$

Proof. To show the first statement, we define bijections between the classes which preserve the number of blocks k as follows. We may assume $k \geq 3$, since all of the patterns contain at least one 3. If $\pi \in P_{n,k}(1231, 1233)$, then π is necessarily of the form $\pi = 1^a 2\alpha 32^b 45 \cdots k$, where $a \geq 1$, $b \geq 0$, and α is a word in $\{1, 2\}$. Let $\pi' = 12 \cdots (k-1)2^{a-1}1^b k\alpha$ and $\pi'' = 12 \cdots (k-1)^{b+1}1^{a-1}k\alpha'$, where α' is obtained from α by replacing each occurrence of the letter 2 with k. Then it may be verified that the mappings $\pi \mapsto \pi'$ and $\pi \mapsto \pi''$ are bijections from $P_{n,k}(1231, 1233)$ to $P_{n,k}(1123, 1233)$ and $P_{n,k}(1123, 1232)$, respectively.

To prove the second statement, suppose $\pi \in P_n(1122, 1232)$ has at least two blocks. Then π necessarily has one of the following four forms, where k denotes the number of blocks in the last three cases:

(1) $\pi = 1\pi'$, where π' contains no 1's, avoids $\{1122, 121\}$ and is nonempty.
(2) $\pi = 1^{r_1} 2 1^{r_2} \cdots k 1^{r_k}$, where $r_1 \geq 2$ and $r_i \geq 0$ if $i \geq 2$.
(3) $\pi = 1\pi' 1^{s_0}(\ell+1)1^{s_1} \cdots k1^{s_{k-\ell}}$, where $2 \leq \ell \leq k$, $s_0 \geq 1$, $s_i \geq 0$ if $i \geq 1$, and π' is a nonempty partition on the letters $\{2, 3, \ldots, \ell\}$ avoiding $\{1122, 121\}$.
(4) $\pi = 12 \cdots (\ell-1)\ell^a 1^{t_0} \ell 1^{t_1} \cdots k1^{t_{k-\ell+1}}$, where $2 \leq \ell \leq k$, $a \geq 1$, $t_0 \geq 1$, $t_i \geq 0$ if $i \geq 1$, and an ℓ appears after the second run of the letter 1.

Thus, we get

$$\sum_{n \geq 0} p_n(1122, 1232)x^n = \frac{1}{1-x} + x(g(x) - 1) + \frac{x^3}{(1-x)(1-2x)}$$
$$+ \frac{x^2}{1-x}(g(x) - 1)\left(1 + \frac{x}{1-2x}\right) + \frac{x^4}{(1-x)^3(1-2x)},$$

where $g(x) = \sum_{n \geq 0} p_n(1122, 121)x^n$. By the table in Section 2.4 above, we have $g(x) = \sum_{i=0}^{3} \left(\frac{x}{1-x}\right)^i$, and substituting this into the last equation gives the second statement. □

Proposition 4.24. *If $n \geq 2$, then $p_n(1123, 1234) = p_n(1122, 1233) = 2^{n-5}(n^2 - n + 14)$.*

Proof. We enumerate both classes directly. If $n \geq 2$, then there are 2^{n-1} members of $P_n(1123, 1234)$ having no 3's as well as 2^{n-1} members of $P_n(1122, 1233)$ having at most one occurrence of the letter 2. So it remains to show that the members of $P_n(1123, 1234)$ in which 3 occurs as well as the members of $P_n(1122, 1233)$ in which 2 occurs at least twice both number $2^{n-5}(n-2)(n+1)$.

To show the first part, let j denote the *largest* index such that the $(j+2)$-nd letter of $\pi \in P_n(1123, 1234)$ is 3. Then the letters to the right of position $j+2$ may constitute any binary word, while those to the left (excepting the first two, which are necessarily $1, 2$) must form a non-decreasing binary word. Upon conditioning further on the number, $\ell+1$, of 3's, we see that the remaining members

of $P_n(1123, 1234)$ number

$$\sum_{j=1}^{n-2} 2^{n-2-j} \sum_{\ell=0}^{j-1} (j-\ell)\binom{j-1}{\ell} = \sum_{j=1}^{n-2} 2^{n-2-j}(j+1)2^{j-2}$$

$$= 2^{n-4}\sum_{j=1}^{n-2}(j+1) = 2^{n-5}(n-2)(n+1),$$

as required.

Members of $P_n(1122, 1233)$ in which 2 occurs at least twice must start $1, 2$, with any letters greater than 2 occurring once. The remaining positions are to be filled by 1's and 2's in such a way that the word comprising the letters in these positions avoids 122 and contains at least one 2. Let a_m denote the number of words over $\{1, 2\}$ of length m avoiding 122 such that 2 occurs at least once. It may be verified that $a_m = \binom{m+1}{2}$. Upon conditioning on the number, ℓ, of elements greater than two occurring, we see that the remaining members of $P_n(1122, 1233)$ number

$$\sum_{\ell=0}^{n-3}\binom{n-2}{\ell}a_{n-2-\ell} = \sum_{\ell=1}^{n-2}\binom{n-2}{\ell}a_\ell$$

$$= \frac{1}{2}\sum_{\ell=1}^{n-2}\binom{n-2}{\ell}(\ell^2+\ell) = 2^{n-5}(n-2)(n+1),$$

which completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4.5. **The cases** $\{1123, 1211\}$ **and** $\{1123, 1222\}$. In order to establish these cases, we first consider a statistic on the members of P_n having at least two blocks as follows. We will say that an *ascent* occurs at position i in $\pi = \pi_1\pi_2\cdots\pi_n \in P_n$ if $\pi_i < \pi_{i+1}$, where $1 \le i \le n-1$. For example, the partition $\pi = 1223413142 \in P_{10}$ has ascents at positions 1, 3, 4, 6, and 8. By the *final ascent*, we will mean the *largest index i* such that $\pi_i < \pi_{i+1}$.

Definition 4.25. *Suppose that the final ascent in $\pi = \pi_1\pi_2\cdots\pi_n \in P_n$ occurs at position m. Then let $fasc$ denote the statistic on P_n defined by $fasc(\pi) = n-m+1$.*

For example, if $\pi = 123241355311 \in P_{12}$, then the final ascent occurs at position 7 and $fasc(\pi) = 12 - 7 + 1 = 6$. Note that $2 \le fasc(\pi) \le n - k + 2$ for all $\pi \in P_{n,k}$, where $n \ge k \ge 2$, the minimum being achieved by any partition whose last two entries form an ascent and the maximum achieved by partitions of the form $12\cdots k\tau$, where τ is any non-increasing k-ary word of length $n - k$. If $n \ge k \ge 2$ and $2 \le t \le n - k + 2$, then let

$$a_{n,k,t} = |\{\pi \in P_{n,k}(1123, 1211): \ fasc(\pi) = t\}|,$$

and define $a_{n,k,t}$ to be zero otherwise. For example, we have $a_{4,3,2} = 2$ since there are two members of $P_{4,3}(1123, 1211)$ having $fasc$ value 2, namely 1223 and 1213, and $a_{4,3,3} = 3$, the partitions in this case being 1231, 1232, and 1233. Furthermore, let $A_{n,k,t}$ denote the subset of partitions enumerated by $a_{n,k,t}$. In the following lemma, we provide an explicit recurrence satisfied by the numbers $a_{n,k,t}$.

Lemma 4.26. *If $3 \le k \le n$ and $2 \le t \le n - k + 2$, then*

(4.6) $$a_{n,k,t} = a_{n-1,k-1,t} + \sum_{j=t-1}^{n-k+1} a_{n-2,k-1,j},$$

with $a_{2,2,2} = 1$ and for $n \geq 3$,

$$(4.7) \qquad a_{n,2,t} = \begin{cases} n - t + 1, & if \quad 3 \leq t \leq n - 1; \\ n - 2, & if \quad t = 2; \\ 2, & if \quad t = n. \end{cases}$$

Proof. First note that members of $A_{n,k,t}$, where $3 \leq k \leq n$ and $2 \leq t \leq n - k + 2$, have either one or two 1's, since such partitions must start with a single 1 and can have at most a single additional 1 coming after a letter greater than one. To show (4.6), first observe that there are $a_{n-1,k-1,t}$ members of $A_{n,k,t}$ containing a single 1, for writing a 1 in front of $\alpha \in A_{n-1,k-1,t}$ (on the letters $\{2, 3, \ldots, k\}$) does not affect the $fasc$ value.

If two 1's occur in $\pi \in A_{n,k,t}$, then π may be formed from $\beta = \beta_1\beta_2 \cdots \beta_{n-2} \in A_{n-2,k-1,j}$, where $t - 1 \leq j \leq n - k + 1$, by first writing a 1 in front of β and then considering cases on j. If $j \geq t$, then write a second 1 just before the $(t-1)$-st letter of β from the right to obtain $\pi = 1\beta_1\beta_2 \cdots \beta_{n-t+1}1\beta_{n-t} \cdots \beta_{n-2}$, and if $j = t - 1$, then write a second 1 at the end of β to obtain $\pi = 1\beta_1\beta_2 \cdots \beta_{n-2}1$. Note that $j \geq t$ in the first case implies no occurrence of 1123 is introduced by the insertion of the second 1 since the final $j - 1 \geq t - 1$ letters of β form a non-increasing word. Also, in the case when $j = t - 1$, the addition of 1 at the end increases the $fasc$ value by one. Thus, the sum in (4.6) conditions on the $fasc$ value of the partition resulting from the removal of the 1's from $\pi \in A_{n,k,t}$ containing two 1's.

We now turn to the case when $k = 2$. Note first that members of $A_{n,2,2}$, $n \geq 3$, are of the form $1\rho12$, where $\rho = 1^a2^b$ with $a, b \geq 0$ and $a + b = n - 3$, whence they number $n - 2$. There are two members of $A_{n,2,n}$, namely, $12 \cdots 2$ and $12 \cdots 21$. Finally, if $3 \leq t \leq n - 1$, then $A_{n,2,t}$ is the set of size $n - t + 1$ comprising the partitions $1^{n-t+1}2^{t-2}1$ and $1^{n-t+1}2^{t-1}$, together with the partitions of the form $1\rho12^{t-1}$, where $\rho = 1^a2^b$ for some $a \geq 0$ and $b \geq 1$ satisfying $a + b = n - t - 1$. This establishes (4.7) and completes the proof. $\qquad \square$

We can now enumerate the partitions avoiding $\{1123, 1211\}$.

Theorem 4.27. *We have*

$$\sum_{n \geq 0} p_n(1123, 1211)t^n = \frac{(1 - t^2)\sqrt{(1 - t)(1 - t - 4t^2)}}{2t^2(1 - 3t + t^2)} - \frac{1 - 3t - 2t^2 + 14t^3 - 15t^4 + 3t^5}{2t^2(1 - t)^2(1 - 3t + t^2)}.$$

Proof. First define the distribution polynomial $A_{n,k}(w) = \sum_{t=2}^{n-k+2} a_{n,k,t}w^t$ if $n \geq k \geq 2$, with $A_{n,k}(w) = 0$ otherwise. Multiplying (4.6) above by w^t, summing over $2 \leq t \leq n - k + 2$, and interchanging summation yields

$$A_{n,k}(w) = A_{n-1,k-1}(w) + \sum_{j=1}^{n-k+1} a_{n-2,k-1,j} \sum_{t=2}^{j+1} w^t$$

$$= A_{n-1,k-1}(w) + \frac{w^2}{1 - w} \sum_{j=2}^{n-k+1} a_{n-2,k-1,j}(1 - w^j)$$

$$(4.8) \qquad = A_{n-1,k-1}(w) + \frac{w^2}{1 - w} \left(A_{n-2,k-1}(1) - A_{n-2,k-1}(w) \right), \qquad n \geq k \geq 3,$$

with $A_{2,2}(w) = w^2$ and

$$(4.9) \qquad A_{n,2}(w) = (n-2)w^2 + \sum_{t=3}^{n-1}(n-t+1)w^t + 2w^n, \qquad n \geq 3.$$

Next define $A_n(v,w) = \sum_{k=2}^n A_{n,k}(w)v^k$ if $n \geq 2$, with $A_1(v,w) = A_0(v,w) = 0$. Multiplying (4.8) by v^k and summing over $3 \leq k \leq n$ yields the recurrence
(4.10)

$$A_n(v,w) - A_{n,2}(w)v^2 = vA_{n-1}(v,w) + \frac{w^2 v}{1-w}(A_{n-2}(v,1) - A_{n-2}(v,w)), \qquad n \geq 3.$$

Define the generating function $A(t,v,w) = \sum_{n \geq 2} A_n(v,w)t^n$. Multiplying (4.10) by t^n and summing over all $n \geq 3$ gives

$$A(t,v,w) - A_2(v,w)t^2 - v^2 \sum_{n \geq 3} A_{n,2}(w)t^n$$

$$= vtA(t,v,w) + \frac{w^2 vt^2}{1-w}(A(t,v,1) - A(t,v,w)),$$

which implies by (4.9) the relation
(4.11)

$$\left(1 - vt + \frac{w^2 vt^2}{1-w}\right)A(t,v,w) = \frac{w^2 v^2 t^2(1 - t(1-w)(1-t))}{(1-t)^2(1-tw)} + \frac{w^2 vt^2}{1-w}A(t,v,1).$$

This type of functional equation may be solved systematically using the *kernel method* (see [2]). Setting the coefficient of $A(t,v,w)$ equal to zero and solving for $w = w_0$ in terms of v and t yields

$$(4.12) \qquad w_0 = \frac{1 - vt - \sqrt{(1-vt)(1-vt-4vt^2)}}{2vt^2}.$$

Note that of the two possible values of $w_0 = w_0(t,v)$, only this one yields a power series in t and v. Setting $w = w_0$ in (4.11) then gives

$$(4.13) \qquad A(t,v,1) = \frac{v(w_0 - 1)(1 - t(1-w_0)(1-t))}{(1-t)^2(1-tw_0)}.$$

Note that $A(t,v,1)$ is the generating function for the cardinality of $P_{n,k}(1123, 1211)$ for $n \geq k \geq 2$. Letting $v = 1$, and including the $k = 0$ and $k = 1$ cases, we see that

$$\sum_{n \geq 0} p_n(1123, 1211)t^n = \frac{1}{1-t} + A(t,1,1)$$

$$= \frac{(1-t^2)\sqrt{(1-t)(1-t-4t^2)}}{2t^2(1-3t+t^2)} - \frac{1 - 3t - 2t^2 + 14t^3 - 15t^4 + 3t^5}{2t^2(1-t)^2(1-3t+t^2)},$$

by (4.12) and (4.13), as desired. $\qquad \square$

Remark: Note that $w_0 = C\left(\frac{vt^2}{1-vt}\right)$, where $C(x) = \frac{1-\sqrt{1-4x}}{2x} = \sum_{n \geq 0} c_n x^n$ is the generating function for the Catalan number $c_n = \frac{1}{n+1}\binom{2n}{n}$.

We now turn to the case of avoiding $\{1123, 1222\}$. For this, we first consider the avoidance of $\{1123, 111\}$. If $n \geq k \geq 2$ and $2 \leq t \leq n - k + 2$, then let

$$b_{n,k,t} = |\{\pi \in P_{n,k}(1123, 111) : \; fasc(\pi) = t\}|,$$

and define $b_{n,k,t}$ to be zero otherwise. For example, we have $b_{4,3,2} = 2$ (the enumerated partitions being 1213 and 1223) and $b_{5,3,3} = 4$ (the partitions being 12132, 12133, 12231 and 12233). Note that $b_{n,k,t} = 0$ for all t if $n > 2k$.

Next, let

$$c_{n,k,t} = |\{\pi = \pi_1\pi_2\cdots\pi_n \in P_{n,k}(1123, 1222): \; fasc(\pi) = t \text{ and } \pi_n \neq 1\}|,$$

if $n \geq k \geq 2$ and $2 \leq t \leq n - k + 2$ and put $c_{n,k,t} = 0$ otherwise. For example, we have $c_{4,3,2} = 2$ (for 1213 and 1223) and $c_{5,3,3} = 3$ (for 12132, 12133, and 12233). In the following lemma, we provide a recurrence for $b_{n,k,t}$ and a relation between $c_{n,k,t}$ and $b_{n,k,t}$.

Lemma 4.28. *If $3 \leq k \leq n$ and $2 \leq t \leq n - k + 2$, then*

$$(4.14) \qquad b_{n,k,t} = b_{n-1,k-1,t} + \sum_{j=t-1}^{n-k+1} b_{n-2,k-1,j}$$

and

$$(4.15) \qquad c_{n,k,t} = b_{n-1,k-1,t} + \sum_{j=t}^{n-k+1} c_{n-1,k,j}.$$

If $k = 2$, we have $b_{n,2,t} = 0$ if $n \geq 5$, $b_{4,2,4} = b_{4,2,3} = b_{4,2,2} = 1$, $b_{3,2,3} = 2$, and $b_{3,2,2} = b_{2,2,2} = 1$. Furthermore,

$$(4.16) \qquad c_{n,2,t} = \begin{cases} n-2, & \text{if} \quad t = 2; \\ 1, & \text{if} \quad t = 3; \\ 0, & \text{if} \quad t \geq 4, \end{cases}$$

if $n \geq 3$, with $c_{2,2,2} = 1$.

Proof. The proof of (4.14) is similar to the one given above for (4.6); note that a partition is restricted to containing one or two 1's since we are now avoiding 111, which also implies $b_{n,2,t} = 0$ if $n \geq 5$. One may verify directly the other conditions for $b_{n,2,t}$.

Let $C_{n,k,t}$ denote the subset of partitions counted by $c_{n,k,t}$. To show (4.15), first note that if $n \geq k \geq 3$, then members of $C_{n,k,t}$ containing a single 1 are of the form $1\pi'$, where π' is a partition on the letters $\{2, 3, \ldots, k\}$ avoiding $\{1123, 111\}$. On the other hand, one may add a 1 just before the $(t-1)$-st letter from the right of $\alpha = \alpha_1\alpha_2\cdots\alpha_{n-1} \in C_{n-1,k,j}$, $t \leq j \leq n - k + 1$, to obtain $\pi = \alpha_1\alpha_2\cdots\alpha_{n-t}1\alpha_{n-t+1}\cdots\alpha_{n-1} \in C_{n,k,t}$ having two or more 1's. Note that $j \geq t$ implies $\alpha_{n-t+1}\cdots\alpha_{n-1}$ is a non-decreasing word and thus contains no 1's since $\alpha_{n-1} > 1$. Therefore, the sum on the right side of (4.15) is seen to count the members $\pi \in C_{n,k,t}$ containing two or more 1's by conditioning on the $fasc$ value of the partition resulting when we remove the right-most 1 from π.

We now consider the case $k = 2$. First, note that there are $n - 2$ members of $C_{n,2,2}$, $n \geq 3$, since they are of the form $1\rho12$, where ρ is all 1's except for possibly a single occurrence of 2. There is a single member of $C_{n,2,3}$ given by $1^{n-2}22$. Finally, there are no members of $C_{n,2,t}$ if $t \geq 4$ since the last letter not 1 implies only 2's may follow the last ascent and these would then number $t - 1 \geq 3$, which is not permitted. This establishes (4.16) and completes the proof. $\qquad\square$

Define the distribution polynomial $B_{n,k}(w) = \sum_{t=2}^{n-k+2} b_{n,k,t}w^t$ if $n \geq k \geq 2$. When $k = 2$, note that $B_{2,2}(w) = w^2$, $B_{3,2}(w) = w^2 + 2w^3$, $B_{4,2}(w) = w^2 + w^3 + w^4$,

and $B_{n,2}(w) = 0$ if $n \geq 5$. If $n \geq 2$, then let $B_n(v,w) = \sum_{k=2}^{n} B_{n,k}(w)v^k$, with $B_0(v,w) = B_1(v,w) = 0$. Finally, define the generating function $B(t,v,w)$ by

$$B(t,v,w) = \sum_{n \geq 2} B_n(v,w)t^n.$$

Note that $B(t,v,w)$ counts all of the members of $P_n(1123, 111)$ according to the number of blocks and the *fasc* value when $n \geq 3$, counts only the partition $\{1\}, \{2\}$ when $n = 2$, and counts neither $\{1\}$ nor the empty partition. There is the following explicit formula for $B(t,v,w)$.

Lemma 4.29. *We have*

$$B(t,v,w) = \frac{w^2(vt^2 - 1)\sqrt{(1-vt)(1-vt-4vt^2)}}{2vt^2((1-w)(1-vt) + w^2vt^2)}$$
$$+ \frac{w^2(1 - vt - 3vt^2 + v^2t^3 + 2v^3(1-w)t^4 + 2w^3(1-2w)t^5 - 2v^3w^3t^6)}{2vt^2((1-w)(1-vt) + w^2vt^2)}.$$

Proof. Multiplying (4.14) by w^t and summing over $2 \leq t \leq n - k + 2$ implies
(4.17)

$$B_{n,k}(w) = B_{n-1,k-1}(w) + \frac{w^2}{1-w}(B_{n-2,k-1}(1) - B_{n-2,k-1}(w)), \qquad n \geq k \geq 3,$$

and multiplying (4.17) by v^k and summing over $3 \leq k \leq n$ yields
(4.18)

$$B_n(v,w) - B_{n,2}(w)v^2 = vB_{n-1}(v,w) + \frac{w^2v}{1-w}(B_{n-2}(v,1) - B_{n-2}(v,w)), \qquad n \geq 3,$$

which is seen to hold for $n = 2$ as well, since $B_0(v,w) = B_1(v,w) = 0$. Multiplying (4.18) by t^n and summing over $n \geq 2$ then gives

$$(4.19) \quad B(t,v,w) - w^2v^2t^2(1 + (1 + 2w)t + (1 + w + w^2)t^2)$$
$$= vtB(t,v,w) + \frac{w^2vt^2}{1-w}(B(t,v,1) - B(t,v,w)).$$

Letting $w = w_0$ in (4.19), where w_0 is given by (4.12) above, implies

$$(4.20) \qquad B(t,v,1) = \frac{w_0^2v^2t^2(1 + (1 + 2w_0)t + (1 + w_0 + w_0^2)t^2)}{1 - vt}.$$

Substituting (4.20) into (4.19), solving for $B(t,v,w)$, and simplifying then yields the required expression. $\qquad \square$

We can now enumerate the partitions avoiding $\{1123, 1222\}$.

Theorem 4.30. *We have*

$$\sum_{n \geq 0} p_n(1123, 1222)t^n = \frac{(1 - t^2)\sqrt{(1-t)(1-t-4t^2)}}{2t^2(1 - 3t + t^2)} - \frac{1 - 3t - 2t^2 + 14t^3 - 15t^4 + 3t^5}{2t^2(1-t)^2(1 - 3t + t^2)}.$$

Proof. Let $C_{n,k}(w) = \sum_{t=2}^{n-k+2} c_{n,k,t}w^t$. Multiplying (4.15) by w^t and summing over $2 \leq t \leq n - k + 2$ yields

$$(4.21) \quad C_{n,k}(w) = B_{n-1,k-1}(w) + \frac{w^2}{1-w}C_{n-1,k}(1)$$
$$- \frac{w}{1-w}C_{n-1,k}(w), \qquad n \geq k \geq 3,$$

with $C_{2,2}(w) = w^2$ and $C_{n,2}(w) = (n-2)w^2 + w^3$ if $n \geq 3$. Let $C_n(v,w) = \sum_{k=2}^{n} C_{n,k}(w)v^k$ if $n \geq 2$. Multiplying (4.21) by v^k and summing over $3 \leq k \leq n$ yields

$$(4.22) \quad C_n(v,w) - C_{n,2}(w)v^2 = vB_{n-1}(v,w) + \frac{w^2}{1-w}(C_{n-1}(v,1) - C_{n-1,2}(1)v^2)$$
$$- \frac{w}{1-w}(C_{n-1}(v,w) - C_{n-1,2}(w)v^2), \qquad n \geq 3.$$

Let $C(t,v,w) = \sum_{n \geq 2} C_n(v,w)t^n$ and $\widetilde{C}(t,v,w) = v^2 \sum_{n \geq 2} C_{n,2}(w)t^n$. Note that

$$\widetilde{C}(t,v,w) = w^2v^2t^2 + \frac{w^3v^2t^3}{1-t} + w^2v^2 \sum_{n \geq 3}(n-2)t^n$$

$$(4.23) \qquad = \frac{w^2v^2t^2(1 - t(1-w)(1-t))}{(1-t)^2}.$$

Multiplying (4.22) by t^n and summing over $n \geq 3$ gives

$$(4.24) \qquad C(t,v,w) - \widetilde{C}(t,v,w) = vtB(t,v,w) + \frac{w^2t}{1-w}(C(t,v,1) - \widetilde{C}(t,v,1))$$
$$- \frac{wt}{1-w}(C(t,v,w) - \widetilde{C}(t,v,w)).$$

Substituting $w = \frac{1}{1-t}$ into (4.24), and rearranging, implies

$$(4.25) \qquad C(t,v,1) = \widetilde{C}(t,v,1) + vt(1-t)B\left(t,v,\frac{1}{1-t}\right).$$

By Lemma 4.29,

$$(4.26) \quad B\left(t,v,\frac{1}{1-t}\right) = \frac{(1-vt^2)\sqrt{(1-vt)(1-vt-4vt^2)}}{2vt^3(1-t-2vt+vt^2)}$$
$$- \frac{1 - (3+v)t + 3t^2 - (1-6v-v^2)t^3 - v(8+3v)t^4 + (3+3v-4v^2)t^5 - v^2(1-2v)t^6}{2vt^3(1-t)^3(1-t-2vt+vt^2)}.$$

Note that $C(t,1,1)$ is the generating function for the number of elements of $P_n(1123,1222)$, $n \geq 2$, ending in a letter greater than one, which implies that $\frac{1}{1-t}C(t,1,1)$ counts all of the members of $P_n(1123,1222)$ having at least two blocks, for adding a string of 1's of arbitrary length to the end does not otherwise affect the enumeration. Thus, we have

$$\sum_{n \geq 0} p_n(1123,1222)t^n = \frac{1}{1-t} + \frac{1}{1-t}C(t,1,1)$$

$$= \frac{1}{1-t} + \frac{1}{1-t}\widetilde{C}(t,1,1) + tB\left(t,1,\frac{1}{1-t}\right)$$

$$= \frac{(1-t^2)\sqrt{(1-t)(1-t-4t^2)}}{2t^2(1-3t+t^2)} - \frac{1 - 3t - 2t^2 + 14t^3 - 15t^4 + 3t^5}{2t^2(1-t)^2(1-3t+t^2)},$$

as required, by (4.23), (4.25), and (4.26). $\qquad \square$

Comparison of Theorems 4.27 and 4.30 reveals that the pairs $\{1123, 1211\}$ and $\{1123, 1222\}$ are equivalent. We have searched for a direct bijection demonstrating this fact and leave it to the reader as an open problem.

4.6. **Table of equivalence classes of** $(4, 4)$**-pairs.** Combining the results of previous sections yields a complete solution to the problem of identifying all of the equivalence classes of $(4, 4)$-pairs. It should be observed that any pattern pair not represented in the table below belongs to a Wilf class of size one, such classes being determined by numerical evidence (note that there are $\binom{15}{2} - 84 = 21$ singleton classes in all).

- $(1121, 1232)$ $\overset{4.10}{\sim}$ $(1112, 1223)$ $\overset{4.10}{\sim}$ $(1121, 1223)$ $\overset{4.10}{\sim}$ $(1211, 1232)$ $\overset{4.16, 4.10}{\sim}$ $(1213, 1223)$ $\overset{4.16}{\sim}$ $(1213, 1234)$ $\overset{4.16}{\sim}$ $(1213, 1231)$ $\overset{4.9}{\sim}$ $(1231, 1234)$ $\overset{2.7}{\sim}$ $(1232, 1234)$ $\overset{2.7}{\sim}$ $(1223, 1234)$ $\overset{2.7}{\sim}$ $(1233, 1234)$ $\overset{4.12}{\sim}$ $(1222, 1233)$ $\overset{4.12}{\sim}$ $(1223, 1232)$ $\overset{4.12}{\sim}$ $(1223, 1233)$ $\overset{4.12}{\sim}$ $(1232, 1233)$ $\overset{4.17, 4.12}{\sim}$ $(1231, 1232)$ $\overset{4.19}{\sim}$ $(1123, 1212)$ $\overset{4.20}{\sim}$ $(1122, 1221)$ $\overset{4.19}{\sim}$ $(1212, 1223)$ $\overset{4.22}{\sim}$ $(1122, 1223)$ $\overset{4.18}{\sim}$ $(1221, 1223)$ $\overset{4.18}{\sim}$ $(1221, 1232)$ $\overset{4.19}{\sim}$ $(1212, 1233)$ $\overset{4.9}{\sim}$ $(1221, 1233)$

- [15] $(1212, 1232) \sim (1112, 1213) \sim (1123, 1223) \sim (1221, 1231) \sim (1123, 1213) \sim (1212, 1213) \sim (1212, 1221) \sim (1222, 1223) \sim (1122, 1212) \sim (1222, 1232) \sim (1211, 1231)$

- $(1112, 1233)$ $\overset{4.9}{\sim}$ $(1211, 1233)$ $\overset{4.9}{\sim}$ $(1121, 1233)$ $\overset{4.10}{\sim}$ $(1112, 1234)$ $\overset{4.9}{\sim}$ $(1121, 1234)$ $\overset{4.9}{\sim}$ $(1211, 1234)$ $\overset{4.13}{\sim}$ $(1222, 1234)$

- $(1213, 1221)$ $\overset{4.16}{\sim}$ $(1213, 1232)$ $\overset{4.16}{\sim}$ $(1213, 1233)$ $\overset{4.9}{\sim}$ $(1231, 1233)$ $\overset{4.23}{\sim}$ $(1123, 1232)$ $\overset{4.23}{\sim}$ $(1123, 1233)$ $\overset{4.23}{\sim}$ $(1122, 1232)$

- [12] $(1112, 1123) \sim (1211, 1212) \sim (1122, 1123) \sim (1121, 1221) \sim (1121, 1212)$

- [11] $(1211, 1221) \sim (1112, 1212) \sim (1212, 1222) \sim (1221, 1222)$

- $(1211, 1222)$ $\overset{4.9}{\sim}$ $(1121, 1222)$ $\overset{4.9}{\sim}$ $(1112, 1222)$

- $(1111, 1121)$ $\overset{4.9}{\sim}$ $(1111, 1211)$ $\overset{4.9}{\sim}$ $(1111, 1112)$

- $(1212, 1234)$ $\overset{4.9}{\sim}$ $(1221, 1234)$

- $(1112, 1232)$ $\overset{4.10}{\sim}$ $(1211, 1223)$

- $(1222, 1231)$ $\overset{4.9}{\sim}$ $(1213, 1222)$

- $(1111, 1223)$ $\overset{4.9}{\sim}$ $(1111, 1232)$

- $(1123, 1211)$ $\overset{4.27, 4.30}{\sim}$ $(1123, 1222)$

- $(1123, 1234)$ $\overset{4.24}{\sim}$ $(1122, 1233)$

- $(1121, 1231)$ $\overset{4.16, 4.17}{\sim}$ $(1121, 1213)$

- $(1111, 1213)$ $\overset{4.9}{\sim}$ $(1111, 1231)$

- $(1111, 1212)$ $\overset{4.9}{\sim}$ $(1111, 1221)$

- $(1121, 1211)$ $\overset{4.14}{\sim}$ $(1112, 1121)$.

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