# SEVERI-BOULIGAND TANGENTS, FRENET FRAMES AND RIESZ SPACES 

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#### Abstract

A compact set $X \subseteq \mathbb{R}^{2}$ has an outgoing Severi-Bouligand tangent unit vector $u$ at some point $x \in X$ iff some principal quotient of the Riesz space $\mathcal{R}(X)$ of piecewise linear functions on $X$ is not archimedean. To generalize this preliminary result, we extend the classical definition of Frenet $k$-frame to any sequence $\left\{x_{i}\right\}$ of points in $\mathbb{R}^{n}$ converging to a point $x$, in such a way that when the $\left\{x_{i}\right\}$ arise as sample points of a smooth curve $\gamma$, the Frenet $k$-frames of $\left\{x_{i}\right\}$ and of $\gamma$ at $x$ coincide. Our method of computation of Frenet frames via sample sequences of $\gamma$ does not require the knowledge of any higher-order derivative of $\gamma$. Given a compact set $X \subseteq \mathbb{R}^{n}$ and a point $x \in \mathbb{R}^{n}$, a Frenet $k$-frame $u$ is said to be a tangent of $X$ at $x$ if $X$ contains a sequence $\left\{x_{i}\right\}$ converging to $x$, whose Frenet $k$-frame is $u$. We prove that $X$ has an outgoing $k$-dimensional tangent of $X$ iff some principal quotient of $\mathcal{R}(X)$ is not archimedean. If, in addition, $X$ is convex, then $X$ has no outgoing tangents iff it is a polyhedron.


## 1. Introduction

In $[10, \S 53$, p. 59 and p.392] and [11, §1, p.99], Severi defined (outgoing) tangents of arbitrary subsets of the euclidean space $\mathbb{R}^{n}$. Subsequently and independently, Bouligand defined the same notion [2, p.32], which today is widely known as "Bouligand tangent". Throughout we will adopt the following equivalent definition, where $\|\cdot\|$ denotes euclidean norm and $\operatorname{conv}(Y)$ is the convex hull of $Y \subseteq \mathbb{R}^{n}$ :

Definition 1.1. [8, pp. 14 and 133] Let $\emptyset \neq X \subseteq \mathbb{R}^{n}$ and $x \in \mathbb{R}^{n}$. A unit vector $u \in \mathbb{R}^{n}$ is a Severi-Bouligand tangent of $X$ at $x$ if $X$ contains a sequence $\left\{x_{i}\right\}$ such that $x_{i} \neq x$ for all $i$, $\lim _{i \rightarrow \infty} x_{i}=x, \quad$ and $\lim _{i \rightarrow \infty}\left(x_{i}-x\right) /\left\|x_{i}-x\right\|=u$. If for some $\mu>0, \operatorname{conv}(x, x+\mu u) \cap X=\{x\}$, we say that $u$ is outgoing.

For an equivalent algebraic handling of tangents, in Section 4 we introduce the Riesz space (=vector lattice) $\mathcal{R}(X)$ of piecewise linear functions on any nonempty compact set $X \subseteq \mathbb{R}^{n}$. When $n=2$, the geometric properties of $X$ are immediately linked to the algebraic properties of $\mathcal{R}(X)$ by the following elementary result (Lemma 4.3): If $\mathcal{R}(X)$ has a non-archimedean principal quotient then $X$ has an outgoing Severi-Bouligand tangent.

In Theorem 5.1 we will extend this result, as well as its converse, to all $n$. To this purpose, in Section 2 we introduce the notion of a Frenet $k$-frame of a sequence $\left\{x_{i}\right\}$ of points in $\mathbb{R}^{n}$, as the natural generalization of the classical Frenet (Jordan) $k$-frame [5, 4] of a curve $\gamma$. Specifically, if the $x_{i}$ arise as sample points of a smooth curve $\gamma$ accumulating at some point $x$ of $\gamma$, then the Frenet $k$-frame of $\left\{x_{i}\right\}$ coincides with the Frenet $k$-frame of $\gamma$ at $x$. This is Theorem 2.2. The proof yields a method to calculate the Frenet $k$-frame of a $C^{k+1}$ curve $\gamma$ at a point $x$ without knowing the derivatives of any parametrization of $\gamma$ : one just takes a sampling sequence $\left\{x_{i}\right\}$ of points of $\gamma$ converging to $x$, and then makes the linear algebra calculations as in the proof of the theorem. To show the wide applicability of our method, Example 2.5 provides a curve $\gamma$ having no Frenet

[^0]$k$-frame at a point $x$, but such that the Frenet $k$-frame of each sequence of points of $\gamma$ converging to $x$ exists and is independent of the parametrization of $\gamma$.

In Section 3 we deal with the relationship between the Frenet $k$-frame $u=\left(u_{1}, \ldots, u_{k}\right)$ of a sequence $\left\{x_{i}\right\}$ in $\mathbb{R}^{n}$ converging to $x$, and any simplex $T \subseteq \mathbb{R}^{n}$ containing $\left\{x_{i}\right\}$. Theorem 3.3 shows that $T$ automatically contains the simplex $\operatorname{conv}\left(x, x+\lambda_{1} u_{1}, \ldots, x+\lambda_{1} u_{1}+\cdots+\lambda_{k} u_{k}\right)$, for some $\lambda_{1}, \ldots, \lambda_{k}>0$. This elementary result will find repeated use in the rest of the paper.

As a $k$-dimensional generalization of the classical Severi-Bouligand tangents, we then say that a Frenet $k$-frame $u$ is tangent at $x$ to a compact set $X \subseteq \mathbb{R}^{n}$ if $X$ contains a sequence $\left\{x_{i}\right\}$ converging to $x$, whose Frenet $k$-frame is $u$. Then Theorem 5.1 provides the desired strengthening of Lemma 4.3, showing that $X$ has no outgoing tangent iff every principal ideal of $\mathcal{R}(X)$ is an intersection of maximal ideals. This latter property is considered in the literature for various classes of structures: For commutative noetherian rings it is known as "von Neumann regularity"; frames having this property are known as "Yosida frames", [7, 2.1]; Chang MV-algebras with this property are said to be "strongly semisimple", [3]. As a corollary of Stone representation ([6, 4.4]), every boolean algebra is strongly semisimple.

Since $\{+,-, \wedge, \vee\}$-reducts of Riesz spaces with strong unit are lattice-ordered abelian groups with strong unit, and the latter are categorically equivalent to MV-algebras, [9, 3.9], following [3] we say that a Riesz space $R$ is strongly semisimple if every principal ideal of $R$ is an intersection of maximal ideals of $R$. Equivalently, every principal quotient of $R$ is archimedean. A large class of examples of strongly semisimple Riesz spaces with totally disconnected maximal spectrum is immediately provided by hyperarchimedean Riesz spaces, [1]. At the other extreme, when $X$ is a polyhedron, $\mathcal{R}(X)$ is strongly semisimple, (see Proposition 6.2).

Using Theorem 5.1, in Theorem 6.4 we prove that a nonempty compact convex subset $X \subseteq \mathbb{R}^{n}$ has no outgoing tangent iff $X$ has only finitely many extreme points iff $X$ is a polyhedron. This shows the naturalness of Definition 4.1 of "outgoing tangent" as a $k$-dimensional extension of the classical Severi-Bouligand tangent. Counterexamples of Theorem 6.4 are easily found in case $X$ is not convex (see Example 6.3).

The only prerequisite for this paper is a working knowledge of elementary polyhedral topology (as given, e.g., by the first chapters of [12]), and of the classical Yosida (Kakutani-Gelfand-Stone) correspondence between points of $X$ and maximal ideals of the Riesz space $\mathcal{R}(X)$. See [6] for a comprehensive account.

## 2. The Frenet frame of a sequence $\left\{x_{i}\right\} \subseteq \mathbb{R}^{n}$

Given two sequences $\left\{p_{i}\right\},\left\{q_{i}\right\} \subseteq \mathbb{R}$, by writing $\lim _{i \rightarrow \infty} p_{i} / q_{i}=r$ we understand that $q_{i} \neq 0$ for each $i$, and $\lim _{i \rightarrow \infty} p_{i} / q_{i}$ exists and equals $r$.

For any vector $y \in \mathbb{R}^{n}$ and linear subspace $L$ of $\mathbb{R}^{n}$, the orthogonal projection of $y$ onto $L$ is denoted

$$
\operatorname{proj}_{L}(y)
$$

For our generalization of Severi-Bouligand tangents we first extend Definition 1.1, replacing the unit vector $u \in \mathbb{R}^{n}$ therein by a $k$-tuple $\left\{u_{1}, \ldots, u_{k}\right\}$ of pairwise orthogonal unit vectors in $\mathbb{R}^{n}$.
Definition 2.1. Given a sequence $\sigma=\left\{x_{i}\right\}$ of points in $\mathbb{R}^{n}$ converging to $x$, and a $k$-tuple $\left(u_{1}, \ldots, u_{k}\right)$ of pairwise orthogonal unit vectors in $\mathbb{R}^{n}$, we say:

- $u_{1}$ is the Frenet 1-frame of $\sigma$ if $u_{1}=\lim _{i \rightarrow \infty}\left(x_{i}-x\right) /\left\|x_{i}-x\right\|$;
- $\left(u_{1}, \ldots, u_{k}\right)$ is the Frenet $k$-frame of $\sigma$ if $\left(u_{1}, \ldots, u_{k-1}\right)$ is the Frenet $(k-1)$-frame of $\sigma$, and

$$
u_{k}=\lim _{i \rightarrow \infty} \frac{x_{i}-x-\operatorname{proj}_{\mathbb{R} u_{1}+\cdots+\mathbb{R} u_{k-1}}\left(x_{i}-x\right)}{\left\|x_{i}-x-\operatorname{proj}_{\mathbb{R} u_{1}+\cdots+\mathbb{R} u_{k-1}}\left(x_{i}-x\right)\right\|}
$$

Following [5], for $[a, b] \subseteq \mathbb{R}$ an interval, suppose $\phi:[a, b] \rightarrow \mathbb{R}^{n}$ is a $C^{k}$ function such that for all $a \leq t<b$, the $k$-tuple of vectors $\left(\phi^{\prime}(t), \phi^{\prime \prime}(t), \ldots, \phi^{(k)}(t)\right)$ forms a linearly independent set in $\mathbb{R}^{n}$. Then the Gram-Schmidt process yields an orthonormal $k$-tuple $\left(v_{1}(t), \ldots, v_{k}(t)\right)$, called the Frenet $k$-frame of $\phi$ at $\phi(t)$.

The terminology of Definition 2.1 is justified by the following result:

Theorem 2.2. Suppose $\phi:[a, b] \rightarrow \mathbb{R}^{n}$ is a $C^{k+1}$ function. Let $a \leq t_{0}<b$ be such that the vectors $\phi^{\prime}\left(t_{0}\right), \phi^{\prime \prime}\left(t_{0}\right), \ldots, \phi^{(k)}\left(t_{0}\right)$ are linearly independent. Then for every sequence $t_{1}, t_{2}, \ldots$ in $\left[t_{0}, b\right] \backslash\left\{t_{0}\right\}$ converging to $t_{0}$, the Frenet $k$-frame of $\left\{\phi\left(t_{i}\right)\right\}$ exists and is equal to the Frenet $k$-frame of $\phi$ at $\phi\left(t_{0}\right)$.

Proof. We can write

$$
\begin{equation*}
\phi(t)=\phi\left(t_{0}\right)+\phi^{\prime}\left(t_{0}\right)\left(t-t_{0}\right)+\frac{\phi^{\prime \prime}\left(t_{0}\right)}{2}\left(t-t_{0}\right)^{2}+\cdots+\frac{\phi^{(k)}\left(t_{0}\right)}{k!}\left(t-t_{0}\right)^{k}+R(t) \tag{1}
\end{equation*}
$$

where the remainder $R:[a, b] \rightarrow \mathbb{R}^{n}$ satisfies

$$
\begin{equation*}
\|R(t)\| \leq M\left(t-t_{0}\right)^{k+1} \text { for some } 0 \leq M \in \mathbb{R} \tag{2}
\end{equation*}
$$

Let $\left(v_{1}, \ldots, v_{k}\right)$ be the Frenet $k$-frame of $\phi$ at $\phi\left(t_{0}\right)$. Then $v_{1}=\phi^{\prime}\left(t_{0}\right) /\left\|\phi^{\prime}\left(t_{0}\right)\right\|$, and for each $1<j \leq k$,

$$
v_{j}=\frac{\phi^{(j)}\left(t_{0}\right)-\operatorname{proj}_{\mathbb{R} v_{1}+\cdots+\mathbb{R} v_{j-1}}\left(\phi^{(j)}\left(t_{0}\right)\right)}{\left\|\phi^{(j)}\left(t_{0}\right)-\operatorname{proj}_{\mathbb{R} v_{1}+\cdots+\mathbb{R} v_{j-1}}\left(\phi^{(j)}\left(t_{0}\right)\right)\right\|}
$$

By induction on $1 \leq j \leq k$ we will prove that the Frenet $j$-frame $\left(u_{1}, \ldots, u_{j}\right)$ of the sequence $\left\{\phi\left(t_{i}\right)\right\}$ (exists and) coincides with the Frenet $j$-frame $\left(v_{1}, \ldots, v_{j}\right)$ of $\phi$ at $\phi\left(t_{0}\right)$.
Basis: Since $\left\|\phi^{\prime}\left(t_{0}\right)\right\| \neq 0$, for all suitably large $i$ we have $\phi\left(t_{i}\right) \neq \phi\left(t_{0}\right)$ and

$$
\begin{aligned}
u_{1} & =\lim _{i \rightarrow \infty} \frac{\phi\left(t_{i}\right)-\phi\left(t_{0}\right)}{\left\|\phi\left(t_{i}\right)-\phi\left(t_{0}\right)\right\|} \\
& =\lim _{i \rightarrow \infty} \frac{\left(\phi\left(t_{i}\right)-\phi\left(t_{0}\right)\right) /\left(t_{i}-t_{0}\right)}{\left\|\left(\phi\left(t_{i}\right)-\phi\left(t_{0}\right)\right) /\left(t_{i}-t_{0}\right)\right\|} \\
& =\frac{\lim _{i \rightarrow \infty}\left(\phi\left(t_{i}\right)-\phi\left(t_{0}\right)\right) /\left(t_{i}-t_{0}\right)}{\left\|\lim _{i \rightarrow \infty}\left(\phi\left(t_{i}\right)-\phi\left(t_{0}\right)\right) /\left(t_{i}-t_{0}\right)\right\|} \\
& =\frac{\phi^{\prime}\left(t_{0}\right)}{\left\|\phi^{\prime}\left(t_{0}\right)\right\|} \\
& =v_{1}
\end{aligned}
$$

Induction Step: By induction hypothesis, for each $1 \leq j<k$ the $j$-tuple $\left(v_{1}, \ldots, v_{j}\right)$ coincides with the Frenet $j$-frame $\left(u_{1}, \ldots, u_{j}\right)$ of the sequence $\left\{\bar{\phi}\left(t_{i}\right)\right\}$. Let the linear subspace $S_{j}$ of $\mathbb{R}^{n}$ be defined by

$$
S_{j}=\mathbb{R} u_{1}+\cdots+\mathbb{R} u_{j}=\mathbb{R} v_{1}+\cdots+\mathbb{R} v_{j}=\mathbb{R} \phi^{\prime}\left(t_{0}\right)+\cdots+\mathbb{R} \phi^{(j)}\left(t_{0}\right)
$$

From (2) we have

$$
\begin{equation*}
\frac{\left\|R(t)-\operatorname{proj}_{S_{j}}(R(t))\right\|}{\left(t-t_{0}\right)^{j+1}} \leq M\left(t-t_{0}\right)^{k-j} \tag{3}
\end{equation*}
$$

For each $l=j+1, \ldots, k$ let us define the vector $\alpha_{l} \in \mathbb{R}^{n}$ by

$$
\begin{equation*}
\alpha_{l}=\frac{\phi^{(l)}\left(t_{0}\right)-\operatorname{proj}_{S_{j}}\left(\phi^{(l)}\left(t_{0}\right)\right)}{l!} \tag{4}
\end{equation*}
$$

whence in particular,

$$
\left\|\alpha_{j+1}\right\|=\frac{\left\|\phi^{(j+1)}\left(t_{0}\right)-\operatorname{proj}_{S_{j}}\left(\phi^{(j+1)}\left(t_{0}\right)\right)\right\|}{(j+1)!} \neq 0
$$

By (1),

$$
\begin{align*}
\phi\left(t_{i}\right)-\phi\left(t_{0}\right)-\operatorname{proj}_{S_{j}}\left(\phi\left(t_{i}\right)-\right. & \left.\phi\left(t_{0}\right)\right)= \\
& \alpha_{j+1}\left(t_{i}-t_{0}\right)^{j+1}+\cdots+\alpha_{k}\left(t_{i}-t_{0}\right)^{k}+R\left(t_{i}\right)-\operatorname{proj}_{S_{j}}\left(R\left(t_{i}\right)\right) . \tag{5}
\end{align*}
$$

From (3)-(5) we get

$$
\begin{aligned}
& u_{j+1}=\lim _{i \rightarrow \infty} \frac{\phi\left(t_{i}\right)-\phi\left(t_{0}\right)-\operatorname{proj}_{S_{j}}\left(\phi\left(t_{i}\right)-\phi\left(t_{0}\right)\right)}{\left\|\phi\left(t_{i}\right)-\phi\left(t_{0}\right)-\operatorname{proj}_{S_{j}}\left(\phi\left(t_{i}\right)-\phi\left(t_{0}\right)\right)\right\|} \\
= & \lim _{i \rightarrow \infty} \frac{\alpha_{j+1}\left(t_{i}-t_{0}\right)^{j+1}+\cdots+\alpha_{k}\left(t_{i}-t_{0}\right)^{k}+R\left(t_{i}\right)-\operatorname{proj}_{S_{j}}\left(R\left(t_{i}\right)\right)}{\left\|\alpha_{j+1}\left(t_{i}-t_{0}\right)^{j+1}+\cdots+\alpha_{k}\left(t_{i}-t_{0}\right)^{k}+R\left(t_{i}\right)-\operatorname{proj}_{S_{j}}\left(R\left(t_{i}\right)\right)\right\|} \\
= & \lim _{i \rightarrow \infty} \frac{\sum_{l=j+1}^{k} \alpha_{l}\left(t_{i}-t_{0}\right)^{l-(j+1)}+\left(R\left(t_{i}\right)-\operatorname{proj}_{S_{j}}\left(R\left(t_{i}\right)\right)\right) \cdot\left(t_{i}-t_{0}\right)^{-(j+1)}}{\left\|\sum_{l=j+1}^{k} \alpha_{l}\left(t_{i}-t_{0}\right)^{l-(j+1)}+\left(R\left(t_{i}\right)-\operatorname{proj}_{S_{j}}\left(R\left(t_{i}\right)\right)\right) \cdot\left(t_{i}-t_{0}\right)^{-(j+1)}\right\|} \\
= & \frac{\alpha_{j+1}}{\left\|\alpha_{j+1}\right\|}=\frac{\phi^{(j+1)}\left(t_{0}\right)-\operatorname{proj}_{S_{j}}\left(\phi^{(j+1)}\left(t_{0}\right)\right)}{\left\|\phi^{(j+1)}\left(t_{0}\right)-\operatorname{proj}_{S_{j}}\left(\phi^{(j+1)}\left(t_{0}\right)\right)\right\|}=v_{j+1} .
\end{aligned}
$$

This concludes the proof.
Remark 2.3. The assumption $\phi \in C^{k+1}$ can be relaxed to $\phi \in C^{k}$, so long as the $k$ th Taylor remainder $R(t)$ satisfies (2).

Remark 2.4. Theorem 2.2 yields a method to calculate the Frenet $k$-frame of a $C^{k+1}$ curve, not involving higher-order derivatives, but taking instead a sampling sequence $\left\{x_{i}\right\}$ of points on the curve, and then making the elementary linear algebra calculations in the proof above.

The wide applicability of this method is shown by the following example:
Example 2.5. Let $\phi:[0,1] \rightarrow \mathbb{R}^{2}$ be defined by $\phi(x)=\left(x, x^{3}\right)$. Then $\phi^{\prime}(0)=(1,0)$ and $\phi^{\prime \prime}(0)=$ $(0,0)$. The Frenet 1 -frame of $\phi$ at $(0,0)$ is the vector $(1,0)$, but $\phi$ has no Frenet 2 -frame at $(0,0)$. And yet, letting $\mathbb{R}(1,0)$ denote the linear subspace of $\mathbb{R}^{2}$ given by the $x$-axis, every sequence $\left\{t_{i}\right\} \in[0,1] \backslash\{0\}$ converging to 0 satisfies

$$
\lim _{i \rightarrow \infty} \frac{\phi\left(t_{i}\right)-\phi(0)-\operatorname{proj}_{\mathbb{R}(1,0)}\left(\phi\left(t_{i}\right)-\phi(0)\right)}{\left\|\phi\left(t_{i}\right)-\phi(0)-\operatorname{proj}_{\mathbb{R}(1,0)}\left(\phi\left(t_{i}\right)-\phi(0)\right)\right\|}=\lim _{i \rightarrow \infty} \frac{\left(0, t_{i}^{3}\right)}{\left\|\left(0, t_{i}^{3}\right)\right\|}=(0,1)
$$

We have shown: There exist a curve $\gamma$ having no Frenet $k$-frame at a point $x$, but the Frenet $k$-frame of every sequence of points of $\gamma$ converging to $x$ exists and is independent of the parametrization of $\gamma$.

Example 2.6. While under the hypotheses of Theorem 2.2 the Frenet $k$-frames of any two sampling sequences of a curve $\gamma$ at a point $x \in \gamma$ are equal, the map $\psi(x)=\left(x, x^{2} \sin (1 / x)\right):[0,1] \rightarrow \mathbb{R}^{2}$ (with the proviso that $\psi(0)=(0,0)$ ), yields an example of a curve $\gamma$ that is not $C^{2}$ and has two sequences $\left\{x_{i}\right\}$ and $\left\{y_{i}\right\}$ of points of $\gamma$ both converging to the same point $(0,0)$ of $\gamma$, but having different Frenet 2-frames.

## 3. Simplexes and Frenet frames

Fix $n=1,2, \ldots$ For any subset $E$ of the euclidean space $\mathbb{R}^{n}$, the convex hull $\operatorname{conv}(E)$ is the set of all convex combinations of elements of $E$. We say that $E$ is convex if $E=\operatorname{conv}(E)$. For any subset $Y$ of $\mathbb{R}^{n}$, the affine hull $\operatorname{aff}(Y)$ of $Y$ is the set of all affine combinations in $\mathbb{R}^{n}$ of elements of $Y$. A set $\left\{y_{1}, \ldots, y_{m}\right\}$ of points in $\mathbb{R}^{n}$ is said to be affinely independent if none of its elements is an affine combination of the remaining elements. The relative interior relint $(C)$ of a convex set $C \subseteq \mathbb{R}^{n}$ is the interior of $C$ in the affine hull of $C$. For $0 \leq d \leq n$, a $d$-simplex $T$ in $\mathbb{R}^{n}$ is the convex hull $\operatorname{conv}\left(v_{0}, \ldots, v_{d}\right)$ of $d+1$ affinely independent points in $\mathbb{R}^{n}$. The vertices $v_{0}, \ldots, v_{d}$ are uniquely determined by $T$. A face of $T$ is the convex hull of a subset $V$ of vertices of $T$. If the cardinality of $V$ is $d$, then $V$ is said to be a facet of $T$.

The positive cone of $Y \subseteq \mathbb{R}^{n}$ at a point $x \in Y$ is the set

$$
\begin{equation*}
\operatorname{Cone}(Y, x)=\left\{y \in \mathbb{R}^{n} \mid x+\rho(y-x) \in Y \text { for some } \rho>0\right\} \tag{6}
\end{equation*}
$$

When $T$ is a simplex, $\operatorname{Cone}(T, x)$ is closed. If $F$ is a face of $T$ and $x \in \operatorname{relint}(F)$ then for each $y \in F$ we have

$$
\begin{equation*}
\operatorname{Cone}(T, x)=\operatorname{aff}(F)+\operatorname{Cone}(T, y) \tag{7}
\end{equation*}
$$

In particular, if $x \in \operatorname{relint}(T)$ then $\operatorname{Cone}(T, x)=\operatorname{aff}(T)$.
Lemma 3.1. Suppose $T \subseteq \mathbb{R}^{n}$ is a simplex and $F$ is a face of $T$.
(a) If $S$ is an arbitrary simplex contained in $T$, and $F \cap \operatorname{relint}(S) \neq \emptyset$, then $S$ is contained in $F$.
(b) A point $z$ lies in relint $(F)$ iff $F$ is the smallest face of $T$ containing $z$.

Proof. (a) Let $F_{1}, \ldots, F_{u}$ be the facets of $T$, with their respective affine hulls $H_{1}, \ldots, H_{u}$. Each $H_{j}$ is the boundary of the closed half-space $H_{j}^{+} \subseteq T$ and of the other closed half-space $H_{j}^{-}$. Without loss of generality, $F_{1}, \ldots, F_{t}$ are the facets of $T$ containing $F$. Then aff $(F)=H_{1} \cap \cdots \cap H_{t}$ and $F=\left(H_{t+1}^{+} \cap \cdots \cap H_{u}^{+}\right) \cap \operatorname{aff}(F)$. By way of contradiction, suppose $x \in F \cap \operatorname{relint}(S)$ and $y \in S \backslash F$. For some $\epsilon>0$ the segment $\operatorname{conv}(x+\epsilon(y-x), x-\epsilon(y-x))$ is contained in $S$. For some hyperplane $H \in\left\{H_{1}, \ldots, H_{t}\right\}$ the point $y$ lies in the open half-space $\operatorname{int}\left(H^{+}\right)=\mathbb{R}^{n} \backslash H^{-}$, where "int" denotes topological interior. Now $x+\epsilon(y-x) \in \operatorname{int}\left(H^{+}\right)$and $x-\epsilon(y-x) \in \operatorname{int}\left(H^{-}\right)$, whence $x-\epsilon(y-x) \notin T$, which contradicts $S \subseteq T$.
(b) This easily follows from (a).

Proposition 3.2. Let $x \in \mathbb{R}^{n}$ and $u_{1}, \ldots, u_{m}$ be linearly independent vectors in $\mathbb{R}^{n}$. Let $\lambda_{1}, \mu_{1}, \ldots, \lambda_{m}, \mu_{m}>0$. Then the intersection of the two $m$-simplexes $\operatorname{conv}\left(x, x+\lambda_{1} u_{1}, \ldots, x+\right.$ $\left.\lambda_{1} u_{1}+\cdots+\lambda_{m} u_{m}\right)$ and $\operatorname{conv}\left(x, x+\mu_{1} u_{1}, \ldots, x+\mu_{1} u_{1}+\cdots+\mu_{m} u_{m}\right)$ is an m-simplex of the form $\operatorname{conv}\left(x, x+\nu_{1} u_{1}, \ldots, x+\nu_{1} u_{1}+\cdots+\nu_{m} u_{m}\right)$ for uniquely determined real numbers $\nu_{1}, \ldots, \nu_{m}>0$.

Proof. We argue by induction on $t=1, \ldots, m$. The cases $t=1,2$ are trivial. Proceeding inductively, for any simplex $W=\operatorname{conv}\left(x, x+\theta_{1} u_{1}, \ldots, x+\theta_{1} u_{1}+\cdots+\theta_{t} u_{t}\right)$, let $W^{\prime}=\operatorname{conv}(x, x+$ $\left.\theta_{1} u_{1}, \ldots, x+\theta_{1} u_{1}+\cdots+\theta_{t-1} u_{t-1}\right)$ and $W^{\prime \prime}=\operatorname{conv}\left(x, x+\theta_{1} u_{1}, \ldots, x+\theta_{1} u_{1}+\cdots+\theta_{t-2} u_{t-2}\right)$. By (7), for each $y \in W^{\prime} \backslash W^{\prime \prime}$ the half-line from $y$ in direction $u_{t}$ intersects $W$ in a segment $\operatorname{conv}\left(y, y+\gamma u_{t}\right)$ for some $\gamma>0$ depending on $y$. Now let

$$
\begin{aligned}
U_{t} & =\operatorname{conv}\left(x, x+\lambda_{1} u_{1}, \ldots, x+\lambda_{1} u_{1}+\cdots+\lambda_{t} u_{t}\right) \\
V_{t} & =\operatorname{conv}\left(x, x+\mu_{1} u_{1}, \ldots, x+\mu_{1} u_{1}+\cdots+\mu_{t} u_{t}\right)
\end{aligned}
$$

We then have

$$
\begin{aligned}
& U_{t-1}=U_{t}^{\prime}=\operatorname{conv}\left(x, x+\lambda_{1} u_{1}, \ldots, x+\lambda_{1} u_{1}+\cdots+\lambda_{t-1} u_{t-1}\right) \\
& V_{t-1}=V_{t}^{\prime}=\operatorname{conv}\left(x, x+\mu_{1} u_{1}, \ldots, x+\mu_{1} u_{1}+\cdots+\mu_{t-1} u_{t-1}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
& U_{t-2}=U_{t}^{\prime \prime}=\operatorname{conv}\left(x, x+\lambda_{1} u_{1}, \ldots, x+\lambda_{1} u_{1}+\cdots+\lambda_{t-2} u_{t-2}\right) \\
& V_{t-2}=V_{t}^{\prime \prime}=\operatorname{conv}\left(x, x+\mu_{1} u_{1}, \ldots, x+\mu_{1} u_{1}+\cdots+\mu_{t-2} u_{t-2}\right)
\end{aligned}
$$

By induction hypothesis, for uniquely determined $\nu_{1}, \ldots, \nu_{t-1}>0$ we can write

$$
U_{t}^{\prime} \cap V_{t}^{\prime}=\operatorname{conv}\left(x, x+\nu_{1} u_{1}, \ldots, x+\nu_{1} u_{1}+\cdots+\nu_{t-1} u_{t-1}\right)
$$

The point $z=x+\nu_{1} u_{1}+\cdots+\nu_{t-1} u_{t-1}$ lies in $U_{t}^{\prime} \backslash U_{t}^{\prime \prime}$. Let $\eta_{1}$ be the largest $\eta$ such that $z+\eta u_{t}$ lies in $U_{t}$. Since $z \in V_{t}^{\prime} \backslash V_{t}^{\prime \prime}$, let similarly $\eta_{2}$ be the largest $\eta$ such that $z+\eta u_{t}$ lies in $V_{t}$. As already noted at the beginning of this proof, the real number $\nu_{t}=\min \left(\eta_{1}, \eta_{2}\right)$ is $>0$. Evidently, $\nu_{t}$ is the largest $\eta$ such that $z+\eta u_{t}$ lies in $U_{t} \cap V_{t}$. We conclude that $U_{t} \cap V_{t}=$ $\operatorname{conv}\left(x, x+\nu_{1} u_{1}, \ldots, x+\nu_{1} u_{1}+\cdots+\nu_{t} u_{t}\right)$.

The following key result will find repeated use in the rest of this paper:
Theorem 3.3. Let $\left(u_{1}, \ldots, u_{k}\right)$ be the Frenet $k$-frame of a sequence $\left\{x_{i}\right\}$ in $\mathbb{R}^{n}$ converging to $x$. Suppose a simplex $T \subseteq \mathbb{R}^{n}$ contains $\left\{x_{i}\right\}$. Then $T$ contains the simplex $\operatorname{conv}\left(x, x+\lambda_{1} u_{1}, \ldots, x+\right.$ $\lambda_{1} u_{1}+\cdots+\lambda_{k} u_{k}$ ), for some $\lambda_{1}, \ldots, \lambda_{k}>0$.

Proof. We will prove the following stronger statement:
Claim. For each $l \in\{1, \ldots, k\}$ there exist $\lambda_{1}, \ldots, \lambda_{l}>0$ such that:
(i) $\operatorname{conv}\left(x, x+\lambda_{1} u_{1}, \ldots, x+\lambda_{1} u_{1}+\cdots+\lambda_{l} u_{l}\right) \subseteq T$ and
(ii) letting $F_{l}$ be the smallest face of $T$ containing the point $z_{l}=x+\lambda_{1} u_{1}+\cdots+\lambda_{l} u_{l}$ (which by Lemma 3.1(b) is equivalent to $\left.z_{l} \in \operatorname{relint}\left(F_{l}\right)\right)$, we have the inclusion $\operatorname{conv}(x, x+$ $\left.\lambda_{1} u_{1}, \ldots, x+\lambda_{1} u_{1}+\cdots+\lambda_{l} u_{l}\right) \subseteq F_{l}$.
The proof is by induction on $l=1, \ldots, k$.
Basis Step $(l=1)$ : Since each $x_{i}$ is in $T$ then $x+\left(x_{i}-x\right) /\left\|x_{i}-x\right\| \in \operatorname{Cone}(T, x)$. Since Cone $(T, x)$ is closed, then $x+u_{1} \in \operatorname{Cone}(T, x)$. From (6) we obtain an $\epsilon>0$ such that $x+\epsilon u_{1} \in T$. Let $\lambda_{1}=\epsilon / 2$. Then $\operatorname{conv}\left(x, x+\lambda u_{1}\right) \subseteq \operatorname{conv}\left(x, x+\epsilon u_{1}\right) \subseteq T$, and (i) follows. Let $F_{1}$ be the smallest face of $T$ containing the point $z_{1}=x+\lambda_{1} u_{1}$. Evidently, $z_{1} \in \operatorname{relint}\left(\operatorname{conv}\left(x, x+\epsilon u_{1}\right)\right)$. By Lemma 3.1(b), $z_{i} \in \operatorname{relint}\left(F_{1}\right)$. By Lemma 3.1(a), $F_{1} \supseteq \operatorname{conv}\left(x, x+\epsilon u_{1}\right) \supseteq \operatorname{conv}\left(x, x+\lambda u_{1}\right)$. This proves (ii) and concludes the proof of the basis step.
Induction Step: For $1 \leq l<k$, induction yields $\lambda_{1}, \ldots, \lambda_{l}>0$ such that, letting $C_{l}=\operatorname{conv}(x, x+$ $\left.\lambda_{1} u_{1}, \ldots, x+\lambda_{1} u_{1}+\cdots+\lambda_{l} u_{l}\right)$ and $z_{l}=x+\lambda_{1} u_{1}+\cdots+\lambda_{l} u_{l}$, we have $C_{l} \subseteq T$. Further, letting $F_{l}$ be the smallest face of $T$ containing $z_{l}$, we have $C_{l} \subseteq F_{l}$, whence $\operatorname{aff}\left(C_{l}\right)=x+\mathbb{R} u_{1}+\cdots+\mathbb{R} u_{l} \subseteq \operatorname{aff}\left(F_{l}\right)$. Since $z_{l} \in \operatorname{relint}\left(F_{l}\right)$ and $x_{i}-x \in \operatorname{Cone}(T, x)$, from (7) we obtain

$$
z_{l}+\frac{x_{i}-x-\operatorname{proj}_{\mathbb{R} u_{1}+\cdots+\mathbb{R} u_{l}}\left(x_{i}-x\right)}{\left\|x_{i}-x-\operatorname{pro}_{\mathbb{R} u_{1}+\cdots+\mathbb{R} u_{l}}\left(x_{i}-x\right)\right\|} \in \operatorname{Cone}\left(T, z_{l}\right)
$$

Cone $\left(T, z_{l}\right)$ is closed, because $z_{l}+u_{l+1} \in \operatorname{Cone}\left(T, z_{l}\right)$. By (6), there exists $\epsilon>0$ such that $z_{l}+\epsilon u_{l+1} \in T$, whence $\operatorname{conv}\left(z_{l}, z_{l}+\epsilon u_{l+1}\right) \subseteq T$. Setting now $\lambda_{l+1}=\epsilon / 2$ and $z_{l+1}=z_{l}+\lambda_{l+1} u_{l+1}$, condition (i) in the claim above follows from the identity

$$
\operatorname{conv}\left(x, x+\lambda_{1} u_{1}, \ldots, x+\lambda_{1} u_{1}+\cdots+\lambda_{l+1} u_{l+1}\right)=\operatorname{conv}\left(C_{l} \cup\left\{z_{l+1}\right\}\right) \subseteq T
$$

Let $F_{l+1}$ be the smallest face of $T$ containing the point $z_{l+1} \in \operatorname{relint}\left(\operatorname{conv}\left(z_{l}, z_{l}+\epsilon u_{l+1}\right)\right)$. By Lemma 3.1(b), $z_{l+1} \in \operatorname{relint}\left(F_{l+1}\right)$. By Lemma 3.1(a),

$$
F_{l+1} \supseteq \operatorname{conv}\left(z_{l}, z_{l}+\epsilon u_{l+1}\right) \supseteq \operatorname{conv}\left(z_{l}, z_{l}+\lambda_{l+1} u_{l+1}\right)
$$

The minimality property of $F_{l}$ yields $F_{l} \subseteq F_{l+1}$. By induction hypothesis, $C_{l} \subseteq F_{l+1}$. In conclusion, $\operatorname{conv}\left(x, x+\lambda_{1} u_{1}, \ldots, x+\lambda_{1} u_{1}+\cdots+\lambda_{l+1} u_{l+1}\right)=\operatorname{conv}\left(C_{l} \cup\left\{z_{l+1}\right\}\right) \subseteq F_{l+1}$, as required to prove (ii) and to complete the proof.

## 4. TANGENTS of $X$, principal ideals of $\mathcal{R}(X)$ : THE CASE $X \subseteq \mathbb{R}^{2}$

For $k=1$ the following definition boils down to Definition 1.1 of Severi-Bouligand tangent vector. As in Definition 1.1, $X$ is an arbitrary nonempty subset of $\mathbb{R}^{n}$.
Definition 4.1. Let $X \subseteq \mathbb{R}^{n}, x \in \mathbb{R}^{n}$ and $u=\left(u_{1}, \ldots, u_{k}\right)$ be a $k$-tuple of pairwise orthogonal unit vectors in $\mathbb{R}^{n}$. Then $u$ is said to be a tangent of $X$ at $x$ if $X$ contains a sequence $\left\{x_{i}\right\}$ converging to $x$, whose Frenet $k$-frame is $u$. We say that $\left\{x_{i}\right\}$ determines $u$. We say that $u$ is outgoing if, in addition, there are $\lambda_{1}, \ldots, \lambda_{k}>0$ such that the simplex $C=\operatorname{conv}\left(x, x+\lambda_{1} u_{1}, \ldots, x+\lambda_{1} u_{1}+\right.$ $\left.\cdots+\lambda_{k} u_{k}\right)$ and its facet $C^{\prime}=\operatorname{conv}\left(x, x+\lambda_{1} u_{1}, \ldots, x+\lambda_{1} u_{1}+\cdots+\lambda_{k-1} u_{k-1}\right)$ have the same intersection with $X$.

The following elementary material on piecewise linear topology [12] is necessary to introduce the Riesz space $\mathcal{R}(X)$ of piecewise linear functions on $X$. In Theorem 5.1 below, the Frenet tangent frames of $X$ will be related to the maximal and principal ideals of $\mathcal{R}(X)$.

A polyhedron $P$ in $\mathbb{R}^{n}$ is a finite union of simplexes in $\mathbb{R}^{n} . P$ need not be convex or connected. Given a polyhedron $P$, a triangulation of $P$ is an (always finite) simplicial complex $\Delta$ such that $P=\bigcup \Delta$. Every polyhedron has a triangulation, [12, 2.1.5]. Given a rational polyhedron $P$ and triangulations $\Delta$ and $\Sigma$ of $P$, we say that $\Delta$ is a subdivision of $\Sigma$ if every simplex of $\Delta$ is contained in a simplex of $\Sigma$. Suppose an $n$-cube $K \subseteq \mathbb{R}^{n}$ is contained in another $n$-cube $K^{\prime} \subseteq \mathbb{R}^{n}$. Then every triangulation $\Delta$ of $K$ has an extension $\Delta^{\prime}$ to a triangulation of $K^{\prime}$, in the sense that $\Delta=\left\{T \in \Delta^{\prime} \mid T \subseteq K\right\}$. A continuous function $f: K \rightarrow \mathbb{R}$ is $\Delta$-linear if it is linear (in the affine
sense) on each simplex of $\Delta$. Via the extension $\Delta^{\prime}, f$ can be extended to a $\Delta^{\prime}$-linear function on $K^{\prime}$. A function $g: K \rightarrow \mathbb{R}$ is piecewise linear if it is $\Delta$-linear for some triangulation $\Delta$ of $K$. We denote by $\mathcal{R}(K)$ the Riesz space of all piecewise linear functions on $K$, with the pointwise operations of the Riesz space $\mathbb{R}$.

More generally, let $X$ be a nonempty compact subset of $\mathbb{R}^{n}$. Let $K \subseteq \mathbb{R}^{n}$ be an (always closed) $n$-cube containing $X$. We momentarily denote by $\mathcal{R}(K) \upharpoonright X$ the Riesz space of restrictions to $X$ of the functions in $\mathcal{R}(K)$. If $L \subseteq \mathbb{R}^{n}$ is an $n$-cube containing $K$, then $\mathcal{R}(K) \upharpoonright X=\mathcal{R}(L) \upharpoonright X$. (For the nontrivial direction, the above mentioned extension property of triangulations yields $\mathcal{R}(L) \upharpoonright K=\mathcal{R}(K)$.) Thus, if both $n$-cubes $K$ and $L$ contain $X$, letting $M \subseteq \mathbb{R}^{n}$ be an $n$-cube containing both $K$ and $L$, we obtain $\mathcal{R}(K) \upharpoonright X=\mathcal{R}(L) \upharpoonright X=\mathcal{R}(M) \upharpoonright X$, independently of the ambient cube $K \supseteq X$. Without fear of ambiguity we may then use the notation $\mathcal{R}(X)$ for the Riesz space of functions thus obtained. Each $f \in \mathcal{R}(X)$ is said to be a piecewise linear function on $X$. It follows that $f$ is continuous.

Lemma 4.2. There is a one-one correspondence $x \mapsto \mathfrak{m}_{x}, \mathfrak{m} \mapsto x_{\mathfrak{m}}$ between maximal ideals $\mathfrak{m}$ of $\mathcal{R}(X)$ and points $x$ of $X$. Specifically, $\mathfrak{m}_{x}$ is the set of all functions in $\mathcal{R}(X)$ vanishing at $x$; conversely, $x_{\mathfrak{m}}$ is the only element in the intersection of the zerosets $Z h=h^{-1}(0)$ of all functions $h \in \mathfrak{m}$.

Proof. The functions in $\mathcal{R}(X)$ separate points, and the constant function 1 is a strong unit in $\mathcal{R}(X)$. Now apply [6, 27.7].

The following elementary result deals with the special case $X \subseteq \mathbb{R}^{2}$. It is an adaptation to Riesz spaces of the MV-algebraic result [3, Theorem 3.1(ii)], and will have a key role in the proof of the much stronger Theorem 5.1.
Lemma 4.3. Let $X \subseteq \mathbb{R}^{2}$ be a nonempty compact set. If the Riesz space $\mathcal{R}(X)$ has a principal ideal that is not an intersection of maximal ideals, then $X$ has an outgoing Severi-Bouligand tangent at some point $x \in X$.

Proof. For every element $e$ of $\mathcal{R}(X)$ let $\langle e\rangle$ denote the principal ideal generated by $e$. Let $g \in \mathcal{R}(X)$ be such that the ideal $\mathfrak{p}=\langle g\rangle$ is not an intersection of maximal ideals of $\mathcal{R}(X)$. Lemma 4.2 yields an element $f \in \mathcal{R}(X)$ such that $f \notin \mathfrak{p}$ and $Z g \subseteq Z f$. Replacing, if necessary, $f$ and $g$ by their absolute values $|f|$ and $|g|$, we may assume $f \geq 0$ and $g \geq 0$. Let $K \subseteq \mathbb{R}^{2}$ be a fixed but otherwise arbitrary closed square containing $X$. By definition of $\mathcal{R}(X)$, there are elements $0 \leq \tilde{f} \in \mathcal{R}(K)$ and $0 \leq \tilde{g} \in \mathcal{R}(K)$ such that $\tilde{f} \upharpoonright X=f$ and $\tilde{g} \upharpoonright X=g$. Since $\tilde{f} \upharpoonright X$ does not belong to $\mathfrak{p}$ then for each $m>0$ there is a point $x_{m} \in X$ such that

$$
\begin{equation*}
\tilde{f}\left(x_{m}\right)>m \cdot \tilde{g}\left(x_{m}\right) \tag{8}
\end{equation*}
$$

Since $X$ is compact, for some $x \in X$ there is a subsequence $\left\{x_{m_{1}}, x_{m_{2}}, \ldots\right\}$ of $\left\{x_{1}, x_{2}, \ldots\right\}$ such that

$$
\begin{equation*}
x_{i} \neq x_{j} \text { for all } i \neq j, \quad \text { and } \quad \lim _{i \rightarrow \infty} x_{m_{i}}=x \tag{9}
\end{equation*}
$$

For each $i=1,2, \ldots$, let the unit vector $u_{i}$ be defined by

$$
u_{i}=\left(x_{m_{i}}-x\right) /\left\|x_{m_{i}}-x\right\| .
$$

Since the unit circumference $S^{1}=\left\{z \in \mathbb{R}^{2} \mid\|z\|=1\right\}$ is compact, it is no loss of generality to assume $\lim _{i \rightarrow \infty} u_{i}=u$, for some $u \in S^{1}$. Therefore, $u$ is a tangent of $X$ at $x$. There remains to be shown that $u$ is outgoing. To this purpose we make the following
Claim. There is a real number $\lambda>0$ such that:
(a) $\tilde{f}$ is (affine) linear on the line segment $\operatorname{conv}(x, x+\lambda u)$;
(b) $\underset{\sim}{\tilde{f}}$ identically vanishes on $\operatorname{conv}(x, x+\lambda u)$;
(c) $\tilde{f}(x+\lambda u) \neq 0$.

As a matter of fact, since each of $x_{m_{1}}, x_{m_{2}}, \ldots$ lies in $K$, by (9) there exists $\delta>0$ such that $\operatorname{conv}(x, x+\delta u) \subseteq K$. An elementary result in polyhedral topology ( $[12,2.2 .4]$ ) yields a triangulation $\Delta$ of $K$ such that both functions $\tilde{f}$ and $\tilde{g}$ are $\Delta$-linear and $\operatorname{conv}(x, x+\delta u)=\bigcup\{T \in$
$\Delta \mid T \subseteq \operatorname{conv}(x, x+\delta u)\}$. Therefore, there exists $\lambda>0$ such that $\operatorname{conv}(x, x+\lambda u) \in \Delta$. We have proved that $\tilde{f}$ is linear in $\operatorname{conv}(x, x+\lambda u)$, and (a) is settled.

To settle (b), since both functions $\tilde{g}$ and $\tilde{f}$ are continuous, we can write

$$
0 \geq \tilde{g}(x)=\lim _{i \rightarrow \infty} \tilde{g}\left(x_{i}\right) \leq \lim _{i \rightarrow \infty} \frac{\tilde{f}\left(x_{i}\right)}{m_{i}}=0
$$

whence $\tilde{g}(x)=g(x)=0$. From $X \cap Z \tilde{g} \subseteq X \cap Z \tilde{f}$ we get $\tilde{f}(x)=f(x)=0$. Since $\Delta$ is finite set, there exists a 2 -simplex $S \in \Delta$ containing infinitely many elements $x_{n_{1}}, x_{n_{2}}, \ldots$ of the set $\left\{x_{m_{1}}, x_{m_{2}}, \ldots\right\}$. By (9), $x \in S$. Further, from $\lim _{i \rightarrow \infty} u_{n_{i}}=u$ and $\operatorname{conv}(x, x+\lambda u) \in \Delta$ it follows that $\operatorname{conv}(x, x+\lambda u) \subseteq S$. Therefore,

$$
\begin{equation*}
S=\operatorname{conv}(x, x+\lambda u, v) \text { for some } v \in S \tag{10}
\end{equation*}
$$

For some $2 \times 1$-matrix $A$ and vector $b \in \mathbb{R}^{2}$ we can write $\tilde{g}(z)=A z+b$ for each $z \in S$. Since $\lim _{i \rightarrow \infty} u_{m_{i}}=u$ and $\tilde{g}(x)=0$, we have the identities

$$
\begin{aligned}
\tilde{g}(x+\lambda u) & =\lambda A u+\tilde{g}(x)=\lim _{i \rightarrow \infty} \frac{\lambda\left(A x_{n_{i}}-A x\right)}{\left\|x_{n_{i}}-x\right\|}=\lim _{i \rightarrow \infty} \frac{\lambda\left(\tilde{g}\left(x_{n_{i}}\right)-\tilde{g}(x)\right)}{\left\|x_{n_{i}}-x\right\|} \\
& =\lim _{i \rightarrow \infty} \frac{\lambda \tilde{g}\left(x_{n_{i}}\right)}{\left\|x_{n_{i}}-x\right\|}=\lim _{i \rightarrow \infty} \frac{\lambda g\left(x_{n_{i}}\right)}{\left\|x_{n_{i}}-x\right\|}
\end{aligned}
$$

Similarly,

$$
\tilde{f}(x+\lambda u)=\lim _{i \rightarrow \infty} \frac{\lambda f\left(x_{n_{i}}\right)}{\left\|x_{n_{i}}-x\right\|},
$$

whence

$$
0 \leq \tilde{g}(x+\lambda u)=\lim _{i \rightarrow \infty} \frac{\lambda g\left(x_{n_{i}}\right)}{\left\|x_{n_{i}}-x\right\|} \leq \lim _{i \rightarrow \infty} \frac{\lambda}{n_{i}} \frac{f\left(x_{n_{i}}\right)}{\left\|x_{n_{i}}-x\right\|}=\tilde{f}(x+\lambda u) \lim _{i \rightarrow \infty} \frac{1}{n_{i}}=0
$$

Since $\tilde{g}$ is linear on $\operatorname{conv}(x, x+\lambda u)$ and $\tilde{g}(x+\lambda u)=0=\tilde{g}(x)$, then (b) follows.
To prove (c), by (8) we get $\tilde{f}\left(x_{n_{i}}\right) \neq 0$ for all $i$, whence $\tilde{g}\left(x_{n_{i}}\right) \neq 0$, because $Z g \subseteq Z f$. Then our assumptions about $S$, together with (10), show that $\tilde{g}(v) \neq 0$. Let the integer $m^{*}$ satisfy the inequality $m^{*} \cdot \tilde{g}(v) \geq \tilde{f}(v)$. If (absurdum hypothesis) $\tilde{f}(x+\lambda u)=0$ then $m^{*} \cdot \tilde{g}(z) \geq \tilde{f}(z)$ for each $z \in S$. In view of (8), this contradicts the existence of infinitely many elements $x_{n_{i}}$ in $S$. Having thus proved (c), our claim is settled.

In conclusion, from (a) and (c) it follows that $\operatorname{conv}(x, x+\lambda u) \cap Z \tilde{f}=\{x\}$. Then from (b) we get

$$
X \cap \operatorname{conv}(x, x+\lambda u)=X \cap Z \tilde{g} \cap \operatorname{conv}(x, x+\lambda u) \subseteq X \cap Z \tilde{f} \cap \operatorname{conv}(x, x+\lambda u)=\{x\}
$$

thus proving that $u$ is an outgoing tangent of $X$ at $x$.

## 5. TANGENTS And Strong SEmisimplicity

Recall that a Riesz space $R$ is said to be strongly semisimple if for every principal ideal $\langle g\rangle$ of $R$ the quotient $R /\langle g\rangle$ is archimedean (i.e., the intersection of the maximal ideals of $R /\langle g\rangle$ is $\{0\}$ ). Equivalently, $\langle g\rangle$ is an intersection of maximal ideals of $R$. (This follows from the canonical one-to-one correspondence between ideals of $R$ containing $\langle g\rangle$, and ideals of $R /\langle g\rangle$.) Since $\{0\}$ is a principal ideal of $R$, if $R$ is strongly semisimple then it is archimedean.

The following result is the promised strengthening of Lemma 4.3:
Theorem 5.1. For any nonempty compact set $X \subseteq \mathbb{R}^{n}$ the following conditions are equivalent:
(i) $X$ has an outgoing tangent at some point $x \in X$.
(ii) The Riesz space $\mathcal{R}(X)$ is not strongly semisimple, i.e., there exists a principal ideal of $\mathcal{R}(X)$ that is not an intersection of maximal ideals.

Proof. Without loss of generality, $X \subseteq[0,1]^{n}$. (This trivially follows because any $n$-cube in $\mathbb{R}^{n}$ is PL-homeomorphic to any other $n$-cube).
(i) $\Rightarrow$ (ii) By Definition 4.1, for some $x \in \mathbb{R}^{n}$ and $k$-tuple $u=\left(u_{1}, \ldots, u_{k}\right)$ of pairwise orthogonal unit vectors in $\mathbb{R}^{n}$, there is a sequence $\left\{x_{i}\right\}$ of points in $\mathbb{R}^{n}$ converging to $x$, such that $u$ is the Frenet $k$-frame of $\left\{x_{i}\right\}$. Further, there are reals $\lambda_{1}, \ldots, \lambda_{k}>0$ such that the simplex $C=\operatorname{conv}(x, x+$ $\left.\lambda_{1} u_{1}, \ldots, x+\lambda_{1} u_{1}+\cdots+\lambda_{k} u_{k}\right)$ and its facet $C^{\prime}=\operatorname{conv}\left(x, x+\lambda_{1} u_{1}, \ldots, x+\lambda_{1} u_{1}+\cdots+\lambda_{k-1} u_{k-1}\right)$ satisfy $C \cap X=C^{\prime} \cap X$.

Let $f_{1}$ and $f_{2}$ be piecewise linear functions defined on $[0,1]^{n}$, taking their values in $\mathbb{R}_{\geq 0}=\{x \in$ $\mathbb{R} \mid x \geq 0\}$ and satisfying the conditions

$$
\begin{equation*}
Z f_{1}=f_{1}^{-1}(0)=C, \quad Z f_{2}=C^{\prime}, \quad \text { and } \quad f_{2} \text { is (affine) linear over } C \tag{11}
\end{equation*}
$$

The existence of $f_{1}$ and $f_{2}$ follows from [12, 2.2.4]. Both restrictions $f_{2} \upharpoonright X$ and $f_{1} \upharpoonright X$ are elements of $\mathcal{R}(X)$. By construction,

$$
\begin{equation*}
Z f_{1} \cap X=Z f_{2} \cap X \tag{12}
\end{equation*}
$$

We claim that the principal ideal $\mathfrak{p}=\left\langle f_{1} \upharpoonright X\right\rangle$ of $\mathcal{R}(X)$ generated by $f_{1} \upharpoonright X$ does not coincide with the intersection of all maximal ideals of $\mathcal{R}(X)$ containing $\mathfrak{p}$.

By (12) together with Lemma 4.2, $f_{2} \upharpoonright X$ belongs to all maximal ideals of $\mathcal{R}(X)$ containing $\mathfrak{p}$. So our claim will be settled once we prove

$$
\begin{equation*}
f_{2} \upharpoonright X \notin \mathfrak{p} \tag{13}
\end{equation*}
$$

To this purpose, arguing by way of contradiction, suppose $f_{2} \upharpoonright X \leq m f_{1} \upharpoonright X$ for some $m=1,2, \ldots$. Since $f_{1}$ and $f_{2}$ are (continuous) piecewise linear, the set $L=\left\{x \in[0,1]^{n} \mid f_{2}(x) \leq m f_{1}(x)\right\}$ is a union of simplexes $T_{1} \cup \cdots \cup T_{r}$. Necessarily for some $j=1, \ldots, r$ the simplex $T_{j}$ contains infinitely many points of the sequence $\left\{x_{i}\right\}$. This subsequence $\left\{x_{t}\right\}$ still converges to $x \in T_{j}$, and $u$ is its Frenet $k$-frame. Theorem 3.3 yields $\mu_{1}, \ldots, \mu_{k}>0$ such that $T_{j}$ contains the simplex $M=\operatorname{conv}\left(x, x+\mu_{1} u_{1}, \ldots, x+\mu_{1} u_{1}+\cdots+\mu_{k} u_{k}\right)$. Now Proposition 3.2 yields uniquely determined $\nu_{1}, \ldots, \nu_{k}>0$ such that

$$
C \cap M=\operatorname{conv}\left(x, x+\nu_{1} u_{1}, \ldots, x+\nu_{1} u_{1}+\cdots+\nu_{k} u_{k}\right)
$$

By (11), $f_{1}$ identically vanishes on $C \cap M$. Further, from $L \supseteq T_{j} \supseteq M \supseteq C \cap M$ and $f_{2} \leq m f_{1}$ on $L$, it follows that $f_{2}=0$ on $C \cap M$. The two simplexes $C \cap M$ and $C$ have the same dimension $k$, and $f_{2}$ is (affine) linear on $C \supseteq C \cap M$. Therefore, $f_{2}=0$ on $C$, which contradicts $Z f_{2}=C^{\prime}$. We have thus proved (13), settled our claim, and completed the proof of $(\mathrm{i}) \Rightarrow(\mathrm{ii})$.
(ii) $\Rightarrow$ (i) By hypothesis, there is a function $f_{1} \in \mathcal{R}\left([0,1]^{n}\right)$ such that the principal ideal $\left\langle f_{1} \upharpoonright X\right\rangle$ of $\mathcal{R}(X)$ generated by the restriction $f_{1} \upharpoonright X$ is not an intersection of maximal ideals of $\mathcal{R}(X)$. Thus there is $f_{2} \in \mathcal{R}\left([0,1]^{n}\right)$ whose restriction $f_{2} \upharpoonright X$ does not belong to the principal ideal $\left\langle f_{1} \upharpoonright X\right\rangle$ generated by $f_{1} \upharpoonright X$, but belongs to all maximal ideals of $\mathcal{R}(X)$ containing $\left\langle f_{1} \upharpoonright X\right\rangle$. By Lemma 4.2, $Z f_{2} \upharpoonright X=Z f_{1} \upharpoonright X$, i.e., $X \cap Z f_{2}=X \cap Z f_{1}$.

Let the map $g: X \rightarrow \mathbb{R}^{2}$ be defined by

$$
\begin{equation*}
g(x)=\left(f_{1}(x), f_{2}(x)\right) \text { for all } x \in X \tag{14}
\end{equation*}
$$

Let $\iota: \mathcal{R}(g(X)) \rightarrow \mathcal{R}(X)$ be defined by $\iota(h)=h \circ g$ for all $h \in \mathcal{R}(g(X))$, where $\circ$ denotes composition. It is easy to see that $\iota$ is a Riesz space homomorphism of $\mathcal{R}(g(X))$ into $\mathcal{R}(X)$. Letting $\pi_{1}, \pi_{2}: \mathbb{R}^{2} \rightarrow \mathbb{R}$ be the canonical projections (=coordinate functions), we have the identities $f_{1} \upharpoonright X=\iota\left(\pi_{1} \upharpoonright g(X)\right)$ and $f_{2} \upharpoonright X=\iota\left(\pi_{2} \upharpoonright g(X)\right)$. Whenever $h \in \mathcal{R}(g(X)), \iota(h)=0$ and $z \in g(X)$, there exists $x \in X$ such that $g(x)=z$. Then $h(z)=h(g(x))=(\iota(h))(x)=0$ and $\iota$ is one-to-one. Actually, $\iota$ is an isomorphism between $\mathcal{R}(g(X))$ and the Riesz subspace of $\mathcal{R}(X)$ generated by $\left\{f_{1} \upharpoonright X, f_{2} \upharpoonright X\right\}$. It follows that the principal ideal $\mathfrak{p}$ of $\mathcal{R}(g(X))$ generated by $\pi_{1} \upharpoonright g(X)$ is not an intersection of maximal ideals of $\mathcal{R}(g(X))$ : specifically, $\pi_{2} \upharpoonright g(X)$ belongs to all maximal ideals containing $\mathfrak{p}$, but does not belong to $\mathfrak{p}$. By Lemma 4.3,

$$
\begin{equation*}
g(X) \text { has a Severi-Bouligand outgoing tangent. } \tag{15}
\end{equation*}
$$

There remains to be proved that $X$ has an outgoing tangent. To help the reader, the long proof is subdivided into two parts.

Part 1: Construction of a tangent $u$ of $X$.
By (15) and Definition 4.1 with $k=1$ (which is the same as Definition 1.1), for some point $y^{*} \in \mathbb{R}^{2}$, unit vector $v^{*} \in \mathbb{R}^{2}$, sequence $\left\{y_{i}\right\} \subseteq \mathbb{R}^{2}$ converging to $y^{*}$, and $\mu>0$, we can write

$$
\begin{equation*}
\lim _{i \rightarrow \infty}\left(y_{i}-y^{*}\right) /\left\|y_{i}-y^{*}\right\|=v^{*} \quad \text { and } \quad \operatorname{conv}\left(y^{*}, y^{*}+\mu v^{*}\right) \cap g(X)=\left\{y^{*}\right\} \tag{16}
\end{equation*}
$$

By (14), $g$ is the restriction to $X$ of the function $f=\left(f_{1}, f_{2}\right):[0,1]^{n} \rightarrow \mathbb{R}^{2}$. Since (each component of) $f$ is piecewise linear, then $f$ is continuous, and both sets $f^{-1}\left(y^{*}\right)$ and $f^{-1}\left(\operatorname{conv}\left(y^{*}, y^{*}+\right.\right.$ $\left.\mu v^{*}\right)$ ) are polyhedra in $[0,1]^{n}$. An elementary result in polyhedral topology ( $[12,2.2 .4]$ ) yields a triangulation $\Delta$ of $[0,1]^{n}$ having the following properties:

- $f$ is (affine) linear over each simplex of $\Delta$,
- $f^{-1}\left(y^{*}\right)=\bigcup\left\{R \in \Delta \mid R \subseteq f^{-1}\left(y^{*}\right)\right\}$, and
- $f^{-1}\left(\operatorname{conv}\left(y^{*}, y^{*}+\mu v^{*}\right)\right)=\bigcup\left\{U \in \Delta \mid U \subseteq f^{-1}\left(\operatorname{conv}\left(y^{*}, y^{*}+\mu v^{*}\right)\right)\right\}$.

For some $n$-simplex $T \in \Delta$, the set $\left\{i \mid f^{-1}\left(y_{i}\right) \cap T \cap X\right\}=\left\{i \mid g^{-1}\left(y_{i}\right) \cap T\right\}$ is infinite. Let $z_{0}, z_{1}, \ldots$ be a converging sequence of elements of $T$ such that $f\left(z_{0}\right), f\left(z_{1}\right), \ldots$ is a subsequence of $y_{0}, y_{1}, \ldots$. Without loss of generality this subsequence coincides with the sequence $\left\{y_{i}\right\}$, and we can write

$$
\begin{equation*}
g\left(z_{i}\right)=y_{i} . \tag{17}
\end{equation*}
$$

Letting $z^{*}=\lim _{i \rightarrow \infty} z_{i}$ we have

$$
\begin{equation*}
z^{*} \in X \cap T \text { and } y^{*}=f\left(z^{*}\right)=g\left(z^{*}\right) \tag{18}
\end{equation*}
$$

The linearity of $f$ on $T$ yields a $2 \times n$ matrix $A$, together with a vector $b \in \mathbb{R}^{2}$ such that for each $t \in$ $T, f(t)=A t+b$.
Claim. For some $k \in\{1, \ldots, n\}$ there is a $k$-tuple of pairwise orthogonal unit vectors $u_{i} \in$ $\mathbb{R}^{n}, \quad(1 \leq i \leq k)$ such that:

- $A u_{j}=0$ for each $1 \leq j<k$,
- $A u_{k} \neq 0$,
- $u=\left(u_{1}, \ldots, u_{k}\right)$ is a tangent of $X$ at $z^{*}$, determined by a suitable subsequence of $z_{0}, z_{1}, \ldots$, in the sense of Definition 4.1.

The vectors $u_{1}, \ldots, u_{k}$ are constructed by the following inductive procedure:
Basis Step: From $A z_{i}+b=y_{i} \neq y^{*}=A z^{*}+b$ it follows that $z_{i} \neq z^{*}$ for each $i$, and hence every vector $z_{i}^{1}=\left(z_{i}-z^{*}\right) /\left\|z_{i}-z^{*}\right\|$ is well defined. Since the $(n-1)$-dimensional unit sphere $S^{n-1} \subseteq \mathbb{R}^{n}$ is compact, it is no loss of generality to assume that the sequence $z_{0}^{1}, z_{1}^{1}, \ldots$ converges to some unit vector $u_{1}$. It follows that $u_{1}$ is a tangent of $X$ at $z^{*}$. If $A u_{1} \neq 0$, upon setting $u=u_{1}$ the claim is proved. If $A u_{1}=0$ we proceed inductively.
Induction Step: Having constructed a tangent $u(l)=\left(u_{1}, \ldots, u_{l}\right)$ of $X$ at $z^{*}$ with $A u_{i}=0$ for each $i \in\{1, \ldots, l\}$, we first observe that $l<n$. (For otherwise, the $u_{j}$ would constitute an orthonormal basis of $\mathbb{R}^{n}$, whence $A$ is the zero matrix, and $A x+b=b$ for each $x \in \mathbb{R}^{n}$, which contradicts $A z_{i}+b \neq A z^{*}+b$.) Let $\rho_{1}, \ldots, \rho_{l}$ be arbitrary real numbers. From

$$
\begin{equation*}
A\left(z^{*}+\rho_{1} u_{1}+\cdots+\rho_{l} u_{l}\right)+b=A\left(z^{*}\right)+b=g\left(z^{*}\right) \neq g\left(z_{i}\right)=A\left(z_{i}\right)+b \tag{19}
\end{equation*}
$$

it follows that no $z_{i}$ lies in the affine space $z^{*}+\mathbb{R} u_{1}+\cdots+\mathbb{R} u_{l}$, i.e., $z_{i}-z^{*} \notin \mathbb{R} u_{1}+\cdots+\mathbb{R} u_{l}$. For each $i$, the unit vector

$$
z_{i}^{l+1}=\frac{z_{i}-z^{*}-\operatorname{proj}_{\mathbb{R} u_{1}+\cdots+\mathbb{R} u_{l}}\left(z_{i}-z^{*}\right)}{\left\|z_{i}-z^{*}-\operatorname{proj}_{\mathbb{R} u_{1}+\cdots+\mathbb{R} u_{l}}\left(z_{i}-z^{*}\right)\right\|}
$$

is well defined. Without loss of generality, we can write $\lim _{i \rightarrow \infty} z_{i}^{l+1}=u_{l+1}$ for some unit vector $u_{l+1} \in \mathbb{R}^{n}$. By construction, $u_{l+1}$ is orthogonal to each of $u_{1}, \ldots, u_{l}$, and the $(l+1)$-tuple
$u(l+1)=\left(u_{1}, \ldots, u_{l}, u_{l+1}\right)$ is a tangent of $X$ at $z^{*}$. In case $A u_{l+1} \neq 0$, upon setting $k=l+1$ and $u=u(l+1)$ we are done. In case $A u_{l+1}=0$, we proceed inductively, with $\left(u_{1}, \ldots, u_{l}, u_{l+1}\right)$ in place of $\left(u_{1}, \ldots, u_{l}\right)$. Our claim is settled, and so is the proof of Part 1.

$$
\text { Part 2: } u \text { is an outgoing tangent of } X .
$$

With the notation of Part 1 , for some $\lambda_{1}, \ldots, \lambda_{k}>0$ we prove the inclusion

$$
\begin{equation*}
\operatorname{conv}\left(z^{*}, z^{*}+\lambda_{1} u_{1}, \ldots, z^{*}+\lambda_{1} u_{1}+\cdots+\lambda_{k} u_{k}\right) \subseteq T \cap f^{-1}\left(\operatorname{conv}\left(y^{*}, y^{*}+\mu v^{*}\right)\right) \tag{20}
\end{equation*}
$$

As a matter of fact, by construction, $u=\left(u_{1}, \ldots, u_{k}\right)$ is a tangent of $X \cap T$ at $z^{*}$. Theorem 3.3 yields real numbers $\epsilon_{1}, \ldots, \epsilon_{k}>0$ such that

$$
\begin{equation*}
\operatorname{conv}\left(z^{*}, z^{*}+\epsilon_{1} u_{1}, \ldots, z^{*}+\epsilon_{1} u_{1}+\cdots+\epsilon_{k} u_{k}\right) \subseteq T \tag{21}
\end{equation*}
$$

Since $A u_{j}=0$ for each $j=1, \ldots, k-1$, from (18)-(19) we obtain the identities

$$
\begin{equation*}
y^{*}=g\left(z^{*}\right)=g(x) \text { for all } x \in \operatorname{conv}\left(z^{*}, z^{*}+\epsilon_{1} u_{1}, \ldots, z^{*}+\epsilon_{1} u_{1}+\cdots+\epsilon_{k-1} u_{k-1}\right) \tag{22}
\end{equation*}
$$

Recalling (17) we can write

$$
\begin{aligned}
0 \neq A u_{k} & =\lim _{i \rightarrow \infty} A z_{i}^{k}=\lim _{i \rightarrow \infty} A\left(\frac{z_{i}-z^{*}-\operatorname{proj}_{\mathbb{R} u_{1}+\cdots+\mathbb{R} u_{l-1}}\left(z_{i}-z^{*}\right)}{\left\|z_{i}-z^{*}-\operatorname{proj}_{\mathbb{R} u_{1}+\cdots+\mathbb{R} u_{l-1}}\left(z_{i}-z^{*}\right)\right\|}\right) \\
& =\lim _{i \rightarrow \infty} \frac{A\left(z_{i}\right)-A\left(z^{*}\right)}{\left\|z_{i}-z^{*}-\operatorname{proj}_{\mathbb{R} u_{1}+\cdots+\mathbb{R} u_{l-1}}\left(z_{i}-z^{*}\right)\right\|} \\
& =\lim _{i \rightarrow \infty} \frac{y_{i}-y^{*}}{\left\|z_{i}-z^{*}-\operatorname{proj}_{\mathbb{R} u_{1}+\cdots+\mathbb{R} u_{l-1}}\left(z_{i}-z^{*}\right)\right\|} \cdot \frac{\left\|y_{i}-y^{*}\right\|}{\left\|y_{i}-y^{*}\right\|} \\
& =\lim _{i \rightarrow \infty} \frac{y_{i}-y^{*}}{\left\|y_{i}-y^{*}\right\|} \cdot \frac{\left\|y_{i}-y^{*}\right\|}{\left\|z_{i}-z^{*}-\operatorname{proj}_{\mathbb{R} u_{1}+\cdots+\mathbb{R} u_{l-1}}\left(z_{i}-z^{*}\right)\right\|}
\end{aligned}
$$

Since $0 \neq v^{*}=\lim _{i \rightarrow \infty}\left(y_{i}-y^{*}\right) /\left\|y_{i}-y^{*}\right\|$, for some $\tau>0$ we obtain

$$
\tau=\lim _{i \rightarrow \infty} \frac{\left\|y_{i}-y^{*}\right\|}{\left\|z_{i}-z^{*}-\operatorname{proj}_{\mathbb{R} u_{1}+\cdots+\mathbb{R} u_{l-1}}\left(z_{i}-z^{*}\right)\right\|} \quad \text { and } \quad A u_{k}=\tau v^{*}
$$

Now the desired $\lambda$ 's in (20) are given by setting $\lambda_{j}=\epsilon_{j}$ for $1 \leq j<k$, and $\lambda_{k}=\min \left\{\epsilon_{k}, \mu / \tau\right\}$. Indeed, letting $C=\operatorname{conv}\left(z^{*}, z^{*}+\lambda_{1} u_{1}, \ldots, z^{*}+\lambda_{1} u_{1}+\cdots+\lambda_{k} u_{k}\right)$, from (21) we obtain

$$
\begin{equation*}
C \subseteq \operatorname{conv}\left(z^{*}, z^{*}+\epsilon_{1} u_{1}, \ldots, z^{*}+\epsilon_{1} u_{1}+\cdots+\epsilon_{k} u_{k}\right) \subseteq T \tag{23}
\end{equation*}
$$

Further, for every $x \in C$ there exists $0 \leq \omega \leq \lambda_{k}$ such that

$$
\begin{equation*}
A x+b=A z^{*}+\omega A u_{k}+b=A z^{*}+b+\omega \tau v^{*}=y^{*}+\omega \tau v^{*} \tag{24}
\end{equation*}
$$

whence $A x+b \in \operatorname{conv}\left(y^{*}, y^{*}+\mu v^{*}\right)$, because $\omega \leq \mu / \tau$. The proof of (20) is complete.
To complete the proof that $\left(u_{1}, \ldots, u_{k}\right)$ is outgoing, letting $C^{\prime}=\operatorname{conv}\left(z^{*}, z^{*}+\lambda_{1} u_{1}, \ldots, z^{*}+\right.$ $\lambda_{1} u_{1}+\cdots+\lambda_{k-1} u_{k-1}$ ), we must show $C^{\prime} \cap X=C \cap X$. By way of contradiction, suppose $x \in(X \cap C) \backslash\left(X \cap C^{\prime}\right)$. Then for suitable $\xi_{1}, \ldots, \xi_{k-1} \geq 0$ and $\xi_{k}>0$, we can write $x=$ $z^{*}+\xi_{1} u_{1}+\cdots+\xi_{k} u_{k}$. By (23), $x \in X \cap T$. Since $\xi_{k}>0$, by (24) we have $g(x)=f(x)=A x+b=$ $y^{*}+\xi_{k} \tau v^{*} \neq y^{*}$. This contradicts the identity $g(x) \in g(X) \cap \operatorname{conv}\left(y^{*}, y^{*}+\mu v^{*}\right)=\left\{y^{*}\right\}$, which follows from (16) and (22).

Having thus proved that the tangent $u$ is outgoing, we have also completed the proof of Part 2 , as well as the proof of the theorem.

## 6. Examples and Further Results

Proposition 6.1. Let $I=\operatorname{conv}(a, b) \subseteq \mathbb{R}$ be an interval, and $\phi: I \rightarrow \mathbb{R}^{n}$ a $C^{2}$ function. Then the Riesz space $\mathcal{R}(\phi(I))$ is strongly semisimple iff $\phi$ is (affine) linear.

Proof. The proof directly follows from Theorems 5.1 and 2.2.
Proposition 6.2. For every polyhedron $P \subseteq \mathbb{R}^{n}$ the Riesz space $\mathcal{R}(P)$ is strongly semisimple, and $P$ has no outgoing tangent.
Proof. For some finite set $\left\{S_{1}, \ldots, S_{m}\right\}$ of simplexes in $\mathbb{R}^{n}$ we can write $P=S_{1} \cup \cdots \cup S_{m}$. If $u$ is a tangent of $P$ at some point $x \in P$ then $u$ is also a tangent of $S_{i}$ at $x$ for some $i=1, \ldots, m$. By Theorem 3.3, $u$ is not an outgoing tangent of $S_{i}$. Thus $u$ is not an outgoing tangent of $P$. Now apply Theorem 5.1.

The following is an example of a strongly semisimple Riesz space $\mathcal{R}(X)$, where $X$ is not a polyhedron:

Example 6.3. Let the set $X \subseteq \mathbb{R}^{2}$ be defined by

$$
X=\{(0,0)\} \cup\{(1 / n, 0) \mid n=1,2, \ldots\} \cup\left\{\left(1 / n, 1 / n^{2}\right) \mid n=1,2, \ldots\right\}
$$

The origin $(0,0)$ is the only accumulation point of $X$. The only tangents of $X$ are given by the vector $(1,0)$ and the pair of vectors $((1,0),(0,1))$. Therefore, $X$ has no outgoing tangents. By Theorem 5.1, the Riesz space $\mathcal{R}(X)$ is strongly semisimple.

However, when the compact set $X \subseteq \mathbb{R}^{n}$ is convex we have:
Theorem 6.4. Let $X \subseteq \mathbb{R}^{n}$ be a nonempty compact convex set. Then the following conditions are equivalent:
(I) The Riesz space $\mathcal{R}(X)$ is strongly semisimple.
(II) $X=\operatorname{conv}\left(x_{1}, \ldots, x_{m}\right)$ for some $x_{1}, \ldots, x_{m} \in \mathbb{R}^{n}$, i.e., $X$ is a polyhedron.
(III) $X$ has no outgoing tangent.

Proof. (III) $\Leftrightarrow$ (I) This is a particular case of Theorem 5.1. (II) $\Rightarrow$ (I) By Proposition 6.2. (I) $\Rightarrow$ (II) Arguing by way of contradiction, assume $\mathcal{R}(P)$ to be strongly semisimple, but $X \neq \operatorname{conv}\left(x_{1}, \ldots, x_{m}\right)$ for any finite set $\left\{x_{1}, \ldots, x_{m}\right\} \subseteq \mathbb{R}^{n}$. Letting $\operatorname{ext}(X)$ denote the set of extreme point of $X$, Minkowski theorem yields the identity $X=\operatorname{conv}(\operatorname{ext}(X))$. Since $X$ is compact, there exists a point $x \in X$ together with a sequence $x_{1}, x_{2}, \ldots$ of extreme points of $X$ such that $\lim _{i \rightarrow \infty} x_{i}=x$ and $x_{i} \neq x_{j}$ for every $i \neq j$.

Claim 1. There exists a subsequence $x_{m_{1}}, x_{m_{2}}, \ldots$ of the sequence $x_{1}, x_{2}, \ldots$, together with a $k$-tuple $\left(u_{1}, \ldots, u_{k}\right)$ of pairwise orthogonal unit vectors in $\mathbb{R}^{n}$ (for some $k \in\{1, \ldots, n\}$ ), having the following properties:
(a) $x_{m_{1}}, x_{m_{2}}, \ldots$ determines the tangent $\left(u_{1}, \ldots, u_{k}\right)$ of $X$ at $x$, in the sense of Definition 4.1.
(b) $\operatorname{aff}\left(x_{m_{1}}, x_{m_{2}}, \ldots\right)=x+\mathbb{R} u_{1}+\cdots+\mathbb{R} u_{k}$.

The vectors $u_{1}, u_{2}, \ldots, u_{k}$ are constructed by the following inductive procedure:
Basis: Since $x_{i} \neq x_{j}$ for each $i \neq j$, then each unit vector $\left(x_{i}-x\right) /\left\|x_{i}-x\right\|$ is well defined. There is a subsequence $x_{m_{1}^{1}}, x_{m_{2}^{1}}, \ldots$ of $x_{1}, x_{2}, \ldots$ and a unit vector $u_{1} \in \mathbb{R}^{n}$ such that $\lim _{i \rightarrow \infty}\left(x_{m_{i}^{1}}-x\right) /\left\|x_{m_{i}^{1}}-x\right\|=$ $u_{1}$. Then $u_{1}$ is a tangent of $X$ at $x$ determined by $x_{m_{1}^{1}}, x_{m_{2}^{1}}, \ldots$.
Induction Step: Let $l \geq 1$ and assume the subsequence $x_{m_{1}^{l}}, x_{m_{2}^{l}}, \ldots$ of $x_{1}, x_{2}, \ldots$ determines the tangent $\left(u_{1}, \ldots, u_{l}\right)$ of $X$ at $x$. If there exists an integer $r$ such that aff $\left(x_{m_{r}^{l}}, x_{m_{r+1}^{l}}, \ldots\right)=$ $x+\mathbb{R} u_{1}+\cdots+\mathbb{R} u_{l}$, then upon setting $k=l$, we are done. If no such $r$ exists, infinitely many vectors in $x_{m_{1}^{l}}, x_{m_{2}^{l}}, \ldots$ do not belong to the affine space $x+\mathbb{R} u_{1}+\cdots+\mathbb{R} u_{l}$. Therefore, for some subsequence $x_{m_{1}^{l+1}}, x_{m_{2}^{l+1}}, \ldots$ and unit vector $u_{l+1} \in \mathbb{R}^{n}$ we can write

$$
\begin{equation*}
u_{l+1}=\lim _{i \rightarrow \infty} \frac{x_{m_{i}^{l+1}}-x-\operatorname{proj}_{\mathbb{R} u_{1}+\cdots+\mathbb{R} u_{l}}\left(x_{m_{i}^{l+1}}-x\right)}{\left\|x_{m_{i}^{l+1}}-x-\operatorname{proj}_{\mathbb{R} u_{1}+\cdots+\mathbb{R} u_{l}}\left(x_{m_{i}^{l+1}}-x\right)\right\|} \tag{25}
\end{equation*}
$$

We then proceed with $\left(u_{1}, \ldots, u_{l+1}\right)$ in place of $\left(u_{1}, \ldots, u_{l}\right)$. Since the affine space aff $\left(x_{m_{1}}, x_{m_{2}}, \ldots\right)$ is contained in $\mathbb{R}^{n}$, this procedure must terminate for some $1 \leq k \leq n$. Claim 1 is settled.

Let us now fix a subsequence $x_{m_{1}}, x_{m_{2}}, \ldots$ of $x_{1}, x_{2}, \ldots$, together with a $k$-tuple $\left(u_{1}, \ldots, u_{k}\right)$ of pairwise orthogonal unit vectors satisfying conditions (a) and (b) in Claim 1.

Claim 2. There are $\lambda_{1}, \ldots, \lambda_{k}>0$ such that the $k$-simplex $C_{k}=\operatorname{conv}\left(x, x+\lambda_{1} u_{1}, \ldots, x+\lambda_{1} u_{1}+\right.$ $\cdots+\lambda_{k} u_{k}$ ) is contained in $X$.

We have already observed that $x \in X$. By Theorem 5.1, the tangent $u_{1}$ of $X$ at $x$ is not outgoing. Hence $\operatorname{conv}\left(x, x+u_{1}\right) \cap X \neq\{x\}$. Let $y \in\left(\operatorname{conv}\left(x, x+u_{1}\right) \cap X\right) \backslash\{x\}$. Thus $y=x+\lambda_{1} u_{1}$ for some $0<\lambda_{1} \leq 1$. Since $X$ is convex, $\operatorname{conv}\left(x, x+\lambda_{1} u_{1}\right) \subseteq X$.

Proceeding inductively, let us assume that $\lambda_{1}, \ldots, \lambda_{l}>0$ are such that the $l$-simplex $C_{l}=$ $\operatorname{conv}\left(x, x+\lambda_{1} u_{1}, \ldots, x+\lambda_{1} u_{1}+\cdots+\lambda_{l} u_{l}\right)$ is contained in $X$, for some $l \in\{1, \ldots, k\}$. If $l=k$ we are done. If $l<k$ let $C_{l+1}^{\prime}=\operatorname{conv}\left(x, x+\lambda_{1} u_{1}, \ldots, x+\lambda_{1} u_{1}+\cdots+\lambda_{l} u_{l}+u_{l+1}\right)$. By construction, $\left(u_{1}, \ldots, u_{l+1}\right)$ is a tangent of $X$ at $x$. Since by hypothesis $\mathcal{R}(X)$ is strongly semisimple, by Theorem $5.1\left(u_{1}, \ldots, u_{l+1}\right)$ is not outgoing, whence there is $y \in\left(C_{l+1}^{\prime} \cap X\right) \backslash C_{l}$. As a consequence, there are $\lambda_{1}^{\prime}, \ldots, \lambda_{l}^{\prime}>0$ and $\lambda_{l+1}>0$ such that $y=x+\lambda_{1}^{\prime} u_{1}+\cdots+\lambda_{l}^{\prime} u_{l}+\lambda_{l+1} u_{l+1}$ and $\lambda_{i}^{\prime} \leq$ $\lambda_{i}$ for each $i \in\{1, \ldots, l\}$. Since $X$ is convex, the set $\operatorname{conv}\left(x, x+\lambda_{1}^{\prime} u_{1}, \ldots, x+\lambda_{1}^{\prime} u_{1}+\cdots+\lambda_{l}^{\prime} u_{l}, y\right)$ is contained in $X$. Setting now (without loss of generality) $\lambda_{i}=\lambda_{i}^{\prime}$, we obtain the inclusion $C_{l+1}=\operatorname{conv}\left(x, x+\lambda_{1} u_{1}, \ldots, x+\lambda_{1} u_{1}+\cdots+\lambda_{l+1} u_{l+1}\right) \subseteq X$, thus completing the inductive step. This procedure terminates after $k$ steps. Claim 2 is settled.

Since the $k$-simplex $C_{k}$ is contained in the affine space $\operatorname{aff}\left(x_{m_{1}}, x_{m_{2}}, \ldots\right)$, and $\left(u_{1}, \ldots, u_{k}\right)$ is the Frenet $k$-frame of the sequence $x_{m_{1}}, x_{m_{2}}, \ldots$, the exists an integer $r^{*}>0$ such that $x_{m_{j}} \in C_{k}$ for each $j=r^{*}, r^{*}+1, \ldots$. By definition, $x_{m_{1}}, x_{m_{2}}, \ldots \in \operatorname{ext}(X)$. By Claim $2, C_{k} \subseteq X$. Thus $x_{m_{r^{*}}}, x_{m_{r^{*}+1}}, \ldots \in \operatorname{ext}\left(C_{k}\right)$. Since $x_{i} \neq x_{j}$ for every $i \neq j$, then the set $\operatorname{ext}\left(C_{k}\right)$ must be infinite, a contradiction. The proof is complete.

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