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Factorization of KdV Schrödinger operators using di erential subresultants

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Abstract

We address the classical factorization problem of a one dimensional Schrödinger operator $-\partial^2 + u - \lambda$, for a stationary potential u of the KdV hierarchy but, in this occasion, a "parameter" λ. Inspired by the more e ective approach of Gesztesy and Holden to the "direct" spectral problem, we give a symbolic algorithm by means of di erential elimination tools to achieve the aimed factorization. Di erential resultants are used for computing spectral curves, and di erential subresultants to obtain the first order common factor. To make our method fully e ective, we design a symbolic algorithm to compute the integration constants of the KdV hierarchy, in the case of KdV potentials that become rational under a Hamiltonian change of variable. Explicit computations are carried for Schrödinger operators with solitonic potentials.

Keywords: Schrödinger operator, factoriazation of ODOs, di erential resultant, di erential subresultant, spectral curve
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1. Introduction

This paper addresses the e ective factorization of the Schrodinger operator

$$L - \lambda = -\partial^2 + u - \lambda \tag{1}$$

for a stationary potential u in a complex variable, say x, and λ a parameter over the field of coefficients. It is well known that whenever the potential satisfies one of the di erential equations

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of the Korteweg de Vries (KdV) hierarchy, this problem is intimately related to the existence of a plane algebraic curve Γ, the spectral curve associated to the operator L. In 1923, J.L. Burchnall and T.W. Chaundy [7] established a correspondence between commuting di erential operators and algebraic curves. They discovered the spectral curve, defined by the so called Burchnall and Chaundy (BC) polynomial. This discovery allowed an algebro-geometric approach to handling the direct and inverse spectral problems for the finite-gap operators, with the spectral data being encoded in the spectral curve and an associated line bundle [16]. In particular, KdV Schrödinger operators (a special case of finite-gap operators) can be treated by the methods in [16], but in this paper we present a di erent approach to the direct spectral problem inspired by the more e ective treatment of Gesztesy and Holden in [14]. We advice for instance [14] for a historic introduction on the subject.

Classically, the spectral curve was computed using a Lenard-type di erential recursion (see [14]), where arbitrary integration constants appeared at each step of the iterative process. In [14] Theorem D.1, the intimate relationship between these integration constants and Γ is shown. Our approach to the problem of computation of constants has the goal of designing an algorithm that depends only on the potential u, but not directly on the spectral curve. For this purpose we describe the flag structure that the constants create, see Section 4.1. In the case of potentials that become rational under a Hamiltonian change of variable [1], we have been able to design the aimed algorithm.

Based on Goodearl's theoretical results [15], we describe the centralizer of L for a KdV potential u. In other words, we determine the essential odd order operator A_{2s+1} of the centralizer of L, that together with L generates the centralizer as a $\mathbb{C}$-algebra $\mathbb{C}[L, A_{2s+1}]$. The potential u satisfies a fixed $\mathbf{KdV}_s$ equation

$$\mathbf{KdV}_s = \mathrm{kdv_s} + \mathrm{c_1 kdv_{s-1}} + \cdots + \mathrm{c_s kdv_0} = 0,$$

with corresponding integration constants $c_1, \ldots, c_s$ in $\mathbb{C}$. Thus for a fixed potential u, the algorithmic determination of the operator A_{2s+1} relies on the algorithmic computation of the constants c_i.

Once we have explicitly obtained the operator A_{2s+1}, a defining equation for Γ can be computed. In fact, E. Previato [25] used di erential resultants to compute spectral curves, opening the door to symbolic computation techniques. The use of these techniques did not transcend so far [14], [19] and their defining polynomials are commonly computed as characteristic polynomials [14], [5], [19]. Di erential resultants for ordinary di erential operators were defined in the 90's by Berkovich and Tsirulik [3] and studied by Chardin [9], who also defined the di erential subresultant sequence; see [18] for a recent report on the subject.

We present a new symbolic algorithm for the factorization of a KdV Schrödinger operator $L - \lambda$ over the field $K(\Gamma)$ of its spectral curve Γ using di erential subresultants. There are other factorization algorithms for linear ordinary di erential operators in the literature, as [6], [29], [28]. But we benefit from the use of the first subresultant since it provides a di erential algebraic formula only in terms of the potential u and the computed constants. In this way the factorization obtained for $L - \lambda$ is written as

$$(\star) \qquad L - \lambda = (-\partial - \phi)(\partial - \phi)$$

with ϕ a quotient of two determinants of matrices with entries di erential polynomials in u. Whenever the spectral curve admits a global parametrization, the algebraic framework that jus-

tifies the correctness of the algorithms allows to develop a parametric version of ($\star$). In the examples of Section 7 we illustrate some of these cases.

A very important requirement in this work is to treat λ as a parameter. The di erential operator $L - \lambda$ is treated first as an operator with coefficients in the field $K(\lambda)$; then, when the spectral curve Γ is considered, as a di erential operator with coefficients in the field $K(\Gamma)$ of rational functions on Γ. Our symbolic factorization structure allows the specialization process to points (λ_0, μ_0) on Γ, recovering the classical factorization of $L - \lambda_0$ at each point (λ_0, μ_0) of Γ_s, see [14]. Another approach to the factorization is carried by means of Darboux transformations and the raising and lowering operators A^+ and A^-, but with this approach one previously needs to compute a set of solutions for a finite set of energy levels, see [1] and the references therein.

The paper is organized as follows. In Section 3, we construct the KdV hierarchy and define di erential subresultants, reviewing its main properties. Section 4 contains our algorithm for computation of the integration constants of the KdV hierarchy. Then in Section 5 we describe the centralizer of L and compute the operator A_{2s+1}. We are ready to review Previato's Theorem, applying it to the computation of the spectral curve of the Lax pair $\{L, A_{2s+1}\}$. Section 6 contains our factorization algorithm for $L - \lambda$ as an operator in $K(\Gamma)[\partial]$. In Section 7, we apply our algorithms to three special families of solitons. A parametric version of the factors is also included for those examples.

We implemented the algorithm for the computation of constants and the factorization algorithm using Maple 18. We used these implementations to compute the examples in Section 7.

2. Notation

We establish some notation to be used throughout the whole manuscript.

Let $\mathbb{N}$ be the set of positive integers including 0. For concepts in di erential algebra we refer the reader to [10], [30] or [20]. Let K be a di erential field of characteristic zero with derivation ∂, whose field of constants C is algebraically closed. Let us consider algebraic variables λ and μ with respect to ∂. Thus $\partial\lambda = 0$ and $\partial\mu = 0$ and we can extend the derivation ∂ of K to the polynomial ring $K[\lambda, \mu]$ by

$$\partial\left(\sum a_{i,j}\lambda^i\mu^j\right) = \sum \partial(a_{i,j})\lambda^i\mu^j, \quad a_{i,j} \in K. \tag{2}$$

Hence $(K[\lambda, \mu], \partial)$ is a di erential ring whose ring of constants is $C[\lambda, \mu]$.

Given a di erential commutative ring E with derivation ∂, let us denote by $E[\partial]$ the ring of di erential operators with coefficients in E and commutation rule

$$[\partial, a] = \partial a - a\partial = \partial(a), a \in E,$$

where ∂a denotes the product in the noncommutative ring $E[\partial]$ and $\partial(a)$ is the image of a by the derivation map. The ring of pseudo-di erential operators in ∂ will be denoted by $E[\partial^{-1}]$ (see [15])

$$E[\partial^{-1}] = \left\{ \sum_{i=-\infty}^{n} a_i\partial^i \mid a_i \in E, n \in \mathbb{Z} \right\},$$

where ∂^{-1} is the inverse of ∂ in $E[\partial^{-1}]$, $\partial^{-1}\partial = \partial\partial^{-1} = 1$.

3. Formal KdV Schrödinger operators

Let us consider a di erential indeterminate u over C. We will call *formal Schrödinger operator* to $L(u) = -\partial^2 + u$ with coefficients in the ring of di erential polynomials

$$C\{u\} = C[u, u', u'', \ldots]$$

where u' stands for $\partial(u)$ and $u^{(n)} = \partial^n(u)$, $n \in \mathbb{N}$.

3.1. KdV polynomials and their Lax pair representations

In this section we will work with the formal Schrödinger operator $L = L(u)$. In a convenient way to be used in this paper, we review well known algorithms to compute the di erential polynomials in u of the KdV hierarchy and the family of di erential operators of its Lax representation. This was studied for the first time in the paper [13]. We follow the normalization in [14], see also [22] for other presentations.

Let us consider the pseudo-di erential operator

$$\mathcal{R} = -\frac{1}{4}\partial^2 + u + \frac{1}{2}u'\partial^{-1} \text{ and its adjoint } \mathcal{R}^* = -\frac{1}{4}\partial^2 + u - \frac{1}{2}\partial^{-1}u'. \tag{3}$$

Observe that $\mathcal{R}^* = \partial^{-1}\mathcal{R}\partial$. The operator $\mathcal{R}^*$ is a recursion operator of the KdV equation (see [22], p. 319). Applying the recursion operator $\mathcal{R}$, we define:

$$\mathrm{kdv_0} := \mathrm{u}', \quad \mathrm{kdv_n} := \mathcal{R}(\mathrm{kdv_{n-1}}), \text{ for } \mathrm{n} \geq 1. \tag{4}$$

Applying $\mathcal{R}^*$ we define:

$$v_0 := 1, \quad v_n := \mathcal{R}^*(v_{n-1}), \text{ for } n \geq 1. \tag{5}$$

Hence for $n \in \mathbb{N}$ it holds

$$2\partial(v_{n+1}) = \mathrm{kdv_n}. \tag{6}$$

We will prove next that for all n, $\mathrm{kdv_n}$ and v_n are di erential polynomials in u, elements of $C\{u\}$. The proof is similar to the one of [22], Theorem 5.31 but we include details for completion, and due to its importance for their symbolic computation, see Section 3.3. We will call the di erential polynomials $\mathrm{kdv_n}$ the *KdV di erential polynomials*.

Lemma 3.1. *The formulas for* $\mathrm{kdv_n}$ *and* v_n *give di erential polynomials in* $C\{u\}$.

Proof. Observe that $\mathcal{R}(\mathrm{kdv_{n-1}})$ is well defined if and only if $\mathrm{kdv_{n-1}}$ is a total derivative. We will prove this by induction on n. It is trivial for $n = 1$ since $\mathrm{kdv_0} = \partial(\mathrm{u})$. Let us assume that $\mathrm{kdv_{n-1}} = \partial(\mathrm{g_{n-1}})$, $g_{n-1} \in C\{u\}$.

Since $\mathcal{R}$ and $\mathcal{R}^*$ are adjoint operators we have $p\mathcal{R}(q) = q\mathcal{R}^*(p) + \partial(a)$, $p, q, a \in C\{u\}$. Thus for $p = u$ and $q = u'$ we get

$$u\mathcal{R}^k(u') = u'(\mathcal{R}^*)^k(u) + \partial(a_k), \quad \text{for } a_k \in C\{u\}.$$

Then

$$u\mathrm{kdv_{n-1}} = (\partial\mathrm{u} - \mathrm{u}\partial)(\mathcal{R}^*)^{\mathrm{n-1}}(\mathrm{u}) + \partial(\mathrm{a_{n-1}}) = -\mathrm{u}\partial(\mathcal{R}^*)^{\mathrm{n-1}}(\mathrm{u}) + \partial(\mathrm{b}), \quad \mathrm{b} \in \mathrm{C\{u\}}$$

which implies that $u\mathrm{kdv}_{n-1} = \partial(b/2)$ is the total derivative of a di erential polynomial in $C\{u\}$. Since

$$\mathcal{R} = \partial \; -\frac{1}{4}\partial + \frac{1}{2}\partial^{-1}u + \frac{1}{2}u\partial^{-1}\Big)\Big($$

we obtain that $\mathrm{kdv}_n = \mathcal{R}(\mathrm{kdv}_{n-1})$ is a total derivative.

The fact that kdv_n is a total derivative and (6) imply that v_{n+1} are also elements of $C\{u\}$. □

As in [14], we define a family of di erential operators in $C\{u\}[\partial]$ of odd order (see also [11], [21])

$$P_1 := \partial, \;\; P_{2n+1} := v_n\partial - \frac{1}{2}\partial(v_n) + P_{2n-1}L, \text{ for } n \geq 1. \tag{7}$$

Observe that

$$P_{2n+1} = \sum_{l=0}^{n}\Big(v_{n-l}\partial - \frac{1}{2}\partial(v_{n-l})\Big)\Big(L^l. \tag{8}$$

The operators P_{2n+1} have the important property that the commutator $[P_{2n+1}, L]$ is a di erential operator in $C\{u\}[\partial]$ but Lemma 3.2 shows that it has order zero, it is the multiplication operator by the kdv_n di erential polynomial. This is the famous Lax representation of kdv_n, see [14], [21]. We will call the di erential operators $P_{2n+1}(u)$ the *KdV di erential operators*.

Lemma 3.2. *For $n \in \mathbb{N}$ it holds $[P_{2n+1}, L] = \mathrm{kdv}_n$.*

Proof. One can easily check that $v_1 = \mathcal{R}^*(1) = u/2$ and $[P_1, L] = u' = 2\partial(v_1)$. We prove the result by induction on n. Since $P_{2n+3} = O + P_{2n-1}L$ where $O = v_{n+1}\partial - \frac{1}{2}\partial(v_{n+1})$, we have

$$[L, P_{2n+3}] = [L, O] + [L, P_{2n+1}]L = [L, O] - 2\partial(v_{n+1})L,$$

with

$$[L, O] = -2\partial(v_{n+1})\partial^2 + (1/2)\partial^3(v_{n+1}) - v_{n+1}u',$$
$$2\partial(v_{n+1})L = -2\partial(v)'_{n+1}\partial^2 + 2\partial(v_{n+1})u.$$

Observe that $\mathcal{R}^* = -\frac{1}{4}\partial^{-1}S$ where $S = \partial^3 - 4u\partial - 2u'$. Thus

$$[L, P_{2n+3}] = (1/2)\partial^3(v_{n+1}) - v_{n+1}u' - 2\partial(v_{n+1})u = (1/2)S(v_{n+1}) = -2\partial\mathcal{R}^*(v_{n+1}) = -2\partial(v_{n+2}).$$

By (6) the result is proved. □

Now let us consider algebraic indeterminates c_n, $n \geq 1$ over C. We define an extended family of *KdV di erential polynomials* $\mathbf{KdV}_n(u, c^n)$, $n \in \mathbb{N}$ in the di erential indeterminate u and the list of algebraic indeterminates $c^n = (c_1, \ldots, c_n)$.

$$\mathbf{KdV}_0 := u', \;\; \mathbf{KdV}_n := \mathrm{kdv}_n + \sum_{l=0}^{n-1}\Big(c_{n-l}\mathrm{kdv}_l, \text{ for } n \geq 1 \tag{9}$$

and an extended family of *KdV di erential operators* whose coefficients are di erential polynomials in u and c^n,

$$\hat{P}_1 := \partial \text{ and } \hat{P}_{2n+1} := P_{2n+1} + \sum_{l=0}^{n-1}\Big(c_{n-l}P_{2l+1}, \text{ for } n \geq 1. \tag{10}$$

One can easily check that

$$[\hat{P}_{2n+1}, L] = \mathbf{KdV}_n = 2\partial(f_{n+1}), \tag{11}$$

for

$$f_0 := v_0 = 1 \text{ and } f_n := v_n + \sum_{l=0}^{n-1} c_{n-l} v_l, \text{ for } n \geq 1. \tag{12}$$

3.2. Di erential resultant and subresultants

Let K be a di erential field as in Section 3. Let us consider di erential operators P and Q in $K[\partial]$ of orders n and m respectively and leading coefficients a_n and b_m. We are interested in the common solutions of the system of linear di erential equations

$$\begin{cases} P = 0 \\ Q = 0 \end{cases}.$$

The tools we have chosen to study this problem are di erential resultant and subresultants. They are an adaptation of the algebraic resultant of two algebraic polynomials in one variable to a noncommutative situation. We summarize next the definition and some important properties of di erential resultants to be used in this work.

3.2.1. Di erential resultant for ODO's and main properties

The Sylvester matrix $S_0(P, Q)$ is the coefficient matrix of the extended system of di erential operators

$$\Xi_0(P, Q) = \{\partial^{m-1}P, \ldots \partial P, P, \partial^{n-1}Q, \ldots, \partial Q, Q\}.$$

Observe that $S_0(P, Q)$ is a squared matrix of size $n+m$ and entries in K. We define the *di erential resultant* of P and Q to be

$$\partial\mathrm{Res}(P, Q) := \det(S_0(P, Q)).$$

Example 3.3. *Given $P = a_2\partial^2 + a_1\partial + a_0$ and $Q = b_3\partial^3 + b_2\partial^2 + b_1\partial + b_0$ in $K[\partial]$,*

$$S_0(P, Q) = \begin{bmatrix} a_2 & a_1 + 2\partial(a_2) & a_0 + 2\partial(a_1) + \partial^2(a_2) & 2\partial(a_0) + \partial^2(a_1) & \partial^2(a_0) \\ 0 & a_2 & a_1 + \partial(a_2) & a_0 + \partial(a_1) & \partial(a_0) \\ 0 & 0 & a_2 & a_1 & a_0 \\ b_3 & b_2 + \partial(b_3) & b_1 + \partial(b_2) & b_0 + \partial(b_1) & \partial(b_0) \\ 0 & b_3 & b_2 & b_1 & b_0 \end{bmatrix}$$

The next propositions state the most relevant properties of the di erential resultant.

Proposition 3.4 ([9])**.** *Let (P, Q) be the left ideal generated by P, Q in $K[\partial]$.*

1. $\partial\mathrm{Res}(P, Q) = AP + BQ$ *with* $A, B \in K[\partial]$, $\mathrm{ord(A)} < \mathrm{m}$, $\mathrm{ord(B)} < \mathrm{n}$, *that is* $\partial\mathrm{Res}(P, Q)$ *belongs to the elimination ideal* $(P, Q) \cap K$.
2. $\partial\mathrm{Res}(P, Q) = 0$ *if and only if* $P = \bar{P}R$, $Q = \bar{Q}R$, *with* $\mathrm{ord(R)} > 0$ *and* $\bar{P}, \bar{Q}, R \in K[\partial]$.

Observe that Proposition 3.4, 1, indicates that $AP + BQ$ is an operator of order zero, the terms in ∂ of degree greater than zero have been eliminated. Furthermore, Proposition 3.4, 2 states that $\partial\mathrm{Res}(P, Q) = 0$ is a condition on the coefficients of the operators that guarantees a right common factor.

Given a fundamental system of solutions $y_1, \dots, y_n$ of $P = 0$, let us denote by $W(y_1, \dots, y_n)$ the Wronskian matrix

$$W(y_1, \dots, y_n) = \begin{bmatrix} y_1 & \cdots & y_n \\ \partial y_1 & \cdots & \partial y_n \\ \vdots & \vdots & \vdots \\ \partial^{n-1} y_1 & \cdots & \partial^{n-1} y_n \end{bmatrix}$$

and by $w(y_1, \dots, y_n)$ its determinant. As in the case of the classical algebraic resultant there is a Poisson formula for $\partial\mathrm{Res}(P, Q)$.

Proposition 3.5 ([9], Theorem 5, see also [25])**.** *Given $P, Q \in K[\partial]$ with respective orders n and m, leading coefficients a_n and b_m and fundamental systems of solutions $y_1, \dots, y_n$ and $z_1, \dots, z_m$ respectively of $P = 0$ and $Q = 0$. It holds,*

$$\partial\mathrm{Res}(P, Q) = (-1)^{nm} a_n^m \frac{w(Q(y_1), \dots, Q(y_n))}{w(y_1, \dots, y_n)} = b_m^n \frac{w(P(z_1), \dots, P(z_m))}{w(z_1, \dots, z_m)}.$$

3.2.2. Subresultant sequence

We introduce next the subresultant sequence for P and Q, which was defined in [9], see also [17]. For $k = 0, 1, \dots, N := \min\{n, m\} - 1$ we define the matrix $S_k(P, Q)$ to be the coefficient matrix of the extended system of di erential operator

$$\Xi_k(P, Q) = \{\partial^{m-1-k} P, \dots \partial P, P, \partial^{n-1-k} Q, \dots, \partial Q, Q\}.$$

Observe that $S_k(P, Q)$ is a matrix with $n + m - 2k$ rows, $n + m - k$ columns and entries in K. For $i = 0, \dots, k$ let $S_k^i(P, Q)$ be the squared matrix of size $n + m - 2k$ obtained by removing the columns of $S_k(P, Q)$ indexed by $\partial^k, \dots, \partial, 1$, except for the column indexed by ∂^i. Whenever there is no room for confusion we denote $S_k(P, Q)$ and $S_k^i(P, Q)$ simply by S_k and S_k^i respectively. The *subresultant sequence* of P and Q is the next sequence of di erential operators in $K[\partial]$:

$$\mathcal{L}_k = \sum_{i=0}^{k} \left(\det(S_k^i) \partial^i, \quad k = 0, \dots, N. \right.$$

In this paper we will only use $\mathcal{L}_1 = \det(S_1^0) + \det(S_1^1)\partial$ where

$$S_1^0 := \mathrm{submatrix}(S_1, \hat{\partial}) \tag{13}$$

and

$$S_1^1 := \mathrm{submatrix}(S_1, \hat{1}) \tag{14}$$

are the submatrices of $S_1 = S_1(P, Q)$ obtained by removing columns indexed by ∂ and 1 respectively.

Recall that $K[\partial]$ is a left Euclidean domain. If ord(P) $\geq$ ord(Q) then $P = qQ + r$ with ord(r) $<$ ord(Q), $q, r \in K[\partial]$. Let us denote by $\gcd(P, Q)$ the greatest common (right) divisor of P and Q.

Theorem 3.6 ([9], Theorem 4. Di erential Subresultant Theorem)**.** *Given di erential operators P and Q in $K[\partial]$, $\gcd(P, Q)$ is a di erential operator of order r if and only if:*

1. *$\mathcal{L}_k$ is the zero operator for $k = 0, 1, , \ldots, r-1$ and,*
2. *$\mathcal{L}_r$ is nonzero.*

Then $\gcd(P, Q) = \mathcal{L}_r$.

Remark 3.7. *From 3.6 we obtain the following consequences.*

1. *Given $\mathcal{L}_r = \gcd(P, Q)$ then $P = \bar{P}\mathcal{L}_r$ and $Q = \bar{Q}\mathcal{L}_r$, $\bar{P}, \bar{Q} \in K[\partial]$.*
2. *The $\gcd(P, Q)$ is nontrivial (it is not in K) if and only if $\mathcal{L}_0 = \partial\mathrm{Res}(P, Q) = 0$.*

We will define resultants and first subresultants of KdV Schrödinger di erential operators. Next we make some remarks in the formal case to be used later when u is specialized to a potential in K.

Remark 3.8. *Let us consider the formal Schrödinger operator $L = -\partial^2 + u$ and the di erential operator $\hat{P}_{2s+1}(u, c^s)$ defined in (10). The following statements hold:*

1. *We have the following formula:*
$$\partial\mathrm{Res}(L - \lambda, \hat{P}_{2s+1} - \mu) = -\mu^2 + R_{2s+1}(u, c^s, \lambda)$$
where $R_{2s+1}(u, c^s, \lambda)$ is a polynomial in $C\{u\}[c^s, \lambda]$.
2. *The determinant of $S_1^1(L - \lambda, \hat{P}_{2s+1} - \mu)$ is a polynomial φ_2 in $C\{u\}[c^s, \lambda]$.*
3. *The determinant of $S_1^0(L - \lambda, \hat{P}_{2s+1} - \mu)$ equals $-\mu - \alpha$, where $\alpha \in C\{u\}[c^s, \lambda]$.*

3.3. Formal examples

We would like to highlight now that all definitions in Section 3.1 are algorithms due to Lemma 3.1. Since u is a di erential indeterminate over C and $\mathrm{kdv_n}$, v_n are di erential polynomials in $C\{u\}$, it is important to note that Lemma 3.1 guarantees that they are well defined, the symbolic integral $\partial^{-1}(\mathrm{kdv_n})$ of $\mathrm{kdv_n}$ can be computed with any software for symbolic computation.

We implemented these definitions with Maple 18. The first iterations of these computations are:

$$\mathrm{kdv}_1 = -\frac{1}{4}\mathrm{u}''' + \frac{3}{2}\mathrm{uu}', \quad \mathrm{kdv}_2 = \frac{1}{16}\mathrm{u}^{(5)} - \frac{5}{8}\mathrm{uu}''' - \frac{5}{4}\mathrm{u}'\mathrm{u}'' + \frac{15}{8}\mathrm{u}^2\mathrm{u}',$$
$$\mathrm{kdv}_3 = \frac{35}{16}\mathrm{u}'\mathrm{u}^3 - \frac{35}{32}\mathrm{u}^2\mathrm{u}''' - \frac{35}{8}\mathrm{uu}'\mathrm{u}'' - \frac{35}{32}(\mathrm{u}')^3 + \frac{21}{32}\mathrm{u}^{(4)}\mathrm{u}' + \frac{7}{32}\mathrm{u}^{(5)}\mathrm{u} + \frac{35}{32}\mathrm{u}'''\mathrm{u}'' - \frac{1}{64}\mathrm{u}^{(7)}$$
$$v_1 = \frac{u}{2}, \quad v_2 = \frac{3}{8}u^2 - \frac{1}{8}u'', \quad v_3 = \frac{5\,u^3}{16} - \frac{5\,u''u}{16} - \frac{5\,(u')^2}{32} + \frac{u^{(4)}}{32}$$

To implement the KdV di erential operators $\hat{P}_{2n+1}$ we used formula (10) and the Maple package OreTools. They are di erential operators in $C\{u\}[\partial]$ the ring of di erential polynomials in u. In fact by (8) we obtain

$$P_3 = -\partial^3 + \frac{3}{2}u\partial + \frac{3}{4}u', \quad P_5 = \partial^5 - \frac{5}{2}u\partial^3 - \frac{15}{4}u'\partial^2 + \frac{15}{8}u^2\partial - \frac{25}{8}u''\partial - \frac{15}{16}u''' + \frac{15}{8}uu',$$
$$P_7 = \frac{105\,u^2u'}{32} - \frac{105\,u'u''}{16} - \frac{105\,u'''u}{32} + \frac{63\,u^{(5)}}{64} + \frac{35\,u^3}{16} - \frac{245\,u'^2}{32} - \frac{175\,u''u}{16} + \frac{161\,u^{(4)}}{32}\Big)\partial$$
$$+ \frac{175\,u'''}{16} - \frac{105\,u'u}{8}\Big)\partial^2 + \frac{105\,u''}{8} - \frac{35\,u^2}{8}\Big)\partial^3 + \frac{35\,u'\partial^4}{4} + \frac{7}{2}\,u\partial^5 - \partial^7.$$

Let us consider the formal Schrödinger operator $L = -\partial^2 + u$ and $\hat{P}_3(u, c^1) = P_3 + c_1 P_1$. Let λ and μ be algebraic indeterminates as in Section 2. The next di erential resultant will be of interest

$$\partial \mathrm{Res}(L - \lambda, \hat{P}_3 - \mu) = -\mu^2 + R_3(u, c^1, \lambda) = -\mu^2 - \lambda^3 - 2c_1\lambda^2 + p_1(u, c_1)\lambda + p_0(u, c_1)$$

where

$$\begin{aligned} p_1(u, c_1) &= \tfrac{1}{4}u'' + \tfrac{3}{4}u^2 + c_1 u - c_1^2 && \text{with } \partial(p_1(u, c_1)) = \mathbf{KdV}_1(u, c^1), \\ p_0(u, c_1) &= \tfrac{1}{16}(u')^2 + \tfrac{1}{4}u^3 - \tfrac{1}{8}u''u - \tfrac{1}{4}u''c_1 + u^2 c_1 + u c_1^2 && \text{with } \partial(p_0(u, c_1)) = \left(\tfrac{u}{2} + c_1\right)\mathbf{KdV}_1(u, c^1). \end{aligned}$$

The di erential subresultant of $L - \lambda$ and $\hat{P}_3 - \mu$ is

$$\mathcal{L}_1 = \det(S_1^0) + \det(S_1^1)\partial = \left(-\mu - \frac{u'}{4}\right) + \left(\frac{u}{2} + c_1 + \lambda\right)\partial$$

with

$$S_1^0 = \begin{bmatrix} -1 & 0 & u' \\ 0 & -1 & u - \lambda \\ -1 & 0 & \frac{3}{4}u' - \mu \end{bmatrix} \text{ and } S_1^1 = \begin{bmatrix} -1 & 0 & u - \lambda \\ 0 & -1 & 0 \\ -1 & 0 & \frac{3}{2}u + c_1 \end{bmatrix}$$

4. Integration constants for KdV potentials

In this section, we specialize u to a potential $\tilde{u}$ in the di erential field K, with field of constants C. First observe that $\mathbf{KdV}_n(\tilde{u}, c^n)$ is equal to zero if there exists a set of constants $\tilde{c}^n \in C^n$ such that $\mathbf{KdV}_n(\tilde{u}, \tilde{c}^n) = 0$.

4.1. Flag of constants for KdV potentials

Having fixed a potential $\tilde{u}$ in K, we will study next the determination of a set of constants $\tilde{c}^n$ satisfying the equation $\mathbf{KdV}_n(\tilde{u}, c^n) = 0$, $n \in \mathbb{N}$, in the set of algebraic variables $c^n = (c_1, \ldots, c_n)$. We will explore the structure of the sets of constants verifying the KdV equations for a given potential $\tilde{u}$. Our method was motivated by [14], Remark 1.5, where the problem is posted but no computational solution is given. We addressed the problem with the goal of giving an algorithm for the computation of constants, that is included in Section 4.2.

Recall that $\mathrm{kdv_n}$ is the di erential polynomial in $C\{u\}$ given by (4). After replacing $u = \tilde{u}$ in $\mathrm{kdv_n}$ we obtain an element of K denoted by $k_n = \mathrm{kdv_n}(\tilde{u})$. Observe that the linear equation in $c_1, \ldots, c_n$

$$\mathbf{KdV}_n(\tilde{u}, c^n) = \mathrm{kdv_n}(\tilde{u}) + \sum_{\ell=0}^{n-1} \mathrm{kdv}_\ell(\tilde{u})\mathrm{c}_{n-\ell} = \mathrm{k_n} + \mathrm{k}_{n-1}\mathrm{c}_1 + \ldots \mathrm{k}_1\mathrm{c}_{n-1} + \mathrm{k}_0\mathrm{c_n} = 0 \qquad (15)$$

determines an affine hyperplane in K^n. Let $\mathcal{H}_n$ be its intersection with C^n

$$\mathcal{H}_n := \{\xi \in C^n \mid \mathbf{KdV}_n(\tilde{u}, \xi) = 0\}.$$

Definition 4.1. *We call a potential $\tilde{u}$ in a di erential field K, a* KdV potential *if there exists $n \geq 1$ such that $\mathcal{H}_n \neq \emptyset$. Let s be the smallest positive integer such that $\mathcal{H}_s \neq \emptyset$, we call s the* KdV level *of $\tilde{u}$. We will write u_s for a KdV potential $\tilde{u}$ of KdV level s.*

Thus the level s of a potential indicates the first equation $\mathbf{KdV}_s = 0$ that is satisfied by u_s for a given set of constants. Furthermore, the next proposition explains that u_s satisfies $\mathbf{KdV}_n = 0$ for all $n > s$. In addition, the choice of constants is unique in the first level but not in the remaining ones.

Proposition 4.2. *Given a potential u_s the following statements are satisfied:*

1. *$\mathcal{H}_s = \{\bar{c}^s\}$ with $\bar{c}^s = (c_1^s, \ldots, c_s^s) \in C^s$.*
2. *For all $n > s$, the C vector space $\mathcal{V}_n := \{\xi \in C^n \mid \mathbf{KdV}_n(u_s, \xi) - k_n = 0\}$ has dimension $n - s$, namely*

$$\mathcal{V}_{s+1} = \langle (1, \bar{c}^s) \rangle, \quad \mathcal{V}_{n+1} = \mathcal{V}_n \oplus \mathcal{W}_n, \text{ with } \mathcal{W}_n = \langle (1, \bar{c}^s, 0, \ldots, 0) \rangle, \tag{16}$$

identifying $\mathcal{V}_n$ with its natural embedding in $\mathcal{V}_{n+1}$ defined by $x \mapsto (0, x)$. Furthermore, there is an infinite flag

$$\mathcal{V}_s \subset \cdots \subset \mathcal{V}_n \subset \cdots, \tag{17}$$

that we call the flag of constants for u_s.

3. *For all $n > s$, we have $\mathcal{H}_n = \bar{c}_n^s + \mathcal{V}_n$, with $\bar{c}_n^s = (c_1^s, \ldots, c_s^s, 0, \ldots, 0) \in C^n$. Furthermore, there is an infinite flag of affine spaces*

$$\mathcal{H}_s \subset \cdots \subset \mathcal{H}_n \subset \cdots, \tag{18}$$

identifying $\mathcal{H}_n$ with its natural embedding in $\mathcal{H}_{n+1}$ defined by $x \mapsto (x, 0)$.

Proof. 1. If there exists $\xi = (\xi_1, \ldots, \xi_s) \neq \bar{c}^s$ in $\mathcal{H}_s$ then for some $1 \leq i \leq s$, $\xi_i - c_i^s \neq 0$ and

$$k_{s-i} + k_{s-i-1} \frac{\xi_{i+1} - c_{i+1}^s}{\xi_i - c_i^s} + \ldots + k_0 \frac{\xi_s - c_s^s}{\xi_i - c_i^s} = 0,$$

contradicting that $\mathcal{H}_{s-i} = \emptyset$.

2. By (4) and (9) we have

$$\mathcal{R}^{n-s}(\mathbf{KdV}_s(u, c^s)) = \mathcal{R}^{n-s}(\mathrm{kdv_s(u)}) + \sum_{\ell=0}^{s-1} \mathrm{c_{s-\ell}} \mathcal{R}^{\mathrm{n-s}}(\mathrm{kdv_\ell(u)}) = \mathrm{kdv_n(u)} + \sum_{\ell=0}^{s-1} \Big(\mathrm{c_{s-\ell} kdv_{n-s+l}(u)}. \tag{19}$$

Let us consider the recursion operator (3) for $u = u_s$, that is $\mathcal{R}_s = -\frac{1}{4}\partial^2 + u_s + \frac{1}{2}u_s'\partial^{-1}$. Replacing u by u_s and c^s by $\bar{c}^s$ in (19) we obtain

$$k_n = -k_{n-1}c_1^s - \cdots - k_{n-s}c_s^s, \tag{20}$$

since $\mathcal{R}_s$ is a linear operator acting on $C\langle u_s \rangle$ and $\mathbf{KdV}_s(u_s, \bar{c}^s) = 0$.
We prove (16) by induction on n. An element $\xi = (\xi_1, \ldots, \xi_{s+1})$ of $\mathcal{V}_{s+1}$ verifies $k_s\xi_1 + \cdots + k_0\xi_{s+1} = 0$, and taking $n = s$ in (20) we get

$$k_{s-1}(\xi_2 - \xi_1 c_1^s) + \cdots + k_0(\xi_{s+1} - \xi_1 c_s^s) = 0.$$

Then $\xi = \xi_1(1, \bar{c}^s)$, because 1. implies that $\mathcal{V}_s$ is the null space. Let us assume that $\mathcal{V}_n$ has basis $\{w_1, \ldots, w_{n-s}\}$. Observe that

$$\mathcal{V}_{n+1} \cap \{\xi \in C^{n+1} \mid \xi_1 = 0\} = \{0\} \times \mathcal{V}_n$$

has basis $\mathcal{B} = \{(0, w_1), \ldots, (0, w_{n-s})\}$. Using (20) we can prove that $\xi \in \mathcal{V}_{n+1}$ verifies

$$k_{n-1}(\xi_2 - \xi_1 c_1^1) + \ldots + k_{n-s}(\xi_{s+1} - \xi_1 c_s^s) + k_{n-s-1}\xi_{s+2} + \cdots + k_0\xi_{n+1} = 0.$$

Let $w = (1, \bar{c}^s, 0, \ldots, 0) \in C^{n+1}$, then $\xi - \xi_1 w \in \{0\} \times \mathcal{V}_n$, which proves that $\{w\} \cup C\{u\}$ is a basis of $\mathcal{V}_{n+1}$ of size $n + 1 - s$.

3. Substituting (20) in $\mathbf{KdV}_n(u_s, c^n)$ gives

$$\mathbf{KdV}_n(u_s, c^n) = k_n(c_1 - c_1^s) + \cdots + k_{n-s}(c_s - c_s^s) + k_{n-s-1}c_{s+1} + \ldots + k_0 c_n,$$

proving that $\mathcal{H}_n = \bar{c}_n^s + \mathcal{V}_n$. Similarly we can prove that given $\xi \in \mathcal{H}_i$, $s \leq i \leq n-1$ then

$$\mathbf{KdV}_{i+1}(u_s, c^{i+1}) = k_i(c_1 - \xi_1) + \cdots + k_1(c_i - \xi_i) + k_0 c_{i+1}.$$

Therefore $\mathcal{H}_i \times \{0\} \subset \mathcal{H}_{i+1}$ and (18) follows.

□

The previous proposition shows that the flag of constants of a KdV potential u_s of KdV level s is determined by $\mathcal{H}_s = \{\bar{c}^s\}$. We call $\bar{c}^s$ the *basic constants vector* of u_s.

Example 4.3. *As a first example, let us consider $\tilde{u} = 6/x^2$ in $K = \mathbb{C}(x)$. One can easily check that $\tilde{u}$ is a KdV potential of level 2. It does not verify $\mathbf{KdV}_1(u, c_1) = kdv_1(u) + c_1 kdv_0(u) = 0$ for any $c_1 \in \mathbb{C}$ but $\mathbf{KdV}_2(\tilde{u}, (0,0)) = kdv_2(\tilde{u}) = 0$ and $\mathbf{KdV}_n(u, c^n) = 0$, $n > 2$ is satisfied by $u = \tilde{u}$ for infinitely many choices of $c^n \in \mathbb{C}^n$. Its basic constant vector is $\bar{c}^2 = (0,0)$. More examples can be found in Section 7.*

The computation of the basic constants vector is algorithmic at least for a big family of potentials, as we explain in the next section.

4.2. Computing the integration constants

We designed an algorithm that decides if a potential $\tilde{u}$ in K is a KdV potential and returns its level s and basic constants vector $\bar{c}^s$. For this purpose we restrict to the case of potentials $\tilde{u}$ that are rational functions in an element η in K. Let $C(\eta)$ be the field of rational functions in η, we assume that $\tilde{u} \in C(\eta)$. Furthermore we assume that $(\eta')^2 \in C(\eta)$. This request is necessary for a hamiltonian algebraization in order to preserve the Galoisian behavior of the factors, see [1]. This situation is satisfied by the three families of KdV potentials that we will use to illustrate all the results of this paper in Section 7.

We can distinguish two cases. If $\eta' \in C(\eta)$ then $C\langle\tilde{u}\rangle \subset C(\eta)$. If $(\eta')^2 \in C(\eta)$ but $\eta' \notin C(\eta)$ then $C\langle\tilde{u}\rangle$ is contained in the linear space $V = C(\eta) \oplus \eta' C(\eta)$.

Lemma 4.4. *Let us consider $a \in V$.*

1. *If $a \in \eta' C(\eta)$ then $a' \in C(\eta)$.*
2. *$a \in C(\eta)$ if and only if $a' \in \eta' C(\eta)$.*

Proof. Clearly, if $a \in \eta' C(\eta)$ then $a' \in C(\eta)$ and also if $a \in C(\eta)$ then $a' \in \eta' C(\eta)$. Let us assume $a' \in \eta' C(\eta)$, with $a = h_0 + h_1\eta'$, $h_0, h_1 \in C(\eta)$ and $h_1 \neq 0$. Then $\partial(a - h_0) = \partial(\eta' h_1) \in C(\eta) \cap \eta' C(\eta) = 0$ thus $\eta' h_1$ is a constant which contradict that $\eta' \notin C(\eta)$. □

Recall that v_n is the di erential polynomial in $C\{u\}$ given by (5). After replacing $u = \tilde{u}$ in v_n we obtain an element of K that will be denoted by $v_n(\tilde{u})$. If $\eta' \in C(\eta)$ then $kdv_n(\tilde{u}) \in C(\eta)$ and also $v_n(\tilde{u}) \in C(\eta)$, for all $n \in \mathbb{N}$.

Lemma 4.5. *Let us consider a potential $\tilde{u} \in C(\eta)$. Then $\mathrm{kdv_n}(\tilde{u}) \in \eta' \mathrm{C}(\eta)$ and $v_n(\tilde{u}) \in C(\eta)$, for all n.*

Proof. Given $\tilde{u} \in C(\eta)$ we can easily prove that $\partial^n(\tilde{u})$ belongs to $\eta' C(\eta)$ for n odd and belongs to $C(\eta)$ for n even. Observe that $v_2(\tilde{u}) \in C(\eta)$ (see Section 3.3) and thus $\mathrm{kdv_1}(\tilde{u}) \in \eta' \mathrm{C}(\eta)$ by Lemma 4.4, since $\mathrm{kdv_1} = \partial(\mathrm{v_2})$. Let us assume that $\mathrm{kdv_n}(\tilde{u}) = \partial(\mathrm{v_{n+1}})(\tilde{u}) \in \eta' \mathrm{C}(\eta)$, then Lemma 4.4 implies $v_{n+1} \in C(\eta)$. Since $\mathrm{kdv_{n+1}} = \mathcal{R}(\mathrm{kdv_n})$ we have

$$\mathrm{kdv_{n+1}}(\tilde{u}) = -\frac{1}{4}\partial^2(\mathrm{kdv_n}(\tilde{u})) + \tilde{u}\mathrm{kdv_n}(\tilde{u}) + \frac{1}{2}\tilde{u}'\mathrm{v_{n+1}}(\tilde{u}),$$

which is the sum of terms in $\eta' C(\eta)$, hence $\mathrm{kdv_{n+1}}(\tilde{u}) \in \eta' \mathrm{C}(\eta)$. □

From the previous lemmas and (15), the next result follows.

Proposition 4.6. *Given $\tilde{u} \in C(\eta)$ then*

$$\mathbf{KdV}_n(\tilde{u}, c^n) = \begin{cases} \frac{p_n(\eta)}{q_n(\eta)} & \text{if } \eta' \in C(\eta), \\ \eta' \frac{p_n(\eta)}{q_n(\eta)} & \text{if } (\eta')^2 \in C(\eta), \quad \eta' \notin C(\eta), \end{cases} \tag{21}$$

where $p_n = \sum_d l_d \eta^d$ and $q_n \in C[\eta]$ are polynomials in η, with l_d linear expressions in $c_1, \ldots, c_n$ over C.

We are ready to give the announced algorithm.

Algorithm 4.7. *(Basic Constants Vector) Let $\eta \in K$ be such that $(\eta')^2 \in C(\eta)$.*

- Given *$\tilde{u} \in C(\eta)$ and $s^* \geq 1$.*
- Decide *if $\tilde{u}$ is a KdV potential of KdV level smaller than or equal to s^* and* return *the KdV level s and its basic constants vector $\bar{c}^s$.*

1. *Set $n := 1$.*
2. *Replace u by $\tilde{u}$ in $\mathbf{KdV}_n(u, c^n)$ to obtain $\frac{p_n(\eta)}{q_n(\eta)}$, as in Proposition 4.6.*
3. *Collect the coefficients in η of p_n to obtain a nonhomogeneous system $\mathcal{S}_n$ of linear equations over C in the unknowns $c_1, \ldots c_n$.*
4. *If $\mathcal{S}_n$ has a solution $\xi \in C^n$,* return *$s := n$ and $\bar{c}^s := \xi$.*
5. *If $n = s^*$* return "it is not a KdV potential up to the required level".
6. *Set $n := n + 1$ and go to Step 2.*

See examples in Sections 7.2 and 7.3.

5. Spectral curves of KdV Schrödinger operators

In this section we will study the centralizers of Schrödinger operators $L_s = L(u_s) = -\partial^2 + u_s$, where u_s is a KdV potential of KdV level s and basics constant vector $\bar{c}^s \in \mathbb{C}^s$, as defined in Section 4.1. We will call L_s a *KdV Schrödinger operator* or $\mathbf{KdV}_s$ for short. This will allow us to define the spectral curve of L_s by means of an operator of order $2s+1$ commuting with L_s. For this purpose we need to study the centralizer of the operator L_s. We will show that this centralizer is generated by L_s and another operator A_{2s+1}. This pair, $\{L_s, A_{2s+1}\}$, will be the one we use to calculate an equation of the spectral curve associated with L_s.

5.1. Centralizers and Burchnall-Chaundy polynomials

To start we summarize some results from [15] about centralizers of di erential operators. Let $P = a_n\partial^n + \cdots + a_1\partial + a_0$ be an operator in $E[\partial]$. Let us denote by $C_E(P)$ the centralizer of P in $E[\partial]$, that is

$$C_E(P) = \{Q \in E[\partial] \mid PQ = QP\}.$$

By [15], Theorem 4.1, if n and a_n are non zero divisors in E then $C_E(P)$ is commutative. Let C^∞ be the ring of infinitely-many times di erentiable complex-valued functions on the real line. By [15], Corollary 4.4, $C_{C^\infty}(P)$ is commutative if and only if there is no nonempty open interval on the real line on which the functions $\partial(a_0), a_1, \ldots, a_n$ all vanish.

Details of the evolution of these results from various previous works are given in [15]. We chose this reference because it simplifies the existing methods and applies them in as wide a context as reasonable. Precursors of the commutativity results are Schur [27], Flanders [12], Krichever [16], Amitsur [2], Carlson and Goodearl [8]. Results describing centralizers $C_R(P)$ as a free module of finite rank appear in [12], [2], [8] and in Ore's well known paper [23].

Recall that a commutative ring E is called reduced if it has no nonzero nilpotent element. Observe that C^∞ is not a field, but it is a reduced ring whose ring of constants is the field $\mathbb{C}$.

Theorem 5.1. *Let E be a reduced di erential ring whose subring F of constants is a field. Let us assume that n is invertible in F and a_n is invertible in E.*

1. *([15], Theorem 4.2) $C_E(P)$ is a commutative integral domain.*
2. *([15], Theorem 1.2) Let X be the set of those i in $\{0, 1, 2, \ldots, n-1\}$ for which $C_E(P)$ contains an operator of order congruent to i module n. For each $i \in X$ choose Q_i such that* $\mathrm{ord}(Q_i) \equiv i \pmod{n}$ *and Q_i has minimal order for this property (in particular $0 \in X$, and $Q_0 = 1$). Then $C_E(P)$ is a free $F[P]$-module with basis $\{Q_i \mid i \in X\}$. Moreover, the rank t of $C_E(P)$ as a free $F[P]$-module is a divisor of n.*

We are ready now to describe the centralizer $C_K(L_s)$ in $K[\partial]$ of the KdV Schrödinger operator L_s. We do so by generalizing an example in [15], Section 1.2. In addition, by [8], Theorem 1.6 we know that $C_K(L_s)$ has rank 2 as a free $C[L_s]$-module.

Replacing u by u_s and c^n by $\bar{c}^s_n = (\bar{c}^s, 0, \ldots, 0)$ in the family of KdV di erential operators $\hat{P}_{2n+1}(u, c^n)$ defined in (10), we obtain a family of di erential operators in $K[\partial]$

$$A_{2n+1} := \hat{P}_{2n+1}(u_s, \bar{c}^s_n), \text{ for all } n \geq s. \tag{22}$$

As a consequence of (11) and Proposition 4.2 we have

$$[A_{2n+1}, L_s] = \mathbf{KdV}_n(u_s, \bar{c}^s_n) = 0, \text{ for all } n \geq s. \tag{23}$$

Thus $A_{2n+1} \in C_K(L_s)$, for all $n \geq s$. The next result shows that A_{2s+1} has an important role in the description of the centralizer of L_s, it is the *di erential operator that determines the centralizer of* L_s.

Theorem 5.2. *Let L_s be a KdV Schrödinger operator. The centralizer of L_s in $K[\partial]$ equals the free $C[L_s]$-module of rank 2 with basis $\{1, A_{2s+1}\}$, that is*

$$C_K(L_s) = \{p_0(L_s) + p_1(L_s)A_{2s+1} \mid p_0, p_1 \in C[L_s]\} = C[L_s]\langle 1, A_{2s+1}\rangle.$$

Proof. We will prove that there does not exist an operator of odd order smaller that $2s + 1$ in $C_K(L_s)$. By Theorem 5.1, 2, this implies that $C_K(L_s) = C[L_s]\langle 1, A_{2s+1}\rangle$.

Let us consider a monic di erential operator $Q \in K[\partial]$ of order $2n+1$ with $n < s$. Let $P_{2n+1}(u)$ be the family of KdV di erential operators defined in (7) and denote by $P^s_{2n+1} := P_{2n+1}(u_s)$. Since $\{P^s_{2i+1}\}_{i\leq n}$ and $\{L^i_s\}_{i\leq n}$ are families of operators in $K[\partial]$ of odd and even orders less than $2n + 1$ respectively, we divide Q by those families and write

$$Q = \sum_{i=0}^{n} q_{2i+1}P^s_{2i+1} + \sum_{i=0}^{n} q_{2i}L^i_s$$

with $q_{2n+1} = 1$ and $q_{2i+1}, q_{2i} \in K$. To compute $[Q, L_s]$, observe that $[a, L_s] = \partial^2(a) + 2\partial(a)\partial$, for $a \in K$ and

$$[q_{2i+1}P^s_{2i+1}, L_s] = -\partial^2(q_{2i+1})P^s_{2i+1} - 2\partial(q_{2i+1})\partial P^s_{2i+1} + q_{2i+1}\mathrm{kdv_i(u_s)}$$

and

$$[q_{2i}L^i_s, L_s] = [q_{2i}, L_s]L^i_s = (\partial^2(q_{2i}) + 2\partial(q_{2i})\partial)L^i_s.$$

Thus in $[Q, L_s]$ the only term of order $2i + 2$ is the leading term of ∂P^s_{2i+1} and the only term of order $2i + 1$ is the leading term of ∂L^i_s. If $[L_s, Q] = 0$ then $\partial(q_{2i}) = 0$ and $\partial(q_{2i+1}) = 0$. Therefore $[q_{2i}L^i_s, L_s] = 0$ and $q_{2i+1} \in C$, $i = 0, \ldots, n$ implies that

$$0 = [Q, L_s] = \sum_{i=0}^{n} q_{2i+1}\mathrm{kdv_i(u_s)}$$

contradicting that u_s has KdV level s. We conclude that $Q \notin C_K(L_s)$, which proves the result. □

A polynomial $f(\lambda, \mu)$ with constant coefficients satisfied by a commuting pair of di erential operators P and Q is called a *Burchnall-Chaundy (BC) polynomial* of P and Q, since the first result of this sort appeared is the 1923 paper [7] by Burchnall and Chaundy. Generalizations (more general rings E) were later studied in [16], [8] and [26]. The next result shows that associated to the centralizer of a di erential operator P there are as many BC polynomials as operators in the centralizer. We will compute these polynomials using di erential resultants, as it will be explained in Section 3.2.

Theorem 5.3. *([15], Theorem 1.13) Let E be a reduced di erential ring whose subring F of constants is a field. Given any operator $Q \in C_E(P)$ there exist polynomials $p_0(P), \ldots, p_{t-1}(P) \in F[P]$ such that*

$$p_0(P) + p_1(P)Q + \cdots + p_{t-1}(P)Q^{t-1} + Q^t = 0.$$

That is, there exists a nonzero polynomial $f_Q(\lambda, \mu) \in F[\lambda, \mu]$ such that $f_Q(P, Q) = 0$.

5.2. Computing spectral curves

The relation between Burchnall and Chaundy polynomials (see Section 5.1) and di erential resultants was given by E. Previato in [25]. Next, we state Previato's theorem in the general case of di erential operators in $K[\partial]$ (5.4) and we give an alternative proof using the Poisson formula for the di erential resultant (Proposition 3.5). Then we will compute BC polynomials of KdV Schrödinger operators. We we will apply Previato's Theorem 5.4 to the computation of the spectral curve of the Lax pair $\{L_s, A_{2s+1}\}$, showing the algebraic structure of the irreducible polynomials $f_s(\lambda,\mu)$ defining the spectral curve Γ_s.

First observe that whenever the operators $P-\lambda$ and $Q-\mu$ have coefficients in the di erential ring $(K[\lambda,\mu],\partial)$ (see Section 2), by means of the di erential resultant, Proposition 3.4, 1, it is ensured that we compute a nonzero polynomial,

$$\partial\mathrm{Res}(P-\lambda, Q-\mu) = a_n^m\mu^n - b_m^n\lambda^m + \cdots \tag{24}$$

in the elimination ideal $(P-\lambda, Q-\mu)\cap K[\lambda,\mu]$. The next result implies that if P and Q commute then

$$\partial\mathrm{Res}(P-\lambda, Q-\mu) \in (P-\lambda, Q-\mu)\cap C[\lambda,\mu].$$

Theorem 5.4 (E. Previato, [25])**.** *Given $P, Q \in K[\partial]$ such that $[P,Q]=0$ then*

$$g(\lambda,\mu) = \partial\mathrm{Res}(P-\lambda, Q-\mu) \in C[\lambda,\mu]$$

and also $g(P,Q)=0$.

Proof. Let $y_1,\dots,y_n$ be a fundamental system of solutions of $(P-\lambda)(Y)=0$. Since $0=[P,Q]=[P-\lambda, Q-\mu]$ we have $(P-\lambda)(Q-\mu)(y_i) = (Q-\mu)(P-\lambda)(y_i) = 0$ then $(Q-\mu)(y_i)$, $i=1,\dots,n$ are solutions of $(P-\lambda)(Y)=0$. Then, there exists a matrix M with entries in the algebraic closure $\mathfrak{C}$ of $C(\lambda,\mu)$ such that there exists a matrix M with entries in $\mathfrak{C}$ such that

$$W((Q-\mu)(y_1),\dots,(Q-\mu)(y_n)) = W(y_1,\dots,y_n)M.$$

By Proposition 3.5,

$$\partial\mathrm{Res}(P-\lambda, Q-\mu) = \frac{w((Q-\mu)(y_1),\dots,(Q-\mu)(y_n))}{w(y_1,\dots,y_n)} = \frac{w(y_1,\dots,y_n)\det(M)}{w(y_1,\dots,y_n)} = \det(M),$$

which belongs to $K[\lambda,\mu]\cap\mathfrak{C} = C[\lambda,\mu]$.

The last statement of this theorem follows from the fact that $g(\lambda,\mu) = \partial\mathrm{Res}(P-\lambda, Q-\mu)$ belongs to the di erential ideal generated by $P-\lambda$ and $Q-\mu$ in $K[\lambda,\mu][\partial]$. Therefore

$$g(\lambda,\mu) = A(P-\lambda) + B(Q-\mu), \text{ with } A, B \in K[\lambda,\mu][\partial].$$

Since P and Q commute then $g(P,Q)=0$. □

The previous theorem shows that BC polynomials (defined in Section 5.1) can be computed using di erential resultants. Let us suppose that $[P,Q]=0$ and let $f(\lambda,\mu)$ be the square free part of $\partial\mathrm{Res}(P-\lambda, Q-\mu) \in C[\lambda,\mu]$ (i.e. the product of the di erent irreducible components of g). The affine plane algebraic curve defined by f

$$\Gamma := \{(\lambda,\mu)\in C^2 \mid f(\lambda,\mu)=0\} \tag{25}$$

is known as the *spectral curve of the pair* $\{P, Q\}$.

Let us suppose that $f(\lambda,\mu)$ is an irreducible polynomial in $K[\lambda,\mu]$ and denote by (f) the prime ideal generated by f in $K[\lambda,\mu]$. As a polynomial in $C[\lambda,\mu]$ is also irreducible and the ideal generated by f in $C[\lambda,\mu]$ is also prime, abusing the notation we will also denote it by (f) and distinguish it by the context. Let us denote by $C(\Gamma)$ and $K(\Gamma)$ the fraction fields of the domains $C[\lambda,\mu]/(f)$ and $K[\lambda,\mu]/(f)$ respectively. Observe that $C(\Gamma)$ and $K(\Gamma)$ are usually interpreted as rational function on Γ.

Remark 5.5. *As di erential operators in $K[\lambda,\mu][\partial]$, the operators $P-\lambda$ and $Q-\mu$ have no common nontrivial solution, see (24), but as elements of $K(\Gamma)[\partial]$ they have a common non constant factor. By Theorem 3.6 the first nonzero subresultant $\mathcal{L}_r = \mathrm{gcrd}(\mathrm{P}-\lambda, \mathrm{Q}-\mu)$ is the greatest common divisor of $P-\lambda$ and $Q-\mu$ in $K(\Gamma)[\partial]$. We will use subresultants in Section 6 to compute factorizations of KdV Schrödinger operators.*

Let us consider the KdV Schrödinger operators $L_s = L(u_s) = -\partial^2 + u_s$, where u_s is a KdV potential of KdV level s and basic constants vector $\bar{c}^s$, as defined in Section 4.1. Let A_{2s+1} be the di erential operator that determines the centralizer of L_s, see (22).

Corollary 5.6. *The spectral curve Γ_s of the pair $\{L_s, A_{2s+1}\}$ is defined by the polynomial in $C[\lambda,\mu]$,*

$$f_s(\lambda,\mu) := \partial\mathrm{Res}(L_s-\lambda, A_{2s+1}-\mu) = -\mu^2 - R_{2s+1}(\lambda),$$

where $R_{2s+1}(\lambda)$ is a polynomial of degree $2s+1$ in $C[\lambda]$. The polynomial $f_s(\lambda,\mu)$ is irreducible in $K[\lambda,\mu]$. In addition, the coefficients of $R_{2s+1}(\lambda)(u,\bar{c}^s,\lambda)$ in Proposition 3.8 are first integrals of $KdV_s(u,\bar{c}^s)$.

Proof. By (23), $[A_{2n+1}, L_s] = \mathbf{KdV}_n(u_s, \bar{c}^s_n) = 0$. Thus by Theorem 5.4, $f_s \in C[\lambda,\mu]$. In addition, by Remark 3.8, $f_s = -\mu^2 - R_{2s+1}(\lambda)$, which can be easily proved to be irreducible in $K[\lambda,\mu]$ because it has odd degree in λ. □

Definition 5.7. The spectral curve of L_s *is defined as the plain algebraic curve Γ_s given by $f_s(\lambda,\mu) = 0$, with f_s as defined in Corollary 5.6.*

6. Factors of KdV Schrödinger operators over spectral curves

Let u_s be a KdV potential of KdV level s and basic constants vector $\bar{c}^s$. Let A_{2s+1} be the di erential operator that determines the centralizer of the KdV Schrödinger $L_s = -\partial^2 + u_s$ as in Theorem 5.2. By Corollary 5.6, the spectral curve Γ_s of the pair $\{L_s, A_{2s+1}\}$ is defined by the irreducible polynomial

$$f_s(\lambda,\mu) = \partial\mathrm{Res}(L_s-\lambda, A_{2s+1}-\mu) = \mu^2 - R_{2s+1}(\lambda) \in C[\lambda,\mu].$$

Let $C(\Gamma_s)$ and $K(\Gamma_s)$ be the fraction fields of the domains $C[\lambda,\mu]/(f_s)$ and $K[\lambda,\mu]/(f_s)$. In this section, we explain how to factor $L_s-\lambda$ as an operator in $K(\Gamma_s)[\partial]$.

6.1. KdV factors on Γ_s

In this section, we consider the operators $L_s - \lambda$ and $A_{2s+1} - \mu$ as elements of $K(\Gamma_s)[\partial]$. Let $\mathcal{L}_1 = \varphi_2\partial + \varphi_1$ be the subresultant of $L_s - \lambda$ and $A_{2s+1} - \mu$ as in Section 3.2.2.

Theorem 6.1. *The greatest common factor of the di erential operators $L_s - \lambda$ and $A_{2s+1} - \mu$ in $K(\Gamma_s)[\partial]$ is the order one operator $\mathcal{L}_1$.*

Proof. Since $\mathcal{L}_0 = \partial\mathrm{Res}(L_s - \lambda, A_{2s+1} - \mu)$ is zero in $K(\Gamma_s)$ by Theorem 3.6 the result follows. □

We can take the monic greatest common factor of $L_s - \lambda$ and $A_{2s+1} - \mu$ to be

$$\partial - \phi_s, \text{ where } \phi_s = -\frac{\varphi_1}{\varphi_2}.$$

The fact that $\partial - \phi_s$ is a right factor implies that

$$L_s - \lambda = (-\partial - \phi_s)(\partial - \phi_s), \text{ in } K(\Gamma_s)[\partial]$$

and moreover ϕ_s is a solution of the Ricatti equation associated to the Schödinger operator $L_s - \lambda$

$$\partial(\phi) + \phi^2 = u_s - \lambda \tag{26}$$

on the spectral curve Γ_s. Therefore, we compute a solution of (26) by means of the di erential subresultant $\mathcal{L}_1$. We will give next some details about ϕ_s.

Lemma 6.2. *The following formula holds*

$$\phi_s = \frac{\mu + \alpha(\lambda)}{\varphi(\lambda)}, \tag{27}$$

where α and φ are nonzero polynomials in $K[\lambda]$. Moreover ϕ_s is nonzero in $K(\Gamma_s)$.

Proof. By (13),(14), we have that $\varphi_1 = det(S_1^0(L_s - \lambda, A_{2s+1} - \mu))$ and $\varphi_2 = det(S_1^1(L_s - \lambda, A_{2s+1} - \mu))$. Now by Remark 3.8, $\varphi_1 = -\mu - \alpha$ and $\alpha, \varphi = \varphi_2$ are nonzero polynomials in $K[\lambda]$. Observe that $\phi_s = 0$ in $K(\Gamma_s)$ if and only if $\mu + \alpha + (f_s) = 0$ in $K[\Gamma_s]$. But this is not possible since f_s, which has degree 2 in μ, is not a factor of $\mu + \alpha$ in $K[\lambda, \mu]$. This proves the last claim. □

To keep notation as simple as possible, we will also write ϕ_s to denote the element ϕ_s in $K(\Gamma_s)$. The next algorithm takes as an input a KdV potential to obtain the factor $\partial - \phi_s$ of $L_s - \lambda$ in $K(\Gamma_s)[\partial]$.

Algorithm 6.3. *(Factorization)*

- Given *u_s a KdV potential of KdV level s and given $\bar{c}^s$ the basic constants vector of u_s.*
- Return *the defining polynomial f_s of the spectral curve Γ_s and the monic greatest common divisor $\partial - \phi_s$ of $L_s - \lambda$ and $A_{2s+1} - \mu$ in $K(\Gamma_s)[\partial]$.*

1. *Define $L_s := -\partial^2 + u_s$ and $A_{2s+1} := \hat{P}_{2s+1}(u_s, \bar{c}^s)$ as in (22).*
2. *Compute $f_s = \partial\mathrm{Res}(L_s - \lambda, A_{2s+1} - \mu)$, the defining polynomial of the spectral curve Γ_s.*
3. *Compute $\mathcal{L}_1 = \varphi_1 + \varphi_2\partial$, the subresultant of $L_s - \lambda$ and $A_{2s+1} - \mu$ as in Section 3.2.2.*
4. *Define $\phi_s := -\frac{\varphi_1}{\varphi_2}$.*

5. *Return f_s and $\partial - \phi_s$.*

Remark 6.4. *Observe that we are computing ϕ_s in closed form as the quotient of two determinants $-\varphi_1/\varphi_2$, which is a well defined function over the spectral curve. As far as we know, there were no algorithms to obtain the factors $\partial - \phi_s$ of $L_s - \lambda$ over the spectral curve. In [14], there are some di erential recursive expressions for the factorization of $L_s - \lambda_0$ for each point (λ_0, μ_0) of Γ_s. Note that our factorization algorithm is defined in $K(\Gamma_s)[\partial]$.*

We would like to obtain a univariate expression of ϕ_s using a parametric representation of Γ_s, whenever it is possible. This will allow us to give a functional representation of ϕ_s and as a byproduct, we will obtain a domain of definition of the solutions of $L_s - \lambda$, see Sections 6.2, and 6.3.

6.2. Factorization for parametrizable spectral curves

Once the factorization problem over $K(\Gamma_s)$ has been solved, in Algorithm 6.3, what remains is to replace (λ, μ) by a parametric representation $(\chi_1(\tau), \chi_2(\tau))$ of Γ_s. We are not aware of a previous work where a global treatment of the factorization is achieved. This procedure strongly depends on the genus of the algebraic curve Γ_s. We summarize next what are the parametrization possibilities (as far as we know) and emphasize on the algorithmic aspects of the process.

A key point to have a one-parameter form factorization algorithm is to obtain a global parametrization of the spectral curve. How complicated is to obtain a global parametrization depends on the genus of the curve. There are algorithms to compute the genus of an algebraic curve [24]. In the case of rational curves there are algorithms to obtain a global parametrization [24]. For elliptic curves we can define a meromorphic parametrization by means of the Weierstrass $\wp$-function. For all other cases, as far as we know, there are no algorithms to obtain a global parametrization. An affine algebraic curve Γ admits at any point $P \in \Gamma$ a local parametrization in the field of Puiseux series, see for instance [24], Section 2.5 but in this paper we would like to talk only about the global treatment of the curve.

If an affine algebraic curve Γ in $\mathbb{C}^2$ is rational (has genus zero) then Γ can be parametrized by rational functions. Let $\aleph(\tau) = (\chi_1(\tau), \chi_2(\tau))$ in $C(\tau)^2$ be a (global) parametrization of Γ, that is:

1. For all $\tau_0 \in C$, but a finite number of exceptions, the point $(\chi_1(\tau_0), \chi_2(\tau_0))$ is on Γ, and
2. for all $(\lambda_0, \mu_0) \in \Gamma$, but a finite number of exceptions, there exists $\tau_0 \in C$ such that $(\lambda_0, \mu_0) = (\chi_1(\tau_0), \chi_2(\tau_0))$.

A rational parametrization $\aleph(\tau)$ of Γ gives an isomorphism from $C(\Gamma)$ to the field of rational functions $C(\tau)$, see [24], Section 4.1. This can be extended to an isomorphism $K(\Gamma) \simeq K(\tau)$. More over, $K(\tau)$ is isomorphic to the fraction field $\mathcal{F} = K(\chi_1(\tau), \chi_2(\tau))$ of the polynomial ring $K[\chi_1(\tau), \chi_2(\tau)]$. Since τ is an algebraic indeterminate over K, by condition 2, it is natural to assume that $\partial(\chi_1(\tau)) = 0$ and $\partial(\chi_2(\tau)) = 0$, which allows to extend the derivation ∂ of K to have a di erential field $(\mathcal{F}, \partial)$.

We define $\tilde{\phi}_s := \rho(\phi_s)$. Observe that $\tilde{\phi}_s$ is a nonzero element of $\mathcal{F}$ since by Lemma 6.2 ϕ_s is nonzero in $K(\Gamma_s)$. We have naturally an isomorphism ρ between the rings of di erential operators $\varrho : K(\Gamma_s)[\partial] \longrightarrow \mathcal{F}[\partial]$ as follows:

$$\varrho\left(\left(\sum_j \left(a_j \partial^j\right)\right)\right) = \sum_j \left(\rho(a_j)\partial^j.\right.$$

For instance $\varrho(L_s - \lambda) = L_s - \chi_1(\tau)$ and $\varrho(\partial - \phi_s) = \partial - \tilde{\phi}_s$. Furthermore, since the isomorphism respects the ring structure, we have

$$L_s - \chi_1(\tau) = (-\partial - \tilde{\phi}_s)(\partial - \tilde{\phi}_s)$$

where $\tilde{\phi}_s$ is a solution of the Ricatti equation $\partial(\phi)+\phi^2 = u_s - \chi_s(\tau)$, since ρ respects the di erential field structure.

6.3. Factors at smooth points of Γ_s

So far in this paper λ and μ were algebraic variables over K, furthermore $\partial\lambda = 0$ and $\partial\mu = 0$. In this section we will talk about the specialization process of (λ, μ) to a point $P_0 = (\lambda_0, \mu_0)$ of the spectral curve Γ_s. In this manner we recover the classical factorization problem of $L_s - \lambda_0$ as an operator in $K[\partial]$, see for instance [14], [1].

Proposition 6.5. *Given $P_0 = (\lambda_0, \mu_0)$ in Γ_s the di erential operators $L_s - \lambda_0$ and $A_{2s+1} - \mu_0$ have a common factor over K. Furthermore*

$$L_s - \lambda_0 = (-\partial - \phi_0)(\partial - \phi_0) \tag{28}$$

where $\phi_0 = \phi_s(P_0)$ with ϕ_s as in (27) and

$$\phi_0 = -\frac{\varphi_1(P_0)}{\varphi_2(P_0)} = \frac{\mu_0 + \alpha(\lambda_0)}{\varphi_2(\lambda_0)} \tag{29}$$

with $\varphi_2(\lambda_0) \neq 0$.

Proof. By Proposition 3.4, 2, the di erential operators $L_s - \lambda_0$ and $A_{2s+1} - \mu_0$ in $K[\partial]$ have a common factor since

$$\partial \mathrm{Res}(L_s - \lambda_0, A_{2s+1} - \mu_0) = f_s(\lambda_0, \mu_0) = 0.$$

With the notation of Lemma 6.2, observe that $\varphi_1(P_0) + \varphi_2(P_0)\partial$ is the subresultant of $L_s - \lambda_0$ and $A_{2s+1} - \mu_0$ as in Section 3.2.2. We will prove next that $\mathcal{L}_1$ is an operator of order one and then by Theorem 3.6, we have the factorization

$$L_s - \lambda_0 = (-\partial - \phi_0)(\partial - \phi_0),$$

where $\phi_0 = \phi_s(P_0)$ and the given formula follows by Lemma 6.2.

Let us suppose that $\mathcal{L}_1$ is the zero operator. Then the second subresultant $\mathcal{L}_2$ equals to $L_s - \lambda_0$. Hence $L_s - \lambda_0$ is a factor of $A_{2s+1} - \mu_0$. That is

$$A_{2s+1} - \mu_0 = Q(L_s - \lambda_0)$$

for some monic di erential operator Q of order $2s - 1$ in $K[\partial]$. Computing the commutator with L_s we obtain

$$0 = [A_{2s+1} - \mu_0, L_s] = [QL_s, L_s] - [\lambda_0, L_s] = [Q, L_s]L_s.$$

Since $K[\partial]$ is a domain $[Q, L_s] = 0$ and Q belong to the centralizer of L_s in $K[\partial]$, which contradicts Theorem 5.2 since Q has even order less than $2s + 1$. We have proved that $\mathcal{L}_1$ is an operator of order one, in other words $\varphi_2(\lambda_0) \neq 0$. □

Remark 6.6. *Observe that if $\phi_0 = 0$ then, due to the Ricatti equation, u_s is the constant potential λ_0, and conversely. From now on we will assume that u_s is not a constant potential.*

We must distinguish two di erent types of points in the curve, the ones with $\mu_0 \neq 0$ and those with $\mu_0 = 0$, that is the finite set

$$Z_s = \Gamma_s \cap (C \times \{0\}) = \{(\lambda, 0) \mid R_{2s+1}(\lambda) = 0\}.$$

Observe that Z_s contains all the affine singular points of Γ_s.

For a given point $P_0 = (\lambda_0, \mu_0) \in C^2$ of the curve Γ_s, we will assume $\mu_0 \neq 0$ from now on. Let us consider ϕ_0 as in (29), in this section we will use the following notation

$$\phi_{0+} = \phi_0 = \frac{\mu_0 + \alpha(\lambda_0)}{\varphi_2(\lambda_0)} \text{ and } \phi_{0-} = \frac{-\mu_0 + \alpha(\lambda_0)}{\varphi_2(\lambda_0)}, \tag{30}$$

pointing out that $\phi_{0+} \neq \phi_{0-}$ since $\mu_0 \neq 0$. Applying Proposition 6.5 to the point $(\lambda_0, -\mu_0)$ we obtain the following factorization of $L_s - \lambda_0$

$$L_s - \lambda_0 = (-\partial - \phi_{0-})(\partial - \phi_{0-}).$$

Let us consider nonzero solutions $_{0+}$ and $_{0-}$ respectively of the di erential equations

$$\partial(\ ) = \phi_{0+}\ \ \text{ and } \partial(\ ) = \phi_{0-}\ \ . \tag{31}$$

Then the equality

$$\frac{w(\ _{0+},\ _{0-})}{_{0+}\ _{0-}} = \phi_{0+} - \phi_{0-} = \frac{2}{\varphi_2(\lambda_0)}\mu_0 \neq 0.$$

implies that $W_0 = w(\ _{0+},\ _{0-}) \neq 0$ in C. Therefore $\{\ _{0+},\ _{0-}\}$ is a fundamental set of solutions of $(L_s - \lambda_0)(\ ) = 0$. Moreover

$$_{0+}\ _{0-} = \frac{\varphi_2(\lambda_0)W_0}{2\mu_0} \in K,$$

hence

$$K\langle\ _{0+},\ _{0-}\rangle = K\langle\ _{0+}\rangle. \tag{32}$$

In the next section we will show by means of examples the type of factors that may appear depending on the type of curve. Even at each smooth point of the spectral curve the field $K\langle\ _{0+}\rangle$ can be very complicated. These situations deserve a more detailed study that we will present in a future work.

7. Schrödinger operators for KdV solitons. Computed examples

Our algorithms 4.7 ,for computation of constants, and 6.3, for the factorization of the Schrödinger operator, are now ready to be implemented with any symbolic computation software, we did it in Maple 18. We will illustrate their performance by means of three well known families of potentials in [31]. The first one is a family of rational potentials, the second one is a family of Rosen-Morse potentials and both are degenerate cases of a third family of hyperelliptic potentials.

Rational	Rosen-Morse	Elliptic
$u_s = \dfrac{s(s+1)}{x^2}$	$u_s = \dfrac{-s(s+1)}{\cosh^2(x)}$	$u_s = s(s+1)\wp(x; g_2, g_3)$

We will factor $L_s - \lambda$, with $L_s = -\partial^2 + u_s$, as an operator in $K(\Gamma_s)[\partial]$, where Γ_s is the spectral curve of L_s.

7.1. Rational KdV solitons

Let us consider the family of rational potentials $u_s = s(s+1)/x^2$, $s \geq 1$, in $K = \mathbb{C}(x)$ with $\partial = d/dx$. It is well known that the KdV level of u_s is s and its basic constants vector $\bar{c}^s = (0, \ldots, 0)$, we checked this result using Algorithm 4.7.

The spectral curve Γ_s is defined by the polynomial $f_s = \mu^2 + \lambda^{2s+1}$. We computed the factor $\partial - \phi_s$ of $L_s - \lambda$ in $K(\Gamma_s)[\partial]$ using Algorithm 6.3. For $s = 1, 2, 3$ the results obtained coincides with the ones in [14], Example 1.30. We show our result for the next level $s = 4$:

$$\phi_4 = -\frac{-\mu\, x^9 + 10\,\lambda^3 x^6 + 270\,\lambda^2 x^4 + 4725\,\lambda\, x^2 + 44100}{x\,(\lambda^4 x^8 + 10\,\lambda^3 x^6 + 135\,\lambda^2 x^4 + 1575\,\lambda\, x^2 + 11025)}.$$

Then, we obtain the factorization:

$$L_4 - \lambda = (-\partial - \phi_4)(\partial - \phi_4)$$

in $K(\Gamma_4)[\partial]$ where $K(\Gamma_4)$ is the fraction field of the domain $K[\lambda, \mu]/(\mu^2 + \lambda^9)$.

Next we observe that the curves Γ_s have all genus zero and a global parametrization is

$$\aleph_s(\tau) = (\chi_1(\tau), \chi_2(\tau)) = \left(-\tau^2, -\tau^{2s+1}\right).$$

Following Section 6.2, the one-parameter form of the factor $\partial - \tilde{\phi}_4$ of

$$L_4 - \chi_1(\tau) = -\partial^2 + \frac{20}{x^2} + \tau^2$$

is given by

$$\tilde{\phi}_4(x, \tau) = -\frac{\tau^9 x^9 - 10\,\tau^6 x^6 + 270\,\tau^4 x^4 - 4725\,\tau^2 x^2 + 44100}{x\,(\tau^8 x^8 - 10\,\tau^6 x^6 + 135\,\tau^4 x^4 - 1575\,\tau^2 x^2 + 11025)}.$$

Then, we obtain the global factorization:

$$L_4 - \chi_1(\tau) = (-\partial - \tilde{\phi}_4)(\partial - \tilde{\phi}_4)$$

in $K(\tau)[\partial] = \mathbb{C}(x, \tau)[\partial]$. A factorization using a global parametrization of the spectral curves is our main contribution to the study of this family of potentials.

7.2. Rosen-Morse KdV solitons

Let us consider the family of Rosen-Morse potentials $u_s = \frac{-s(s+1)}{\cosh^2(x)}$, $s \geq 1$, in the di erential field $K = \mathbb{C}(e^x) = \mathbb{C}\langle \cosh(x) \rangle$ with $\partial = d/dx$.

We show how to obtain the basic constants vector $\bar{c}^s$ for level $s = 3$ using Algorithm 4.7. We observe that u_s belongs to $C(\eta)$ with $\eta = \cosh(x)$ and that $(\eta')^2 = \eta^2 - 1$, thus the hypothesis of the algorithm hold. For the first three iterations of the algorithm, the system $\mathcal{S}_n$, $n = 0, 1, 2$ has no solution. In fact, we have

$$\mathbf{KdV}_0(u_3) = -24\frac{\eta'}{\eta^3}, \quad \mathbf{KdV}_1(u_3, c^1) = -24\frac{\eta'}{\eta^5}\left(-15 + (-1 + c_1)\eta^2\right),$$
$$\mathbf{KdV}_2(u_3, c^2) = 12\frac{\eta'}{\eta^7}\left(-225 + (30c_1 - 150)\eta^2 + (-2c_2 + 2c_1 - 2)\eta^4\right).$$

Thus $\mathbf{KdV}_0(u_3) \neq 0$ and $\mathbf{KdV}_n(u_3, \bar{c}^n) \neq 0$ for all $\bar{c}^n \in \mathbb{C}^n$, $n = 1, 2$. For $n = 3$ we obtain

$$\begin{aligned}\mathbf{KdV}_3(u_3, c^3) =& \eta' \frac{p_3(\eta)}{q_3(\eta)} \\ =& -12\frac{\eta'}{\eta^7}\left(-3150 + 225c_1 + (150c_1 - 630 - 30c_2)\eta^2 + (-2c_2 + 2c_1 + 2c_3 - 2)\eta^4\right).\end{aligned}$$

From the coefficients in η of $p_3(\eta)$ we obtain the triangular system

$$\mathcal{S}_3 = \{-3150 + 225c_1 = 0,\ \ 150c_1 - 630 - 30c_2 = 0,\ \ -2c_2 + 2c_1 + 2c_3 - 2 = 0\}.$$

The unique solution of this system is the basic constant vector $\bar{c}^3 = (14, 49, 36)$. Then, u_3 is a solution of the di erential equation

$$\mathbf{KdV}_3(u, \bar{c}^3) = \mathrm{kdv}_3 + 14\mathrm{kdv}_2 + 49\mathrm{kdv}_1 + 36\mathrm{kdv}_0 = 0.$$

The defining polynomial f_s of Γ_s is known to be equal to $f_s = \mu^2 + \lambda^2 \prod_{\kappa=1}^{s}(\lambda + \kappa^2)^2$, see for instance [14], Example 1.31. We checked these results using our implementation of the di erential resultant $\partial\mathrm{Res}(L_s - \lambda, A_{2s+1} - \mu)$.

The next table shows the level s, the basic constant vector $\bar{c}^s$, computed with Algorithm 4.7, and the computation of the factor $\partial - \phi_s$ using the Factorization Algorithm 6.3 for the operator $L_s - \lambda$ in $K(\Gamma_s)[\partial]$:

s	$\bar{c}^s$	ϕ_s
1	(1)	$\dfrac{\mu\cosh(x)^3 + \sinh(x)}{\cosh(x)(\lambda\cosh(x)^2 + \cosh(x)^2 - 1)}$
2	$(5, 4)$	$\dfrac{\mu\cosh(x)^5 + 3\cosh(x)^2\sinh(x)\lambda + 12\sinh(x)\cosh(x)^2 - 18\sinh(x)}{(\cosh(x)^4\lambda^2 + 5\cosh(x)^4\lambda + 4\cosh(x)^4 - 3\lambda\cosh(x)^2 - 12\cosh(x)^2 + 9)\cosh(x)}$
3	$(14, 49, 36)$	$\dfrac{\mu + \alpha(\lambda)}{\varphi(\lambda)}$

with

$$\alpha = \frac{6\cosh(x)^4\sinh(x)\lambda^2 + 78\cosh(x)^4\sinh(x)\lambda - 90\cosh(x)^2\sinh(x)\lambda + a}{\cosh(x)^7},$$

$$a = 27\sinh(x)(8\cosh(x)^4 - 30\cosh(x)^2 + 25),$$

$$\varphi = \frac{\cosh(x)^6\lambda^3 + 14\cosh(x)^6\lambda^2 + 49\cosh(x)^6\lambda - 6\cosh(x)^4\lambda^2 - 78\cosh(x)^4\lambda + 45\cosh(x)^2\lambda + b}{\cosh(x)^6},$$

$$b = 9\sinh(x)^2(4\cosh(x)^4 - 20\cosh(x)^2 + 25).$$

All the curves Γ_s for this family are rational, in particular they admit a polynomial global parametrization

$$\aleph_s(\tau) = (\chi_1(\tau), \chi_2(\tau)) = \left(-\tau^2, -\tau\prod_{\kappa=1}^{s}\left(\tau^2 - \kappa^2\right)\right). \tag{33}$$

The next table shows $\tilde{\phi}_s$:

s	$\tilde{\phi}_s$
1	$\dfrac{\left(\tau^2-\tau\right)w^2+\left(2\,\tau^2-4\right)w+\tau^2+\tau}{((\tau-1)\,w+\tau+1)\,(w+1)}$
2	$\dfrac{a_3(\tau)w^3+b_2(\tau)w^2+a_1(\tau)w+a_0(\tau)}{(b_2(\tau)w^2+b_1(\tau)w+b_0(\tau))\,(w+1)}$
3	$\dfrac{c_4(\tau)w^4+c_3(\tau)w^3+c_2(\tau)w^2+c_1(\tau)w+c_0(\tau)}{(d_3(\tau)w^3+d_2(\tau)w^2+d_1(\tau)w+d_0(\tau))\,(w+1)}$

where $w = e^{2x}$,

$$a_3 = -\tau^3 - 3\,\tau^2 - 2\,\tau,\ \ a_2 = -3\,\tau^3 - 3\,\tau^2 + 18\,\tau + 24,\ \ a_1 = -3\,\tau^3 + 3\,\tau^2 + 18\,\tau - 24,$$
$$a_0 = -\tau^3 + 3\,\tau^2 - 2\,\tau,\ \ b_2 = \tau^2 + 3\,\tau + 2,\ \ b_1 = 2\,\tau^2 - 8,\ \ b_0 = \tau^2 - 3\,\tau + 2.$$

and

$$c_4 = \tau^4 - 6\,\tau^3 + 11\,\tau^2 - 6\,\tau,\ \ c_3 = 4\,\tau^4 - 12\,\tau^3 - 40\,\tau^2 + 168\,\tau - 144,\ \ c_2 = 6\,\tau^4 - 102\,\tau^2 + 432,$$
$$c_1 = 4\,\tau^4 + 12\,\tau^3 - 40\,\tau^2 - 168\,\tau - 144,\ \ c_0 = \tau^4 + 6\,\tau^3 + 11\,\tau^2 + 6\,\tau,\ \ d_3 = \tau^3 - 6\,\tau^2 + 11\,\tau - 6,$$
$$d_2 = 3\,\tau^3 - 6\,\tau^2 - 27\,\tau + 54,\ \ d_1 = 3\,\tau^3 + 6\,\tau^2 - 27\,\tau - 54,\ \ d_0 = \tau^3 + 6\,\tau^2 + 11\,\tau + 6.$$

Hence, we obtain the global factorization by means of the global parametrization (33):

$$L_s - \chi_1(\tau) = (-\partial - \tilde{\phi}_s)(\partial - \tilde{\phi}_s)$$

in $K(\tau)[\partial] = \mathbb{C}(e^x, \tau)[\partial]$. These factorizations, using a global parametrization of the spectral curves for this family of potentials, are new as far as we know.

7.3. Elliptic and Hyperelliptic KdV solitons

Next we consider the family of elliptic potentials $u_s = s(s+1)\wp(x; g_2, g_3)$, $s \geq 1$, where $\wp$ is the Weierstrass $\wp$-function for g_2, g_3, satisfying $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$. In this case $K = \mathbb{C}\langle\wp\rangle = \mathbb{C}(\wp, \wp')$ with $\partial = d/dx$.

The requirements of Algorithm 4.7 are satisfied since u_s belongs to $\mathbb{C}(\eta)$ for $\eta = \wp$ and $(\eta')^2 = 4\eta^3 - g_2\eta - g_3 \in C(\eta)$. Thus we used Algorithm 4.7 to compute $\bar{c}^s$. For $s = 1, 2$ we could check that the results obtained coincide with the ones in [14], Example 1.32.

Next, we show our computations for $s = 3$. Using Algorithm 4.7, we checked that $\mathbf{KdV}_0(u_3) \neq 0$ and $\mathbf{KdV}_n(u_3, \bar{c}^n) \neq 0$ for all $\bar{c}^n \in \mathbb{C}^n$, $n = 1, 2$. From

$$\begin{aligned}\mathbf{KdV}_3(u_3, c^3) =& \eta' \frac{p_3(\eta)}{q_3(\eta)}\\ =& \eta' \left((-5670g_2 - 360c_2)\eta^2 - 2700c_1\eta - 153g_2c_1 - 1782g_3 - 24c_3\right)\end{aligned}$$

we obtain the triangular linear system in c_1, c_2 and c_3

$$\mathcal{S}_3 = \{-5670g_2 - 360c_2 = 0,\ \ 2700c_1 = 0,\ \ -153g_2c_1 - 1782g_3 - 24c_3 = 0\},$$

whose unique solution is $\bar{c}^3 = (0, -63g_2/4, -297g_3/4)$. Then, u_3 is a solution of the di erential equation

$$\mathbf{KdV}_3(u,\bar{c}^3) = \mathrm{kdv}_3 - \frac{63g_2}{4}\mathrm{kdv}_1 - \frac{297g_3}{4}\mathrm{kdv}_0 = 0.$$

Then, we compute the defining polynomial f_s of Γ_s with our implementation of the di erential resultant $\partial\mathrm{Res}(L_s - \lambda, A_{2s+1} - \mu)$. Here we obtain the polynomial $f_3(\lambda,\mu) = \mu^2 + R_7(\lambda)$ where

$$R_7 = \frac{1}{16}\lambda(-16\lambda^6 + 504g_2\lambda^4 + 2376g_3\lambda^3 - 4185g_2^2\lambda^2 + 3375g_2^3 - 36450g_2g_3\lambda - 91125g_3^2).$$

Using Algorithm 6.3, we computed the factor $\partial - \phi_s$ of the operator $L_s - \lambda$ in $K(\Gamma_s)[\partial]$. For $s = 1, 2$ the results coincide with the ones obtained in [14], Example 1.32. We show here

$$\phi_3 = \frac{\mu + \wp'\left(\frac{675}{2}\wp^2 - \frac{225}{8}g_2 + 45\wp\lambda + 3\lambda^2\right)}{\lambda^3 + 6\wp\lambda^2 + (45\wp^2 - 15g_2)\lambda - 225\wp'^2}$$

where $\wp$ and $\wp'$ denote $\wp(x; g_2, g_3)$ and $\wp'(x, g_2, g_3)$ respectively. Then, we obtain the factorization:

$$L_3 - \lambda = (-\partial - \phi_3)(\partial - \phi_3)$$

in $K(\Gamma_3)[\partial]$ where $K(\Gamma_3)$ is the fraction field of the domain $K[\lambda,\mu]/(\mu^2 + R_7(\lambda))$.

It is well known that the curves Γ_s for this family are not rational, they have genus s. In the case of the elliptic potential $u_1 = 2\wp(x; g2, g3)$ one can easily prove that $\aleph_1(\tau) = \left(-\wp(\tau), \frac{1}{2}\wp'(\tau)\right)$ is a global parametrization of the spectral curve Γ_1 whose defining polynomial is the irreducible polynomial $f_1 = -\mu^2 - \lambda^3 + (1/4)g_2\lambda - (1/4)g_3$. In this case

$$\tilde{\phi}_1 = \frac{\frac{-1}{2}(\wp'(x) - \wp'(\tau))}{\wp(x) - \wp(\tau)}.$$

Hence, we obtain the global factorization by means of the given global parametrization:

$$L_1 + \wp(\tau) = (-\partial - \tilde{\phi}_1)(\partial - \tilde{\phi}_1)$$

in $\mathbb{C}\langle\wp(x), \wp(\tau)\rangle[\partial]$. For $s \geq 2$, as far as we know there are no e ective algorithms to compute a global parametrization $(\chi_1(\tau), \chi_2(\tau))$ of Γ_s. This is a difficult open problem. Some contributions have been made in this direction, for instance by Y.V. Brezhnev in [4].

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