ON THE q-DYSON ORTHOGONALITY PROBLEM

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ABSTRACT. By combining the Gessel–Xin method with plethystic substitutions, we obtain a recursion for a symmetric function generalization of the q-Dyson constant term identity also known as the Zeilberger–Bressoud q-Dyson theorem. This yields a constant term identity which generalizes the non-zero part of Kadell's orthogonality ex-conjecture and a result of Károlyi, Lascoux and Warnaar.
Keywords: q-Dyson constant term identity, Kadell's orthogonality conjecture, symmetric function

1. INTRODUCTION

The study of constant term identities can be traced back to a 1962 paper on random matrices and the theory of statistical levels of complex systems by Freeman Dyson [5]. In the course of this work he conjectured that for non-negative integers $a_0, \dots, a_n$,

$$\operatorname*{CT}_{x} \prod_{0\le i\ne j\le n} (1 - x_i/x_j)^{a_i} = \frac{(a_0 + \cdots + a_n)!}{a_0! \cdots a_n!}, \tag{1.1}$$

where $\operatorname*{CT}_{x}$ denotes taking the constant term with respect to $x := (x_0, \dots, x_n)$. Dyson's conjecture was soon proved by Gunson [11] and Wilson [26]. Subsequently, an elegant proof using Lagrange interpolation was given by Good [9], and much later, Zeilberger gave a combinatorial proof using tournaments [29]. These days Dyson's ex-conjecture is usually referred as the Dyson constant term identity.

In this introduction, we first briefly review the history of the Dyson constant term identity and some of its generalizations. Then we state our main result, a symmetric function generalization of the Dyson constant term identity, related to Kadell's orthogonality (ex-)conjecture. We conclude the introduction by outlining the main new ideas used in this paper.

In 1975 Andrews [1] conjectured the following q-analogue of (1.1):

$$\operatorname*{CT}_{x} \prod_{0\le i<j\le n} (x_i/x_j; q)_{a_i} (qx_j/x_i; q)_{a_j} = \frac{(q;q)_{a_0+\cdots+a_n}}{(q;q)_{a_0}(q;q)_{a_1}\cdots(q;q)_{a_n}}, \tag{1.2}$$

where $(z;q)_k := (1-z)(1-zq)\dots(1-zq^{k-1})$ is a q-shifted factorial. Andrews' q-Dyson conjecture was first proved in 1985 by Zeilberger and Bressoud [30], who generalized Zeilberger's method of tournaments mentioned above. Twenty years later Gessel and Xin [8] gave a second proof using formal Laurent series, and then, in 2014, Károlyi and Nagy [17]

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discovered a very short and elegant proof using multivariable Lagrange interpolation. Finally, an inductive proof was found by Cai [3] by adding additional parameters to the problem.

In 1982, Macdonald realised that the equal parameter case of the q-Dyson identity, i.e., $a_0 = a_1 = \cdots = a_n = k$, can be formulated as a combinatorial identity for the root system A_n. This led him to conjecture a constant term identity for arbitrary root systems [20]:

$$\mathrm{CT} \prod_{\alpha \in R^+} (e^{-\alpha}; q)_k (q e^{\alpha}; q)_k = \prod_{i=1}^{r} \begin{bmatrix} d_i k \\ k \end{bmatrix}.$$

Here R is a reduced irreducible finite root system of rank r, R^+ is the set of positive roots, $d_1, \dots, d_r$ are the degrees of the fundamental invariants, and $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$ is a q-binomial coefficient. Initially, many cases of Macdonald's conjecture were proven on a case by case basis [2, 7, 12, 14, 30]. A uniform proof for $q = 1$ was first found by Opdam [24] using hypergeometric shift operators. Eventually, a case-free proof of the full conjecture was given by Cherednik [4] based on his double affine Hecke algebra. For more on the extensive literature of Macdonald's constant term conjecture we refer the reader to [6] and references therein.

Let $\lambda = (\lambda_0, \dots, \lambda_n)$ be a partition. In 1988, Macdonald [21, 22] introduced a family of symmetric functions $P_\lambda(q,t) = P_\lambda(x_0, \dots, x_n; q, t)$, now called Macdonald polynomials. Given the scalar product on the ring of symmetric functions in $x_0, \dots, x_n$

$$\langle f, g \rangle_{q,q^k} := \frac{1}{(n+1)!} \mathop{\mathrm{CT}}_{x} f(x_0, \dots, x_n) g(x_0^{-1}, \dots, x_n^{-1}) \prod_{0 \le i \neq j \le n} (x_i/x_j; q)_k,$$

Macdonald established the orthogonality

$$\big\langle P_\lambda(q, q^k), P_\mu(q, q^k) \big\rangle_{q,q^k} = 0 \quad \text{if } \lambda \neq \mu,$$

for k a positive integer. Moreover, he showed that the quadratic norm evaluation is given by [22, page 373]

$$\big\langle P_\lambda(q, q^k), P_\lambda(q, q^k) \big\rangle = \prod_{0 \le i < j \le n} \frac{(q^{\lambda_i - \lambda_j + 1 + (j-i)k}; q)_{k-1}}{(q^{\lambda_i - \lambda_j + 1 + (j-i-1)k}; q)_{k-1}}.$$

For $\lambda = 0$ the Macdonald polynomials trivialise to 1, so that

$$\big\langle 1, 1 \big\rangle_{q,q^k} = \prod_{i=1}^{n+1} \begin{bmatrix} ik - 1 \\ k - 1 \end{bmatrix}.$$

By a simple transformation, it is not difficult to show that this is equivalent to the equal parameter case of (1.2). This provides a satisfactory explanation for the $a_0 = a_1 = \cdots = a_n = k$ case of the q-Dyson constant term identity in terms of orthogonal polynomials. Finding a similar such explanation for the full q-Dyson identity is an important open problem.

The first step towards a resolution of this problem was made by Kadell [15], who formulated an orthogonality conjecture which we will describe next. Let $X = (x_0, x_1, \dots)$ be an alphabet of countably many variables. Then the rth complete symmetric function $h_r(X)$ may be defined in terms of its generating function as

$$\sum_{r\geq 0} z^r h_r(X) = \prod_{i\geq 0} \frac{1}{1-zx_i}. \tag{1.3}$$

More generally, for the complete symmetric function indexed by a composition (or partition) $v=(v_0,v_1,\dots,v_k)$

$$h_v := h_{v_0}\cdots h_{v_k}.$$

For $a:=(a_0,a_1,\dots,a_n)$ a sequence of non-negative integers, let $x^{(a)}$ denote the alphabet

$$x^{(a)} = (x_0, x_0q, \dots, x_0q^{a_0-1}, \dots, x_n, x_nq, \dots, x_nq^{a_n-1}) \tag{1.4}$$

of cardinality $|a|:=a_0+\dots+a_n$, and define the generalized q-Dyson constant term

$$D_{v,\lambda}(a) = \operatorname*{CT}_x x^{-v} h_\lambda\big(x^{(a)}\big) \prod_{0\leq i<j\leq n} (x_i/x_j;q)_{a_i}(qx_j/x_i;q)_{a_j}. \tag{1.5}$$

Here $v=(v_0,\dots,v_n)\in\mathbb{Z}^{n+1}$, x^v denotes the monomial $x_0^{v_0}\cdots x_n^{v_n}$ and λ is a partition such that $|v|=|\lambda|$. (Note that if $|v|\neq|\lambda|$ then $D_{v,\lambda}(a)=0$.) For the constant term (1.5), Kadell formulated the following conjecture [15, Conjecture 4].

Conjecture 1.1. *For r a positive integer and v a composition such that $|v|=r$,*

$$D_{v,(r)}(a) = \begin{cases} \dfrac{q^{\sum_{i=k+1}^n a_i}(1-q^{a_k})(q^{|a|};q)_r}{(1-q^{|a|})(q^{|a|-a_k+1};q)_r} \displaystyle\prod_{i=0}^n \begin{bmatrix} a_i+\dots+a_n \\ a_i \end{bmatrix} & \text{if } v=(0^k,r,0^{n-k}), \\[2ex] 0 & \text{otherwise.} \end{cases} \tag{1.6}$$

In fact Kadell only considered $v=(r,0^n)$ in his conjecture, but the more general statement given above is what was proved by Károlyi, Lascoux and Warnaar in [16, Theorem 1.3] using multivariable Lagrange interpolation and key polynomials. If for a sequence $u=(u_0,\dots,u_n)$ of integers we denote by u^+ the sequence obtained from u by ordering the u_i in weakly decreasing order (so that u^+ is a partition if u is a composition), then Károlyi et al. also proved a closed-form expression for $D_{v,v^+}(a)$ in the case when v is a composition all of whose parts have multiplicity one, i.e., $v_i\neq v_j$ for all $0\leq i<j\leq n$. Subsequently, Cai [3] gave an inductive proof of Kadell's conjecture. He also showed that the following more general orthogonality holds.

Theorem 1.2. *Let $v\in\mathbb{Z}^{n+1}$ and λ a partition such that $|v|=|\lambda|$. If $D_{v,\lambda}(a)$ is non-vanishing, then $v^+\geq\lambda$ in dominance order.*

We note that the converse of Theorem 1.2 also appears to be true. This is trivially the case for $n=0$, and for $n=1$ we used Maple to verify that $D_{v,\lambda}(a)\neq 0$ for $|v|\leq 23$ and $v^+\geq\lambda$.

In this paper, we are concerned with the $\lambda=v^+$ case of $D_{v,\lambda}(a)$. For this case we obtain a recursion for $D_{v,\lambda}(a)$ provided that the largest part of v occurs with multiplicity one. Given a sequence $s=(s_0,\dots,s_n)$ and an integer k such that $0\leq k\leq n$, define $s^{(k)}:=(s_0,\dots,s_{k-1},s_{k+1},\dots,s_n)$.

Theorem 1.3. *Let $v=(v_0,\dots,v_n)$ be a composition such that its largest part has multiplicity one in v. Fix a non-negative integer k by $v_k=\max\{v\}$. Then*

$$D_{v,v^+}(a)=q^{\sum_{i=k+1}^n a_i}\begin{bmatrix}v_k+|a|-1\\ a_k-1\end{bmatrix}D_{v^{(k)},(v^{(k)})^+}\big(a^{(k)}\big). \tag{1.7}$$

For example, if $v=(0,2,3,2,1)$, then $v^+=(3,2,2,1,0)$, $k=2$ and $v^{(2)}=(0,2,2,1)$. If all the non-zero parts of v have multiplicity one, then we can iterate (1.7). Together with the q-Dyson identity (1.2) this yields a closed-form formula for $D_{v,v^+}(a)$.

Corollary 1.4. *Let $v=(v_0,\dots,v_n)$ be a composition all of whose positive parts have multiplicity one, and set $l:=\ell(v)$, the number of the non-zero parts of v. Let $\sigma\in\mathfrak{S}_{n+1}$ be any permutation for which $\sigma(v):=(v_{\sigma(0)},\dots,v_{\sigma(n)})=v^+$. Then*

$$D_{v,v^+}(a)=q^c\prod_{i=0}^{l-1}\begin{bmatrix}v_{\sigma(i)}+|a|-a_{\sigma(0)}-\cdots-a_{\sigma(i-1)}-1\\ a_{\sigma(i)}-1\end{bmatrix}\prod_{i=l}^{n}\begin{bmatrix}a_{\sigma(i)}+\cdots+a_{\sigma(n)}\\ a_{\sigma(i)}\end{bmatrix}, \tag{1.8}$$

where

$$c=\sum_{i=0}^{l-1}\sum_{\substack{j=\sigma(i)+1\\ j\notin\{\sigma(0),\dots,\sigma(i-1)\}}}^{n}a_j.$$

Clearly, there are $(n-l+1)!$ admissible permutations $\sigma\in\mathfrak{S}_{n+1}$. Since the product

$$\prod_{i=l}^{n}\begin{bmatrix}a_{\sigma(i)}+\cdots+a_{\sigma(n)}\\ a_{\sigma(i)}\end{bmatrix}=\frac{(q;q)_{a_{\sigma(l)}+\cdots+a_{\sigma(n)}}}{(q;q)_{a_{\sigma(l)}}\cdots(q;q)_{a_{\sigma(n)}}}$$

is symmetric in $a_{\sigma(l)},\dots,a_{\sigma(n)}$ each such σ results in the same expression for $D_{v,v^+}(a)$. When $l=1$ Corollary 1.4 reduces to the non-zero part of Kadell's (ex)-conjecture, and when $l=n$ or $l=n+1$ it reduces to a result by Károlyi, Lascoux and Warnaar [16, Proposition 4.5].

The method employed to prove Theorem 1.3 is based on the well-known fact that two polynomials of degree at most d are equal if they are equal at $d+1$ distinct points. This method was used previously to prove several constant term identities, such as in the Gessel–Xin proof of the q-Dyson identity [8] or in the proof of what are known as first-layer formulas for q-Dyson products [19].

It is not difficult to show that $D_{v,v^+}(a)$ is a polynomial of degree $v_k+|a|-a_k$ in q^{a_k}. Assuming the conditions of Theorem 1.3, it is also not hard to show that this polynomial vanishes if $-a_k\in\{0,1,\dots,v_k+|a|-a_k-1\}$. However, since $D_{v,v^+}(a)$ is not actually defined for negative integer values of a_k, we need to extend the definition to all integers a_k. For this, we require the theory of iterated Laurent series, developed in [27]. In the field of iterated Laurent series, $D_{v,v^+}(a)$ is well-defined for all $a_k\in\mathbb{Z}$ and can again be viewed as a polynomial in q^{a_k}. To prove the above vanishing properties of $D_{v,v^+}(a)$ (again with v as in the theorem), we combine the Gessel–Xin method with plethystic substitutions, a powerful tool from the theory of symmetric functions. It trivially follows that the right-hand side of (1.7) satisfies the same polynomiality and vanishing properties. By degree considerations, one may conclude that the left and right-hand sides of (1.7) are equal if they agree at one additional point.

The remainder of this paper is organised as follows. In the next section we introduce some basic notation used throughout this paper. In Sections 3 and 4 we introduce the two main tools used in this paper—plethystic notation and substitutions, and iterated Laurent series respectively. In Section 5 we give a proof of Theorem 1.3.

2. Basic notation

In this section we introduce some basic notation used throughout this paper.

For $v = (v_0, v_1, \dots, v_n)$ a sequence, we write $|v|$ for the sum of its entries, i.e., $|v| = v_0 + \dots + v_n$. Moreover, if $v \in \mathbb{R}^{n+1}$ then we write v^+ for the sequence obtained from v by ordering its elements in weakly decreasing order. If all the entries of v are non-negative integers, we refer to v as a (weak) composition. A partition is a sequence $\lambda = (\lambda_0, \lambda_1, \dots)$ of non-negative integers such that $\lambda_0 \geq \lambda_1 \geq \cdots$ and only finitely-many λ_i are positive. The length of a partition λ, denoted $\ell(\lambda)$ is defined to be the number of non-zero λ_i (such λ_i are known as the parts of λ). We adopt the convention of not displaying the tails of zeros of a partition. We say that $|\lambda| = \lambda_0 + \lambda_1 + \cdots$ is the size of the partition λ. We adopt the standard dominance order on the set of partitions of the same size. If λ, μ are partitions such that $|\lambda| = |\mu|$ then $\lambda \geq \mu$ if $\lambda_0 + \dots + \lambda_i \geq \mu_0 + \dots + \mu_i$ for all $i \geq 0$. Similarly, for two integer sequences $v = (v_0, \dots, v_n)$ and $u = (u_0, \dots, u_m)$, we write $v \geq u$ if $v_0 + \dots + v_i \geq u_0 + \dots + u_i$ for all $i \geq 0$, where we set $v_i = 0$ for $i > n$ and $u_j = 0$ for $j > m$. Note here that we do not require that $|v| = |w|$. As usual, we write $\lambda > \mu$ if $\lambda \geq \mu$ but $\lambda \neq \mu$, and $v > u$ if $v \geq u$ but $v \neq u$.

The infinite q-shifted factorial is defined as

$$(z)_\infty = (z;q)_\infty := \prod_{i=0}^{\infty} (1 - zq^i),$$

where, typically, we suppress the base q. Then, for k an integer,

$$(z)_k = (z;q)_k := \frac{(z;q)_\infty}{(zq^k;q)_\infty}.$$

Note that

$$(z)_k = \begin{cases} (1-z)(1-zq)\cdots(1-zq^{k-1}) & \text{if } k \geq 0, \\ \dfrac{1}{(1-zq^k)(1-zq^{k+1})\cdots(1-zq^{-1})} & \text{if } k < 0. \end{cases}$$

Using the above we can define the q-binomial coefficient as

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q^{n-k+1})_k}{(q)_k}$$

for n an arbitrary integer and k a non-negative integer.

3. Plethystic notation

Plethystic or λ-ring notation is a device to facilitate computations in the ring of symmetric functions. The notion was introduced by Grothendieck [10] in the study of Chern classes. Nowadays plethystic notation has become an indispensable computational tool for organizing and manipulating intricate relationships between symmetric functions. In this section, we briefly introduce plethystic notation and substitutions. For more details, see [13, 18, 23, 25].

Denote by $\Lambda_{\mathbb{F}}$ the ring of symmetric functions in countably many variables with coefficients in a field $\mathbb{F}$. For an alphabet $X = (x_0, x_1, \dots)$, we additively write $X := x_0 + x_1 + \cdots$, and use plethystic brackets to indicate this additive notation:

$$f(X) = f(x_0, x_1, \dots) = f[x_0 + x_1 + \cdots] = f[X], \quad \text{for } f \in \Lambda_{\mathbb{F}}.$$

For r a positive integer, let p_r be the power sum symmetric function in the alphabet X, defined by

$$p_r = \sum_{i \geq 0} x_i^r.$$

In addition, we set $p_0 = 1$. For a partition $\lambda = (\lambda_0, \lambda_1, \dots)$, let

$$p_\lambda = p_{\lambda_0} p_{\lambda_1} \cdots .$$

The p_r are algebraically independent over $\mathbb{Q}$, and the p_λ form a basis of $\Lambda_{\mathbb{Q}}$ [22]. That is,

$$\Lambda_{\mathbb{Q}} = \mathbb{Q}[p_1, p_2, \dots].$$

Now we introduce a consistent arithmetic on alphabets in terms of the basis of power sums. In particular, a power sum whose argument is the sum, difference or Cartesian product of two alphabets X and Y is defined as

$$p_r[X + Y] = p_r[X] + p_r[Y], \tag{3.1a}$$

$$p_r[X - Y] = p_r[X] - p_r[Y], \tag{3.1b}$$

$$p_r[XY] = p_r[X] p_r[Y]. \tag{3.1c}$$

For example, for the alphabets $X = x_1 + x_2 + \cdots$ and $Y = y_1 + y_2 + \cdots$, the sum of X and Y is $X + Y = x_1 + x_2 + \cdots + y_1 + y_2 + \cdots$. In general we cannot give meaning to division by an arbitrary alphabet and only division by $1 - t$ (the difference of two one-letter alphabets with "letters" 1 and t respectively) is meaningful. In particular

$$p_r\left[\frac{X}{1-t}\right] = \frac{p_r[X]}{1 - t^r}. \tag{3.2}$$

Note that the alphabet $1/(1-t)$ may be interpreted as the infinite alphabet $1 + t + t^2 + \cdots$. Indeed, by (3.1a) and (3.1c)

$$p_r[X(1 + t + t^2 + \cdots)] = p_r[X] \sum_{k=0}^{\infty} p_r[t^k] = p_r[X] \sum_{k=0}^{\infty} t^{kr} = \frac{p_r[X]}{1 - t^r}.$$

Having the above rules for plethystic substitutions we can view symmetric functions as operators acting on alphabets, and by carrying out complicated substitutions we can turn simple algebraic identities into much more complicated ones. For example, since

$$\sum_{i\le j} x_i x_j = \tfrac{1}{2}\Big(\sum_i x_i\Big)^2 + \tfrac{1}{2}\sum_i x_i^2$$

we have

$$h_2 = \tfrac{1}{2}\big(p_1^2 + p_2\big)$$

as an identity in the algebra of symmetric functions. Consequently,

$$h_2[X] = \tfrac{1}{2}(p_1^2[X] + p_2[X])$$

where X can be *any* alphabet, obtained by combining the rules of addition, subtraction, multiplication and division described in (3.1) and (3.2).

For r a positive integer, let the elementary symmetric function be defined as

$$e_r = \sum_{0\le i_1<\cdots<i_r} x_{i_1}x_{i_2}\cdots x_{i_r}.$$

Also set $e_0 = 1$. By the definition of the elementary symmetric function, one can observe the following simple fact: For r a positive integer and X an alphabet of finitely many variables,

$$e_r[X] = 0 \qquad \text{if } |X| < r, \tag{3.3}$$

where $|X|$ denotes the cardinality of X. This simple fact plays an important role in proving vanishing properties of expressions of the form $D_{v,\lambda}(a)$.

Finally, we need the following two basic plethystic identities. One can find proofs in [13, Theorem 1.27].

Proposition 3.1. *Let X and Y be two alphabets. For r a non-negative integer,*

$$h_r[X+Y] = \sum_{i=0}^{r} h_i[X]h_{r-i}[Y], \tag{3.4}$$

$$h_r[-X] = (-1)^r e_r[X]. \tag{3.5}$$

4. Constant term evaluations using iterated Laurent series

In this section we introduce some essential ingredients of the field of iterated Laurent series and describe a basic lemma for extracting constant terms from rational functions. Throughout this paper we let $K = \mathbb{C}(q)$ and work in the field of iterated Laurent series $K\langle\langle x_n, x_{n-1}, \dots, x_0\rangle\rangle = K((x_n))((x_{n-1}))\cdots((x_0))$, unless specified otherwise. Elements of $K\langle\langle x_n, x_{n-1}, \dots, x_0\rangle\rangle$ are regarded first as Laurent series in x_0, then as Laurent series in x_1, and so on. The reason the field $K\langle\langle x_n, x_{n-1}, \dots, x_0\rangle\rangle$ is highly suitable for proving constant term identities is explained in [8]. For a more detailed account of the properties of this field, see [27] and [28]. Crucial in what is to follow is that the field $K(x_0, \dots, x_n)$ of rational functions in the variables $x_0, \dots, x_n$ with coefficients in K forms a subfield of $K\langle\langle x_n, x_{n-1}, \dots, x_0\rangle\rangle$, so that every rational function is identified with its unique Laurent series expansion.

The following series expansion of $1/(1-cx_i/x_j)$ for $c \in K \setminus \{0\}$ forms a key ingredient in our approach:

$$\frac{1}{1-cx_i/x_j} = \begin{cases} \displaystyle\sum_{l\geq 0} c^l (x_i/x_j)^l & \text{if } i<j, \\ \displaystyle -\sum_{l<0} c^l (x_i/x_j)^l & \text{if } i>j. \end{cases}$$

Thus, the constant term in x_i of $1/(1-cx_i/x_j)$ is 1 if $i<j$ and 0 if $i>j$. That is,

$$\operatorname*{CT}_{x_i} \frac{1}{1-cx_i/x_j} = \begin{cases} 1 & \text{if } i<j, \\ 0 & \text{if } i>j, \end{cases} \tag{4.1}$$

where, for $f \in K\langle\langle x_n, x_{n-1}, \dots, x_0\rangle\rangle$, we use the notation $\operatorname*{CT}_{x_i} f$ to denote taking the constant term of f with respect to x_i. An important property of the constant term operators defined this way is their commutativity:

$$\operatorname*{CT}_{x_i} \operatorname*{CT}_{x_j} f = \operatorname*{CT}_{x_j} \operatorname*{CT}_{x_i} f.$$

This implies that the operation of taking the constant term in $K\langle\langle x_n, x_{n-1}, \dots, x_0\rangle\rangle$ is well-defined.

The following lemma is a basic tool for extracting constant terms from rational functions and has appeared previously in [8].

Lemma 4.1. *For a positive integer m, let $p(x_k)$ be a Laurent polynomial in x_k of degree at most $m-1$ with coefficients in $K\langle\langle x_n, \dots, x_{k-1}, x_{k+1}, \dots, x_0\rangle\rangle$. Let $0 \leq i_1 \leq \cdots \leq i_m \leq n$ such that all $i_r \neq k$, and define*

$$f = \frac{p(x_k)}{\prod_{r=1}^{m}(1-x_k/c_r x_{i_r})}, \tag{4.2}$$

where $c_1, \dots, c_m \in K \setminus \{0\}$ such that $c_r \neq c_s$ if $x_{i_r} = x_{i_s}$. Then

$$\operatorname*{CT}_{x_k} f = \sum_{\substack{r=1 \\ i_r>k}}^{m} \big(f\,(1-x_k/c_r x_{i_r})\big)\Big|_{x_k=c_r x_{i_r}}. \tag{4.3}$$

5. Proof of Theorem 1.3

To prove Theorem 1.3, which is a recursion for $D_{v,v^+}(a)$, we shall first prove a similar recursion — see Theorem 5.1 below — for a more general constant term, denoted $D_{v,\lambda}(a,m)$ and defined in (5.2) below. As shown in Section 5.1, using a cyclic action γ on $D_{v,\lambda}(a,m)$, Theorem 5.1 implies Theorem 1.3.

5.1. **The constant term $D_{v,\lambda}(a,m)$.** In this subsection we define $D_{v,\lambda}(a,m)$ mentioned above and show that it suffices to consider this constant term for those $v=(v_0,\dots,v_n)\in\mathbb{Z}^{n+1}$ for which $\max\{v\}=v_0$.

For $m\in\{0,1,\dots,n+1\}$ and $a=(a_0,\dots,a_n)$ a composition, define the alphabet

$$x_m^{(a)} := (x_0q^{-1},x_0,\dots,x_0q^{a_0-2},\dots,x_{m-1}q^{-1},x_{m-1},\dots,x_{m-1}q^{a_{m-1}-2},\\ x_m,x_mq,\dots,x_mq^{a_m-1},\dots,x_n,x_nq,\dots,x_nq^{a_n-1}).$$

Note that, plethystically,

$$x_m^{(a)} = \sum_{i=0}^{n} \frac{1-q^{a_i}}{1-q}\, x_i q^{-\chi(i<m)}, \tag{5.1}$$

where χ is the truth function. For $v=(v_0,\dots,v_n)\in\mathbb{Z}^{n+1}$, λ a partition such that $|\lambda|=|v|$, and m and a as above, we define the constant term

$$D_{v,\lambda}(a,m) := \operatorname*{CT}_x x^{-v} h_\lambda\big(x_m^{(a)}\big) \prod_{0\le i<j\le n} (x_i/x_j)_{a_i}(qx_j/x_i)_{a_j}. \tag{5.2}$$

Clearly, the alphabet $x^{(a)}$ and constant term $D_{v,\lambda}(a)$, defined in (1.4) and (1.5) respectively, are given by $x^{(a)}=x_0^{(a)}$ and $D_{v,\lambda}(a)=D_{v,\lambda}(a,0)$. By the homogeneity of the complete symmetric function h_λ and the fact that $x_{n+1}^{(a)}=x_0^{(a)}/q$,

$$D_{v,\lambda}(a,n+1) = q^{-|\lambda|} D_{v,\lambda}(a,0). \tag{5.3}$$

Hence, it suffices to restrict the range of m to $0\le m\le n$ or $1\le m\le n+1$.

As in [19], for $f\in K\langle\!\langle x_n,x_{n-1},\dots,x_0\rangle\!\rangle$, define the cyclic action γ by

$$\gamma\big(f(x_0,x_1,\dots,x_n)\big) = f(x_1,x_2,\dots,x_n,x_0/q).$$

Then $\operatorname*{CT}_x f = \operatorname*{CT}_x \gamma(f)$, and, for $0\le m\le n$,

$$\gamma\big(D_{v,\lambda}(a,m)\big) = q^{v_n} D_{\gamma^{-1}(v),\lambda}\big(\gamma^{-1}(a),m+1\big), \tag{5.4}$$

where $\gamma(v):=(v_1,\dots,v_n,v_0)$. For $k\in\{0,1,\dots,n\}$, by applying (5.4) exactly $n+1-k$ times and also using (5.3) we find that

$$D_{v,\lambda}(a,m) = \begin{cases} q^{v_k+\dots+v_n} D_{\gamma^{-(n+1-k)}(v),\lambda}\big(\gamma^{k-n-1}(a),m'\big) & \text{if } m\le k,\\ q^{-v_0-\dots-v_{k-1}} D_{\gamma^{-(n+1-k)}(v),\lambda}\big(\gamma^{k-n-1}(a),m'\big) & \text{if } m>k, \end{cases} \tag{5.5}$$

where $1\le m'\le n+1$ and $m'\equiv m-k \pmod{n+1}$. In particular, if k is an integer such that $\max\{v\}=v_k$, then $\gamma^{-(n+1-k)}(v)=(v_k,\dots,v_n,v_0,\dots,v_{k-1})$ has the property that its first part is its largest part. Hence (5.5) allows us to assume without loss of generality that $v_0=\max\{v\}$. Also assuming that v is a composition such that $v_0>v_i$ for all $1\le i\le n$ we will prove the following theorem.

Theorem 5.1. *For $v=(v_0,\dots,v_n)$ a composition such that $v_0=\max\{v\}$ has multiplicity one in v, and $m\in\{1,2,\dots,n+1\}$,*

$$D_{v,v^+}(a,m)=q^{\sum_{i=1}^{m-1}a_i-v_0}\begin{bmatrix}v_0+|a|-1\\ a_0-1\end{bmatrix}D_{v^{(0)},(v^{(0)})^+}\big(a^{(0)},m-1\big). \tag{5.6}$$

This theorem, together with (5.5), implies Theorem 1.3 in a few simple steps.

Proof of Theorem 1.3. Let $k\in\{0,1,\dots,n\}$ be fixed by $v_k=\max\{v\}$. Taking $m=0$ in (5.5) we have

$$D_{v,v^+}(a)=q^{v_k+\cdots+v_n}D_{\gamma^{-(n+1-k)}(v),v^+}\big(\gamma^{k-n-1}(a),n+1-k\big). \tag{5.7}$$

Here $\gamma^{-(n+1-k)}(v)$ has the property that its first part, v_k, is its unique largest part. Thus we can apply (5.6) to obtain

$$\begin{aligned}D_{\gamma^{-(n+1-k)}(v),v^+}&\big(\gamma^{k-n-1}(a),n+1-k\big)\\ &=q^{\sum_{i=k+1}^{n}a_i-v_k}\begin{bmatrix}v_k+|a|-1\\ a_k-1\end{bmatrix}D_{\gamma^{-(n-k)}(v^{(k)}),(v^{(k)})^+}\big(\gamma^{k-n}(a^{(k)}),n-k\big),\end{aligned}$$

where $\gamma^{-(n-k)}(v^{(k)})=(v_{k+1},\dots,v_n,v_0,\dots,v_{k-1})$ and $\gamma^{k-n}(a^{(k)})=(a_{k+1},\dots,a_n,a_0,\dots,a_{k-1})$. Using (5.7) by taking $(k,v,a)\mapsto(k+1,v^{(k)},a^{(k)})$, we have

$$D_{v^{(k)},(v^{(k)})^+}(a^{(k)})=q^{\sum_{i=k+1}^{n}v_i}D_{\gamma^{-(n-k)}(v^{(k)}),(v^{(k)})^+}\big(\gamma^{k-n}(a^{(k)}),n-k\big).$$

As a result,

$$D_{\gamma^{-(n+1-k)}(v),v^+}\big(\gamma^{k-n-1}(a),n+1-k\big)=q^{\sum_{i=k+1}^{n}a_i-\sum_{i=k}^{n}v_i}\begin{bmatrix}v_k+|a|-1\\ a_k-1\end{bmatrix}D_{v^{(k)},(v^{(k)})^+}(a^{(k)}).$$

Substituting this into (5.7), we finally obtain

$$D_{v,v^+}(a)=q^{\sum_{i=k+1}^{n}a_i}\begin{bmatrix}v_k+|a|-1\\ a_k-1\end{bmatrix}D_{v^{(k)},(v^{(k)})^+}\big(a^{(k)}\big),$$

completing the proof. □

5.2. **Outline of the proof of Theorem 5.1.** Our proof of Theorem 5.1 is quite lengthy and involved, and before presenting the full details we briefly outline the three key steps.

(1) **Polynomiality** — We will show that, for fixed non-negative integers $a_1,\dots,a_n$, the constant term $D_{v,\lambda}(a,m)$ is a polynomial in q^{a_0} of degree at most $a_1+\cdots+a_n+v_0$.

(2) **Determination of roots** — We will show that $D_{v,v^+}(a,m)$ vanishes for $-a_0\in\{0,1,\dots,a_1+\cdots+a_n+v_0-1\}$ if $v=(v_0,\dots,v_n)$ is a composition such that $v_0=\max\{v\}$ has multiplicity one in v.

(3) **Value at $a_0 = 1$** — Assuming the conditions of Theorem 5.1, we will show that $D_{v,v^+}(a,m)$ evaluated at $a_0 = 1$ can be expressed as the same constant term with $(n,m) \mapsto (n-1, m-1)$. That is

$$D_{v,v^+}(a,m)|_{a_0=1} = q^{\sum_{i=1}^{m-1} a_i - v_0} D_{v^{(0)},(v^{(0)})^+}\big(a^{(0)}, m-1\big).$$

The details of these key steps will be presented in the subsections 5.3, 5.5 and 5.6 respectively. Subsection 5.4 prepares some technical preliminaries needed in Subsection 5.5.

5.3. **Polynomiality.** As mentioned above, the aim of this subsection is to prove that the constant term $D_{v,\lambda}(a,m)$ is a polynomial in q^{a_0} of degree at most $a_1 + \cdots + a_n + v_0$.

We begin by recalling [19, Lemma 2.2].

Lemma 5.2. *Let $L(x_1, \dots, x_n)$ be an arbitrary Laurent polynomial. Then, for fixed non-negative integers $a_1, \dots, a_n$, and t an integer not exceeding $a_1 + \cdots + a_n$,*

$$\operatorname*{CT}_x x_0^t L(x_1, \dots, x_n) \prod_{0 \le i < j \le n} (x_i/x_j)_{a_i} (qx_j/x_i)_{a_j} \tag{5.8}$$

is a polynomial in q^{a_0} of degree at most $a_1 + \cdots + a_n - t$. Moreover, if $t > a_1 + \cdots + a_n$, then the constant term (5.8) *vanishes.*

We remark that the correct interpretation of the above lemma is that, for $t \le a_1 + \cdots + a_n$, there exists a polynomial $P(x)$ of degree at most $a_1 + \cdots + a_n - t$ such that, for all non-negative integers a_0,

$$\operatorname*{CT}_x x_0^t L(x_1, \dots, x_n) \prod_{0 \le i < j \le n} (x_i/x_j)_{a_i} (qx_j/x_i)_{a_j} = P(q^{a_0}).$$

Using Lemma 5.2 it is not hard to show that the constant term $D_{v,\lambda}(a,m)$ is a polynomial in q^{a_0} for fixed non-negative integers $a_1, \dots, a_n$. This is the content of the next proposition.

Proposition 5.3. *Let $a_1, \dots, a_n$ be fixed non-negative integers and $D_{v,\lambda}(a,m)$ be defined as in* (5.2)*. If $-v_0 \le a_1 + \cdots + a_n$, then $D_{v,\lambda}(a,m)$ is a polynomial in q^{a_0} of degree at most $a_1 + \cdots + a_n + v_0$. If $-v_0 > a_1 + \cdots + a_n$ then $D_{v,\lambda}(a,m) = 0$.*

Proof. We write

$$x_m^{(a)} = \frac{1-q^{a_0}}{1-q}\, x_0 q^{-\chi(0<m)} + \hat{x}_m^{(a)},$$

where

$$\hat{x}_m^{(a)} := \sum_{i=1}^{n} \frac{1-q^{a_i}}{1-q}\, x_i q^{-\chi(i<m)}.$$

Then, by repeated use of (3.4) with

$$X \mapsto \frac{1-q^{a_0}}{1-q}\, x_0 q^{-\chi(0<m)} \quad \text{and} \quad Y \mapsto \hat{x}_m^{(a)}$$

and the homogeneity of the complete symmetric function, the constant $D_{v,\lambda}(a,m)$ can be expanded as

$$D_{v,\lambda}(a,m)=\sum_k q^{-\chi(0<m)|k|}h_k\Big[\frac{1-q^{a_0}}{1-q}\Big] \\ \times \operatorname*{CT}_x \frac{x_0^{|k|-v_0}}{x_1^{v_1}\cdots x_n^{v_n}}h_{\lambda-k}\big[\hat{x}_m^{(a)}\big]\prod_{0\le i<j\le n}(x_i/x_j)_{a_i}(qx_j/x_i)_{a_j}, \tag{5.9}$$

where $k:=(k_0,\dots,k_{\ell(\lambda)-1})$ is a composition and the sum is over $0\le k_i\le\lambda_i$ for $0\le i\le \ell(\lambda)-1$. Note that, generally, k and $\lambda-k$ are compositions, not partitions. By Lemma 5.2 the constant term in (5.9) vanishes if $a_1+\cdots+a_n+v_0<|k|$, and is a polynomial in q^{a_0} of degree at most $a_1+\cdots+a_n+v_0-|k|$ if $a_1+\cdots+a_n+v_0\ge|k|$. Together with the fact that $h_k[(1-z)/(1-q)]$ is a polynomial in z of degree $|k|$[1], each summand in (5.9) is either a polynomial in q^{a_0} of degree at most $a_1+\cdots+a_n+v_0$ or is 0. Moreover, if $a_1+\cdots+a_n+v_0<0$, then every constant term in (5.9) vanishes and $D_{v,\lambda}(a,m)=0$. □

5.4. **Preliminaries for the determination of the roots of $D_{v,\lambda}(a,m)$.** In this subsection we prepare some general results used in the next section to determine the roots of $D_{v,\lambda}(a,m)$.

Lemma 5.4. *For s a positive integer, let $(b_1,\dots,b_{s+1})$ and $(k_1,\dots,k_s)$ be compositions such that $1\le k_i\le b_1+\cdots+b_{s+1}$ for $1\le i\le s$. Then at least one of the following holds:*

1. *$1\le k_i\le b_i$ for some i with $1\le i\le s$;*
2. *$-b_j\le k_i-k_j\le b_i-1$ for some $1\le i<j\le s$;*
3. *there exists a permutation $w\in\mathfrak{S}_s$ and a composition $(t_1,\dots,t_s)$ such that*

$$k_{w(j)}-k_{w(j-1)}=b_{w(j)}+t_j \quad \text{for } 1\le j\le s. \tag{5.10}$$

Here $k_0=w(0):=0$, the t_i satisfy $\sum_{j=1}^s t_j\le b_{s+1}$ and $t_j>0$ if $w(j-1)<w(j)$ for $1\le j\le s$.

When $b_{s+1}=0$ case (3) can not occur. Indeed, if (3) were to hold with $b_{s+1}=0$ this would imply $\sum_{j=1}^s t_j\le 0$, contradicting the fact that $t_1>0$. This special case of the lemma corresponds to [8, Lemma 4.2] by Gessel and Xin. If (3) holds with $b_{s+1}=1$, then $t_1=1$ and $t_j=0$ for $2\le j\le s$. Hence $w(j-1)>w(j)$ for $2\le j\le s$ so that $w=(s,\dots,2,1)$ and $k_i=1+\sum_{j=i}^s b_j$ for all i. This special case of the lemma appeared previously as [19, Lemma 3.2]. We finally remark that (1) and (3) can not hold simultaneously. If (3) were to hold, then by (5.10) we have $k_{w(j)}\ge b_{w(j)}+1$ for all j, contradicting to (1). Also, it is not hard to show that (2) and (3) can not hold simultaneously.

Proof. We prove the lemma by showing that if (1) and (2) fail then (3) must hold.

Assume that (1) and (2) are both false. Then we construct a weighted tournament T on the complete graph on s vertices, labelled $1,\dots,s$, as follows. For the edge (i,j) with $1\le i<j\le s$ we draw an arrow from j to i and attach a weight b_i if $k_i-k_j\ge b_i$. If, on

[1]It is easily shown that $h_r[(1-z)/(1-q)]=(z)_r/(q)_r$, see e.g., [22, page 27].

the other hand, $k_i - k_j \le -b_j - 1$ then we draw an arrow from i to j and attach the weight $b_j + 1$. Note that the weight of each edge of a tournament is non-negative.

We call a directed edge from i to j ascending if $i < j$. It is immediate from our construction that (i) the weight of the edge $i \to j$ is less than or equal $k_j - k_i$, and (ii) the weight of an ascending edge is positive.

We will use (i) and (ii) to show that any of the above-constructed tournaments is acyclic and hence transitive. As consequence of (i), the weight of a directed path from i to j in T, defined as the sum of the weights of its edges, is at most $k_j - k_i$. Proceeding by contradiction, assume that T contains a cycle C. By the above, the weight of C must be non-positive, and hence 0. Since C must have at least one ascending edge, which by (ii) has positive weight, the weight of C is positive, a contradiction.

Since each T is transitive, there is exactly one directed Hamilton path P in T, corresponding to a total order of the vertices. Assume P is given by

$$P = w(1) \to w(2) \to \cdots \to w(s-1) \to w(s),$$

where we have suppressed the edge weights. Then $k_{w(s)} - k_{w(1)} \ge b_{w(2)} + \cdots + b_{w(s)}$, and thus

$$\begin{aligned} k_{w(s)} &\ge k_{w(1)} + b_{w(2)} + \cdots + b_{w(s)} \\ &\ge b_{w(1)} + 1 + b_{w(2)} + \cdots + b_{w(s)} \\ &= b_1 + \cdots + b_s + 1. \end{aligned} \tag{5.11}$$

Together with the assumption that $k_{w(s)} \le b_1 + \cdots + b_{s+1}$ this implies that P has at most $b_{s+1} - 1$ ascending edges. Let $(t_1, \dots, t_s)$ be a composition such that (5.10) holds. When $j = 1$ this gives $k_{w(1)} = b_{w(1)} + t_1$. Since (1) does not hold, $k_{w(1)} \ge b_{w(1)} + 1$, so that $t_1 > 0$. For $2 \le j \le s$, if $w(j-1) \to w(j)$ is an ascending edge, then t_j is a positive integer. That is, for $2 \le j \le s$ if $w(j-1) < w(j)$ then $t_j > 0$. Since

$$\begin{aligned} \sum_{j=1}^{s} (k_{w(j)} - k_{w(j-1)}) = k_{w(s)} &= b_{w(1)} + \cdots + b_{w(s)} + t_1 + \cdots + t_s \\ &= b_1 + \cdots + b_s + t_1 + \cdots + t_s \le b_1 + \cdots + b_s + b_{s+1}, \end{aligned}$$

we have $t_1 + \cdots + t_s \le b_{s+1}$. This completes the proof of the assertion that (3) must hold if both (1) and (2) fail. ∎

Our next proposition concerns alphabets of the form $x_m^{(a)}$ as defined in (5.1).

Proposition 5.5. *For s a positive integer, let $(b_1, \dots, b_{s+1})$ and $(k_1, \dots, k_s)$ be compositions such that $1 \le k_i \le b_1 + \cdots + b_{s+1}$ for $1 \le i \le s$. If the k_i are such that (3) of Lemma 5.4 holds, then for m a non-negative integer*

$$- \sum_{i=0}^{s} \frac{1 - q^{b_i}}{1 - q} \, x_i q^{-\chi(i<m)} \Bigg|_{\substack{b_0 = -\sum_{i=1}^{s+1} b_i, \\ x_i = q^{k_s - k_i},\, 0 \le i \le s}} = q^{n_1} + \cdots + q^{n_{b_{s+1}}}, \tag{5.12}$$

where $\{n_1, \dots, n_{b_{s+1}}\}$ is a set of integers determined by m and the b_i and k_j.

We remark that the set $\{n_1,\dots,n_{b_{s+1}}\}$ can be explicitly determined. However, since the precise values of the n_i play no role in the following, we have omitted them from the above statement. Indeed, the important fact about the right-hand side is that, viewed as an alphabet, has cardinality b_{s+1}.

Proof. Denote the left-hand side of (5.12) by L. Carrying out the substitutions

$$b_0 \mapsto -\sum_{i=1}^{s+1} b_i \quad \text{and} \quad x_i \mapsto q^{k_s-k_i} \text{ for } 0\le i\le s$$

in

$$-\sum_{i=0}^{s} \frac{1-q^{b_i}}{1-q}\, x_i q^{-\chi(i<m)},$$

we obtain

$$\begin{aligned} L &= -\frac{q^{k_s}}{1-q}\biggl(\bigl(1-q^{-\sum_{i=1}^{s+1} b_i}\bigr)q^{-\chi(0<m)} + \sum_{i=1}^{s}(1-q^{b_i})q^{-k_i-\chi(i<m)}\biggr) \\ &= -\frac{q^{k_s}}{1-q}\biggl(\bigl(1-q^{-\sum_{i=1}^{s+1} b_i}\bigr)q^{-\chi(0<m)} + \sum_{i=1}^{s}(1-q^{b_{w(i)}})q^{-k_{w(i)}-\chi(w(i)<m)}\biggr), \end{aligned}$$

where $w\in\mathfrak{S}_s$ is any permutation such that (5.10) holds. By summing that equation over j from 1 to i, we find

$$k_{w(i)} = \sum_{j=1}^{i}(b_{w(j)}+t_j) \quad \text{for } 1\le i\le s.$$

Hence

$$L = -\frac{q^{k_s}}{1-q}\biggl(\bigl(1-q^{-\sum_{i=1}^{s+1} b_i}\bigr)q^{-\chi(0<m)} + \sum_{i=1}^{s}(1-q^{b_{w(i)}})q^{-\sum_{j=1}^{i}(b_{w(j)}+t_j)-\chi(w(i)<m)}\biggr).$$

By rearranging the terms in the above expression this may be written as

$$\begin{aligned} L = \frac{q^{k_s}}{1-q}\biggl(& q^{-\sum_{i=1}^{s+1} b_i-\chi(0<m)}\Bigl(1-q^{b_{s+1}-\sum_{j=1}^{s} t_j+\chi(0<m)-\chi(w(s)<m)}\Bigr) \\ &+ \sum_{i=1}^{s} q^{b_{w(i)}-\sum_{j=1}^{i}(b_{w(j)}+t_j)-\chi(w(i)<m)}\Bigl(1-q^{t_i-\chi(w(i-1)<m)+\chi(w(i)<m)}\Bigr)\biggr). \end{aligned}$$

Next we will show that

$$b_{s+1}-\sum_{j=1}^{s} t_j+\chi(0<m)-\chi(w(s)<m)\in\mathbb{N} \tag{5.13a}$$

and

$$t_i-\chi(w(i-1)<m)+\chi(w(i)<m)\in\mathbb{N} \quad \text{for } 1\le i\le s, \tag{5.13b}$$

where $\mathbb{N}=\{0,1,2,\dots\}$. Since $w(s)>0$, it is clear that $\chi(0<m)\geq\chi(w(s)<m)$. Together with the condition $\sum_{j=1}^{s}t_j\leq b_{s+1}$, this implies that (5.13a) holds. To also show that (5.13b) holds, it suffices to show that if $\chi(w(i)<m)=0$ and $\chi(w(i-1)<m)=1$ for some i, then t_i is a positive integer. If $\chi(w(i)<m)=0$ and $\chi(w(i-1)<m)=1$, then $w(i)\geq m$ and $w(i-1)<m$ respectively. Hence $w(i-1)<w(i)$. It follows that $t_i>0$ by the conditions on the t_i in item (3) of Lemma 5.4. Consequently, (5.13b) holds as well. Since (5.13a) and (5.13b) hold, and $(1-q^n)/(1-q)=1+\dots+q^{n-1}$ for $n\in\mathbb{N}$, we may conclude that $L=q^{n_1}+\dots+q^{n_p}$ where p is given by

$$\begin{aligned}p&=b_{s+1}-\sum_{j=1}^{s}t_j+\chi(0<m)-\chi(w(s)<m)+\sum_{i=1}^{s}\Big(t_i-\chi(w(i-1)<m)+\chi(w(i)<m)\Big)\\&=b_{s+1},\end{aligned}$$

completing the proof. □

5.5. **Determination of the roots of $D_{v,v^+}(a,m)$.** In this subsection, we will determine all the roots of $D_{v,v^+}(a,m)$ if v is a composition and $v_0=\max\{v\}$ has multiplicity one in v. More precisely, $D_{v,v^+}(a,m)$ vanishes for $-a_0\in\{0,1,\dots,a_1+\dots+a_n+v_0-1\}$.

Since $D_{v,\lambda}(a,m)$ is a polynomial in q^{a_0} by Proposition 5.3, we can extend the definition of a_0 to all integers. In this subsection, we are concerned with $D_{v,v^+}(a,m)$ for a_0 a non-positive integer. Thus, for simplicity we denote the Laurent series of $D_{v,v^+}(a,m)$ by $Q(-a_0)$ if a_0 is a non-positive integer. That is, for d a non-negative integer

$$\begin{aligned}Q(d)=x^{-v}h_{v^+}&\bigg[\frac{1-q^{-d}}{1-q}x_0q^{-\chi(0<m)}+\sum_{i=1}^{n}\frac{1-q^{a_i}}{1-q}x_iq^{-\chi(i<m)}\bigg]\\&\times\prod_{i=1}^{n}\frac{(qx_i/x_0)_{a_i}}{(q^{-d}x_0/x_i)_d}\prod_{1\leq i<j\leq n}(x_i/x_j)_{a_i}(qx_j/x_i)_{a_j},\end{aligned}\tag{5.14}$$

and

$$\operatorname*{CT}_x Q(d)=D_{v,v^+}(a,m)|_{a_0=-d}.$$

Therefore, instead of determining the roots of $D_{v,v^+}(a,m)$, we prove

$$\operatorname*{CT}_x Q(d)=0\quad\text{for } d\in\{0,1,\dots,a_1+\dots+a_n+v_0-1\}.$$

Here and in the following of this subsection, we assume that v is a composition, $v_0=\max\{v\}$ has multiplicity one in v, and $m\in\{0,1,\dots,n+1\}$, unless specified otherwise. Furthermore, we assume n is a positive integer, since the $n=0$ case for $Q(d)$ is trivial.

We begin by showing that $\mathrm{CT}_x\,Q(0)=0$. Since v_0 is the unique largest part of v, it is a positive integer. By a degree consideration in x_0 of $Q(0)$, it is easy to see that $\mathrm{CT}_x\,Q(0)=0$. In the remainder of this subsection, we will prove $\mathrm{CT}_x\,Q(d)=0$ for $d\in\{1,\dots,a_1+\dots+a_n+v_0-1\}$ by combining the Gessel–Xin method with plethystic substitutions. The main process of the Gessel–Xin method is to recursively apply Lemma 4.1 to a rational function of the form (4.2) to extract the constant term in one variable each time, until eliminating all the variables of the rational function.

To apply Lemma 4.1 to $Q(d)$, we need to show that $Q(d)$ is of the form (4.2) with respect to x_0. The denominator of $Q(d)$ — $\prod_{i=1}^{n}(q^{-d}x_0/x_i)_d$ — is of the form

$$\prod_{r=1}^{nd}(1-x_0/c_r x_{i_r}),$$

which has degree nd in x_0. Here $c_1,\dots,c_{nd}\in K\setminus\{0\}$ satisfy $c_r\neq c_s$ if $x_{i_r}=x_{i_s}$. To get the degree in x_0 of the numerator of $Q(d)$, we need the next result.

Proposition 5.6. *Let d and r be non-negative integers, and $\{n_1,\dots,n_d\}$ be a set of integers. For z an arbitrary letter and Y an alphabet independent of z,*

$$h_r\big[-(q^{n_1}+\cdots+q^{n_d})z+Y\big] \tag{5.15}$$

is a polynomial in z of degree at most $\min\{r,d\}$. In particular, if $Y=0$ and $d<r$ then (5.15) *vanishes.*

Proof. By (3.4) we can expand (5.15) as

$$\sum_{i=0}^{r} z^i h_i\big[-(q^{n_1}+\cdots+q^{n_d})\big]h_{r-i}[Y].$$

By (3.5) it becomes

$$\sum_{i=0}^{r}(-z)^i e_i\big[q^{n_1}+\cdots+q^{n_d}\big]h_{r-i}[Y].$$

Since $e_i\big[q^{n_1}+\cdots+q^{n_d}\big]=0$ for $i>d$ by (3.3), the above sum reduces to

$$\sum_{i=0}^{\min\{r,d\}}(-z)^i e_i\big[q^{n_1}+\cdots+q^{n_d}\big]h_{r-i}[Y],$$

which is a polynomial in z of degree at most $\min\{r,d\}$.

If $Y=0$ then (5.15) becomes

$$h_r\big[-(q^{n_1}+\cdots+q^{n_d})z\big]=(-z)^r e_r\big[q^{n_1}+\cdots+q^{n_d}\big].$$

The above equation holds by (3.5). It vanishes for $d<r$ by (3.3). □

For d a positive integer

$$\frac{1-q^{-d}}{1-q}=-(q^{-1}+q^{-2}+\cdots+q^{-d}).$$

Thus, for r a non-negative integer

$$h_r\left[\frac{1-q^{-d}}{1-q}x_0 q^{-\chi(0<m)}+\sum_{i=1}^{n}\frac{1-q^{a_i}}{1-q}x_i q^{-\chi(i<m)}\right]$$

is of the form (5.15) with

$$x_0 \mapsto z, \quad \sum_{i=1}^{n} \frac{1-q^{a_i}}{1-q} x_i q^{-\chi(i<m)} \mapsto Y, \quad n_i = -i - \chi(0<m) \text{ for } i = 1, \dots, d.$$

By Proposition 5.6 it is a polynomial in x_0 of degree at most $\min\{r, d\}$. It follows that the Laurent polynomial in x_0 of the numerator of $Q(d)$

$$x^{-v} h_{v^+} \left[\frac{1-q^{-d}}{1-q} x_0 q^{-\chi(0<m)} + \sum_{i=1}^{n} \frac{1-q^{a_i}}{1-q} x_i q^{-\chi(i<m)} \right] \prod_{i=1}^{n} (qx_i/x_0)_{a_i} \cdot D$$

has degree

$$\sum_{i=0}^{n} \min\{v_i, d\} - v_0$$

in x_0. Here

$$D := \prod_{1 \le i < j \le n} (x_i/x_j)_{a_i} (qx_j/x_i)_{a_j}$$

is independent of x_0. For n a positive integer the degree

$$\sum_{i=0}^{n} \min\{v_i, d\} - v_0 \le v_1 - v_0 + nd < nd.$$

The last inequality holds because $v_1 < v_0$ by the fact that v_0 is the unique largest part of v. The above inequality shows that the degree in x_0 of the numerator of $Q(d)$ — $\sum_{i=0}^{n} \min\{v_i, d\} - v_0$ — is strictly less than nd, the degree in x_0 of the denominator of $Q(d)$. Therefore, $Q(d)$ is of the form (4.2). Then, by applying Lemma 4.1 to $Q(d)$ with respect to x_0 we obtain

$$\operatorname*{CT}_{x_0} Q(d) = \sum_{\substack{0<u_1\le n, \\ 1\le k_1 \le d}} Q(d \mid u_1; k_1), \tag{5.16}$$

where

$$Q(d \mid u_1; k_1) = Q(d) \Big(1 - \frac{x_0}{x_{u_1} q^{k_1}}\Big)\Big|_{x_0 = x_{u_1} q^{k_1}}.$$

For each term in (5.16) we extract the constant term in x_{u_1}, and then perform further constant term extractions, eliminating one variable at each step. In order to keep track of the terms we obtain, we introduce some notation from [8].

Let f be a rational function of $x_0, \dots, x_n$. For s a positive integer, let $k = (k_1, \dots, k_s)$ and $u = (u_1, \dots, u_s)$ be compositions such that $0 < u_1 < \cdots < u_s \le n$. Define $E_{u,k} f$ to be the result of replacing x_{u_i} in f with $x_{u_s} q^{k_s - k_i}$ for $i = 0, \dots, s-1$, where $u_0 = k_0 := 0$. Then, for d a positive integer and $0 < k_i \le d$,

$$Q(d \mid u; k) := Q(d \mid u_1, \dots, u_s; k_1, \dots, k_s) = E_{u,k} \bigg(Q(d) \prod_{i=1}^{s} \Big(1 - \frac{x_0}{x_{u_i} q^{k_i}}\Big) \bigg). \tag{5.17}$$

Note that the product on the right-hand side of (5.17) cancels all the factors in the denominator of $Q(d)$ that would be taken to zero by $E_{u,k}$.

Lemma 5.7. *Let v be a composition such that v_0 is its unique largest part, $a_1,\dots,a_n$ be non-negative integers and $m\in\{0,1,\dots,n+1\}$. For $d\in\{1,\dots,a_1+\cdots+a_n+v_0-1\}$, the rational functions $Q(d\mid u;k)$ defined as in (5.17) have the following properties:*

(i) *If $1\le k_i\le a_{u_1}+\cdots+a_{u_s}$ for all i with $1\le i\le s\le n$, then $Q(d\mid u;k)=0$.*
(ii) *If $k_i>a_{u_1}+\cdots+a_{u_s}$ for some $i\in\{1,\dots,s\}$, then*

$$\operatorname*{CT}_{x_{u_s}} Q(d\mid u;k)=\begin{cases}\displaystyle\sum_{\substack{u_s<u_{s+1}\le n,\\ 1\le k_{s+1}\le d}} Q(d\mid u_1,\dots,u_s,u_{s+1};k_1,\dots,k_s,k_{s+1}) & \text{for } u_s<n;\\ 0 & \text{for } u_s=n.\end{cases} \tag{5.18}$$

In particular,

$$\operatorname*{CT}_{x_n} Q(d\mid 1,\dots,n;k_1,\dots,k_n)=0. \tag{5.19}$$

Proof of (i). Taking $b_i\mapsto a_{u_i}$ for $i=1,\dots,s$ and $b_{s+1}=0$ in Lemma 5.4, we have the following result. If $1\le k_i\le a_{u_1}+\cdots+a_{u_s}$ for $i=1,\dots,s$, then either $1\le k_i\le a_{u_i}$ for some i, or $-a_{u_j}\le k_i-k_j\le a_{u_i}-1$ for some $i<j$. If $1\le k_i\le a_{u_i}$ for some i, then $Q(d\mid u;k)$ has the factor

$$E_{u,k}\Big((qx_{u_i}/x_0)_{a_{u_i}}\Big)=\Big(\frac{x_{u_s}q^{k_s-k_i}}{x_{u_s}q^{k_s}}q\Big)_{a_{u_i}}=(q^{1-k_i})_{a_{u_i}}=0.$$

If $-a_{u_j}\le k_i-k_j\le a_{u_i}-1$ for some $i<j$, then $Q(d\mid u;k)$ has the factor

$$E_{u,k}\Big((x_{u_i}/x_{u_j})_{a_{u_i}}(qx_{u_j}/x_{u_i})_{a_{u_j}}\Big)=E_{u,k}\Big(q^{\binom{a_{u_j}+1}{2}}(-x_{u_j}/x_{u_i})^{a_{u_j}}(q^{-a_{u_j}}x_{u_i}/x_{u_j})_{a_{u_i}+a_{u_j}}\Big),$$

which is equal to

$$q^{\binom{a_{u_j}+1}{2}}(-q^{k_i-k_j})^{a_{u_j}}(q^{k_j-k_i-a_{u_j}})_{a_{u_i}+a_{u_j}}=0.$$

Proof of (ii). Note that since $d\ge k_i$ for all i, the hypothesis implies that $d>a_{u_1}+\cdots+a_{u_s}$. Let

$$d=\sum_{i=1}^{s}a_{u_i}+b \tag{5.20}$$

for a positive integer b. Then $1\le k_i\le\sum_{i=1}^{s}a_{u_i}+b$ for all $i=1,\dots,s$. If we take $b_i\mapsto a_{u_i}$ for $1\le i\le s$ and $b_{s+1}\mapsto b$ in Lemma 5.4, then at least one of the following three cases holds:

(1) $1\le k_i\le a_{u_i}$ for some i with $1\le i\le s$;
(2) $-a_{u_j}\le k_i-k_j\le a_{u_i}-1$ for some $1\le i<j\le s$;
(3) there exists a permutation $w\in\mathfrak{S}_s$ and a composition $(t_1,\dots,t_s)$ such that

$$k_{w(j)}-k_{w(j-1)}=a_{u_{w(j)}}+t_j\quad\text{for } 1\le j\le s.$$

Here $k_0=w(0):=0$, the t_i satisfy $\sum_{j=1}^{s}t_j\le b$ and $t_j>0$ if $w(j-1)<w(j)$ for $1\le j\le s$.

If either (1) or (2) holds, then $Q(d\mid u;k)=0$ for $1\le s\le n$ by the same argument as that in part (i). In addition, (5.18) holds if the k_i satisfy (1) or (2) since both sides vanish. It remains to show that (5.18) holds if the k_i satisfy (3). We discuss this according to the following three cases: (a) $s=n$; (b) $1\le s<n$ and $u_s=n$; (c) $1\le s<n$ and $u_s<n$.

If $s=n$, then $u_i=i$ for $i=1,\dots,n$. In this case, we prove (5.18) by showing that

$$\operatorname*{CT}_{x_n} Q(d\mid 1,\dots,n;k_1,\dots,k_n)=0 \tag{5.21}$$

for the k_i satisfy (3) and $d>a_1+\cdots+a_n$. By (5.20) if $s=n$ then $b=d-a_1-\cdots-a_n$. Together with $d<a_1+\cdots+a_n+v_0$ yields $b<v_0$ for $s=n$. If the k_i satisfy (3), then by Proposition 5.5 with $s\mapsto n, b_i\mapsto a_i$ for $i=1,\dots,n$ and $b_{n+1}\mapsto b$,

$$x_m^{(a)}\Big|_{\substack{a_0=-\sum_{i=1}^n a_i-b,\\ x_i=x_nq^{k_n-k_i},0\le i\le n}}$$

is of the form

$$-(q^{n_1}+\cdots+q^{n_b})x_n.$$

Here $\{n_1,\dots,n_b\}$ is a set of integers. It follows that

$$h_{v_0}\big(x_m^{(a)}\big)\Big|_{\substack{a_0=-\sum_{i=1}^n a_i-b,\\ x_i=x_nq^{k_n-k_i},0\le i\le n}} \tag{5.22}$$

is of the form

$$h_{v_0}\big[-x_n(q^{n_1}+\cdots+q^{n_b})\big],$$

which vanishes for $b<v_0$ by Proposition 5.6. Therefore, $Q(d|1,\dots,n;k_1,\dots,k_n)=0$ because it has (5.22) as a factor. Consequently, (5.21) holds.

To prove (5.18) for $1\le s<n$ and the k_i satisfy (3), we need Proposition 5.8. It shows that $Q(d\mid u;k)$ is a rational function of the form (4.2) with respect to x_{u_s}. Then we can apply Lemma 4.1 to eliminate the variable x_{u_s} in $Q(d\mid u;k)$.

If $1\le s<n$ and $u_s=n$, then applying Lemma 4.1 yields $\mathrm{CT}_{x_n} Q(d\mid u;k)=0$, since there is no variable in $Q(d\mid u;k)$ with index larger than n. Therefore, (5.18) holds for this case.

For $1\le s<n$ and $u_s<n$, (5.18) holds if we can show that

$$\operatorname*{CT}_{x_{u_s}} Q(d\mid u;k)=\sum_{\substack{u_s<u_{s+1}\le n,\\ 1\le k_{s+1}\le d}} Q(d\mid u_1,\dots,u_s,u_{s+1};k_1,\dots,k_s,k_{s+1}). \tag{5.23}$$

For any rational function F of x_{u_s} and integers j and z, let $T_{j,z}F$ be the result of replacing x_{u_s} with $x_jq^{z-k_s}$ in F. Since $Q(d\mid u;k)$ is a rational function of the form (4.2) with respect to x_{u_s} by Proposition 5.8, applying Lemma 4.1 gives

$$\operatorname*{CT}_{x_{u_s}} Q(d\mid u;k)=\sum_{\substack{u_s<u_{s+1}\le n\\ 1\le k_{s+1}\le d}} T_{u_{s+1},k_{s+1}}\bigg(Q(d\mid u;k)\Big(1-\frac{x_{u_s}q^{k_s}}{x_{u_{s+1}}q^{k_{s+1}}}\Big)\bigg). \tag{5.24}$$

To prove (5.23), it suffices to show that

$$Q(d\mid u';k')=T_{u_{s+1},k_{s+1}}\bigg(Q(d\mid u;k)\Big(1-\frac{x_{u_s}q^{k_s}}{x_{u_{s+1}}q^{k_{s+1}}}\Big)\bigg),$$

where $u' = (u_1, \ldots, u_s, u_{s+1})$ and $k' = (k_1, \ldots, k_s, k_{s+1})$. The equality follows easily from the identity

$$T_{u_{s+1},k_{s+1}} \circ E_{u,k} = E_{u',k'}. \tag{5.25}$$

To see that (5.25) holds, we have

$$(T_{u_{s+1},k_{s+1}} \circ E_{u,k})\, x_{u_i} = T_{u_{s+1},k_{s+1}} \left(x_{u_s} q^{k_s - k_i} \right) = x_{u_{s+1}} q^{k_{s+1}-k_i} = E_{u',k'}\, x_{u_i},$$

and if $j \notin \{u_0, u_1, \ldots, u_s\}$ then $(T_{u_{s+1},k_{s+1}} \circ E_{u,k})\, x_j = x_j = E_{u',k'}\, x_j$. □

We complete the proof of Lemma 5.7 by proving the next proposition.

Proposition 5.8. *Let $a_1, \ldots, a_n$ be non-negative integers. Let s, d, u, k and $Q(d \mid u; k)$ be defined as in* (5.17) *such that $1 \le s < n$ and $d > a_{u_1} + \cdots + a_{u_s}$. If the k_i satisfy (3) in the proof of Lemma 5.7, then $Q(d \mid u; k)$ is a rational function of the form* (4.2) *with respect to x_{u_s}.*

Proof. Write $Q(d \,|\, u; k)$ as N/D, in which N (the "numerator") is

$$E_{u,k} \Bigg(h_{v^+} \left[\frac{1-q^{-d}}{1-q} x_0 q^{-\chi(0<m)} + \sum_{i=1}^{n} \frac{1-q^{a_i}}{1-q} x_i q^{-\chi(i<m)} \right] \times x^{-v} \prod_{j=1}^{n} (q x_j / x_0)_{a_j} \prod_{\substack{1 \le i,j \le n \\ j \ne i}} (q^{\chi(i>j)} x_i / x_j)_{a_i} \Bigg),$$

and D (the "denominator") is

$$E_{u,k} \left(\prod_{j=1}^{n} (q^{-d} x_0 / x_j)_d \Big/ \prod_{i=1}^{s} (1 - q^{-k_i} x_0 / x_{u_i}) \right).$$

Since $d > a_{u_1} + \cdots + a_{u_s}$, let $d = a_{u_1} + \cdots + a_{u_s} + b$ for a positive integer b. Notice that

$$E_{u,k} \left(h_{v^+} \left[\frac{1-q^{-d}}{1-q} x_0 q^{-\chi(0<m)} + \sum_{i=1}^{n} \frac{1-q^{a_i}}{1-q} x_i q^{-\chi(i<m)} \right] \right) \tag{5.26}$$

can be written as

$$h_{v^+}(x_m^{(a)}) \Big|_{\substack{a_0 = -d = -\sum_{i=1}^{s} a_{u_i} - b, \\ x_{u_i} = x_{u_s} q^{k_s - k_i}, 0 \le i \le s}}.$$

It is of the form

$$h_{v^+} \left[-x_{u_s} (q^{n_1} + \cdots + q^{n_b}) + Y \right]$$

by Proposition 5.5 with $b_i \mapsto a_{u_i}$ for $i = 1, \ldots, s$ and $b_{s+1} \mapsto b$ if the k_i satisfy (3) in the proof of Lemma 5.7. Here $\{n_1, \ldots, n_b\}$ is a set of integers and $Y = \sum_{i \notin U} (1 - q^{a_i}) x_i q^{-\chi(i<m)}/(1-q)$ is an alphabet independent of x_{u_s}, where $U := \{u_0, u_1, \ldots, u_s\}$ and $u_0 = 0$. Thus, (5.26) is a

polynomial in x_{u_s} of degree at most $\sum_{i=0}^{n}\min\{v_i,b\}$ by Proposition 5.6. It follows that the parts of N contributing to the degree in x_{u_s},

$$E_{u,k}\Bigg(h_{v^+}\bigg[\frac{1-q^{-d}}{1-q}x_0q^{-\chi(0<m)}+\sum_{i=1}^{n}\frac{1-q^{a_i}}{1-q}x_iq^{-\chi(i<m)}\bigg]\\ \times\prod_{i\in U}x_i^{-v_i}\prod_{i=1}^{s}\prod_{j\notin U}(q^{\chi(u_i>j)}x_{u_i}/x_j)_{a_{u_i}}\Bigg),$$

has degree at most

$$\sum_{i=0}^{n}\min\{v_i,b\}-\sum_{i\in U}v_i+(n-s)(a_{u_1}+\cdots+a_{u_s}).$$

The parts of D contributing to the degree in x_{u_s} are

$$E_{u,k}\bigg(\prod_{j\notin U}(q^{-d}x_0/x_j)_d\bigg),$$

which has degree $(n-s)d$. Let TD be the difference between the degrees in x_{u_s} of N and D.

$$\begin{aligned}TD:&=\sum_{i=0}^{n}\min\{v_i,b\}-\sum_{i\in U}v_i+(n-s)(a_{u_1}+\cdots+a_{u_s}-d)\\ &=\sum_{i=0}^{n}\min\{v_i,b\}-\sum_{i\in U}v_i-(n-s)b.\end{aligned}$$

Denote $\lambda=(\lambda_0,\dots,\lambda_n)=v^+$. Since v_0 is the unique largest part of v, $\lambda_0=v_0>\lambda_i$ for $i=1,\dots,n$. For $1\le s<n$

$$TD\le\sum_{i=n-s}^{n}\min\{\lambda_i,b\}-\lambda_0-\sum_{i=n-s+1}^{n}\lambda_i\le\sum_{i=n-s}^{n}\lambda_i-\lambda_0-\sum_{i=n-s+1}^{n}\lambda_i=\lambda_{n-s}-\lambda_0<0.$$

Consequently, $Q(d\mid u;k)$ is a rational function of the form (4.2) with respect to x_{u_s}. ☐

Now we are ready to determine the roots of $D_{v,v^+}(a,m)$.

Lemma 5.9. *Let $(a_1,\dots,a_n)$ and $v=(v_0,\dots,v_n)$ be compositions such that $v_0=\max\{v\}$ has multiplicity one in v. For $-a_0\in\{0,1,\dots,a_1+\cdots+a_n+v_0-1\}$ and $m\in\{0,1,\dots,n+1\}$,*

$$D_{v,v^+}(a,m)=0. \tag{5.27}$$

Note that $D_{v,v^+}(a,m)$ is a polynomial in q^{a_0} of degree at most $a_1+\cdots+a_n+v_0$ for fixed non-negative integers $a_1,\dots,a_n$ by Proposition 5.3. Assuming the conditions of Lemma 5.9, for $D_{v,v^+}(a,m)$ viewed as a polynomial in q^{a_0}, we find all its roots.

Proof. Since $\operatorname*{CT}_x Q(-a_0)=D_{v,v^+}(a,m)$, we prove the lemma by showing that

$$\operatorname*{CT}_x Q(d)=0$$

for $d \in \{0, \dots, a_1 + \cdots + a_n + v_0 - 1\}$ under the assumptions of this lemma. We have shown that $\operatorname{CT}_x Q(0) = 0$.

We prove by induction on $n - s$ that

$$\operatorname{CT}_x Q(d \mid u; k) = 0 \quad \text{for } d \in \{1, \dots, a_1 + \cdots + a_n + v_0 - 1\}.$$

When $s = 0$ this is what we need. Note that taking constant term with respect to a variable that does not appear has no effect. We may assume that $s \le n$ and $0 < u_1 < \cdots < u_s \le n$, since otherwise $Q(d \mid u; k)$ is not defined. If $s = n$ then $u_i = i$ for $i = 1, 2, \dots, n$. Thus,

$$\operatorname{CT}_x Q(d \mid u_1, \dots, u_n; k_1, \dots, k_n) = \operatorname{CT}_x Q(d \mid 1, \dots, n; k_1, \dots, k_n) = 0$$

for $d \in \{1, \dots, a_1 + \cdots + a_n + v_0 - 1\}$ by (5.19).

Now suppose $0 \le s < n$. If part (i) of Lemma 5.7 applies, then $Q(d \mid u; k) = 0$. Otherwise, part (ii) of Lemma 5.7 applies and (5.18) holds. Therefore, applying CT to both sides of (5.18) gives

$$\operatorname{CT}_x(d \mid u; k) = \begin{cases} \displaystyle\sum_{\substack{u_s < u_{s+1} \le n \\ 1 \le k_{s+1} \le d}} \operatorname{CT}_x Q(d \mid u_1, \dots, u_s, u_{s+1}; k_1, \dots, k_s, k_{s+1}) & \text{for } u_s < n, \\ 0 & \text{for } u_s = n. \end{cases}$$

By induction, every term in the above sum is zero, and so is the sum. $\square$

Note that we can obtain a more general result by the similar argument as that about $D_{v,v^+}(a, m)$ in this subsection: Let $\lambda = (\lambda_0, \lambda_1, \dots)$ be a partition and $v = (v_0, v_1, \dots, v_n)$ be a composition such that $\lambda \ge v^+$, $v_0 = \max\{v\}$ and $\lambda_0 > \max\{v_i \mid i = 1, \dots, n\}$. Then

$$D_{v,\lambda}(a, m) = 0 \quad \text{for } -a_0 \in \{0, \dots, a_1 + \cdots + a_n + \lambda_0 - 1\}. \tag{5.28}$$

Furthermore, for $D_{v,\lambda}(a, m)$ viewed as a polynomial in q^{a_0}, if $\lambda_0 > v_0$ then the number of roots exceeds its degree. It follows that the polynomial $D_{v,\lambda}(a, m)$ is identically zero. Together with (5.4), we can conclude that $D_{v,\lambda}(a, m) \equiv 0$ for a partition λ and a composition v such that $\lambda \ge v^+$ and $\lambda_0 > \max\{v\}$. This contains the vanishing part of Conjecture 1.1 as a special case. For $v \in \mathbb{Z}^{n+1}$, the argument about $D_{v,\lambda}(a, m)$ in this subsection is no longer valid in general. For these cases, Cai obtained an orthogonality result, see Proposition 5.10 in the next subsection.

5.6. **The value of $D_{v,v^+}(a, m)$ at $a_0 = 1$.** To determine $D_{v,v^+}(a, m)$, the last step is to obtain its one non-vanishing value at an additional point. In this subsection, we characterize the value of $D_{v,v^+}(a, m)$ at $a_0 = 1$, and complete the proof of Theorem 5.1. We need a few results first.

By (5.4), it is easy to see that Theorem 1.2 by Cai is equivalent to the next result.

Proposition 5.10. *Let $v \in \mathbb{Z}^{n+1}$ and λ a partition such that $|v| = |\lambda|$. If $D_{v,\lambda}(a, m)$ is non-vanishing, then $v^+ \ge \lambda$.*

Note that we have a way to avoid using Proposition 5.10 hinted by Cai's result in this paper, but the method is too complicated to present here.

By Proposition 5.10, we show that the following constant terms vanish.

Lemma 5.11. *Let $(a_1,\dots,a_n)$ and $(v_1,\dots,v_n)$ be compositions. For r an integer such that $r>\max\{v_i\mid i=1,\dots,n\}+1$,*

(5.29)
$$\operatorname*{CT}_x \frac{h_r[\hat{x}_m^{(a)}]}{x_0^r x_1^{v_1}\cdots x_n^{v_n}}\prod_{i=1}^n h_{v_i}[x_0/q+\hat{x}_m^{(a)}](1-x_0/x_i)(qx_i/x_0)_{a_i}\prod_{1\le i<j\le n}(x_i/x_j)_{a_i}(qx_j/x_i)_{a_j}=0,$$

where

$$\hat{x}_m^{(a)}:=\sum_{i=1}^n \frac{1-q^{a_i}}{1-q}\,x_i q^{-\chi(i<m)},$$

and $m\in\{1,2,\dots,n+1\}$.

Proof. By repeated use of (3.4) with $X\mapsto x_0/q$ and $Y\mapsto \hat{x}_m^{(a)}$ we can expand $\prod_{i=1}^n h_{v_i}[x_0/q+\hat{x}_m^{(a)}]$ as

$$\sum_{\substack{0\le k_i\le v_i\\ 1\le i\le n}}(x_0/q)^{\sum_{i=1}^n k_i}\prod_{i=1}^n h_{v_i-k_i}[\hat{x}_m^{(a)}].$$

Together with the expansion

$$\prod_{i=1}^n(1-x_0/x_i)=\sum_{s=0}^n\sum_{1\le t_1<\cdots<t_s\le n}(-1)^s\frac{x_0^s}{x_{t_1}\cdots x_{t_s}},$$

the constant term in (5.29) becomes

$$\sum(-1)^s q^z \operatorname*{CT}_x\frac{h_\mu[\hat{x}_m^{(a)}]}{x_0^{\Delta}x_1^{v_1}\cdots x_n^{v_n}x_{t_1}\cdots x_{t_s}}\prod_{i=1}^n(qx_i/x_0)_{a_i}\prod_{1\le i<j\le n}(x_i/x_j)_{a_i}(qx_j/x_i)_{a_j}, \tag{5.30}$$

where

$$z=-\sum_{i=1}^n k_i,\ \ \mu=(r,v_1-k_1,\dots,v_n-k_n)^+,\ \text{ and }\Delta=r-s-\sum_{i=1}^n k_i,$$

and the sum is over all integers k_i, t_i and s such that $0\le k_i\le v_i$ for $1\le i\le n$, $0\le s\le n$ and $1\le t_1<\cdots<t_s\le n$. Let C be a constant term in (5.30). We show that every C equals 0 according to the sign of Δ.

If $\Delta>0$, then $C=0$ by a degree consideration in x_0.

If $\Delta=0$, then

$$C=\operatorname*{CT}_x\frac{h_\mu[\hat{x}_m^{(a)}]}{x_1^{v_1}\cdots x_n^{v_n}x_{t_1}\cdots x_{t_s}}\prod_{i=1}^n(qx_i/x_0)_{a_i}\prod_{1\le i<j\le n}(x_i/x_j)_{a_i}(qx_j/x_i)_{a_j}.$$

By a degree consideration in x_0, it reduces to

$$C = \operatorname*{CT}_x \frac{h_\mu[\hat{x}_m^{(a)}]}{x_1^{v_1}\cdots x_n^{v_n}x_{t_1}\cdots x_{t_s}} \prod_{1\le i<j\le n}(x_i/x_j)_{a_i}(qx_j/x_i)_{a_j}.$$

Let $w=(w_1,\dots,w_n)$ be the vector such that $x_1^{w_1}\cdots x_n^{w_n} = x_1^{v_1}\cdots x_n^{v_n}x_{t_1}\cdots x_{t_s}$. Then C can be written as

$$D_{w,\mu}(a^{(0)}, m-1).$$

Since the largest part of μ is r, which exceeds the largest part of w (that is at most $\max\{v_i|i=1,\dots,n\}+1$), $w^+\ge\mu$ can not hold. Thus, we can conclude $C=0$ by Proposition 5.10.

For $\Delta<0$, let $w=(w_0,w_1,\dots,w_n)$ be the vector such that $x^w = x_0^{\Delta}x_1^{v_1}\cdots x_n^{v_n}x_{t_1}\cdots x_{t_s}$. Then

$$C = D_{w,\mu}(a,m)\mid_{a_0=0}.$$

Because of the same reason as that for the $\Delta=0$ case, $w^+\ge\mu$ can not hold. Thus, $D_{w,\mu}(a,m)=0$ by Proposition 5.10. It follows that

$$C = D_{w,\mu}(a,m)\mid_{a_0=0} = 0.$$

In summary, every summand in (5.30) equals 0 and so is the sum. □

Using Lemma 5.11 and the generating function of complete symmetric functions (1.3), we can characterize the value of $D_{v,v^+}(a,m)$ at $a_0=1$ for v a composition such that $v_0=\max\{v\}$ has multiplicity one in v.

Lemma 5.12. *Let $v=(v_0,\dots,v_n)$ be a composition such that v_0 is its unique largest part. For $m\in\{1,2,\dots,n+1\}$*

$$D_{v,v^+}(a,m)|_{a_0=1} = q^{-v_0+\sum_{i=1}^{m-1}a_i}D_{v^{(0)},(v^{(0)})^+}(a^{(0)},m-1). \tag{5.31}$$

Proof. By the definition of $D_{v,v^+}(a,m)$, for $m\in\{1,2,\dots,n+1\}$

$$D_{v,v^+}(a,m)|_{a_0=1} = \operatorname*{CT}_x \frac{h_{v^+}\big[x_0/q+\hat{x}_m^{(a)}\big]}{x^v}\prod_{i=1}^n(1-x_0/x_i)(qx_i/x_0)_{a_i}D_n(a^{(0)}),$$

where $\hat{x}_m^{(a)} := \sum_{i=1}^n x_i(1-q^{a_i})q^{-\chi(i<m)}/(1-q)$ and $D_n(a^{(0)}) := \prod_{1\le i<j\le n}(x_i/x_j)_{a_i}(qx_j/x_i)_{a_j}$. By (3.4) with $X\mapsto x_0/q$ and $Y\mapsto\hat{x}_m^{(a)}$, expand $h_{v_0}[x_0/q+\hat{x}_m^{(a)}]$ as

$$\sum_{r=0}^{v_0}(x_0/q)^{v_0-r}h_r[\hat{x}_m^{(a)}].$$

Then

$$D_{v,v^+}(a,m)|_{a_0=1} = q^{-v_0}\sum_{r=0}^{v_0}\operatorname*{CT}_x \frac{h_r[\hat{x}_m^{(a)}]h_{(v^{(0)})^+}[x_0/q+\hat{x}_m^{(a)}]}{(x_0/q)^r x_1^{v_1}\cdots x_n^{v_n}}\prod_{i=1}^n(1-x_0/x_i)(qx_i/x_0)_{a_i}D_n(a^{(0)}).$$

If $r > v_0$, then $r > \max\{v_i \mid i = 1, \dots, n\} + 1$ because v_0 is the unique largest part of v. Hence, by Lemma 5.11 the constant term in the above sum equals 0 if $r > v_0$. It follows that $D_{v,v^+}(a,m)|_{a_0=1}$ can be written as

$$q^{-v_0} \operatorname*{CT}_x \sum_{r=0}^{\infty} \frac{h_r[\hat{x}_m^{(a)}] h_{(v^{(0)})^+}[x_0/q + \hat{x}_m^{(a)}]}{(x_0/q)^r x_1^{v_1} \cdots x_n^{v_n}} \prod_{i=1}^{n} (1 - x_0/x_i)(qx_i/x_0)_{a_i} D_n(a^{(0)}).$$

By the generating function of complete symmetric functions (1.3) this becomes

$$q^{-v_0} \operatorname*{CT}_x \frac{h_{(v^{(0)})^+}[x_0/q + \hat{x}_m^{(a)}] \prod_{i=1}^{n} (1 - x_0/x_i)(qx_i/x_0)_{a_i} D_n(a^{(0)})}{\prod_{i=1}^{m-1} (x_i/x_0)_{a_i} \prod_{i=m}^{n} (qx_i/x_0)_{a_i} x_1^{v_1} \cdots x_n^{v_n}}.$$

Cancelling the same factors yields

$$\begin{aligned} q^{-v_0} \operatorname*{CT}_x & \frac{h_{(v^{(0)})^+}[x_0/q + \hat{x}_m^{(a)}]}{x_1^{v_1} \cdots x_n^{v_n}} \prod_{i=1}^{m-1} \frac{(1 - x_0/x_i)(1 - q^{a_i} x_i/x_0)}{1 - x_i/x_0} \prod_{i=m}^{n} (1 - x_0/x_i) D_n(a^{(0)}) \\ &= q^{-v_0} \operatorname*{CT}_x \frac{h_{(v^{(0)})^+}[x_0/q + \hat{x}_m^{(a)}]}{x_1^{v_1} \cdots x_n^{v_n}} \prod_{i=1}^{m-1} (q^{a_i} - x_0/x_i) \prod_{i=m}^{n} (1 - x_0/x_i) D_n(a^{(0)}). \end{aligned}$$

By a degree consideration in x_0, it further reduces to

$$q^{-v_0 + \sum_{i=1}^{m-1} a_i} \operatorname*{CT}_x \frac{h_{(v^{(0)})^+}[\hat{x}_m^{(a)}]}{x_1^{v_1} \cdots x_n^{v_n}} D_n(a^{(0)}),$$

which can be written as

$$q^{-v_0 + \sum_{i=1}^{m-1} a_i} D_{v^{(0)},(v^{(0)})^+}(a^{(0)}, m-1). \qquad \square$$

Now we obtain all the ingredients for characterizing $D_{v,v^+}(a,m)$ if v is a composition such that $v_0 = \max\{v\}$ has multiplicity one in v, and $m \in \{1, 2, \dots, n+1\}$. By Proposition 5.3, $D_{v,v^+}(a,m)$ is a polynomial in q^{a_0} of degree at most $a_1 + \cdots + a_n + v_0$ for fixed non-negative integers $a_1, \dots, a_n$. By Lemma 5.12

$$D_{v,v^+}(a,m)|_{a_0=1} = q^{-v_0 + \sum_{i=1}^{m-1} a_i} D_{v^{(0)},(v^{(0)})^+}(a^{(0)}, m-1).$$

By Lemma 5.9

$$D_{v,v^+}(a,m) = 0$$

for $-a_0 \in \{0, 1, \dots, a_1 + \cdots + a_n + v_0 - 1\}$. Hence, the above properties uniquely determine $D_{v,v^+}(a,m)$ as

$$D_{v,v^+}(a,m) = q^{-v_0 + \sum_{i=1}^{m-1} a_i} \begin{bmatrix} v_0 + |a| - 1 \\ a_0 - 1 \end{bmatrix} D_{v^{(0)},(v^{(0)})^+}(a^{(0)}, m-1). \tag{5.32}$$

This completes the proof of Theorem 5.1.

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