LIMITS FOR EMBEDDING DISTRIBUTIONS

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ABSTRACT. In this paper, we first establish a central limit theorem which is new in probability, then we find and prove that, under some conditions, the embedding distributions of H-linear family of graphs with spiders are asymptotic normal distributions. As corollaries, the asymptotic normality for the embedding distributions of path-like sequence of graphs with spiders and the genus distributions of ladder-like sequence of graphs are given. We also prove that the limit of Euler-genus distributions is the same as that of crosscap-number distributions. The results here can been seen a version of central limit theorem in topological graph theory.

1. INTRODUCTION

A *graph* is a pair $G = (V, E)$, where $V = V(G)$ is the set of vertices, and $E = E(G)$ is the set of edges. In topological graph theory, a graph is permitted to have both loops and multiple edges. A *surface* S is a compact connected 2-dimensional manifold without boundary. The *orientable surface* $O_k (k \geq 0)$ can be obtained from a sphere with k handles attached, where k is called the *genus* of O_k, and the *non-orientable surface* $N_j (j \geq 1)$ with j crosscaps, where j is called the *crosscap-number* of N_j. The *Euler-genus* γ^E of a surface S is given by

$$\gamma^E = \begin{cases} 2k, & \text{if } S = O_k, \\ j, & \text{if } S = N_j. \end{cases}$$

We use S_i to denote the surface S with Euler-genus i, for $i \geq 0$.

A graph G is *embeddable* into a surface S if it can be drawn in the surface such that any edge does not pass through any vertex and any two edges do not cross each other. If G is embedded on the surface S, then the components of $S - G$ are the *faces* of the embedding. A graph embedding is called a *2-cell cellular embedding* if any simple closed curve in that face can be continuously deformed or contracted in that face to a single point. All graph embeddings in the paper are 2-cell cellular embeddings.

A *rotation at a vertex* v of a graph G is a cyclic ordering of the edge-ends incident at v. A *(pure) rotation system* ρ of a graph G is an assignment of a rotation at every vertex of G. A *general rotation system* for a graph G is a pair (ρ, λ), where ρ is a rotation system and λ is a map on $E(G)$ with values in $\{0, 1\}$. If $\lambda(e) = 1$, then the edge e is said to be *twisted*; otherwise $\lambda(e) = 0$, and we call the edge e *untwisted*. If $\lambda(e) = 0$, for all $e \in E(G)$, then the general rotation system (ρ, λ) is a pure rotation system. It is well-known that any graph embedding can be described by a general rotation system. Let T be a spanning tree of G, a *T-rotation system* (ρ, λ) of G is a general rotation system (ρ, λ) such that $\lambda(e) = 0$, for every edge $e \in E(T)$. For a

fixed spanning tree T, two embeddings of G are considered to be *equivalent* if their T-rotation systems are combinatorially equivalent. It is known that there is a sequence of vertex-flips that transforms a general rotation system into a T-rotation system.

The number of *(distinct) cellular embeddings* of a graph G on the surfaces O_k, N_j, and S_i are denoted by $\gamma_k(G), \tilde{\gamma}_j(G)$, and $\varepsilon_i(G)$, respectively. By the *genus distribution* of a graph G we mean the sequence

$$\gamma_0(G), \gamma_1(G), \gamma_2(G), \cdots,$$

and the *genus polynomial* of G is

$$\Gamma_G(x) = \sum_{k=0}^{\infty} \gamma_k(G) x^k.$$

Similarly, we have *the crosscap-number distribution* $\{\tilde{\gamma}_i(G)\}_{n=1}^{\infty}$ and *the Euler-genus distribution* $\{\varepsilon_i(G)\}_{n=1}^{\infty}$. The *crosscap-number polynomial* $\tilde{\Gamma}_G(x)$ and the *Euler-genus polynomial* $\mathcal{E}_G(x)$ of G are defined analogously.

For a deeper discussion of the above concepts, we may refer the reader to [3, 4]. The following assumption will be needed throughout the paper. When we say *embedding distribution* of a graph G, we mean its genus distribution, crosscap-number distribution or Euler-genus distribution.

Usually, the embedding distribution of a graph G with tractable size can been calculated explicitly, we still concern the global feature of the embedding distribution of G. For example: **(1)** *Log-concavity.* For this aspect, we refer to [10, 12, 13, 14, 29] etc. **(2)** *Average genus, average crosscap-number* and *average Euler-genus.* The *average genus* of graph G is given by

$$\gamma_{avg}(G) \quad = \quad \frac{\Gamma'_G(1)}{\Gamma_G(1)} = \sum_{k=0}^{\infty} \frac{k \cdot \gamma_k(G)}{\Gamma_G(1)},$$

the *average crosscap-number* $\tilde{\gamma}_{avg}(G)$ and *average Euler-genus* $\varepsilon_{avg}(G)$ of a graph G is similarly defined. The study of average genus, average crosscap-number and average Euler-genus received many attentions in topological graph theory. For researches on this aspect, one can see [2, 26, 33] etc. There is also a notation of *variance.* For example, see that in [30]. The *variance* of the genus distribution of the graph G is given by

$$\gamma_{var}(G) = \sum_{k=0}^{\infty} \left(k - \gamma_{avg}(G)\right)^2 \cdot \frac{\gamma_k(G)}{\Gamma_G(1)}.$$

We define the variance of crosscap-number distribution $\tilde{\gamma}_{var}(G)$ and of Euler-genus distribution $\varepsilon_{var}(G)$ similarly.

The motivation of this article is as follows. Let $\{G_n^{\circ}\}_{n=1}^{\infty}$ be a sequence of linear family of graphs with spiders whose definition is given in subsection 3.1, and we denote the embedding distribution of graph G_n° by $\{p_i(n)\}_{i=0}^{\infty}$. The normalized sequence of $\{p_i(n)\}_{i=0}^{\infty}$ is

$$\frac{p_i(n)}{\sum_{k=0}^{\infty} p_k(n)}, \quad i = 0, 1, \cdots.$$

Then, the above sequence is a distribution in probability, we denote it by F_n. One problem appears, when n is big enough, whether the distribution F_n will look like some well-known distribution in probability. If the answer is yes, then it demonstrates the outline of embedding distribution for graph G_n° when n is big enough. In the point of mathematics, this is to seek the limit for F_n or the embedding distribution of graph G_n°.

In this paper, we make researches on the embedding distributions which are closely related to the above problem. Under some weak conditions, we show the embedding distributions (genus, crosscap-number or Euler-genus distributions) of G_n° are *asymptotically normal distribution* when n tends to infinity. We say the embedding distributions (genus, crosscap-number or Euler-genus distributions) of G_n° are *asymptotically normal distribution* with mean μ_n and variance σ_n^2 if

$$\lim_{n\to\infty} \sup_x \left| \sum_{i \le \sigma_n x + \mu_n} p_i(n) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2} \mathrm{d}t \right| = 0.$$

Since normal distributions have many very good properties, the genus distributions (crosscap-number or Euler-genus distributions) of G_n° also have many good properties when n is big enough. Such as: (1) *Symmetry*, normal distributions are symmetric around their mean. (2) *Normal distributions are defined by two parameters, the mean μ and the standard deviation σ.* Approximately 95% of the area of a normal distribution is within two standard deviations of the mean. This implies that the genus distributions of G_n° are mainly concentrated on the interval $(\gamma_{avg}(G_n^\circ) - 2\sqrt{\gamma_{var}(G_n^\circ)}, \gamma_{avg}(G_n^\circ) + 2\sqrt{\gamma_{var}(G_n^\circ)}]$ when n is big enough. Similar results also hold for crosscap-number and Euler-genus distributions. We also show that the genus distributions (crosscap-number or Euler-genus distributions) of G_n° are not always asymptotically normal distribution. This is the first time someone prove the embedding distributions of some families of graphs are asymptotically normal.

In Section 2, we establish a central limit theorem which is also new in probability. In Section 3, we apply this central limit theorem to the embedding distributions of G_n° and give their limits. In Section 4, some examples are demonstrated.

2. A central limit theorem

For a non-negative integer sequence $\{p_i(n)\}_{i=0}^\infty$, let $P_n(x) = \sum_{i=0}^\infty p_i(n)x^i, x \in \mathbb{R}$. In this section, we always assume $P_n(x)$ satisfies a kth-order homogeneous linear recurrence relation

$$(2.1) \qquad P_n(x) = b_1(x)P_{n-1}(x) + b_2(x)P_{n-2}(x) + \cdots + b_k(x)P_{n-k}(x),$$

where $b_j(x)(1 \le j \le k)$ are polynomials with integer coefficients. We define a polynomial associated with (2.1)

$$(2.2) \qquad F(x,\lambda) = \lambda^k - b_1(x)\lambda^{k-1} - b_2(x)\lambda^{k-2} \cdots - b_{k-1}(x)\lambda - b_k(x).$$

Obviously, for any $x \in \mathbb{R}$, there exist some $r = r(x) \in \mathbb{N}$, $m_1(x), \cdots, m_r(x) \in \mathbb{N}$ with $\sum_{i=1}^r m_i(x) = k$, and numbers $\lambda_i(x), i = 1, \cdots, r$ with $|\lambda_1(x)| \ge |\lambda_2(x)| \ge |\lambda_3(x)| \ge \cdots |\lambda_r(x)|$ such that

$$(2.3) \qquad F(x,\lambda) = (\lambda - \lambda_1(x))^{m_1(x)} \cdots (\lambda - \lambda_r(x)^{m_r(x)}.$$

And the general solution to (2.1) is given by

$$(2.4) \qquad P_n(x) = \sum_{i=1}^r \lambda_i^n(x)\big(a_{i,0}(x) + a_{i,1}(x)n + \cdots + a_{i,m_i(x)-1}(x)n^{m_i(x)-1}\big).$$

Let

$$(2.5) \qquad e = \frac{\lambda_1'(1)}{D}, \quad v = \frac{-\big(\lambda_1'(1)\big)^2 + D \cdot \lambda_1''(1) + D \cdot \lambda_1'(1)}{D^2}.$$

where $D = \lambda_1(1)$. For any $n \in \mathbb{N}$, let X_n be a random variable with distribution

$$\mathbb{P}(X_n = i) = \frac{p_i(n)}{P_n(1)}, \quad i = 0, 1, \cdots,$$

The remainder of this section is devoted to the proof of the following theorem.

Theorem 2.1. *Let $P_n(x) = \sum_{i=0}^{\infty} p_i(n)x^i, n \geq k+1$ be polynomials satisfying (2.1). At $x = 1$, suppose the multiplicity of maximal root for polynomial (2.2) is 1. Then the following results hold depending on the value of v.*

Case I: $v > 0$. *The law of X_n is asymptotically normal with mean $e \cdot n$ and variance $v \cdot n$ when n tends to infinity. That is,*

$$\lim_{n \to \infty} \sup_{x \in \mathbb{R}} \left| \mathbb{P}\left(\frac{X_n - e \cdot n}{\sqrt{v \cdot n}} \leq x \right) - \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} \mathrm{d}u \right| = 0.$$

In particular, we have

$$\lim_{n \to \infty} \sup_{x \in \mathbb{R}} \left| \frac{1}{P_n(1)} \sum_{0 \leq i \leq x\sqrt{v \cdot n} + e \cdot n]} p_i(n) - \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} \mathrm{d}u \right| = 0.$$

Case II: $v < 0$. *This case is impossible to appear.*

Case III: $v = 0$. *For any $\alpha > \frac{1}{3}$, the law of $\frac{X_n - e \cdot n}{n^\alpha}$ is asymptotically one-point distribution concentrated at 0. In more accurate words, the following holds.*

$$\lim_{n \to \infty} \mathbb{P}\left(\frac{X_n - e \cdot n}{n^\alpha} \leq x \right) = \begin{cases} 1, & \text{if } x \geq 0, \\ 0, & \text{else.} \end{cases}$$

Furthermore, if all these functions $b_1(x), \cdots, b_k(x)$ are constant, then the limits of the law of X_n is a discrete distribution. That is, for some $\kappa \in \mathbb{N}$ and $\omega_j, j = 0, \cdots, \kappa$ with $\sum_{j=1}^{\kappa} \omega_j = 1$, we have

$$(2.6) \qquad \lim_{n \to \infty} \mathbb{P}(X_n = j) = \omega_j, j = 0, \cdots, \kappa.$$

Proof. One arrives at that

$$(2.7) \qquad D = \lambda_1(1) > |\lambda_2(1)| \geq |\lambda_3(1)| \geq \cdots |\lambda_r(1)| \text{ and } m_1(1) = 1.$$

By [21] and the smooth of F, for $i = 1, \cdots, r$, $\lambda_i(x)$ are continuous. Thus, for some $\delta > 0$, we have

$$(2.8) \qquad \lambda_1(x) > |\lambda_2(x)| \geq |\lambda_3(x)| \geq \cdots |\lambda_r(x)|, \quad \forall x \in (1 - \delta, 1 + \delta).$$

For $m_1(x)$ and $\lambda_1(x)$, we have the following fact.

Fact: for some $\delta > 0$, we have $\lambda_1(x)$ is smooth on $(1 - \delta, 1 + \delta)$ and

$$(2.9) \qquad m_1(x) = 1, \quad \forall x \in (1 - \delta, 1 + \delta).$$

One easily sees that

$$F(1, \lambda) = (\lambda - \lambda_1(1))(\lambda - \lambda_2(1))^{m_2(1)} \cdots (\lambda - \lambda_r(1))^{m_r(1)}.$$

and

$$(2.10) \qquad \frac{\partial F(x, \lambda)}{\partial \lambda} \bigg|_{x=1, \lambda=\lambda_1(1)} \neq 0.$$

Actually, $\lambda_1(x)$ can be seen an implicit function decided by

$$F(x, \lambda) = 0.$$

By the smooth of F and the implicit function theorem, $\lambda_1(x)$ is smooth on $(1 - \varepsilon, 1 + \varepsilon)$ for some $\varepsilon > 0$. By the smooth of F and (2.10), for some $\varepsilon > 0$, we have

$$\frac{\partial F(x, \lambda)}{\partial \lambda} \neq 0, \quad \forall x \in (1 - \varepsilon, 1 + \varepsilon), \forall \lambda \in (\lambda_1(1) - \varepsilon, \lambda_1(1) + \varepsilon)$$

which yields the desired result (2.9).

Combining (2.7), the general solution to (2.1) is given by

$$
\begin{aligned}
P_n(x) = {}& a(x)\lambda_1^n(x) \\
& + \sum_{i=2}^{r} \lambda_i^n(x)\big(a_{i,0}(x) + a_{i,1}(x)n + \cdots + a_{i,m_i(x)-1}(x)n^{m_i(x)-1}\big).
\end{aligned}
\tag{2.11}
$$

We consider the following three different cases.

Case I: $v > 0$. Let $Y_n = \frac{X_n - en}{\sqrt{vn}}$ and $\phi_{Y_n}(t) = \mathbb{E}e^{\mathrm{i}tY_n}$ be the characteristic function of Y_n, where i is a complex number with $\mathrm{i}^2 = -1$.

In order to prove

$$\lim_{n \to \infty} \sup_{x \in \mathbb{R}} \left| \mathbb{P}(Y_n \leq x) - \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} \mathrm{d}u \right| = 0,$$

by [22, Chapter 1] and the continuity theory([9, Chapter 15]) for characteristic function in probability, we only need to prove

$$\lim_{n \to \infty} \phi_{Y_n}(t) = \lim_{n \to \infty} \mathbb{E}e^{\mathrm{i}\cdot t \frac{X_n - en}{\sqrt{vn}}} = \int_{\mathbb{R}} e^{\mathrm{i}\cdot tu} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} \mathrm{d}u = e^{-\frac{t^2}{2}}, \quad \forall t. \tag{2.12}$$

We will give a proof of this.

Let $a_n = \frac{1}{\sqrt{vn}}, b_n = \sqrt{\frac{e^2 n}{v}}$ and $y = e^{a_n t}$. By these definitions, one easily sees that

$$
\begin{aligned}
\frac{n\lambda_1'(1)}{D}a_n - b_n &= ne\frac{1}{\sqrt{vn}} - \sqrt{\frac{e^2 n}{v}} = 0, \\
n\frac{\lambda_1'(1)D + \lambda_1''(1)D - \lambda'(1)^2}{2D^2}a_n^2 &= n \cdot \frac{v}{2} \cdot \frac{1}{vn} = \frac{1}{2}.
\end{aligned}
\tag{2.13}
$$

By Taylor formula, we have

$$\ln\frac{\lambda_1(y)}{D} = \frac{\lambda_1'(1)}{D}(y - 1) + \frac{1}{2D^2}\big[\lambda_1''(1)D - (\lambda_1'(1))^2\big](y - 1)^2 + o((y-1)^2),$$

and

$$y = 1 + a_n t + \frac{a_n^2 t^2}{2} + o(a_n^2 t^2).$$

Therefore, by $\lim_{n\to\infty} y = 1$, it holds that

$$
\begin{aligned}
\ln\frac{\lambda_1(y)}{D} &= \frac{\lambda_1'(1)}{D}\big(a_n t + \frac{1}{2}a_n^2 t^2 + o(a_n^2 t^2)\big) \\
&\quad + \frac{1}{2D^2}\big[\lambda_1''(1)D - (\lambda_1'(1))^2\big]\big(a_n t + \frac{1}{2}a_n^2 t^2 + o(a_n^2 t^2)\big)^2 \\
&\quad + o\Big(\big(a_n t + \frac{1}{2}a_n^2 t^2 + o(a_n^2 t^2)\big)^2\Big). \\
&= \frac{\lambda_1'(1)}{D}\big(a_n t + \frac{1}{2}a_n^2 t^2\big) + \frac{1}{2D^2}\big[\lambda_1''(1)D - (\lambda_1'(1))^2\big]a_n^2 t^2 + o(\frac{1}{n}),
\end{aligned}
$$

where in the last equality, we have used $a_n^2 t^2 = \frac{t^2}{vn}$. By (2.7) and the above equality, we obtain

$$
\begin{aligned}
\lim_{n\to\infty}\mathbb{E}e^{t\frac{X_n-en}{\sqrt{vn}}} &= \lim_{n\to\infty}\mathbb{E}e^{a_n t X_n - b_n t} = \lim_{n\to\infty}\frac{P_n(e^{a_n t})e^{-b_n t}}{P_n(1)} = \lim_{n\to\infty}\frac{P_n(y)e^{-b_n t}}{a(1)D^n} \\
&= \lim_{n\to\infty}\frac{\lambda_1^n(x)a(y)}{a(1)D^n}e^{-b_n t} = \lim_{n\to\infty}\frac{\lambda_1^n(y)}{D^n}e^{-b_n t} = \lim_{n\to\infty}\exp\{n\ln\frac{\lambda_1(y)}{D} - b_n t\} \\
&= \lim_{n\to\infty}\exp\Big\{\frac{n\lambda_1'(1)}{D}\big(a_n t + \frac{1}{2}a_n^2 t^2\big) + \frac{n}{2D^2}\big[\lambda_1''(1)D - (\lambda_1'(1))^2\big]a_n^2 t^2 - b_n t + n\cdot o(\frac{1}{n})\Big\} \\
&= e^{\frac{t^2}{2}},
\end{aligned}
$$

where in the last equality, we have used (2.5).

In the above equality, we replace t by $\mathrm{i}t$. Then, we get

$$
\lim_{n\to\infty}\mathbb{E}e^{\mathrm{i}\cdot t\frac{X_n-en}{\sqrt{vn}}} = e^{-\frac{t^2}{2}}
$$

which finishes the proof of (2.12).

Case II: $v < 0$. Let $Y_n = \frac{X_n-en}{\sqrt{-vn}}$ and $\phi_{Y_n}(t) = \mathbb{E}e^{\mathrm{i}tY_n}$. Set $a_n = \frac{1}{\sqrt{-vn}}, b_n = \sqrt{\frac{e^2 n}{-v}}$ and $y = e^{a_n t}$.

By Taylor formula, we have

$$
\ln\frac{\lambda_1(y)}{D} = \frac{\lambda_1'(1)}{D}\big(a_n t + \frac{1}{2}a_n^2 t^2\big) + \frac{1}{2D^2}\big[\lambda_1''(1)D - (\lambda_1'(1))^2\big]a_n^2 t^2 + o(\frac{1}{n}).
$$

Then, by (2.7), we have

$$
\begin{aligned}
\lim_{n\to\infty}\mathbb{E}e^{t\frac{X_n-en}{\sqrt{-vn}}} &= \lim_{n\to\infty}\mathbb{E}e^{a_n t X_n - b_n t} = \lim_{n\to\infty}\frac{P_n(e^{a_n t})e^{-b_n t}}{a(1)D^n} = \lim_{n\to\infty}\frac{P_n(y)e^{-b_n t}}{a(1)D^n} \\
&= \lim_{n\to\infty}\frac{\lambda_1^n(y)a(y)}{a(1)D^n}e^{-b_n t} = \lim_{n\to\infty}\frac{\lambda_1^n(y)}{D^n}e^{-b_n t} = \lim_{n\to\infty}\exp\{n\ln\frac{\lambda_1(y)}{D} - b_n t\} \\
&= \lim_{n\to\infty}\exp\Big\{\frac{n\lambda_1'(1)}{D}\big(a_n t + \frac{1}{2}a_n^2 t^2\big) + \frac{n}{2D^2}\big[\lambda_1''(1)D - (\lambda_1'(1))^2\big]a_n^2 t^2 - b_n t + n\cdot o(\frac{1}{n})\Big\} \\
&= e^{-\frac{t^2}{2}}.
\end{aligned}
$$

In the above equality, we replace t by $\mathrm{i}t$ and get

$$
(2.14) \qquad\qquad \lim_{n\to\infty}\mathbb{E}e^{\mathrm{i}\cdot t\frac{X_n-en}{\sqrt{vn}}} = e^{\frac{t^2}{2}}.
$$

On the other hand, by the properties of characteristic function, one sees

$$
\Big|\mathbb{E}e^{\mathrm{i}\cdot t\frac{X_n-en}{\sqrt{vn}}}\Big| \le 1
$$

which contradicts with (2.14) and gives the desired result.

Case III: $v = 0$. Let $Y_n = \frac{X_n - en}{n^\alpha}$, $a_n = n^{-\alpha}$, $b_n = en^{1-\alpha}$ and $y = e^{a_n t}$.

By Taylor formula and $v = 0$, we have

$$
\begin{aligned}
\ln \frac{\lambda_1(y)}{D} &= \frac{\lambda_1'(1)}{D}(y-1) + \frac{1}{2D^2}\left[\lambda_1''(1)D - (\lambda_1'(1))^2\right](y-1)^2 + O((y-1)^3) \\
&= \frac{\lambda_1'(1)}{D}\left(a_n t + \frac{1}{2}a_n^2 t^2\right) + \frac{1}{2D^2}\left[\lambda_1''(1)D - (\lambda_1'(1))^2\right]a_n^2 t^2 + O(n^{-3\alpha}) \\
&= \frac{\lambda_1'(1)}{D}a_n t + O(n^{-3\alpha}) = en^{-\alpha}t + O(n^{-3\alpha}).
\end{aligned}
$$

Therefore,

$$
\lim_{n\to\infty} \exp\{n \ln \frac{\lambda_1(y)}{D} - b_n t\} = \lim_{n\to\infty} \exp\{n \cdot en^{-\alpha}t + n \cdot O(n^{-3\alpha}) - en^{1-\alpha}t\} = 1.
$$

Then, by (2.7), we have

$$
\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}e^{t\frac{X_n - en}{n^\alpha}} = \lim_{n\to\infty} \mathbb{E}e^{ta_n X_n - b_n t} = \lim_{n\to\infty} \frac{P_n(y)e^{-b_n t}}{a(1)D^n} = \lim_{n\to\infty} \frac{P_n(y)e^{-b_n t}}{a(1)D^n} \\
&= \lim_{n\to\infty} \frac{\lambda_1^n(y)a(y)}{a(1)D^n}e^{-b_n t} = \lim_{n\to\infty} \frac{\lambda_1^n(y)}{D^n}e^{-b_n t} = \lim_{n\to\infty} \exp\{n \ln \frac{\lambda_1(y)}{D} - b_n t\} = 1.
\end{aligned}
$$

In the above equality, we replace t by $\mathrm{i}t$ and get

$$
\lim_{n\to\infty} \mathbb{E}e^{\mathrm{i}\cdot t \frac{X_n - en}{n^\alpha}} = 1
$$

which yields the desired result.

Now we give a proof of (2.6). Noting (2.11), we have

$$
P_n(x) = a(x)\lambda_1^n + \sum_{i=2}^{r}(a_{i,0}(x) + a_{i,1}(x)n + \cdots + a_{i,m_i(x)-1}(x)n^{m_i(x)-1})\lambda_i^n,
$$

where $m_i(x) = m_i, i = 2, \cdots, r$ are constants. Since $P_n(x)$ is a polynomial of x and $\lambda_i, i = 1, \cdots, r$ are constant, $a(x)$ is also a polynomial of x. Assume

$$
a(x) = \sum_{j=0}^{\kappa} c_j x^j.
$$

then,

$$
\lim_{n\to\infty} \mathbb{E}e^{tX_n} = \lim_{n\to\infty} \frac{P_n(e^t)}{P_n(1)} = \lim_{n\to\infty} \frac{a(e^t)D^n}{a(1)D^n} = \frac{a(e^t)}{a(1)} = \sum_{j=0}^{\kappa} \omega_j e^{jt},
$$

where $\omega_j = \frac{c_j}{a(1)}$. In the above equality, we replace t by $\mathrm{i}t$ and get

(2.15)
$$
\lim_{n\to\infty} \mathbb{E}e^{\mathrm{i}\cdot tX_n} = \sum_{j=0}^{\kappa} \omega_j e^{\mathrm{i}jt}.
$$

By the continuity theory([9, Chapter 15]) for characteristic function in probability, we obtain the desired result. $\qquad\square$

By (2.6), the condition $v > 0$ is necessary to ensure the asymptotic normality. In the end of this section, we give a remark here to explain the e, v appeared in Theorem 2.1.

Remark 2.2. *In the special case, when $P_n(x) = a(x)\lambda_1^n(x)$, it holds that*

$$\mathbb{E}X_n = \frac{P_n'(1)}{P_n(1)} = e \cdot n + O(1)$$

and

$$Var(X_n) = \frac{P_n''(1) + P_n'(1)}{P_n(1)} - \left(\frac{P_n'(1)}{P_n(1)}\right)^2 = v \cdot n + O(1).$$

Since $\lambda_1(x) > \lambda_i(x), i = 2, \cdots, k$, we can expect that for the $P_n(x)$ in (2.11) and $P_n(x) = a(x)\lambda_1^n(x)$, they have the same asymptotic mean and variance when n tends to infinity.

3. THE LIMITS FOR EMBEDDING DISTRIBUTIONS

In this section, we consider the limit for embedding distributions of H-*linear family of graphs with spiders* $\{G_n^{\circ}\}_{n=1}^{\infty}$. In subsection 3.1, we give a definition of G_n°. In subsection 3.2, we briefly describe the production matrix. Then we give the limit of embedding distribution for graph G_n° in subsection 3.3. Finally, we demonstrate the relation between the limit of crosscap-number distributions and Euler-genus distributions in subsection 3.4.

3.1. H-**linear family of graphs with spiders.** The definition of H-*linear family of graphs with spiders*, gave by Chen and Gross [6], is a generalization of H-linear family of graphs introduced by Stahl [27]. Suppose H is a connected graph. Let $U = \{u_1, \cdots, u_s\}$ and $V = \{v_1, \cdots, v_s\}$ be two disjoint subsets of $V(H)$. For $i = 1, 2, \cdots$, suppose H_i is a copy of H and let $f_i : H \to H_i$ be an isomorphism. For each $i \geq 1$ and $1 \leq j \leq s$, we set $u_{i,j} = f_i(u_j)$ and $v_{i,j} = f_i(v_j)$. An H-*linear family of graphs*, denoted by $\mathcal{G} = \{G_n\}_{n=1}^{\infty}$, is defined as follows:

- $G_1 = H_1$.
- G_n is constructed by G_{n-1} and H_n be amalgamating the vertex $v_{n-1,j}$ of G_{n-1} with the vertex $u_{n,j}$ of H_n for $j = 1, \cdots, s$.
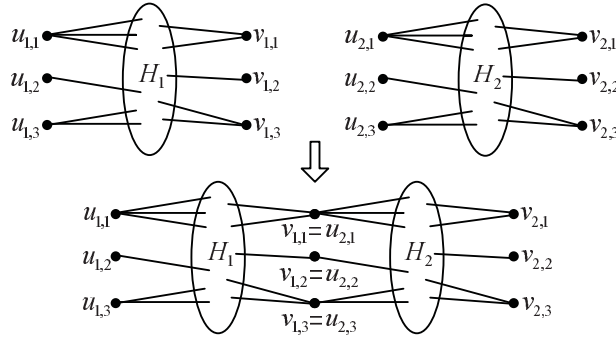

FIGURE 3.1. The graph G_2 in a generic H-linear sequence

Figure 3.1 shows an example for the graphs H_1, H_2 and G_2.

Now, we inroduce the definition of H-linear family of graphs with spiders. For $1 \leq j \leq s$, let $(J_j, t_{j,i'})$ and $(\overline{J_j}, \overline{t_{j,i'}})$ be graphs in which $\{t_{j,i'}\}, \{\overline{t_{j,i'}}\}$, respectively, are sets of root-vertices. For $1 \leq i \leq s$, I_i and $\bar{I}_i$ are subsets of $\{1, \cdots, s\}$. A graph $\{G_n^{\circ}\}_{n=1}^{\infty}$ is constructed from G_n by amalgamating the vertex $u_{1,i}$ of G_n with the vertex $t_{j,i'}$ of J_j for $j \in I_i$, and amalgamating the vertex $v_{n,i}$ of G_n with $\overline{t_{j,i'}}$ of $\overline{J_j}$ for $j \in \bar{I}_i$. The graphs $(J_j, t_{j,i'})$ and $(\overline{J_j}, \overline{t_{j,i'}})$ are called *spiders* for the sequence $\mathcal{G}$. The resulting sequence of graphs is said to be an H-*linear family of graphs*

with spiders and is denoted by $\mathcal{G}^o$. We call $\mathcal{G}^\circ$ ring-like, if there is a spider among $J_1, J_2, \ldots, J_s$ that coincide with a spider among $\overline{J}_1, \overline{J}_2, \ldots, \overline{J}_s$. The graphs in Figure 3.2 demonstrate an example of ring-like, in which $s = 1$ and $\overline{J}_1 = J_1$.
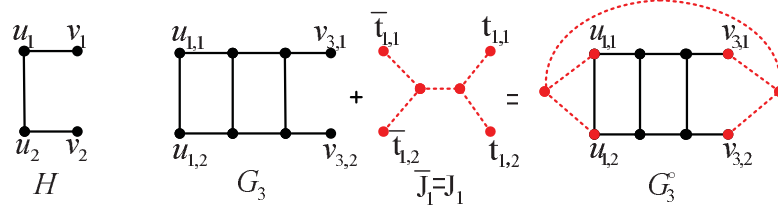

FIGURE 3.2. Using a spider to construct G_3°

3.2. **Production matrix.** By permutation-partition pairs, Stahl [27] showed that the calculation of genus polynomial of G_n can be converted to a (transfer) matrix method. Such matrices are also called production matrices [11] or transfer matrix [20] (using different techniques and methods). Here we briefly describe the production matrix of $\mathcal{G}$ (or $\mathcal{G}^\circ$). For more details on this, see [20, 11] etc. We refer to [3] for face-tracing algorithm.

We suppose that there are k embedding types for the graph G_n with roots $u_{1,1}, u_{1,2}, \ldots, u_{1,s}$, $v_{n,1}, v_{n,2}, \ldots, v_{n,s}$. For $1 \leq j \leq k$, let $\gamma_k^j(G_n)$ be the number of embeddings of G_n in O_k with type j and

$$\Gamma_{G_n}^j(x) = \sum_{i \geq 0} \gamma_i^j(G_n) x^i.$$

From the definition H-linear family of graphs and face-tracing algorithm, we obtain

$$(\Gamma_{G_n}^1(x), \Gamma_{G_n}^2(x), \cdots, \Gamma_{G_n}^k(x))^T = M(x) \cdot (\Gamma_{G_{n-1}}^1(x), \Gamma_{G_{n-1}}^2(x), \cdots, \Gamma_{G_{n-1}}^k(x))^T.$$

(3.1)

where α^T is the transpose of the vector α and

$$M(x) = \begin{bmatrix} m_{1,1}(x) & m_{1,2}(x) & \cdots & m_{1,k}(x) \\ m_{2,1}(x) & m_{2,2}(x) & \cdots & m_{2,k}(x) \\ \vdots & \vdots & \ddots & \vdots \\ m_{k,1}(x) & m_{k,2}(x) & \cdots & m_{k,k}(x) \end{bmatrix}$$

is the transfer matrix [20] or production matrix [11] of genus distribution of $\mathcal{G}$ (or $\mathcal{G}^\circ$). In [11], the authors showed that the production matrix $M(x)$ can been calculated with a computer program.

Let

$$V_{G_n}(x) = (\Gamma_{G_n}^1(x), \Gamma_{G_n}^2(x), \cdots, \Gamma_{G_n}^k(x))^T,$$

another property for the genus polynomial of H-linear sequence with spiders is that there exists a k-dimensional row-vector

$$V = (v_1(x), \cdots, v_k(x))$$

such that $\Gamma_{G_n^\circ}(x) = V \cdot V_{G_n}(x)$. Note that if there are no spiders, that is $\mathcal{G}^\circ = \mathcal{G}$, then $V = (1, 1, \cdots, 1)$.

For the case of Euler-genus polynomials, there also has the production matrix of Euler-genus distribution of $\mathcal{G}$ (or $\mathcal{G}^\circ$). We take [4] as an example of this.

Example 3.1. Suppose P_n is the path graph on n vertices. An *ladder graph L_n* is obtained by taking the graphical cartesian product of the path graph P_n with P_2, i.e. $L_n = P_n \square P_2$. From Section 3 in [4], the 2×2 production matrix of Euler-genus distribution of the ladder graph is given by

$$M(x) = \left[\begin{array}{cc} 2 & 4 \\ 2x + 4x^2 & 4x \end{array} \right].$$

It follows that

$$M(1) = \left[\begin{array}{cc} 2 & 4 \\ 6 & 4 \end{array} \right].$$

The *maximum genus*, *maximum non-orientable genus* and *maximum Euler-genus* of a graph G, denoted by $\gamma_{\max}(G)$, $\tilde{\gamma}_{\max}(G)$ and $\varepsilon_{\max}(G)$, respectively, are given by $\gamma_{\max}(G) = \max\{i | \gamma_i(G) > 0\}$, $\tilde{\gamma}_{\max}(G) = \max\{i | \tilde{\gamma}_i(G) > 0\}$ and $\varepsilon_{\max}(G) = \max\{i | \varepsilon_i(G) > 0\}$ respectively. One can see that $\varepsilon_{\max}(G) = \max\{2\gamma_{\max}(G), \tilde{\gamma}_{\max}(G)\}$. A *cactus graph*, also called a *cactus tree*, is a connected graph in which any two graph cycles have no vertex in common. Recall that a graph G with orientable maximum genus 0 if and only if G is the cactus graph, and a graph H of maximum Euler-genus 0 if and only if H is homeomorphic to the path graph P_n on n vertices for $n > 1$.

The following two basic properties are followed by the definition of H-linear family of graphs with spiders.

Proposition 3.1. *For $1 \leq j \leq k$, $\sum_{i=1}^{k} m_{ij}(1)$ are all the same constant D. Moreover, for any $n \geq 2$, we have $\frac{P_n(1)}{P_{n-1}(1)} = D$, where $P_n(x) = \Gamma_{G_n^\circ}(x)$ or $\mathcal{E}_{G_n^\circ}(x)$.*

Remark 3.2. *In the proof of Theorem 3.5 below, we will see D have the same meaning as that appeared in Section 2, so we use the same notation.*

Proposition 3.3. *Suppose $M(x)$ is the production matrix of genus distribution (Euler-genus distribution) of $\mathcal{G}$, then $M(x)$ is a constant if and only if the maximum genus (maximum Euler-genus) of G_n equals $0, \forall n \in \mathbb{N}$.*

3.3. **The limits for embedding distributions of H-linear families of graphs with spiders.** A square matrix $A = (a_{i,j})_{i,j=1}^k$ is said to be non-negative if

$$a_{i,j} \geq 0, \quad \forall i, j = 1, \cdots, k.$$

Let A be a non-negative $k \times k$ matrix with maximal eigenvalue r and suppose that A has exactly h eigenvalues of modulus r. The number h is called the *index* of imprimitivity of A. If $h = 1$, the matrix A is said to be *primitive*; otherwise, it is imprimitive. A square matrix $A = (a_{i,j})_{i,j=1}^k$ is said to a stochastic matrix if

$$\sum_{i=1}^{k} a_{ij} = 1, j = 1, \cdots, k.$$

The following property of primitive stochastic matrix can be found in Proposition 9.2 in [1].

Proposition 3.4. *Every eigenvalue λ of a stochastic matrix A satisfies $|\lambda| \leq 1$. Furthermore, if the stochastic matrix A is primitive, then all other eigenvalues of modulus are less than 1, and algebraic multiplicity of 1 is one.*

Theorem 3.5. *Consider the sequence of graphs $\mathcal{G}^\circ = \{G_n^\circ\}_{n=1}^\infty$. Let $\{p_i(n), i = 0, 1 \cdots, \}$ be the genus polynomial (Euler-genus polynomial) of graph G_n°, $P_n(x)$ be the genus polynomial (Euler-genus polynomial) of G_n° and $M(x)$ be the production matrix for $\mathcal{G}^\circ$. If the matrix $M(1)$ is primitive, then the results of Theorem 2.1 hold. Furthermore, if $M(x)$ is a constant, then the limit of the law for embedding distributions of G_n° is a discrete distribution.*

Proof. Suppose that the characteristic polynomial of the production matrix $M(x)$ is

$$F(x, \lambda) = \lambda^k - b_1(x)\lambda^{k-1} - b_2(x)\lambda^{k-2} \cdots - b_{k-1}(x)\lambda - b_k(x),$$

where $b_j(x)(1 \le j \le k)$ are polynomials with integer coefficients. We also assume $F(x, \lambda) = (\lambda - \lambda_1(x))^{m_1(x)} \cdots (\lambda - \lambda_r(x))^{m_r(x)}$. Then, by the results in [6] ([4]), the sequence of genus polynomials (Euler-genus polynomial) of H-linear family of graphs with spiders G_n° satisfy the following k^{th}-order linear recursion

$$P_n(x) = b_1(x)P_{n-1}(x) + b_2(x)P_{n-2}(x) + \cdots + b_k(x)P_{n-k}(x).$$

By Proposition 3.1 and Proposition 3.4, if $M(1)$ is primitive, we have

$$D = \lambda_1(1) > |\lambda_2(1)| \ge |\lambda_3(1)| \ge \cdots |\lambda_r(1)| \text{ and } m_1(1) = 1.$$

For any $n \in \mathbb{N}$, we denote the embedding distribution of graph G_n° by $\{p_i(n), i = 0, 1, 2, \cdots, \}$ and let X_n be a random variable with distribution

$$\mathbb{P}(X_n = i) = \frac{p_i(n)}{P_n(1)}, \quad i = 0, 1, \cdots,$$

and

(3.2) $$e = \frac{\lambda_1'(1)}{D}, \quad v = \frac{-\left(\lambda_1'(1)\right)^2 + D \cdot \lambda_1''(1) + D \cdot \lambda_1'(1)}{D^2}.$$

So following the lines in the proof of Theorem 2.1, we finish our proof. Furthermore if $M(x)$ is a constant, then all these functions $b_1(x), \cdots, b_k(x)$ are constant. Noting the case III in Theorem 2.1, this theorem follows. $\square$

The primitive of the matrix $M(1)$ is very important in our proof. For this, we give the following example.

Example 3.2. Let

$$M(x) = \begin{bmatrix} x + 1 & 0 \\ 0 & 2x \end{bmatrix}.$$

$M(1)$ is imprimitive. By calculation, we obtain

$$\lambda_1(x) = \frac{3x + 1 + |x - 1|}{2}, \quad \lambda_2(x) = \frac{3x + 1 - |x - 1|}{2}.$$

In this case, we even don't have the differentiability of $\lambda_1(x)$ at $x = 1$.

Remark 3.6. *As pointed by Stahl [27], in all known cases, the production matrix $M(x)$ for the genus distributions of any H-linear family of graphs is primitive at $x = 1$. Currently, we don't know whether this is true for general (or most) linear families of graphs.*

In the rest of this subsection, we apply Theorem 3.5 to path-like and ladder-like sequences of graphs.

A vertex with degree 1 is called a *pendant vertex*, and the edge incident with that vertex is called a *pendant edge*. If a pendant vertex u of a graph G is chosen to be a root, then the vertex u is called a *pendant root*. Suppose (H, u, v) is a connected graph with two pendant roots u, v. For $i = 1, 2, \ldots, n$, let (H_i, u_i, v_i) be a copy of (H, u, v). By the way in subsection 3.1, we construct a (H, u, v)-linear family of graphs and a (H, u, v)-linear family of graphs with spiders $(J_1, t_{1,i'})$ and $(\overline{J_1}, \overline{t_{1,i'}})$, where $(J_1, t_{1,i'})$ and $(\overline{J_1}, \overline{t_{1,i'}})$ be two connected graphs with roots $t_{1,i'}, \overline{t_{1,i'}}$ respectively. For the (H, u, v)-linear family of graphs with spiders or not, they have the same production matrix. Therefore, we use the same notation $\{P_n^H\}_{n=1}^{\infty}$ to denote them. For convenience, we call the graph P_n^H *path-like*. Figure 3.3 demonstrate the graphs H and P_n^H, the shadow part of H can be any connected graph.
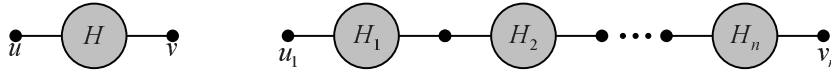

FIGURE 3.3. Graph H (left), and path-like graph P_n^H (right)

Corollary 3.7. *The genus distributions (Euler-genus distributions) of the path-like sequence of graphs $\{P_n^H\}_{n=1}^{\infty}$ with spiders are asymptotic normal distribution if the maximum genus (maximum Euler-genus) of (H, u, v) is greater than 0.*

Proof. Let (H, u, v) be a graph with two pendant roots u and v. We introduce the following two *partial genus distributions (partial Euler-genus distributions)* for (H, u, v). Let $d_i(H)$ be the number of embeddings of (H, u, v) in the surface O_i (S_i) such that u and v lie on different face-boundary walks. In this case, we say that the embedding has type d. Similarly, let $s_i(H)$ be the number of embeddings of (H, u, v) in O_i (S_i) such that u, v lie on the same face-boundary walk, and we call the embedding has type s. The two partial genus polynomials (partial Euler-genus polynomials) of (H, u, v) are given by $D_H(x) = \sum_{i \geq 0} d_i(H)x^i$, and $S_H(x) = \sum_{i \geq 0} s_i(H)x^i$. Clearly,

$$(3.3) \qquad P_H(x) = D_H(x) + S_H(x),$$

where $P_H(x)$ is the genus polynomial (Euler-genus polynomial) of (H, u, v). By face-tracing and Euler formula, we have the following claim.

Claim: If the graph $(P_{n-1}^H, u_1, v_{n-1})$ and (H_n, u_n, v_n) embed on surfaces O_i (S_i) and O_j (S_j), respectively, then the graph P_n^H embeds on O_{i+j} (S_{i+j}).

By using the claim above, we will build recurrence formulas for the partial genus polynomials of P_n^H. There are four cases.

Case 1: If both the embeddings of P_{n-1}^H and (H_n, u_n, v_n) have type d, then the embedding of P_n^H has type d. This case contributes to $D_{P_n^H}(x)$ the term $D_{P_{n-1}^H}(x)D_{H_n}(x)$.

Case 2: If the embeddings of P_{n-1}^H and (H_n, u_n, v_n) have type d and s, respectively, then the embedding of P_n^H has type d. This case contributes to $D_{P_n^H}(x)$ the term $D_{P_{n-1}^H}(x)S_{H_n}(x)$.

Case 3: If the embedding of P_{n-1}^H has type s and the embedding of (H_n, u_n, v_n) has type d, then the embedding of P_n^H has type d, this case contributes to $D_{P_n^H}(x)$ the term $S_{P_{n-1}^H}(x)D_{H_n}(x)$.

Case 4: If both the embeddings of P_{n-1}^H and (H_n, u_n, v_n) have type s, then the embedding of P_n^H has type s. This case contributes to $S_{P_n^H}(x)$ the term $S_{P_{n-1}^H}(x)S_{H_n}(x)$.

The following linear recurrence system of equations summarizes the four cases above.

$$(3.4) \qquad D_{P_n^H}(x) \ = \ (D_{H_n}(x) + S_{H_n}(x)) \, D_{P_{n-1}^H}(x) + D_{H_n}(x)S_{P_{n-1}^H}(x),$$

$$(3.5) \qquad S_{P_n^H}(x) \ = \ S_{H_n}(x)S_{P_{n-1}^H}(x).$$

Rewriting the equations above as

$$\begin{bmatrix} D_{P_n^H}(x) \\ S_{P_n^H}(x) \end{bmatrix} = \begin{bmatrix} D_{H_n}(x) + S_{H_n}(x) & D_{H_n}(x) \\ 0 & S_{H_n}(x) \end{bmatrix} \begin{bmatrix} D_{P_{n-1}^H}(x) \\ S_{P_{n-1}^H}(x) \end{bmatrix}.$$

Since (H_n, u_n, v_n) be a copy of (H, u, v), thus the production matrix $M(x)$ of the genus (Euler-genus) distributions of $\{P_n^H\}_{n=1}^\infty$ is

$$\begin{bmatrix} D_H(x) + S_H(x) & D_H(x) \\ 0 & S_H(x) \end{bmatrix}.$$

For simplicity of writing, we use $D(x)$, $S(x)$ and $P(x)$ to denote $D_H(x)$, $S_H(x)$ and $P_H(x)$, respectively. Obviously, the eigenvalues of matrix $M(x)$ are given by

$$\lambda_1(x) = D(x) + S(x), \quad \lambda_2(x) = S(x).$$

Since the graph H is connected, $S(x) = 0$ is impossible. We make the following discussions on the $D(x)$

If $D(x) = 0$, then H is the path graph P_m on m vertices ($m \geq 2$). In this case, the maximum genus (maximum Euler-genus) of (H, u, v) is equal 0.

Now we consider the case $D(x) \neq 0$. Under this situation, $\lambda_1(1) > \lambda_2(1)$ and the matrix $M(1)$ is primitive. By direct calculation, we get

$$e \ = \ \frac{D'(1) + S'(1)}{D(1) + S(1)},$$

$$v \ = \ \frac{-(D'(1) + S'(1))^2 + (D(1) + S(1))(D''(1) + S''(1) + D'(1) + S'(1))}{(D(1) + S(1))^2}.$$

We assume $P(x) = \sum_m c_m x^m$. By *Cauchy-Schwarz inequality*,

$$\left(\sum_m m c_m \right)^2 \leq \sum_m m^2 c_m \cdot \sum_m c_m = \left(\sum_m (m^2 - m)c_m + \sum_m m c_m \right) \cdot \sum_m c_m,$$

which implies $P'(1)^2 \leq (P''(1) + P'(1)) \cdot P(1)$. By this inequality and (3.3), one easily sees that $v \geq 0$. Therefore, $v = 0$ is equivalent to the above Cauchy-Schwarz inequality becomes equality, that is

$$\left(\sum_m m c_m \right)^2 = \sum_m m^2 c_m \cdot \sum_m c_m.$$

Since $P(x) \neq 0$, we have $c_m > 0$ for some $m \geq 0$. The above equality is also equivalent to that for some $x \in \mathbb{R}$

$$m\sqrt{c_m} + x\sqrt{c_m} = 0, \quad \forall m \geq 0.$$

Therefore, $v = 0$ if and only if $\gamma_{max}(H) = \gamma_{min}(H)$ $(\varepsilon_{\max}(H) = \varepsilon_{\min}(H))$. Noting that, a known fact in the topological graph theory says that $\gamma_{max}(H) = \gamma_{min}(H)$ $(\varepsilon_{\max}(H) = \varepsilon_{\min}(H))$ implies that H is the cactus graph (H is the path graph), we complete our proof.
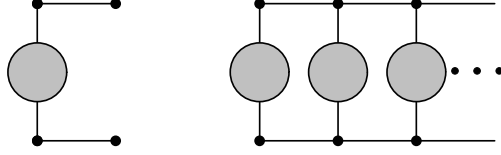
$\square$

FIGURE 3.4. Graph H (left), and ladder-like graph L_n^H (right)

Given any graph (H, u, v) whose root-vertices u and v are both 1-valent, as in Figure 3.4, we construct a *ladder-like sequence of graphs* $\{(L_n^H, u_n, v_n)\}_{n=1}^{\infty}$ [7]. The shadow part of H can be any connected graph, and the ladder-like sequences are a special case of H-linear family of graphs.

Corollary 3.8. *The genus distributions of the ladder-like sequence of graphs $\{L_n^H\}_{n=1}^{\infty}$ are asymptotic normal distribution.*

Proof. By [7], the production matrix for genus distributions of the ladder-like sequence of graphs $L_1^H, L_2^H, L_3^H, \cdots$ is

$$M(x) = p(x) \begin{bmatrix} 4x & 2x & 0 \\ 0 & 0 & 0 \\ 0 & 2x & 4x \end{bmatrix} + q(x) \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 4 \\ 4x & 2x & 0 \end{bmatrix},$$

where $p(x), q(x) \in \mathbb{Z}(x)$ are the partial genus polynomials for H. Also by [7], $q(x) = 0$ is impossible.

Obviously, $D = 4(p(1) + q(1))$ and the eigenvalues of matrix $M(1)$ are given by

$$\lambda_1 = 4(p(1) + q(1)), \ \lambda_2 = 4p(1) - 2q(1), \lambda_3 = 0.$$

By $p(1) \geq 0$ and $q(1) > 0$, one easily sees that

$$\lambda_1 > |\lambda_2|, \quad \lambda_1 > |\lambda_3|$$

and the matrix $M(1)$ is primitive.

With the help of a computer, one arrives at that

$$e = \frac{3\left(p'(1) + q'(1)\right) + 3p(1) + q(1)}{3(p(1) + q(1))},$$

$$v = \frac{4}{27(p(1) + q(1))} \Big[2q(1)^2 - 27\big(p'(1) + q'(1)\big)^2$$
$$+ 9q(1)\Big(3p''(1) + 3q''(1) + 3q'(1) + 7p'(1)\Big)$$
$$+ p(1)\Big(27p''(1) + 27p'(1) + 27q''(1) - 9q'(1) + 14q(1)\Big)\Big].$$

In the rest of this corollary, we will prove $v > 0$. Assume

$$p(x) = \sum_m a_m x^m, \quad q(x) = \sum_m b_m x^m.$$

Using *Cauchy-Schwarz inequality* again, we see that $p'(1)^2 \leq (p''(1) + p'(1)) \cdot p(1)$ and $q'(1)^2 \leq (q''(1) + q'(1)) \cdot q(1)$. Therefore, in order to prove $v > 0$, it is suffice to show that

$$54p'(1)q'(1) + 9p(1)q'(1) < 2q(1)^2 + 9q(1)\Big(3p''(1) + 7p'(1)\Big) + p(1)\Big(27q''(1) + 14q(1)\Big),$$

which is equivalent to

$$\sum_{i,j} \left[54ija_ib_j + 9a_ijb_j\right] < 2q(1)^2 + \sum_{i,j} \left[27(i^2-i) + 63i + 27(j^2-j) + 14\right]a_ib_j.$$

The above inequality is due to

$$2q(1)^2 + \sum_{i,j}\left[27(i-j)^2 + 36(i-j) + 14\right]a_ib_j = 2q(1)^2 + \sum_{i,j}\left(27\left(i-j+\frac{2}{3}\right)^2 + 2\right)a_ib_j > 0.$$

We complete the proof of $v > 0$.

By Theorem 3.5, the genus distributions of the ladder-like sequence $\{L_n^H\}_{n=1}^\infty$ are asymptotic normal distribution with mean $e \cdot n$ and variance $v \cdot n$. $\qquad\square$

3.4. Limits for crosscap-number distributions of graphs.
We demonstrate the relationship between the limits of crosscap-number distributions and Euler-genus distributions.

Theorem 3.9. *Let* $a_n = \frac{\Gamma_{G_n^\circ}(1)}{\mathcal{E}_{G_n^\circ}(1)}, b_n = 1 - a_n$. *If* $\lim\limits_{n\to\infty} a_n = 0$, *we have*

$$(3.6) \qquad \lim_{n\to\infty}\sup_{x\in\mathbb{R}}\left|\sum_{0\le i\le x}\frac{\varepsilon_i(G_n^\circ)}{\mathcal{E}_{G_n^\circ}(1)} - \sum_{0\le i\le x}\frac{\tilde{\gamma}_i(G_n^\circ)}{\tilde{\Gamma}_{G_n^\circ}(1)}\right| = 0,$$

which implies that the limits of crosscap-number distributions are the same as that of Euler-genus distributions.

Proof. Since $a_n = \frac{\Gamma_{G_n^\circ}(1)}{\mathcal{E}_{G_n^\circ}(1)}$, then $b_n = 1 - a_n = \frac{\tilde{\Gamma}_{G_n^\circ}(1)}{\mathcal{E}_{G_n^\circ}(1)}$. Let $\mathbb{U}_n, \tilde{\mathbb{U}}_n, \mathbb{W}_n$ be three random variables with distributions given by the genus, crosscap-number and Euler-genus respectively, that is

$$\mathbb{P}(\mathbb{U}_n = i) = \frac{\gamma_i(G_n^\circ)}{\Gamma_{G_n^\circ}(1)}, \ \ \mathbb{P}(\tilde{\mathbb{U}}_n = i) = \frac{\tilde{\gamma}_i(G_n^\circ)}{\tilde{\Gamma}_{G_n^\circ}(1)}, \ \ \mathbb{P}(\mathbb{W}_n = i) = \frac{\varepsilon_i(G_n^\circ)}{\mathcal{E}_{G_n^\circ}(1)}, \ \ i = 0, 1 \cdots.$$

Since

$$\mathcal{E}_G(x) = \Gamma_G(x^2) + \tilde{\Gamma}_G(x),$$

for any $i \in \mathbb{N}$, we have

$$\mathcal{E}_{G_n^\circ}(1) \cdot \mathbb{P}(\mathbb{W}_n = i) = \varepsilon_i(G_n^\circ) = \gamma_{\frac{i}{2}}(G_n^\circ) + \tilde{\gamma}_i(G_n^\circ)$$

$$= \mathbb{P}(\mathbb{U}_n = \frac{i}{2}) \cdot \Gamma_{G_n^\circ}(1) + \mathbb{P}(\tilde{\mathbb{U}}_n = i) \cdot \tilde{\Gamma}_{G_n^\circ}(1),$$

here $\gamma_{\frac{i}{2}}(G_n^\circ)$ is defined as 0 when i is an odd number. Therefore, it holds that

$$\mathbb{P}(\mathbb{W}_n = i) = a_n\mathbb{P}(2\mathbb{U}_n = i) + b_n\mathbb{P}(\tilde{\mathbb{U}}_n = i)$$

By this, for any $x \ge 0$, we have

$$\mathbb{P}(\mathbb{W}_n \le x) - \mathbb{P}(\tilde{\mathbb{U}}_n \le x) = a_n\mathbb{P}(2\mathbb{U}_n \le x) + (b_n - 1)\mathbb{P}(\tilde{\mathbb{U}}_n \le x).$$

Thus

$$\left|\mathbb{P}(\mathbb{W}_n \le x) - \mathbb{P}(\tilde{\mathbb{U}}_n \le x)\right| \le a_n + (1 - b_n).$$

By the definitions of $\mathbb{W}_n, \tilde{\mathbb{U}}_n$, the above inequality implies (3.6). $\qquad\square$

By (3.6), once the limits of Euler-genus distributions is obtained, the limits of crosscap-number distributions is also known.

With the same method as that in Theorem 3.9, we obtain the following corollary.

Corollary 3.10. *Let $\{G_n\}_{n=1}^{\infty}$ be any sequence of graphs, which is not required to be H-linear family of graphs with spiders. If $\lim\limits_{n\to\infty} \frac{\Gamma_{G_n}(1)}{\mathcal{E}_{G_n}(1)} = \frac{1}{2^{\beta(G_n)}} = 0$ or $\lim\limits_{n\to\infty} \beta(G_n) = \infty$, we have*

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} \left| \sum_{0\leq i\leq x} \frac{\varepsilon_i(G_n)}{\mathcal{E}_{G_n}(1)} - \sum_{0\leq i\leq x} \frac{\tilde{\gamma}_i(G_n)}{\tilde{\Gamma}_{G_n}(1)} \right| = 0.$$

4. More examples and some research problems

The graphs, which have explicit formulas for their embedding distributions, mainly are linear families of graphs with spiders, see [4, 6, 8, 11, 29] for details. There are many linear families of graphs with spiders which satisfy the conditions of Theorem 2.1 or Theorem 3.5. However, we give a few examples to demonstrate the theorems, including the famous Möbius ladders, Ringel ladders and Circular ladders.

Example 4.1. Let Y_n be the iterated-claw graph of Figure 4.1.

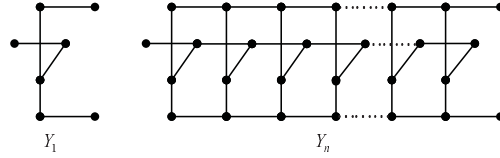Y_1 Y_n

FIGURE 4.1. The claw Y_1 (left), and the iterated-claw graph Y_n (right)

The genus polynomial for the iterated-claw graph Y_n [14] is given by

$$\Gamma_{Y_n}(x) = 20x\Gamma_{Y_{n-1}}(x) - 8(-3x + 8x^2)\Gamma_{Y_{n-2}}(x) - 384x^3\Gamma_{Y_{n-3}}(x).$$

Since

$$\begin{aligned} F(x,\lambda) &= \lambda^3 - 20x\lambda^2 + 8(8x^2 - 3x)\lambda + 384x^3, \\ F(1,\lambda) &= (\lambda - 16)\big(\lambda - 2(1-\sqrt{7})\big)\big(\lambda - 2(\sqrt{7}+1)\big), \end{aligned}$$

the conditions of Theorem 2.1 hold for $P_n(x) = \Gamma_{Y_n}(x)$ and $\lambda_1(1) = D = 16$. By the derivative rule of implicit function and with the help of Maple, one sees

$$e = \frac{6}{7}, \quad v = \frac{8}{147} > 0.$$

For the constants e, v given above, we have

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} \left| \frac{1}{\Gamma_{Y_n}(1)} \sum_{0\leq i\leq x\sqrt{v\cdot n}+e\cdot n} \gamma_i(Y_n) - \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} \mathrm{d}u \right| = 0.$$

By [4], the Euler-genus polynomials of Y_n satisfy the following three-order recurrence relation

$$\mathcal{E}_{Y_n}(x) = 2(3x + 28x^2)\mathcal{E}_{Y_{n-1}}(x) - 16(-3x^2 - 12x^3 + 4x^4)\mathcal{E}_{Y_{n-2}}(x) - 3072x^6\mathcal{E}_{Y_{n-3}}(x).$$

Set $P_n(x) = \mathcal{E}_{Y_n}(x)$ and

$$F(x,\lambda) = \lambda^3 - 2(3x + 28x^2)\lambda^2 + 16(-3x^2 - 12x^3 + 4x^4)\lambda + 3072x^6.$$

Since $F(1,\lambda) = (\lambda - 64)(\lambda - 6)(\lambda + 8)$, the conditions of Theorem 2.1 hold and $D = \lambda_1(1) = 64$. By the derivative rule of implicit function and with the help of Maple, one sees

$$e = \frac{160}{87}, v = \frac{269092}{1975509} > 0.$$

For the constants e, v given above, the Euler-genus distributions of Y_n are asymptotic normal distribution with mean $e \cdot n$ and variance $v \cdot n$. By Theorem 3.9, the crosscap-number distributions of Y_n are also asymptotic normal distribution with mean $e \cdot n$ and variance $v \cdot n$.

Example 4.2. Let $G_n = P_n \square P_3$ be the *grid graph* [17]. The genus polynomials for the grid graph G_n are given by the recursion

$$\Gamma_{G_n}(x) = (1 + 30x)\Gamma_{G_{n-1}}(x) - 42(-x + 4x^2)\Gamma_{G_{n-2}}(x)$$
$$-72(x^2 + 14x^3)\Gamma_{G_{n-3}}(x) + 1728x^4\Gamma_{G_{n-4}}(x).$$

With the help of Maple, the conditions of Theorem 2.1 hold for $P_n(x) = \Gamma_{G_n}(x)$ and

$$D = \lambda_1(1) = 24, \ e = \frac{34}{41}, \ v = \frac{4816}{68921} > 0.$$

The Euler-genus polynomials for the grid graphs G_n [4] satisfy the recursion

$$\mathcal{E}_{G_n}(x) = (1 + 11x + 84x^2)\mathcal{E}_{G_{n-1}}(x) + 12x^2(7 + 30x - 28x^2)\mathcal{E}_{G_{n-2}}(x)$$
$$-288x^4(1 + 4x + 32x^2)\mathcal{E}_{G_{n-3}}(x) + 27648x^8\mathcal{E}_{G_{n-4}}(x).$$

With the help of Maple, the conditions of Theorem 2.1 hold for $P_n(x) = \mathcal{E}_{G_n}(x)$ and

$$D = \lambda_1(1) = 96, \ e = \frac{5488}{3037}, \ v = \frac{4819233780}{28011371653} > 0.$$

By the discussions above, the embedding distributions of the grid graph G_n are asymptotic normal distributions.

Example 4.3. Suppose that n is a positive integer. Let C_n be the cycle graph on n vertices. The Ringel ladder R_n is obtained by adding an edge joining the two vertices of the leftmost edge and rightmost edge of the ladder graph L_n. A *circular ladder* CL_n is the graphical Cartesian product $CL_n = C_n \square P_2$. The *Möbius ladder* ML_n is formed from an $2n$-cycle by adding edges connecting opposite pairs of vertices in the cycle. i.e., the Möbius ladder can be described as a circular ladder with a half-twist. It is known in [6] that the Ringel ladder, circular ladder, and Möbius ladder are ring-like families of graphs. Let H_n be the Ringel ladder R_n, circular ladder CL_n, or Möbius ladder ML_n.

By Theorem 3.1 in [6], the genus polynomials for H_n [6] satisfy the recursion

$$\Gamma_{H_n}(x) = 4\Gamma_{H_{n-1}}(x) + (-5 + 20x)\Gamma_{H_{n-2}}(x) + (-56x + 2)\Gamma_{H_{n-3}}(x)$$
$$-4(32x - 11)x\Gamma_{H_{n-4}}(x) + 8(28x - 1)x\Gamma_{H_{n-5}}(x)$$
$$+32(8x - 3)x^2\Gamma_{H_{n-6}}(x) - 256x^3\Gamma_{H_{n-7}}(x).$$

Since

$$F(x, \lambda) = \lambda^7 - 4\lambda^6 - (-5 + 20x)\lambda^5 - (-56x + 2)\lambda^4$$
$$+4(32x - 11)x\lambda^3 - 8(28x - 1)x\lambda^2 - 32(8x - 3)x^2\lambda + 256x^3$$
$$= (\lambda - 1)(\lambda + 2\sqrt{x})(\lambda - 2\sqrt{x})(\lambda - 1 + \sqrt{1 + 8x})(\lambda - 1 - \sqrt{1 + 8x})$$
$$* \left(\lambda - \frac{1}{2}(1 - \sqrt{32x + 1})\right)\left(\lambda - \frac{1}{2}(1 + \sqrt{32x + 1})\right),$$

the conditions of Theorem 2.1 hold for $P_n(x) = \Gamma_{H_n}(x)$ and $\lambda_1(x) = \sqrt{8x + 1} + 1, D = \lambda_1(1) = 4$. With the help of Maple, one easily sees

$$e = \frac{1}{3}, \ v = \frac{2}{27} > 0.$$

By Theorem 2.1, for the constants e, v given above , we have

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} \left| \frac{1}{\Gamma_{H_n}(1)} \sum_{0\le i\le x\sqrt{v\cdot n}+e\cdot n} \gamma_i(H_n) - \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} \mathrm{d}u \right| = 0.$$

The Euler-genus polynomials for H_n [4] are given by the recursion

$$\begin{aligned}
\mathcal{E}_{H_n}(x) &= (12x+4)\mathcal{E}_{H_{n-1}}(x) + (-12x^2 - 34x - 5)\mathcal{E}_{H_{n-2}}(x) \\
&\quad + (-240x^3 - 20x^2 + 26x + 2)\mathcal{E}_{H_{n-3}}(x) \\
&\quad + 4(80x^3 + 128x^2 + 14x - 1)x\mathcal{E}_{H_{n-4}}(x) \\
&\quad + 16(112x^3 + 8x^2 - 14x - 1)x^2\mathcal{E}_{H_{n-5}}(x) \\
&\quad - 128(8x^2 + 18x + 3)x^4\mathcal{E}_{H_{n-6}}(x) \\
&\quad - 2048(2x+1)x^6\mathcal{E}_{H_{n-7}}(x).
\end{aligned}$$

Since

$$\begin{aligned}
F(x, \lambda) &= \lambda^7 - (12x+4)\lambda^6 - (-12x^2 - 34x - 5)\lambda^5 - (-240x^3 - 20x^2 + 26x + 2)\lambda^4 \\
&\quad - 4(80x^3 + 128x^2 + 14x - 1)x\lambda^3 - 16(112x^3 + 8x^2 - 14x - 1)x^2\lambda^2 \\
&\quad + 128(8x^2 + 18x + 3)x^4\lambda + 2048(2x+1)x^6, \\
F(1, \lambda) &= (\lambda+2)^2(\lambda-3)(\lambda-4)(\lambda-8)\left(\lambda - \frac{1}{2}(5 - \sqrt{89})\right)\left(\lambda - \frac{1}{2}(5 + \sqrt{89})\right),
\end{aligned}$$

the conditions of Theorem 2.1 hold for $P_n(x) = \mathcal{E}_{H_n}(x)$ and $D = \lambda_1(1) = 8$. With the help of Maple, we have

$$\lambda_1(x) = \sqrt{20x^2 + 4x + 1} + 2x + 1, \quad e = \frac{4}{5}, \quad v = \frac{22}{125} > 0.$$

By Theorem 2.1, for the constants e, v given above, the Euler-genus distributions of H_n is asymptotic normal distribution with mean $e \cdot n$ and variance $v \cdot n$.

4.1. **Some researches problems.** In the end of this paper, we demonstrate some research problems.

Question 4.1. *In our paper, the limits of the embedding distributions for graphs are normal distributions or some discrete distributions. Can we prove that the limit of the embedding distributions for any H-linear family of graphs is a normal distribution or a discrete distribution. Furthermore, if the maximum genus (Euler-genus) $\varepsilon_{\max}(G)$ of H is great than 0, do we have the limit for the genus distributions (Euler-genus distribution) for any H-linear family of graphs is a normal distribution?*

Question 4.2. *Suppose that $\{\widetilde{G}_n\}_{n=1}^{\infty}$ is a family of graphs with $\beta(\widetilde{G}_n) \to \infty$ (orientable maximum genus $\gamma_M(\widetilde{G}_n) \to \infty$). Are the crosscap-number distributions (genus distributions) of $\widetilde{G}_n$ asymptotic normal.*

A *bouquet of circles* B_n is define as a graph with one vertex and n edges. A *dipole* D_n is a graph with two vertices joining by n multiple edges. In [15], Gross, Robbins, and Tucker obtained a second-order recursion for the genus distributions of B_n. The genus distributions of D_n were obtained by Rieper in [25], and independently by Kwak and Lee in [16]. The *wheel* W_n is a graph formed by connecting a single vertex to each of the vertices of a n-cycle. The genus distribution of W_n was obtained in [5] by Chen, Gross and Mansour. The following special case of question 4.2 is may not hard.

Question 4.3. *Are the embedding distributions of B_n, D_n and W_n asymptotic normal?*

The following question is for the complete graph K_n on n vertices and complete bipartite graph $K_{m,n}$.

Question 4.4. *Are the embedding distributions of K_n, and $K_{m,n}$ asymptotic normal?*

Another question is the following.

Question 4.5. *Is the embedding distribution of a random graph on n vertices asymptotic normal?*

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