# AVOIDING A PAIR OF PATTERNS IN MULTISETS AND COMPOSITIONS 

VÍT JELÍNEK, TOUFIK MANSOUR, JOSÉ L. RAMÍREZ, AND MARK SHATTUCK


#### Abstract

In this paper, we study the Wilf-type equivalence relations among multiset permutations. We identify all multiset equivalences among pairs of patterns consisting of a pattern of length three and another pattern of length at most four. To establish our results, we make use of a variety of techniques, including Ferrers-equivalence arguments, sorting by minimal/maximal letters, analysis of active sites and direct bijections. In several cases, our arguments may be extended to prove multiset equivalences for infinite families of pattern pairs. Our results apply equally well to the Wilf-type classification of compositions, and as a consequence, we obtain a complete description of the Wilf-equivalence classes for pairs of patterns of type $(3,3)$ and $(3,4)$ on compositions, with the possible exception of two classes of type $(3,4)$.


## 1. Introduction

A multiset is an unordered collection of elements which may be repeated. A multiset of height $k$ is a multiset $S$ whose elements are positive integers and whose largest element is $k$. A finite multiset may be represented by $S=1^{a_{1}} 2^{a_{2}} \cdots k^{a_{k}}$, where $a_{i} \geq 0$ is the multiplicity of $i$ in $S$, i.e., the number of copies of $i$ in $S$. We call a multiset of height $k$ reduced if each member of $[k]=\{1,2, \ldots, k\}$ appears at least once in $S$, or equivalently, each multiplicity $a_{i}$ is at least 1. We will assume, unless otherwise noted, that the multisets we work with are reduced. The size of a multiset $S$, denoted $|S|$, is the sum of the multiplicities of its elements.
A multipermutation of a multiset $S=1^{a_{1}} 2^{a_{2}} \cdots k^{a_{k}}$ is an arrangement of the elements of $S$ into a sequence. We identify such a multipermutation with a word $\pi=\pi_{1} \pi_{2} \cdots \pi_{n}$ that has exactly $a_{i}$ occurrences of each symbol $i \in[k]$. The height of $\pi$, denoted $\operatorname{ht}(\pi)$, is the height of the underlying multiset, i.e., the maximum of $\pi_{1}, \ldots, \pi_{n}$.
For two multipermutations $\rho=\rho_{1} \cdots \rho_{\ell}$ and $\pi=\pi_{1} \cdots \pi_{n}$, we say that $\pi$ contains $\rho$, if $\pi$ has a subsequence $\pi_{i(1)} \pi_{i(2)} \cdots \pi_{i(\ell)}$ whose elements have the same relative order as $\rho$, i.e., $\pi_{i(a)}<\pi_{i(b)}$ if and only if $\rho_{a}<\rho_{b}$ and $\pi_{i(a)}>\pi_{i(b)}$ if and only if $\rho_{a}>\rho_{b}$ for every $a, b \in[\ell]$. If $\pi$ does not contain $\rho$, it avoids $\rho$. In this context, $\rho$ is usually referred to as a pattern.
For a word $\pi=\pi_{1} \pi_{2} \cdots \pi_{n}$ of height $k$, its reversal is the word $\pi^{r}=\pi_{n} \pi_{n-1} \cdots \pi_{1}$, and its complement is the word $\pi^{c}=k+1-\pi_{1}, k+1-\pi_{2}, \ldots, k+1-\pi_{n}$. Note that the reversal represents the same multiset as $\pi$, while the complement may represent a different one.
For a multiset $S$ and a multipermutation $\rho$, we let $\operatorname{Av}(S ; \rho)$ denote the set of all the multipermutations of $S$ that avoid $\rho$, and we let $\operatorname{av}(S ; \rho)$ be the cardinality of $\operatorname{Av}(S ; \rho)$. Two multipermutations $\rho$ and $\sigma$ are $m$-equivalent, denoted by $\rho \stackrel{m}{\sim} \sigma$, if for every multiset $S$, $\operatorname{av}(S ; \rho)$ equals av $(S ; \sigma)$.

[^0]The notion of $m$-equivalence can be straightforwardly extended to sets of patterns. For instance, suppose that $P$ is a set of multipermutations. We let $\operatorname{Av}(S ; P)$ be the set of multipermutations of $S$ that avoid all the patterns contained in $P$, and we let av $(S ; P)$ be its cardinality. We again call two sets $P$ and $Q$ of multipermutations $m$-equivalent, denoted $P \stackrel{m}{\sim} Q$, if $\operatorname{av}(S ; P)=\operatorname{av}(S ; Q)$ for every multiset $S$. To avoid clutter, we often omit nested braces and write, e.g., $\operatorname{Av}(S ; \pi, \rho)$ instead of $\operatorname{Av}(S ;\{\pi, \rho\})$.
We may easily observe that each multipermutation $\rho$ is $m$-equivalent to its reversal $\rho^{r}$, and that for every pair $\rho$ and $\sigma$ of $m$-equivalent patterns, we also have $\rho^{r} \stackrel{m}{\sim} \sigma^{r}$ and $\rho^{c} \stackrel{m}{\sim} \sigma^{c}$. Moreover, these symmetry relations can be generalized in an obvious manner to equivalences involving sets of patterns.
The notion of $m$-equivalence has been previously studied by Jelínek and Mansour [6], who called it 'strong equivalence'. They focused on the classification of this equivalence for single patterns of fixed size, and they characterized the $m$-equivalence classes of patterns of size at most six. From their results, we will use here the following fact [6, Lemma 2.4].
Fact 1.1 (Jelínek and Mansour [6]). For any $k$, all the patterns that consist of a single symbol ' 1 ', a single symbol ' 3 ' and $k-2$ symbols ' 2 ' are $m$-equivalent.
The main purpose of this paper is to classify the $m$-equivalence for sets of patterns containing a pattern of size three and a pattern of size at most four. This extends earlier results concerning avoidance by multisets of a single permutation [8] or word [4] pattern of length three. With the help of computer enumeration, we identified the plausible $m$-equivalences and have managed to verify all of these $m$-equivalence. This also yields all of the non-singleton $(3,3)$ and $(3,4)$ Wilf-equivalence classes for compositions, up to at most two sporadic cases. Many of our results are based on arguments that generalize to larger patterns. However, to keep the presentation simple, we mostly state our theorems and proofs for the special case of patterns of size up to four, which is our main focus. We point out the possible generalizations separately as remarks.

## 2. Avoidance results for multisets

We may represent words of height $k$ and length $n$ as binary matrices with $k$ rows and $n$ columns and exactly one 1 -cell in each column. We assume that the rows of a matrix are numbered bottom-to-top, and the columns are numbered left-to-right. For a multipermutation $\sigma$ of height $k$, let $M(\sigma)$ be the $k \times n$ matrix with a 1-cell in row $i$ and column $j$ if and only if the $j$-th letter of $\sigma$ is equal to $i$. For example,

$$
M(31321)=\left[\begin{array}{lllll}
1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1
\end{array}\right]
$$

Conversely, if $M$ is a matrix with exactly one 1-cell in each column and at least one 1-cell in each row, then there is a unique reduced multipermutation $\sigma$ such that $M=M(\sigma)$. If there is no risk of confusion, we will identify a multipermutation $\sigma$ with its corresponding matrix $M(\sigma)$, and we will say, for instance, that two matrices $M$ and $M^{\prime}$ are $m$-equivalent, if they represent two $m$-equivalent multipermutations.
The Ferrers diagram (or Ferrers shape) is an array of boxes (called cells) arranged into down-justified columns, which have nonincreasing length. A filling of a Ferrers diagram is an assignment of zeros and ones into its cells. A filling is column-sparse if every column has at most one 1-cell. A filling is sparse if every row and every column has at most one 1-cell.

A transversal filling, or a transversal, is a filling in which every row and every column has exactly one 1-cell. In this paper, we only deal with column-sparse fillings and their various restrictions. We treat binary matrices, i.e. matrices containing only values 0 and 1 , as fillings of rectangular Ferrers diagrams.
The deletion of the $i$-th column in a Ferrers diagram $F$ is the operation that removes from $F$ all the boxes in the $i$-th column, and then shifts the boxes in columns $i+1, i+2, \ldots$ one unit to the left in order to fill the created gap. Note that the deletion transforms $F$ into a smaller Ferrers diagram. Deletion of a row is defined analogously.
A filling $\phi$ of a Ferrers diagram $F$ contains a binary matrix $M$ if $\phi$ can be transformed into $M$ via a sequence of deletions of rows and columns, possibly followed by changing some 1 -cells of $\phi$ into 0 -cells. If $\phi$ does not contain $M$, we say that $\phi$ avoids $M$. Notice that a multipermutation $\sigma$ contains a multipermutation $\rho$ if and only if the matrix $M(\sigma)$, understood as a Ferrers diagram, contains $M(\rho)$ in the sense defined above.
We say that two binary matrices $M_{1}$ and $M_{2}$ are strongly Ferrers-equivalent, denoted $M_{1} \stackrel{s f}{\sim}$ $M_{2}$, if for every Ferrers shape $F$, there is a bijection between $M_{1}$-avoiding and $M_{2}$-avoiding column-sparse fillings of $F$ that preserves the number of 1-cells in each row and column. We also say that $M_{1}$ and $M_{2}$ are Ferrers-equivalent for transversals, or FT-equivalent for short, if for every Ferrers diagram $F$, the number of its $M_{1}$-avoiding transversals is equal to the number of its $M_{2}$-avoiding transversals. This relation is denoted by $M_{1} \stackrel{f t}{\sim} M_{2}$. For a pair of multipermutations $\sigma$ and $\rho$, we will often write $\sigma \stackrel{s f}{\sim} \rho$ or $\sigma \stackrel{f t}{\sim} \rho$ for $M(\sigma) \stackrel{s f}{\sim} M(\rho)$ and $M(\sigma) \stackrel{f t}{\sim} M(\rho)$, respectively. As with $m$-equivalence, we will also extend strong Ferrersequivalence and FT-equivalence from individual patterns to sets of patterns.
Clearly, if two patterns (or sets of patterns) $M_{1}$ and $M_{2}$ are strongly Ferrers-equivalent, then they are also FT-equivalent. Moreover, as the next simple lemma shows, FT-equivalence can always be extended from transversals to general sparse fillings, provided the two patterns have no zero rows or columns.

Lemma 2.1. Suppose that $M_{1}$ and $M_{2}$ are FT-equivalent matrices, and that every row and column of $M_{1}$ and of $M_{2}$ contains at least one 1-cell. Then for every Ferrers diagram $F$, there is a bijection between $M_{1}$-avoiding and $M_{2}$-avoiding sparse fillings of $F$, which has the additional property of preserving the number of 1-cells in each row and column of $F$.

Proof. Let $F$ be a Ferrers shape, and let $\phi$ be an $M_{1}$-avoiding sparse filling of $F$. We delete all the rows and columns of $F$ that have no 1-cell in $\phi$. This transforms the filling $\phi$ of the diagram $F$ into an $M_{1}$-avoiding transversal $\phi^{\prime}$ of a Ferrers diagram $F^{\prime}$. Since $M_{1}$ and $M_{2}$ are FT-equivalent, there is a bijection $B$ that maps $M_{1}$-avoiding transversals of $F^{\prime}$ into $M_{2^{-}}$ avoiding transversals of $F^{\prime}$. We define $\psi^{\prime}:=B\left(\phi^{\prime}\right)$. We then reinsert the rows and columns we deleted in the first step into $\psi^{\prime}$, filling the newly inserted boxes by zeros. This transforms $\psi^{\prime}$ into a sparse filling $\psi$ of the original diagram $F$. Since $M_{2}$ has a 1-cell in every row and column, the insertion of an all-zero row or column into $\psi^{\prime}$ cannot create an occurrence of $M_{2}$. Thus, $\psi$ is an $M_{2}$-avoiding sparse filling of $F$, and we may easily observe that the transformation $\phi \mapsto \psi$ is the required bijection.

Recall that a multipermutation $\sigma$ contains a multipermutation $\rho$ if and only if the matrix $M(\sigma)$ contains $M(\rho)$. Thus, for a multiset $S=1^{a_{1}} 2^{a_{2}} \cdots k^{a_{k}}$ of size $n$ and a pattern $\rho$, there is a bijective correspondence between the set $\operatorname{Av}(S ; \rho)$ of $\rho$-avoiding multipermutations of $S$ and the set of all the $M(\rho)$-avoiding matrices of shape $k \times n$ having exactly one 1-cell
in each column and exactly $a_{i} 1$-cells in row $i$, for each $i \in[k]$. It follows that for two multipermutations $\sigma$ and $\tau, \sigma \stackrel{s f}{\sim} \tau$ implies $\sigma \stackrel{m}{\sim} \tau$.
If $\rho$ is a word and $k$ an integer, we denote by $\rho+k$ the word obtained by increasing each letter of $\rho$ by $k$. Recall that the height $\operatorname{ht}(\rho)$ of a word $\rho$ is the maximum value appearing in $\rho$. For two words $\rho$ and $\tau$, we let their direct sum $\rho \oplus \tau$ be the concatenation of $\rho$ and $\tau+\operatorname{ht}(\rho)$. For instance, for $\rho=122$ and $\tau=312$, we have $\rho \oplus \tau=122534$. For a set of patterns $P=\left\{\alpha^{1}, \alpha^{2}, \ldots, \alpha^{m}\right\}$ and a pattern $\beta$, we write $P \oplus \beta$ as a shorthand for the set $\left\{\alpha^{1} \oplus \beta, \alpha^{2} \oplus \beta, \ldots, \alpha^{m} \oplus \beta\right\}$.
An important feature of the various flavors of Ferrers-equivalence is that they are closed with respect to direct sums. This follows from a standard argument appearing, among others, in the works of Backelin, West and Xin [1, Proposition 2.3] or of Stankova and West [10, Proposition 1] in the context of permutations, and later in the works of Jelínek and Mansour [6, Lemma 2.1] and [5, Lemma 14] in the more general setting of words. We omit repeating the argument here, and merely state the required result as a fact.

Fact $2.2([1,5,6,10])$. Let $P$ and $P^{\prime}$ be two sets of multipermutations, and let $\rho$ be another multipermutation. If $P$ and $P^{\prime}$ are strongly Ferrers-equivalent, then $P \oplus \rho$ and $P^{\prime} \oplus \rho$ are also strongly Ferrers-equivalent. Likewise, if $P$ and $P^{\prime}$ are FT-equivalent, then $P \oplus \rho$ and $P^{\prime} \oplus \rho$ are also FT-equivalent.
2.1. Results based on Ferrers-equivalence arguments. We now state several known results on various forms of Ferrers-equivalence which will be useful for our purposes. The first such result is the strong Ferrers-equivalence, for any $k$, of the increasing pattern $12 \cdots k$ and the decreasing pattern $k(k-1) \cdots 1$. This equivalence has been established by Backelin et al. [1] for transversal fillings, and Krattenthaler [7] then obtained more general results which imply the strong Ferrers-equivalence of the two patterns.
Fact 2.3 (Krattenthaler [7]). For any $k$, the pattern $12 \cdots k$ is strongly Ferrers-equivalent to $k(k-1) \cdots 1$.

Another family of strongly Ferrers-equivalent patterns has been found by Jelínek and Mansour [5, Lemma 39].
Fact 2.4 (Jelínek and Mansour [5]). For any $i, j \geq 0$, the pattern $2^{i} 12^{j}$ is strongly Ferrersequivalent to $12^{i+j}$.

The next result, due to Stankova and West [10], is specific to FT-equivalence.
Fact 2.5 (Stankova and West [10]). The patterns 312 and 231 are FT-equivalent.
In the statement of Fact 2.5, FT-equivalence cannot be directly replaced with strong Ferrersequivalence, as was pointed out by Guo et al. [3]. However, Guo et al. [3] have found a different way of generalizing Fact 2.5 to a strong Ferrers-equivalence result, which we now state.

Fact 2.6 (Guo et al. [3]). We have the following strong Ferrers-equivalences for sets of patterns:

- $\{231,221\} \stackrel{s f}{\sim}\{312,212\}$ and
- $\{231,121\} \stackrel{s f}{\sim}\{312,211\}$.

There is another, simpler way to translate an arbitrary FT-equivalence result into a strong Ferrers-equivalence, which involves the pattern 11. Clearly, a multipermutation $\sigma$ avoids 11 if and only if each of its elements has multiplicity 1, i.e., $\sigma$ is actually a permutation. Similarly, a column-sparse filling of a Ferrers diagram avoids $M(11)$ if and only if each row has at most one 1 -cell, that is, the filling is sparse.
A pair of patterns $\sigma, \rho$ is said to be Wilf-equivalent, denoted $\sigma \stackrel{w}{\sim} \rho$, if for every $n$, the number of permutations of $[n]$ that avoid $\sigma$ is the same as the number of those that avoid $\rho$. Intuitively speaking, strong Ferrers-equivalence refines $m$-equivalence in the same way as FT-equivalence refines Wilf-equivalence. This intuition is made more rigorous by the next easy observation, which can be easily deduced from the definitions and from Lemma 2.1. We omit its proof.

Observation 2.7. For any two multipermutations $\sigma$ and $\rho$, if $\sigma \stackrel{w}{\sim} \rho$ then $\{\sigma, 11\} \stackrel{m}{\sim}\{\rho, 11\}$, and if $\sigma \stackrel{f t}{\sim} \rho$ then $\{\sigma, 11\} \stackrel{s f}{\sim}\{\rho, 11\}$.

A similar observation states that $m$-equivalence, as well as strong Ferrers-equivalence, is preserved when we add a pattern $1^{r}$ for any $r \geq 2$.
Observation 2.8. For any two patterns $\tau$ and $\tau^{\prime}$ and for any $k \geq 2, \tau \stackrel{m}{\sim} \tau^{\prime} \operatorname{implies}\left\{1^{k}, \tau\right\} \stackrel{m}{\sim}$ $\left\{1^{k}, \tau^{\prime}\right\}$, and $\tau \stackrel{s f}{\sim} \tau^{\prime}$ implies $\left\{1^{k}, \tau\right\} \stackrel{s f}{\sim}\left\{1^{k}, \tau^{\prime}\right\}$.

By combining the previous facts and observations, we obtain the following equivalences among pairs involving a pattern of size 3 and a pattern of size 4 .

Proposition 2.9. The following equivalences hold:
(a) $\{122,1111\} \stackrel{m}{\sim}\{212,1111\}$,
(b) $\{111,1223\} \stackrel{m}{\sim}\{111,1232\} \stackrel{m}{\sim}\{111,1322\} \stackrel{m}{\sim}\{111,2123\} \stackrel{m}{\sim}\{111,2132\} \stackrel{m}{\sim}\{111,2213\}$,
(c) $\{111,1233\} \stackrel{m}{\sim}\{111,2133\}$,
(d) $\{111,1234\} \stackrel{m}{\sim}\{111,1243\} \stackrel{m}{\sim}\{111,1432\} \stackrel{m}{\sim}\{111,2134\} \stackrel{m}{\sim}\{111,2143\} \stackrel{m}{\sim}\{111,3214\}$,
(e) $\{123,1111\} \stackrel{m}{\sim}\{132,1111\} \stackrel{\sim}{\sim}\{213,1111\}$,
(f) $\{123,1112\} \stackrel{m}{\sim}\{213,1112\}$,
(g) $\{112,1234\} \stackrel{m}{\sim}\{112,2134\} \stackrel{m}{\sim}\{112,3214\}$,
(h) $\{112,2314\} \stackrel{m}{\sim}\{112,3124\}$.

Proof. Parts (a) to (e) all use Obs. 2.8 to conclude the $m$-equivalence in conjunction with Fact 2.4 for (a), Fact 1.1 for (b) and Facts 2.2 and 2.3 for (c)-(e). Part (f) and the equivalence $\{112,1234\} \stackrel{m}{\sim}\{112,3214\}$ make use of Fact 2.3 and Obs. 2.8 first and then Fact 2.2. The equivalence $\{112,1234\} \stackrel{m}{\sim}\{112,2134\}$ follows from the strong Ferrers-equivalence of 123 and 213, together with Obs. 2.8 and Fact 2.2. Finally, part (h) follows from combining Fact 2.5, Obs. 2.7 and Fact 2.2 in that order.
2.2. Sorting minimal/maximal letter technique. In this subsection, we prove some equivalences by defining bijections which reorder the relevant pattern-avoiding multiset permutations, expressed as words.

Theorem 2.10. The following pair of patterns are m-equivalent:
(1) $\{111,1221\} \stackrel{m}{\sim}\{111,2112\}$,
(2) $\{112,1211\} \stackrel{m}{\sim}\{121,1112\}$.

Proof. (1) Fix a multiset $S=1^{b_{1}} \cdots k^{b_{k}}$, and let $\mathcal{A}$ and $\mathcal{B}$ denote the sets $\operatorname{Av}(S ; 111,1221)$ and $\operatorname{Av}(S ; 111,2112)$, respectively. Given $\lambda \in \mathcal{A}$, let $a_{1}>\cdots>a_{r}$ denote the set of letters within $\lambda$ which occur twice. Let $\rho \in \mathcal{B}$ be obtained from $\lambda$ by replacing each occurrence $a_{i}$ with $a_{r+1-i}$ for $1 \leq i \leq r$, leaving all other letters unchanged in their positions. Note that these other letters must occur once and therefore cannot affect the avoidance of any of the patterns we consider. It is then seen that the mapping $\lambda \mapsto \rho$ is a bijection between $\mathcal{A}$ and $\mathcal{B}$, as desired.
(2) Equivalently, we show $\{122,2212\} \stackrel{m}{\sim}\{212,1222\}$. Let us write $P_{A}=\{122,2212\}$ and $P_{B}=\{212,1222\}$. With $S$ as above, we will describe a bijection between $\operatorname{Av}\left(S ; P_{A}\right)$ and $\operatorname{Av}\left(S ; P_{B}\right)$. We proceed by induction on the number $k=\operatorname{ht}(S)$. Clearly, if $k=1$, the required bijection is the identity mapping, since there is only one multipermutation of $S$, and it avoids all patterns with two or more symbols.
Define now the multiset $S^{\prime}=1^{b_{1}} \cdots(k-1)^{b_{k-1}}$ obtained by removing all copies of $k$ from $S$. By induction, there is a bijection $f_{k-1}$ between $\operatorname{Av}\left(S^{\prime} ; P_{A}\right)$ and $\operatorname{Av}\left(S^{\prime} ; P_{B}\right)$. Let $\lambda^{\prime}$ be a multipermutation from $\operatorname{Av}\left(S^{\prime} ; P_{A}\right)$. We want to insert $b_{k}$ copies of $k$ into $\lambda^{\prime}$ to create a $P_{A^{-}}$ avoiding multipermutation $\lambda$ of $S$. If $b_{k}=1$, then we may insert the symbol $k$ in any position of $\lambda^{\prime}$ while preserving $P_{A}$-avoidance. If $b_{k}=2$, then we observe that one of the copies of $k$ must be the leftmost symbol of $\lambda$ in order to preserve $P_{A}$-avoidance, while the other can be placed arbitrarily. If $b_{k} \geq 3$, then all the symbols $k$ in $\lambda$ must appear consecutively at the leftmost $b_{k}$ positions.
Similarly, when extending a multipermutation $\rho^{\prime} \in \operatorname{Av}\left(S^{\prime} ; P_{B}\right)$ to a multipermutation $\rho \in$ $\operatorname{Av}\left(S ; P_{B}\right)$, we proceed as follows: if $b_{k}=1$, the symbol $k$ can be placed arbitrarily, if $b_{k}=2$, the only restriction is that the two symbols $k$ must appear consecutively, and if $b_{k} \geq 3$, then the symbols $k$ must form the leftmost $b_{k}$ symbols of $\rho$.
The above description shows that in both the $P_{A}$-avoiding and the $P_{B}$-avoiding multipermutations of $S$, the position of all the symbols $k$ is uniquely determined by the position of the rightmost copy of $k$. This yields a straightforward bijection $f_{k}$ between $\operatorname{Av}\left(S ; P_{A}\right)$ and $\operatorname{Av}\left(S ; P_{B}\right)$, defined as follows: fix a $\lambda \in \operatorname{Av}\left(S ; P_{A}\right)$, remove from $\lambda$ all the occurrences of $k$ to obtain a $\lambda^{\prime} \in \operatorname{Av}\left(S^{\prime} ; P_{A}\right)$, define $\rho^{\prime}=f_{k-1}\left(\lambda^{\prime}\right) \in \operatorname{Av}\left(S^{\prime} ; P_{B}\right)$, and finally, let $\rho$ be the unique member of $\operatorname{Av}\left(S ; P_{B}\right)$ in which the rightmost occurrence of $k$ appears at the same position as the rightmost occurrence of $k$ in $\lambda$. We easily see that this provides the required bijection.

Remark: Extending the bijections described above shows more generally

$$
\begin{array}{cc}
\left\{1^{i+1}, \tau\right\} \stackrel{m}{\sim}\left\{1^{i+1}, \tau^{c}\right\}, & i \geq 2, \\
\left\{112,121^{i-1}\right\} \stackrel{m}{\sim}\left\{121,1^{i} 2\right\}, & i \geq 3,
\end{array}
$$

where $\tau$ denotes any permutation of the multiset $1^{i} \cdots k^{i}$ and $\tau^{c}$ is the complement of $\tau$.
2.3. Equivalences by analysis of active sites. In this subsection, we establish several equivalences by considering active sites within multiset permutations and the associated generating trees for the patterns in question. An active site of a parent multipermutation is in general a position in which we may insert one or more copies of a letter in producing its offspring without introducing a given set of patterns. In some instances, it will be convenient to modify this definition somewhat to accommodate the patterns in question. Throughout, we consider permutations of multisets of $[k]$, though at times it will be more convenient notationally to insert either successively smaller or larger letters into a parent permutation in
producing its offspring. For examples of the generating tree method applied to the avoidance problem on ordinary permutations, see, e.g., [11, 12].
We first establish the equivalence of $\{112,2212\}$ and $\{121,2122\}$ via an active site analysis where we successively insert smaller and smaller letters into a parent permutation.

Theorem 2.11. The sets of patterns $\{112,2212\}$ and $\{121,2122\}$ are $m$-equivalent, that is

$$
\{112,2212\} \stackrel{m}{\sim}\{121,2122\} .
$$

Proof. Let $S=k^{a_{1}}(k-1)^{a_{2}} \ldots 1^{a_{k}}$, where $k \geq 1$ and $a_{1}, \ldots, a_{k} \geq 1$ are fixed. We first enumerate members $\pi \in \operatorname{Av}(S ; 112,2212)$. To do so, we consider the various partial permutations $\pi_{i} \in \operatorname{Av}\left(S_{i} ; 112,2212\right)$, where $S_{i}=k^{a_{1}} \cdots(k-i+1)^{a_{i}}$ for $1 \leq i \leq k$. We form the permutations $\pi_{i+1}$ by inserting $a_{i+1}$ copies of $t=k-i$ appropriately into the $\pi_{i}$. By an active site, within a permutation $\pi_{i}$ of the stated form where $1 \leq i \leq k$, we mean a position where one may insert a single copy of the letter $t$ without introducing an occurrence of 2212 (where one may assume $a_{k+1}=1$ in the case $i=k$ ). It is understood that if $a_{i+1}>1$, then all other letters $t$ are to be added at the very end of $\pi_{i}$ in order to avoid 112 .
Let $v_{i}=2$ if $a_{i}=1$ and $v_{i}=3$ if $a_{i} \geq 2$ for $1 \leq i \leq k$. We now show by induction on $i$ that each $\pi_{i} \in \operatorname{Av}\left(S_{i} ; 112,2212\right)$ has exactly $s_{i}=1+\sum_{j=1}^{i}\left(v_{j}-1\right)$ active sites. The $i=1$ case is apparent since there are $v_{1}$ (active) sites in the composition $\pi_{1}=k^{a_{1}}$ corresponding to the very first and very last positions of $\pi_{1}$ for all exponents $a_{1}$ and also to the position directly after the first $k$ if $a_{1} \geq 2$. Now assume that the hypothesis is true in the $i$-case for some $1 \leq i<k$ and we show it holds in the $(i+1)$-case. If $a_{i+1}=1$, then a single $t$ can be inserted into any one of the sites of some $\pi_{i}$ without introducing either pattern, and it is seen that regardless of where $t$ is inserted, the number of sites increases by one (essentially, one of the present sites is split into two). Also, replacing $i$ with $i+1$ in $s_{i}$ raises its value by one since $a_{i+1}=1$, which accounts for the additional site. If $a_{i+1}>1$, then there are two new sites introduced by the insertion of the letters $t$, i.e., one directly following the leftmost added $t$ and another at the very end following the last $t$. Since we have $s_{i+1}-s_{i}=2$ in this case, the induction is complete.
As all $\pi_{i}$ have the same number $s_{i}$ of active sites for each $i$ (with this number depending only on $S$ ), we have that the number of possible $\pi_{i+1}$ is given by the product of $s_{i}$ with the number of $\pi_{i}$ for each $i$. Thus, the number of possible $\pi=\pi_{k} \in \operatorname{Av}(S ; 112,2212)$ of the stated form is given by $\prod_{i=1}^{k-1} s_{i}$. A similar argument whose main details we describe briefly shows that there are the same number of $\rho \in \operatorname{Av}(S ; 121,2122)$. Let $\rho_{i} \in \operatorname{Av}\left(S_{i} ; 121,2122\right)$ for $1 \leq i \leq k$ and consider forming the $\rho_{i+1}$ from the various $\rho_{i}$ by inserting copies of $t$ appropriately. Define active site analogously except that now we insert all letters $u_{i+1}$ at the site as a single run (so as to avoid 121). Reasoning by induction as before, one can show for each $i$ that there are $s_{i}$ sites in all $\rho_{i} \in \operatorname{Av}\left(S_{i} ; 121,2122\right)$, which implies the same product formula as above for the number of possible $\rho=\rho_{k}$.
Theorem 2.12. We have $\{112,2122\} \stackrel{m}{\sim}\{121,1222\}$.
Proof. We show that $|\operatorname{Av}(S ; 112,2122)|=|\operatorname{Av}(S ; 121,1222)|$, where $S$ is as in the preceding proof. Let $\pi_{i}$ denote an arbitrary member of $\operatorname{Av}\left(S_{i} ; 112,2122\right)$, where $S_{i}$ is as before. By an offspring of $\pi_{i}$, we mean some $\pi_{i+1} \in \operatorname{Av}\left(S_{i+1} ; 112,2122\right)$ that can be obtained from $\pi_{i}$ by inserting $a_{i+1}$ copies of $t=k-i$ appropriately. Define an active site of $\pi_{i}$ to be a position in which a (single) $t$ may be inserted without introducing 2122. Note that an offspring of $\pi_{i}$ is produced when a single $t$ is added at an active site and all other $t$ are added at the end.

Suppose $1 \leq i<k$ and that $\pi_{i}$ has exactly $\ell$ (active) sites. We consider the nature of the offspring of $\pi_{i}$ based on cases for the exponent $a_{i+1}$. If $a_{i+1}=1$ or 2 , then one may verify that each of the $\ell$ offspring of $\pi_{i}$ has $\ell+1$ or $\ell+2$ sites, respectively. If $a_{i+1} \geq 3$, first note in this case that every site of $\pi_{i}$ to the right of the leftmost $t$ is lost, as all offspring in this case must end in at least two letters $t$. Allowing the leftmost $t$ to occur in each of the possible positions, it is seen that there is exactly one offspring of $\pi_{i}$ that has $j$ sites for each $j \in[3, \ell+2]$.
Now consider forming $\rho_{i+1} \in \operatorname{Av}\left(S_{i+1} ; 121,1222\right)$ for $1 \leq i<k$ from $\rho_{i} \in \operatorname{Av}\left(S_{i} ; 121,1222\right)$. Define offspring and (active) site analogously as before. Suppose that $\rho_{i}$ has $\ell$ sites and we describe its offspring. If $a_{i+1}=1$ or 2 , then it is seen again that each offspring of $\rho_{i}$ has $\ell+1$ or $\ell+2$ sites, respectively. If $a_{i+1} \geq 3$, then inserting the run $t^{a_{i+1}}$ into a site effectively nullifies all sites of $\rho_{i}$ occurring to the left of the run. Thus, the number of sites in the offspring of $\rho_{i}$ ranges from 3 to $\ell+2$ in this case. Comparing the offspring of the various $\pi_{i}$ and $\rho_{i}$, one can show by induction on $i$ (upon considering cases based on the exponent $a_{i+1}$ ) that the number of members of $\operatorname{Av}\left(S_{i} ; 112,2122\right)$ and having exactly $r$ sites is the same as the corresponding number of members of $\operatorname{Av}\left(S_{i} ; 121,1222\right)$ for all $r \geq 2$. Allowing $r$ to vary over all possible values then implies the desired result.
Theorem 2.13. We have $\{112,2121\} \stackrel{m}{\sim}\{121,1122\}$.
Proof. Let $\pi, \pi_{i}, \rho, \rho_{i}, S, S_{i}$ for $1 \leq i \leq k$ and offspring be defined as in the proof of Theorem 2.11, but now in conjunction with the pattern sets $\{112,2121\}$ and $\{121,1122\}$, respectively. By an active site in $\pi_{i} \in \operatorname{Av}\left(S_{i} ; 112,2121\right)$, we mean a position in which one can insert a single letter $t$ such that no occurrence of 2121 arises when another copy of $t$ is appended to the end of the resulting multipermutation. Let act $(\lambda)$ denote the number of (active) sites of a multipermutation $\lambda$ and we will make use of this notation in subsequent proofs. Note that inserting a single $t$ (i.e., when $a_{i+1}=1$ ) into any position of $\pi_{i}$ introduces neither 112 nor 2121 and changes the act statistic value (always increasing it by one) if and only if the $t$ is inserted into a present site of $\pi_{i}$. On the other hand, if $a_{i+1}>1$, then inserting $t$ into a site $x$ of $\pi_{i}$ not the last (and placing $a_{i+1}-1$ copies of $t$ at the end) nullifies all sites of $\pi_{i}$ to the right of $x$, with $x$ effectively preserved; moreover, the site at the very end of $\pi_{i}$ is in essence replaced by a site at the very end of $\pi_{i+1}$.
Suppose now act $\left(\pi_{i}\right)=\ell$ where $1 \leq i<k$. By the previous observations, if $a_{i+1}=1$, then $\pi_{i}$ has $\ell$ offspring with $\ell+1$ sites and $m_{i}-\ell+1$ with $\ell$ sites, where $m_{i}=a_{1}+\cdots+a_{i}$. If $a_{i+1}>1$, then it is seen that the set of act values in the $\ell$ offspring of $\pi_{i}$ comprise the interval $[2, \ell+1]$.
Now define an active site in $\rho_{i} \in \operatorname{Av}\left(S_{i} ; 121,1122\right)$ to be a position of $\rho_{i}$ in which one can insert a run of $t$ of length two or more without introducing 1122, with the corresponding statistic again denoted by act. Upon considering cases based on whether $a_{i+1}=1$ or $a_{i+1}>1$, one can show that the set of act values of the offspring of $\rho_{i}$ where act $\left(\rho_{i}\right)=\ell$ is the same as those of the offspring of $\pi_{i}$ above. By induction on $i$ (the $i=1$ case trivial), the corresponding act statistics on $\operatorname{Av}\left(S_{i} ; 112,2121\right)$ and $\operatorname{Av}\left(S_{i} ; 121,1122\right)$ are identically distributed for all $1 \leq i \leq k$. Taking $i=k$ in particular implies the desired equivalence of patterns.

Theorem 2.14. We have $\{112,2312\} \stackrel{m}{\sim}\{121,1223\} \stackrel{m}{\sim}\{121,2213\}$.
Proof. We again make use of the same notation. For the first pattern set, let us define an active site to be a position of $\pi_{i} \in \operatorname{Av}\left(S_{i} ; 112,2312\right)$ in which a single $t$ may be inserted without introducing 2312 , where any remaining $t$ must be added at the end of $\pi_{i}$. Suppose $\operatorname{act}\left(\pi_{i}\right)=\ell$, where $1 \leq i<k$. If $a_{i+1}=1$, then each offspring of $\pi_{i}$ is seen to have $\ell+1$ sites.

On the other hand, if $a_{i+1} \geq 2$, then inserting the leftmost $t$ into any site $y$ other than the last destroys all sites of $\pi_{i}$ to the right of $y$. Note that $y$ itself is split into two sites, with a new site reemerging at the end of $\pi_{i+1}$ corresponding to the final added $t$. Thus as $y$ varies, one gets offspring whose act values comprise the interval $[3, \ell+1]$. If all of the letters $t$ are added at the very end of $\pi_{i}$, then every site of $\pi_{i}$ is preserved with each position directly following a $t$ active as well in this case, which implies that the offspring will have $\ell+a_{i+1}$ sites altogether.
Now suppose $\rho_{i} \in \operatorname{Av}\left(S_{i} ; 121,1223\right)$ with act $\left(\rho_{i}\right)=\ell$ or $\gamma_{i} \in \operatorname{Av}\left(S_{i} ; 121,2213\right)$ with act $\left(\gamma_{i}\right)=$ $\ell$, where active sites are defined analogously. One can show by comparable reasoning as before that if $a_{i+1}=1$, then the offspring of both $\rho_{i}$ and $\gamma_{i}$ all have act values of $\ell+1$, whereas if $a_{i+1} \geq 2$, then the values comprise the set $[3, \ell+1] \cup\left\{\ell+a_{i+1}\right\}$. By induction on $i$ (the $i=1$ case trivial), it is seen that the various act statistics defined on the sets consisting of the possible $\pi_{i}, \rho_{i}$ or $\gamma_{i}$ are identically distributed for $1 \leq i \leq k$, which in particular implies the desired equivalences.
It is also possible to establish the second equivalence via a bijection. It is instructive to describe such a bijection since it will be seen to preserve further statistics within the framework of multiset equivalence. It suffices to define a bijection $f$ between the set of permutations of $1^{b_{1}} \cdots k^{b_{k}}$ that avoid $\{121,1223\}$ and those that avoid $\{121,2213\}$, where $k \geq 1$ and $b_{1}, \ldots, b_{k} \geq 1$ are fixed. Suppose $\alpha=x_{1} \cdots x_{m}$, expressed as a word, is a permutation belonging to the former set, where $m=b_{1}+\cdots+b_{k}$. We first decompose $\alpha$ as $\alpha=\alpha^{(1)} k \alpha^{(2)}$, where $\alpha^{(2)}$ contains no $k$. Define $\widetilde{\alpha}$ by $\widetilde{\alpha}=\operatorname{rev}\left(\alpha^{(1)}\right) k \alpha^{(2)}$, where $\operatorname{rev}\left(\alpha^{(1)}\right)$ denotes the reversal of $\alpha^{(1)}$. If $\alpha^{(2)}=\varnothing$, then set $f(\alpha)=\widetilde{\alpha}$. Otherwise, let $k_{1}$ denote the largest letter occurring in $\alpha^{(2)}$ and suppose $\alpha^{(2)}=\alpha^{(3)} k_{1} \alpha^{(4)}$, where $\alpha^{(4)}$ contains no $k_{1}$.
We now introduce the following definition. Suppose $w=w_{1} w_{2} \cdots$ is a $k$-ary word and $i \in[k]$. Then we will refer to a (maximal) string of consecutive letters in $w$ all of which belong to $[i, k]$ as an $i$-upper run and a string all of whose letters belong to $[i-1]$ as an $i$-lower run. We consider a left-to-right scan of the $k_{1}$-upper and $k_{1}$-lower runs of $\lambda=\operatorname{rev}\left(\alpha^{(1)}\right) k \alpha^{(3)} k_{1}$. Let $\delta_{1}, \ldots, \delta_{t}$ denote the distinct $k_{1}$-lower runs in $\alpha^{(3)}$, where $t=0$ is possible. Then $\alpha^{(3)}$ can be decomposed as $\alpha^{(3)}=\rho_{0} \delta_{1} \rho_{1} \cdots \delta_{t} \rho_{t}$ if $t>0$, with $\alpha^{(3)}=\rho_{0}$ if $t=0$, where $\rho_{0}, \ldots, \rho_{t}$ are $k_{1}$-upper runs with (only) $\rho_{0}$ and $\rho_{t}$ possibly empty. Similarly, let $\operatorname{rev}\left(\alpha^{(1)}\right)=\tau_{0} \sigma_{1} \tau_{1} \cdots \sigma_{s} \tau_{s}$ if $s>0$, with $\operatorname{rev}\left(\alpha^{(1)}\right)=\tau_{0}$ if $s=0$, where $\sigma_{1}, \ldots, \sigma_{s}$ are $k_{1}$-lower runs, $\tau_{0}, \ldots, \tau_{s}$ are $k_{1}$-upper runs and $\tau_{0}, \tau_{s}$ are possibly empty.
We now define a multiset $\lambda^{*}$ derived from $\lambda$ as follows. If $s>t>0$, then let $\lambda^{*}$ be defined as

$$
\lambda^{*}=\tau_{0} \operatorname{rev}\left(\delta_{t}\right) \tau_{1} \operatorname{rev}\left(\delta_{t-1}\right) \tau_{2} \cdots \operatorname{rev}\left(\delta_{1}\right) \tau_{t} \sigma_{1} \tau_{t+1} \sigma_{2} \tau_{t+2} \cdots \sigma_{s-t} \tau_{s} k \rho_{0} \sigma_{s-t+1} \rho_{1} \cdots \sigma_{s} \rho_{t} k_{1}
$$

where it is seen that this definition may be extended to the case when $s=t>0$, with $\lambda^{*}=\lambda$ if $t=0$ for all $s$. If $t>s>0$, then let $\lambda^{*}$ be given by

$$
\begin{aligned}
\lambda^{*}= & \tau_{0} \operatorname{rev}\left(\delta_{t}\right) \tau_{1} \operatorname{rev}\left(\delta_{t-1}\right) \tau_{2} \cdots \operatorname{rev}\left(\delta_{t-s+1}\right) \tau_{s} k \rho_{0} \operatorname{rev}\left(\delta_{t-s}\right) \rho_{1} \operatorname{rev}\left(\delta_{t-s-1}\right) \rho_{2} \cdots \operatorname{rev}\left(\delta_{1}\right) \rho_{t-s} \\
& \sigma_{1} \rho_{t-s+1} \sigma_{2} \rho_{t-s+2} \cdots \sigma_{s} \rho_{t} k_{1},
\end{aligned}
$$

which can be extended to the case when $t>s=0$. That is, if $\lambda^{\prime}$ is obtained from $\lambda$ by replacing the runs $\delta_{1}, \ldots, \delta_{t}$ with $\operatorname{rev}\left(\delta_{t}\right), \ldots, \operatorname{rev}\left(\delta_{1}\right)$ in that order, then $\lambda^{*}$ is obtained from $\lambda^{\prime}$ by repositioning the $k_{1}$-lower runs to the right of the last $k$ so that they now occur prior to those to the left of this $k$, maintaining the order of the $k_{1}$-upper runs (as well as the order of the letters within all runs). Note that the $k_{1}$-upper runs of $\lambda^{*}$ are the same as those in $\lambda$,
with the relative order of $k_{1}$-lower and $k_{1}$-upper runs in a left-to-right scan also seen to be the same.
If $\alpha^{(4)}$ is empty, then set $f(\alpha)=\lambda^{*}$. Note that by reordering the $k_{1}$-lower runs as described, we have eliminated any possible occurrences of 2213 in which the 3 can correspond to the terminal $k_{1}$. If $\alpha^{(4)}$ is non-empty, then let $k_{2}<k_{1}$ be the largest letter of $\alpha^{(4)}$ and write $\alpha^{(4)}=\alpha^{(5)} k_{2} \alpha^{(6)}$, where $\alpha^{(6)}$ contains no $k_{2}$. Then consider any $k_{2}$-lower runs within the $\alpha^{(5)}$ section of $\tau=\lambda^{*} \alpha^{(5)} k_{2}$. We arrange the $k_{2}$-lower runs of $\tau$ such that the reversals of those in $\alpha^{(5)}$ occur (in reverse order) prior to the others, as we did with the $\delta_{i}$ 's above in $\lambda$. If $\tau^{*}$ denotes the resulting word, then set $f(\alpha)=\tau^{*}$ if $\alpha^{(6)}=\varnothing$.
Otherwise, we continue in the manner described until $\alpha^{(2 i)}=\varnothing$ for some $i \geq 4$, setting $f(\alpha)$ equal to the word that results after applying the procedure described above for a final time. By construction, it is seen that $f(\alpha)$ avoids 121, as does each word arising from an intermediate step of the algorithm. One may verify also that $f(\alpha)$ avoids 2213 ; note that it suffices to check that $f(\alpha)$ contains no 2213 in which the 3 corresponds to a (strict) right-left maximum. To reverse $f$, consider successively the right-left maxima, starting with the last letter and working back to the rightmost $k$, where we reverse each step of the algorithm described above starting with the last. Note that this may be done since the values of right-left maxima are preserved by each step of the algorithm and hence by $f$.

Remark: From the preceding proof, we have in particular that the multiset equivalence of $\{121,1223\}$ and $\{121,2213\}$ respects the last letter and right-left maxima statistics.

Theorem 2.15. We have $\{121,1322\} \stackrel{m}{\sim}\{112,1232\} \stackrel{m}{\sim}\{112,2132\}$.
Proof. Proceeding as in the prior proof and using the same notation, consider permutations of $k^{a_{1}} \cdots 1^{a_{k}}$ that avoid either $\{121,1322\},\{112,1232\}$ or $\{112,2132\}$. If $a_{i+1}=1$ and $\operatorname{act}\left(\pi_{i}\right)=\ell$, then all offspring of $\pi_{i}$ have act value $\ell+1$ for each pattern set. If $a_{i+1}>1$, then the act values of the offspring of $\pi_{i}$ are seen to comprise the interval $\left[a_{i+1}+1, a_{i+1}+\ell\right]$ in each case. Note that when avoiding $\{121,1322\}$, inserting $a_{i+1}>1$ copies of $t$ into a site $v$ of $\pi_{i}$ destroys all sites to the left of $v$ while splitting $v$ into $a_{i+1}+1$ sites. If avoiding $\{112,1232\}$, inserting the leftmost $t$ into a site of $\pi_{i}$, not the last, is seen to destroy all sites to its left when $a_{i+1}>1$. Similar reasoning applies to $\{112,2132\}$ except that (non-terminal) sites to the right are destroyed. Since the pattern sets obey the same rules with regard to the number of sites in offspring, the result follows.

Remark: Extending the previous proof shows more generally

$$
\left\{121,132^{r}\right\} \stackrel{m}{\sim}\left\{112,1232^{r-1}\right\} \stackrel{m}{\sim}\left\{112,2132^{r-1}\right\}, \quad r \geq 1
$$

One can provide a proof of the following result analogous to the previous ones by modifying appropriately the definition of a site. However, we find it more instructive to give a bijective argument which makes use of a certain encoding of the offspring and suggests how one might go about finding bijective proofs of other comparable results.

Theorem 2.16. We have $\{112,1231\} \stackrel{m}{\sim}\{121,1132\}$.
Proof. Let $\pi$ be a multipermutation of the multiset $S=k^{a_{1}} \cdots 1^{a_{k}}$. Given $1 \leq i \leq k$, let $S_{i}=k^{a_{1}} \cdots(k-i+1)^{a_{i}}$ and $\pi_{i} \in \operatorname{Av}\left(S_{i} ; 112,1231\right)$ be obtained from $\pi$ by considering only the relative positions of the parts $k, \ldots, k-i+1$. We construct an encoding for $\pi$ as follows. Let $\pi_{i}=\alpha_{1} \cdots \alpha_{m_{i}}$ as a word where $m_{i}=a_{1}+\cdots+a_{i}$ and $p_{i}$ denote the index $r$ such that
$\alpha_{r}<\alpha_{r+1}$ with $r$ maximal, if it exists (i.e., $p_{i}$ corresponds to the rightmost ascent bottom of $\pi_{i}$ ), with $p_{i}=0$ otherwise (i.e., if $\pi_{i}$ is decreasing, perhaps weakly). We seek to form $\pi_{i+1}$ from $\pi_{i}$ for $i \geq 1$ by making an appropriate insertion of the $a_{i+1}$ copies of $t=k-i$.
Note that in forming $\pi_{i+1}$ from $\pi_{i}$ when $p_{i}>0$ that there are exactly $p_{i}$ positions to the left of the rightmost ascent bottom of $\pi_{i}$ in which to insert $t$. Observe further that if a $t$ is inserted anywhere to the left of $p_{i}$ in $\pi_{i}$, then necessarily $a_{i+1}=1$, for otherwise $\pi_{i+1}$ would contain an occurrence of 1231 as the remaining copies of $t$ are forced to occur at the very end (in particular, to the right of the last ascent).
Let $\ell$ denote the smallest index $j>1$ such that $p_{j}>0$, assuming such $j$ exists. Then $p_{1}=$ $\cdots=p_{\ell-1}=0$ implies $\pi_{\ell-1}$ is decreasing (i.e., $\pi_{\ell-1}=k^{a_{1}} \cdots(u+1)^{a_{\ell-1}}$ where $u=k-\ell+1$ ). Then $p_{\ell}>0$ and $\pi_{\ell}$ avoiding 112 implies $\pi_{\ell}$ contains only one ascent, with $p_{\ell}$ determining the position of that ascent. Note that $\pi_{\ell}$ is of the form $\pi_{\ell}=k^{a_{1}} \cdots x \cdots(u+1)^{a_{\ell-1}} u^{a_{\ell}-1}$, where $x$ corresponds to the position of the leftmost letter $u$ (i.e., $x$ corresponds to the ( $p_{\ell}$ )-th entry of $\pi_{\ell}$ where $1 \leq p_{\ell} \leq m_{\ell-1}$ ).
Now assume $\ell \leq i \leq k-1$. If it is the case that both $p_{i+1}=p_{i}+1$ and $a_{i+1}=1$, then we must also specify, in addition to the value of $p_{i+1}$, some element of the set $\left[p_{i}+1\right]$, which gives the entry number of the only $t$ in $\pi_{i+1}$. Note that when $a_{i+1}=1$, we have $p_{i+1}=p_{i}+1$ if and only if the $t$ is inserted to the left of or within the rightmost ascent of $\pi_{i}$, as the position of the rightmost ascent bottom is shifted to the right by one place in this case. Thus, the position number of $t$ must belong to $\left[p_{i}+1\right]$. Let $\rho$ denote the vector $\left(p_{1}, \ldots, p_{k}\right)$ consisting of the various $p_{i}$ values, where in addition an element of $\left[p_{i}+1\right]$ is specified parenthetically in the $i$-th component for each $i \in[\ell, k-1]$ such that $p_{i+1}=p_{i}+1$ and $a_{i+1}=1$. One can verify that the sequence $p_{i}$ satisfies the following succession rules for $\ell \leq i \leq k-1$ :
$p_{i+1}= \begin{cases}p_{i}+1, & \text { if the (single) } t \text { is inserted anywhere to the left of position } p_{i} \text { in } \pi_{i} ; \\ p_{i}+s, & \text { if the leftmost } t \text { is inserted in s-th position to the right of position } p_{i} \text { in } \pi_{i}, \\ & \text { but not at the very end of } \pi_{i} ; \\ p_{i}, & \text { if the run of } t \text { is inserted at the very end of } \pi_{i} .\end{cases}$
From this, it is seen that $\pi$ can be reconstructed from $\rho$, starting with a run of $k$ and successively inserting $k-1, \ldots, 1$. Note that in cases when there is no additional specified member of $\left[p_{i}+1\right]$, the position of the added $t$ is determined completely by $p_{i+1}$ alone (i.e., when $p_{i+1} \neq p_{i}+1$ or when $p_{i+1}=p_{i}+1$ and $a_{i+1}>1$ ).
Now let $\lambda$ be a multipermutation of $S$ that avoids $\{121,1132\}$. Let $\lambda_{1}=k^{a_{1}}$ and $\lambda_{i+1} \in$ $\operatorname{Av}\left(S_{i+1} ; 121,1132\right)$ for $i \geq 1$ be formed from $\lambda_{i} \in \operatorname{Av}\left(S_{i} ; 121,1132\right)$ by inserting $a_{i+1}$ copies of $t$ as before. If $1 \leq i \leq k$, then let $\lambda_{i}=\beta_{1} \cdots \beta_{m_{i}}$ as a word and let $p_{i}^{\prime}$ denote the index $r$ such that $\beta_{r}>\beta_{r+1}$ with $r$ maximal, if it exists (i.e., $p_{i}^{\prime}$ corresponds to the rightmost descent top of $\lambda_{i}$ ), with $p_{i}^{\prime}=0$ otherwise (i.e., if $\lambda_{i}$ is increasing, perhaps weakly). Suppose $\ell>1$ is determined by $p_{1}^{\prime}=\cdots=p_{\ell-1}^{\prime}=0$, with $p_{\ell}^{\prime}>0$ (assuming such $\ell$ exists). Then the position of the (only) run of $u$ in $\lambda_{\ell}$ is such that the first $u$ corresponds to the ( $p_{\ell}^{\prime}+1$ )-st entry, where $1 \leq p_{\ell}^{\prime} \leq m_{\ell-1}$.
Suppose $\ell \leq i \leq k-1$. If $p_{i+1}^{\prime}=p_{i}^{\prime}$, then the run of $t$ is inserted so that it separates the letters of the rightmost descent of $\lambda_{i}$. If $a_{i+1}=1$ and $p_{i+1}^{\prime}=p_{i}^{\prime}+1$, then additionally we must specify some number $x$ in $\left[p_{i}^{\prime}+1\right]$, which indicates the position of the lone inserted $t$ letter if $x<p_{i}^{\prime}+1$, with $t$ inserted directly following the rightmost descent bottom of $\lambda_{i}$ if $x=p_{i}^{\prime}+1$. Let $\rho^{\prime}=\left(p_{1}^{\prime}, \ldots, p_{k}^{\prime}\right)$, where in addition an element of $\left[p_{i}^{\prime}+1\right]$ is specified parenthetically in the $i$-th component for each $i \in[\ell, k-1]$ satisfying the stated requirements. Note that $p_{i+1}^{\prime}$
for $\ell \leq i \leq k-1$ satisfies
$p_{i+1}^{\prime}= \begin{cases}p_{i}^{\prime}+1, & \text { if the (single) } t \text { is inserted to the left of position } p_{i}^{\prime} \text { in } \lambda_{i} ; \\ p_{i}^{\prime}+s, & \text { if the } t \text { run is inserted in s-th position to the right of the }\left(p_{i}^{\prime}+1\right) \text {-st entry } \\ p_{i}^{\prime}, & \text { of } \lambda_{i} ; \\ \text { if the } t \text { run is inserted between the }\left(p_{i}^{\prime}\right) \text {-th and }\left(p_{i}^{\prime}+1\right) \text {-st entries of } \lambda_{i} .\end{cases}$
From this, we see that $\lambda$ can be recovered from $\rho^{\prime}$. Note that the $p_{i}$ and $p_{i}^{\prime}$ for $i>\ell$ satisfy equivalent recurrences and hence yield the same set of possible vectors $\rho$ and $\rho^{\prime}$ (along with any parenthetical elements).
Thus, one may define a bijection between multipermutations avoiding $\{112,1231\}$ and those avoiding $\{121,1132\}$ by starting with $\pi$ from the former set and producing its vector $\rho$. Then read the entries from $\rho$ as a vector $\rho^{\prime}$ for constructing members of the latter set using the multiset of parts from $\pi$. This yields $\pi^{\prime}$ belonging to the latter set and the mapping $\pi \mapsto \pi^{\prime}$ is seen to be a bijection. Note that this mapping preserves all part size multiplicities, and in particular, the number of parts.

Theorem 2.17. We have $\{123,1121\} \stackrel{m}{\sim}\{132,1211\} \stackrel{m}{\sim}\{213,1121\}$.
Proof. Let $S=k^{a_{k}} \cdots 1^{a_{1}}$ and $S_{i}=k^{a_{k}} \cdots i^{a_{i}}$ for $1 \leq i \leq k$. We show that permutations of $S$ avoiding any of the three sets of patterns are equinumerous. We first consider $\{123,1121\}$. Consider forming members $\pi_{i} \in \operatorname{Av}\left(S_{i} ; 123,1121\right)$ from $\pi_{i+1} \in \operatorname{Av}\left(S_{i+1} ; 123,1121\right)$ for $1 \leq$ $i<k$ by inserting $i$ 's appropriately into $\pi_{i+1}$. Note that all of the $i$ 's must be inserted to the right of the rightmost ascent bottom of $\pi_{i+1}$, lest a 123 would arise. Furthermore, if $a_{i}>1$, the $i$ 's must be inserted either as a single run or as two runs in which the first is a single $i$. Suppose act $\left(\pi_{i+1}\right)=\ell$, meaning that the length of its final decreasing sequence is $\ell-1$. If $a_{i}=1$, then $\pi_{i+1}$ has offspring with $2,3, \ldots, \ell+1$ sites. If $a_{i}>1$, then for each $j \in[2, \ell]$, we have that $\pi_{i+1}$ has $\ell+1-j$ offspring with $j$ sites and not ending in $i$. These offspring are obtained by inserting $a_{i}-1$ letters $i$ in the $j$-th rightmost site and a single $i$ in any of the $\ell+1-j$ sites to the left of and including the first insertion point. On the other hand, there are $\ell$ offspring of $\pi_{i+1}$ ending in $i$, with their act values comprising the interval $\left[a_{i}+1, a_{i}+\ell\right]$. To see this, note that the final run of $i$ contributes $a_{i}-1$ sites in addition to those occurring at the point just when the leftmost $i$ has been inserted into $\pi_{i+1}$.
In the case of avoiding $\{132,1211\}$, a site corresponds to a position to the right of the rightmost descent top. Thus $\operatorname{act}(\pi)=\ell$ in this case means that the final increasing run is of length $\ell-1$. A similar analysis as before which we leave to the reader reveals that the same rules are followed concerning the number of sites in offspring and yields the first equivalence.
We use similar notation as before when discussing members of $\operatorname{Av}(S ; 213,1121)$. Note that a site of $\pi_{i} \in \operatorname{Av}\left(S_{i} ; 213,1121\right)$ where $i>1$ corresponds either to the first or last position of $\pi_{i}$ or to a position directly following any letter $x$ such that $\pi_{i}=\alpha x \beta$, where $\beta$ is nonempty and $\min (\alpha \cup\{x\}) \geq \max (\beta)$. Then inserting an $i$ into a site of some $\pi_{i+1}$, not the last, destroys all sites to its right except for the last, whereas inserting an $i$ at the end clearly adds a site. Furthermore, if $a_{i}>1$ and act $\left(\pi_{i+1}\right)=\ell$, then there are $\ell-j$ offspring of $\pi_{i+1}$ having $j$ sites and two runs of $i$ for each $j \in[2, \ell]$, where the second run is not at the end. The offspring for which there are two runs of $i$ where the second run occurs at the end have act values comprising $\left[a_{i}+1, a_{i}+\ell-1\right]$, whereas those containing a single run of $i$ have values in $[2, \ell] \cup\left\{a_{i}+\ell\right\}$. Combining these observations implies the second equivalence and completes the proof.

Theorem 2.18. We have $\{123,1112\} \stackrel{m}{\sim}\{132,1121\}$ and $\{123,1211\} \stackrel{m}{\sim}\{213,1211\}$.
Proof. We use the same notation as in the proof of Theorem 2.17 above and form permutations $\pi_{i} \in \operatorname{Av}\left(S_{i} ; 123,1112\right)$ from $\pi_{i+1} \in \operatorname{Av}\left(S_{i+1} ; 123,1112\right)$ by inserting $a_{i}$ letters $i$ and similarly for $\{132,1121\}$. Note that sites of $\pi_{i+1}$ in the case of avoiding $\{123,1112\}$ or $\{132,1121\}$ correspond to possible points of insertion to the right of the rightmost ascent bottom or descent top, respectively. Suppose act $\left(\pi_{i+1}\right)=\ell$ in both and we count sites in the offspring. If $a_{i}=1$, then in either case the offspring have $2,3, \ldots, \ell+1$ sites. If $a_{i}>1$, then for $\{123,1112\}$, we consider cases based on whether or not there are two letters $i$ occurring in positions to the left of the rightmost letter in $[i+1, k]$. If so, then for each $j \in\{0,1, \ldots, \ell-2\}$, there are $\ell-1-j$ offspring that have $a_{i}+j$ sites, upon inserting an $i$ in the ( $\ell-1-j$ )-th site from the left, inserting a second $i$ anywhere to its left in any site and then adding the remaining $a_{i}-2$ letters $i$ at the end of $\pi_{i+1}$. If not, then at most a single $i$ occurs in a site other than the very last, which results in offspring with act values in $\left[a_{i}+1, a_{i}+\ell\right]$. Combining the previous cases, we have for $a_{i}>1$,

$$
\begin{equation*}
\ell \rightsquigarrow a_{i}(\ell-1), a_{i}+j(\ell-j) \text { for } j \in[\ell-1], a_{i}+\ell(1), \quad \ell \geq 2, \tag{*}
\end{equation*}
$$

where the multiplicities of the offspring (having the specified number of sites) are denoted in parentheses.
On the other hand, if $\operatorname{act}\left(\pi_{i+1}\right)=\ell$ for $\pi_{i+1} \in \operatorname{Av}\left(S_{i+1} ; 132,1121\right)$ and $a_{i}>1$, we consider cases on the offspring $\pi_{i}$ based on whether or not two or more $i$ 's occur in positions prior to the rightmost letter in $[i+1, k]$. If so, then $\pi_{i}$ cannot end in $i$ and consider further cases on the position of the second leftmost inserted $i$. If it occurs in the $j$-th leftmost site of $\pi_{i+1}$ for $1 \leq j \leq \ell-1$, then there are $j-1$ offspring $\rho$ for which $\operatorname{act}(\rho)=a_{i}+\ell-j$ and a single $\rho$ with act $(\rho)=a_{i}+\ell-j+1$. If there is at most one $i$ occurring prior to the rightmost letter in $[i+1, k]$, then it is seen that there are $\ell-1$ possible offspring $\rho$ with $\operatorname{act}(\rho)=a_{i}$ and a single $\rho$ with act $(\rho)=a_{i}+1$. Combining these cases demonstrates that the offspring of $\pi_{i+1}$ have the same distribution of act values as those in $\left(^{*}\right)$ above when $a_{i}>1$, which implies the first equivalence.
Similar reasoning applies to the second equivalence. Note that sites in the case of avoiding $\{213,1211\}$ correspond to positions in the parent permutation $\pi$ wherein one can insert a single $i$ without introducing 213 and therefore correspond to the very beginning or end of $\pi$ or directly after $x$, where $x$ is such $\pi=A x B$ and $\min (A \cup\{x\}) \geq \max (B)$. Further, the $i$ 's must occur as one or two runs for both pattern sets, where a second run is a single letter, for otherwise 1211 would arise. For both sets of patterns, one may verify the following succession rules: $\ell \rightsquigarrow 2(\ell-1), a_{i}+\ell(1), j(\ell-j+2)$ for $j \in[3, \ell+1]$ if $a_{i}>1$, with $\ell \rightsquigarrow 2,3, \ldots, \ell+1$ if $a_{i}=1$, which yields the second equivalence.

Combining the first equivalence of Theorem 2.18 with a symmetric version of the equivalence $\{123,1222\} \stackrel{m}{\sim}\{132,1222\}$ from Proposition 2.9 yields $\{123,1112\} \stackrel{m}{\sim}\{132,1121\} \stackrel{m}{\sim}$ $\{213,1112\}$.
2.4. Equivalences involving doubly labeled offspring. To establish the equivalences in this subsection, it will be convenient to label the offspring by a vector $(a, b)$ tracking two kinds of active sites.

Theorem 2.19. The following equivalences hold: $\{121,1243\} \stackrel{m}{\sim}\{121,2143\},\{112,1243\} \stackrel{m}{\sim}$ $\{112,2143\}$ and $\{121,1234\} \stackrel{m}{\sim}\{121,2134\}$.

Proof. Let $S_{i}=k^{a_{k}} \ldots i^{a_{i}}$ for $1 \leq i \leq k$. For the first equivalence, we form $\pi_{i} \in \operatorname{Av}\left(S_{i} ; \alpha, \beta\right)$ from $\pi_{i+1} \in \operatorname{Av}\left(S_{i+1} ; \alpha, \beta\right)$ by inserting $a_{i}$ letters $i$ for $1 \leq i<k$, where $\{\alpha, \beta\}=\{121,1243\}$ or $\{121,2143\}$. We first consider $\{121,1243\}$. Suppose $\rho \in \operatorname{Av}\left(S_{i} ; 121,1243\right)$ can be written as $\rho=\gamma x y \gamma^{\prime}$, where $x>y$ and $y \gamma^{\prime}$ is (weakly) increasing, i.e., $x$ is the rightmost descent top of $\rho$. Label $\rho$ by $(a, b)$, where $a$ denotes the number of active sites to the left of $x$ and $b$ is the number to the right of $x$. We will refer to the sites accounted for by $a$ and $b$ as primary and secondary, respectively. Furthermore, we take $a=0$ if $\rho$ contains no descents, i.e., if $\rho$ is increasing. Given $\pi_{i+1} \in \operatorname{Av}\left(S_{i+1} ; 121,1243\right)$ with label $(a, b)$, we determine the labels of its $a+b$ offspring $\rho$. If $a>0$, then

$$
(a, b) \rightsquigarrow(1, b),(2, b), \ldots,(a, b),\left(a, a_{i}+b\right),\left(a+1, a_{i}+b-1\right), \ldots,\left(a+b-1, a_{i}+1\right) .
$$

To establish this rule, first note that all $i$ 's must be inserted as a single run in $\pi_{i+1}$. Furthermore, if $x y$ represents the rightmost descent of $\pi_{i+1}$, then inserting one or more $i$ 's in the $j$-th site to the left of $x$ destroys all sites to the left. The $j$-th primary site itself is in essence preserved since the position directly following the final added $i$ is still active. Letting $j$ range over $[a]$ accounts for $(1, b), \ldots,(a, b)$. Otherwise, the $i$ 's are added to the right of $x$, i.e., within the final increasing sequence of $\pi_{i+1}$. In this case, if the $i$ 's are inserted within the $j$-th position to the right of $x$, where $1 \leq j \leq b$, then the rightmost descent shifts to the right by $j-1$ positions and thus the first $j-1$ secondary sites become primary. Further, it is seen that there are now $a_{i}+b-j+1$ secondary sites. Letting $j$ vary over [b] then accounts for $\left(a, a_{i}+b\right),\left(a+1, a_{i}+b-1\right), \ldots,\left(a+b-1, a_{i}+1\right)$.
On the other hand, if $a=0$, then $b=a_{i+1}+\cdots+a_{k}+1$, with $\pi_{i+1}$ increasing. In this case, if the run of $i$ is added directly following $z$ in $\pi_{i+1}$, then all positions to the left (right) of $z$ are primary (secondary). This implies the succession rule

$$
(0, b) \rightsquigarrow\left(0, a_{i}+b\right),\left(1, a_{i}+b-1\right), \ldots,\left(b-1, a_{i}+1\right) .
$$

One may verify that the same succession rules for $(a, b)$ when $a>0$ and $a=0$ are followed when avoiding $\{121,2143\}$. Note that insertion of $i$ into the $j$-th site to the left of $x$ now destroys all primary sites to its right (instead of to its left).
For the second equivalence, we make use of the same labels $(a, b)$ as before, but now in conjunction with the relevant patterns. If $a>0$, then we have
$(a, b) \rightsquigarrow\left(b, a_{i}\right),\left(b+1, a_{i}\right), \ldots,\left(a+b-1, a_{i}\right),\left(a+b, a_{i}\right)(b-1$ times $),\left(a+b-1, a_{i}+1\right), \quad a_{i}>1$, with

$$
(a, b) \rightsquigarrow(1, b),(2, b), \ldots,(a, b),(a, b+1),(a+1, b), \ldots,(a+b-1,2), \quad a_{i}=1 .
$$

If $a=0$, then $b=a_{i+1}+\cdots+a_{k}+1=k-i+1$ since $a_{i+1}=\cdots=a_{k}=1$ in order for $\pi_{i+1}$ to be increasing, for otherwise a 112 would arise. Thus, we have the rules

$$
(0, b)=(0, k-i+1) \rightsquigarrow\left(b-1, a_{i}+1\right) \text { and }\left(b, a_{i}\right)(b-1 \text { times }), \quad a_{i}>1,
$$

and

$$
(0, b)=(0, k-i+1) \rightsquigarrow(0, b+1),(1, b), \ldots,(b-1,2), \quad a_{i}=1 .
$$

One may verify that both pattern sets $\{112,1243\}$ and $\{112,2143\}$ obey the preceding succession rules.
For the final equivalence, we show alternatively $\{212,1234\} \stackrel{m}{\sim}\{212,1243\}$. We form $\pi_{i} \in$ $\operatorname{Av}\left(S_{i} ; \alpha, \beta\right)$ from $\pi_{i-1} \in \operatorname{Av}\left(S_{i-1} ; \alpha, \beta\right)$ for $i>1$, where $S_{i}=1^{b_{1}} \cdots i^{b_{i}}$ and $\{\alpha, \beta\}=$ $\{212,1234\}$ or $\{212,1243\}$. We use the labels $(a, b)$, where $a$ and $b$ now denote the number of sites to the right of or to the left of the leftmost ascent top $x$, respectively. Sites of
the former or of the latter kind will be described as primary or secondary, respectively, and we take $a=0$ if a permutation is (weakly) decreasing. Note that, in the case of avoiding $\{212,1234\}$, a position to the right of $x$ is a (primary) site if it occurs to the left of all letters $z$ playing the role of a 3 in some occurrence of 123 and that all positions to the left of $x$ are (secondary) sites. We have the following succession rule if $a>0$ :
$(a, b) \rightsquigarrow(1, b),(2, b), \ldots,(a, b),\left(a+b+b_{i}-2,2\right),\left(a+b+b_{i}-3,3\right), \ldots,\left(a+b_{i}, b\right),\left(a, b+b_{i}\right)$.
To see this, note that the first $a$ offspring of $\pi_{i-1}$ listed account for the case when the $i$ 's are placed in a primary site (which destroys all sites to the right). If the letters $i$ are placed at the beginning of $\pi_{i-1}$, then one gets $\left(a, b+b_{i}\right)$, whereas if the $i$ 's are placed in the $j$-th site from the left where $2 \leq j \leq b$, then one gets $\left(a+b+b_{i}-j, j\right)$ since in this case the final $b-j$ secondary sites become primary. This accounts for the remaining offspring. If $a=0$, then $b=b_{1}+\cdots+b_{i-1}+1$ and $\pi_{i-1}$ is decreasing, which implies

$$
(0, b) \rightsquigarrow\left(0, b+b_{i}\right),\left(2, b+b_{i}-2\right),\left(3, b+b_{i}-3\right), \ldots,\left(b, b_{i}\right) .
$$

Note that $\left(0, b+b_{i}\right)$ accounts for the case when all $i$ 's are added at the beginning, whereas inserting the $i$ 's directly after the $j$-th letter of $\pi_{i-1}$ from the left gives $\left(j+1, b+b_{i}-j-1\right)$ for $1 \leq j \leq b-1$. By similar arguments, one can show that $\{212,1243\}$ follows the same succession rules, which implies the third equivalence.
2.5. Equivalence of $\{132,2213\}$ and $\{213,1322\}$. We apply an active site analysis in proving the equivalence of $\{132,2213\}$ and $\{213,1322\}$ for multisets which requires a further enumeration of several classes of offspring arising from a parent permutation having a fixed number $\ell$ of sites.
Theorem 2.20. We have $\{132,2213\} \stackrel{m}{\sim}\{213,1322\}$.
Proof. We show $|\operatorname{Av}(S ; 132,2213)|=|\operatorname{Av}(S ; 213,1322)|$, where $S=1^{b_{1}} \cdots k^{b_{k}}$. Let $\pi_{i}$ denote a permutation of $S_{i}=1^{b_{1}} \cdots i^{b_{i}}$ for $1 \leq i \leq k$ avoiding either pattern set. We form all possible $\pi_{i}$ from $\pi_{i-1}$ for $i>1$ by inserting $b_{i}$ copies of $i$. Suppose act $\left(\pi_{i-1}\right)=\ell$ and we consider the act values of its offspring. It is seen that both pattern sets obey the succession rule $\ell \rightsquigarrow 2, \ldots, \ell+1$ if $b_{i}=1$; henceforth, assume $b_{i}>1$.
We now consider the offspring $\pi_{i} \in \operatorname{Av}\left(S_{i} ; 132,2213\right)$ of $\pi_{i-1} \in \operatorname{Av}\left(S_{i-1} ; 132,2213\right)$. First note a letter $i$ may be inserted into $\pi_{i-1}$ at the very beginning or directly following $x$ such that $\pi_{i-1}=A x B$, where $\min (A \cup\{x\}) \geq \max (B), A \cup\{x\}$ avoids 221 and $B$ is possibly empty. Suppose that the sites of $\pi_{i-1}$ are labeled 1 to $\ell$, starting with the leftmost. Let $j$ and $r$ denote respectively the numbers of the sites in which the leftmost and second leftmost letter $i$ are inserted, where $1 \leq j \leq r \leq \ell$. To enumerate the offspring $\pi_{i}$, we consider several cases on $j$ and $r$. First suppose $j=r=1$, and let $s$ denote the number of additional letters $i$ inserted into this site, where $0 \leq s \leq b_{i}-2$. Note that only the beginning or positions coming directly after the added $i$ are active in the offspring in this case, due to avoidance of 2213 . Thus, there are $\binom{b_{i}-s+\ell-4}{\ell-2}$ possible offspring $\rho$ with $\operatorname{act}(\rho)=s+3$ for each $s$ if $\ell \geq 2$, with a single offspring corresponding to $s=b_{i}-2$ if $\ell=1$.
If $j=1$ and $r>1$, then all possible offspring $\rho$ containing at least three runs of $i$ have act value 2 . To see this, note that any position of $\rho$ beyond the first letter in $[i-1]$ and to the left of the rightmost added $i$ cannot be a site (due to 132), while any position to the right of, but not directly following, the last $i$ in the second run of $i$ isn't a site either (due to 2213). Thus, the only possible sites are directly before or after the initial $i$. If $\rho$ has only two runs, then the position directly following the last $i$ is also a site so that $\rho$ has three sites in this
case. Thus, there are $\binom{b_{i}-2+\ell-r}{\ell-r}-1 \operatorname{possible} \rho$ with $\operatorname{act}(\rho)=2$, and one with $\operatorname{act}(\rho)=3$. Now suppose $j=r>1$. By similar reasoning as before, we have act $(\rho)=1$ for which there are at least two runs, as only the initial position can then be a site. If there is a single run of $i$ in $\rho$, then $\operatorname{act}(\rho)=2$ in this case since there is also a site directly following the last $i$. Thus, there are $\binom{b_{i}-2+\ell-r}{\ell-r}-1$ possible $\rho$ with $\operatorname{act}(\rho)=1$, and one with $\operatorname{act}(\rho)=2$. A similar analysis reveals that this is also the case for all $j$ and $r$ such that $1<j<r \leq \ell$.
We now determine the number of offspring corresponding to each act value. Combining the prior cases, and considering all possible $j$ for each $r$ in the last case, we have that the total number of $\rho$ for which $\operatorname{act}(\rho)=1$ is given by

$$
\sum_{r=2}^{\ell}(r-1)\left(\binom{b_{i}-2+\ell-r}{\ell-r}-1\right)=\sum_{r=0}^{\ell-2}(\ell-r-1)\binom{b_{i}-2+r}{r}-\binom{\ell}{2}
$$

Simplifying further, and making use of the upper summation formula $\sum_{k=m}^{n}\binom{k}{m}=\binom{n+1}{m+1}$ (see, e.g., [2, p. 174]), gives

$$
\begin{aligned}
& (\ell-1)\binom{b_{i}+\ell-3}{b_{i}-1}-\binom{\ell}{2}-\sum_{r=0}^{\ell-2} r\binom{b_{i}-2+r}{b_{i}-2} \\
& =(\ell-1)\binom{b_{i}+\ell-3}{b_{i}-1}-\binom{\ell}{2}-\left(b_{i}-1\right) \sum_{r=1}^{\ell-2}\binom{b_{i}-2+r}{b_{i}-1} \\
& =(\ell-1)\binom{b_{i}+\ell-3}{b_{i}-1}-\left(b_{i}-1\right)\binom{b_{i}+\ell-3}{b_{i}}-\binom{\ell}{2} .
\end{aligned}
$$

Also, combining the various cases above implies that the number of $\rho$ for which act $(\rho)=2$ is given by

$$
\ell-1+\binom{\ell-1}{2}+\sum_{r=2}^{\ell}\left(\binom{b_{i}-2+\ell-r}{\ell-r}-1\right)=\binom{b_{i}+\ell-3}{b_{i}-1}+\binom{\ell-1}{2}
$$

and the number for which $\operatorname{act}(\rho)=3$ by $\binom{b_{i}+\ell-4}{\ell-2}+\ell-1$, assuming $\ell \geq 2$. Further, if $b_{i} \geq 3$, then the first case above yields $\binom{b_{i}-j+\ell-1}{\ell-2}$ possible $\rho$ such that act $(\rho)=j$ for $4 \leq j \leq b_{i}+1$. Therefore, if $\ell, b_{i} \geq 2$, we have the following succession rules:

$$
\ell \rightsquigarrow\left\{\begin{array}{l}
1 \quad\left((\ell-1)\binom{b_{i}+\ell-3}{b_{i}-1}-\left(b_{i}-1\right)\binom{b_{i}+\ell-3}{b_{i}}-\binom{\ell}{2}\right), \\
2 \\
3 \quad\left(\binom{b_{i}+\ell-3}{b_{i}-1}+\binom{\ell-1}{2}\right), \\
3 \quad\left(\binom{b_{i}+\ell-4}{b_{i}-2}+\ell-1\right), \\
j \in\left[4, b_{i}+1\right] \quad\left(\binom{b_{i}+\ell-j-1}{b_{i}-j+1}\right),
\end{array}\right.
$$

where the number of offspring corresponding to the given act value is given parenthetically. Note that the last case does not apply if $b_{i}=2$. Also, from the first case above, we have $1 \rightsquigarrow b_{i}+1$ for all $b_{i} \geq 2$.
We now consider $\{213,1322\}$ and form $\pi_{i} \in \operatorname{Av}\left(S_{i} ; 213,1322\right)$ from $\pi_{i-1} \in \operatorname{Av}\left(S_{i-1} ; 213,1322\right)$. First note that a site of $\pi_{i-1}$ is a position, including at the very beginning, that lies within the initial increasing run with the restriction that inserting $i$ does not introduce 1322. To
elucidate, suppose $\pi_{i-1}=a_{1} \cdots a_{m} \pi^{\prime}$, where $a_{1} \leq \cdots \leq a_{m-1}$ and $a_{m-1}>a_{m}$. Note that any letter for which there are at least two occurrences to the right of $a_{m-1}$ cannot belong to $\left[a_{1}+1, a_{m-1}-1\right]$, for otherwise $\pi_{i-1}$ would contain 1322 of the form $a_{1} a_{m} x x$. If some letter in $\left[a_{m-1}+1, i-1\right]$ is repeated, then $\operatorname{act}\left(\pi_{i-1}\right)=1$, so assume that this is not the case. On the other hand, suppose that a letter $a_{i}=s>a_{1}$ where $i<m$ occurs at least twice. For each such $s$, define $p=p_{s} \geq 2$ such that $s=a_{p}$, with $p$ the second-to-largest possible index. Consider the value of $s$, say $s^{*}$, for which $p$ is maximal and denote this particular $p$ by $p^{*}$. Then any position to the left of $a_{p^{*}}$ except for the beginning is eliminated as a possible site. Thus, the sites of $\pi_{i-1}$ other than the beginning must comprise a set of consecutive positions, with this set nonempty if $p^{*} \leq m-1$.
Let $j$ and $r$ be as before, but now in conjunction with $\{213,1322\}$, and we consider the same cases. If $j=r=1$, then similar reasoning yields the same result as before in this case. If $j=1$ and $r>1$, then each position of offspring $\rho$ beyond the second letter is eliminated as a site due to 213, with the first two positions active. This gives $\binom{b_{i}-2+\ell-r}{b_{i}-2}$ possible $\rho$ for every $r$, each with $\operatorname{act}(\rho)=2$. For the last two cases, assume for now $b_{i} \geq 3$. If $j=r>1$, then $\operatorname{act}(\rho)=3$ if offspring $\rho$ possesses a single run of $i$, since in this case the positions directly following the final two $i$ 's are active in addition to the initial site. If the $i$ 's occur as two runs in $\rho$, with the second run of length one, of which there are $\ell-r$ possible $\rho$, then $\operatorname{act}(\rho)=2$ as the only non-initial site directly follows the penultimate added letter $i$. For all other $\rho\left(\binom{b_{i}-2+\ell-r}{b_{i}-2}-(\ell-r+1)\right.$ possibilities), we have act $(\rho)=1$ since all positions beyond the first letter prior to the second-to-last added $i$ are eliminated as sites (due to 1322) as are all positions to the right of, but not directly following, the first run of $i$ (due to 213). If $1<j<r \leq \ell$, we similarly get $\binom{b_{i}-2+\ell-r}{b_{i}-2}$ possible $\rho$ each with act $(\rho)=1$ since $b_{i} \geq 3$.
Combining the previous cases yields the same succession rules with regard to avoiding the pattern set $\{213,1322\}$ when $b_{i} \geq 3$ as those given above for $\{132,2213\}$. Note that if $b_{i}=2$, then one gets a single $\rho$ with $\operatorname{act}(\rho)=3$ for each $r$ in the $j=r>1$ case and $\operatorname{act}(\rho)=2$ for each $1<j<r$. Thus when $b_{i}=2$ and $\ell \geq 2$, one gets $\ell \rightsquigarrow 1(0), 2\left(\binom{\ell}{2}\right), 3(\ell)$, with the indicated multiplicities, which is in accord with the prior formula when $b_{i}=2$. Finally, for all $b_{i} \geq 2$, we have $1 \rightsquigarrow b_{i}+1$. Comparing the various successions rules above implies the desired result.

### 2.6. Equivalences by variations of the Simion-Schmidt correspondence.

Theorem 2.21. We have $\{123,1322\} \stackrel{m}{\sim}\{132,1223\}$.
Proof. Let $S=1^{b_{1}} \cdots k^{b_{k}}$. Consider a natural extension of the Simion-Schmidt correspondence [9] which will establish $|\operatorname{Av}(S ; 132)|=|\operatorname{Av}(S ; 123)|$ that is defined as follows. Let $\ell_{1}>\cdots>\ell_{r}=1$ denote the set of left-right minima (lr min) of $\lambda \in \operatorname{Av}(S ; 132)$. Then we may write $\lambda=\ell_{1} \lambda^{(1)} \cdots \ell_{r} \lambda^{(r)}$, where there is no $\ell_{i}$ in the section $\lambda^{(j)}$ for $1 \leq j<i$. A letter belonging to some $\lambda^{(i)}$ that is not equal to $\ell_{i}$ will be described as red, with all other letters of $\lambda$ being blue. Note that one or more copies of an $\operatorname{lr} \min \ell_{j}$ is red if $\ell_{j} \in \lambda^{(i)}$ for some $i>j$, with all other copies of $\ell_{j}$ blue (including the leftmost). Also, $\lambda$ avoiding 132 implies that the red letters occurring in each $\lambda^{(i)}$ are increasing. Let $\lambda^{\prime}$ be obtained from $\lambda$ by considering the subsequence $S$ of $\lambda$ comprising all of its red letters and rewriting the entries of $S$ in decreasing order, leaving the blue letters unchanged in their positions. One may verify that the mapping $\lambda \mapsto \lambda^{\prime}$ is a bijection. Note that if an $\operatorname{lr} \min x$ of $\lambda$ occurs as a blue letter with multiplicity $p \geq 1$ and as a red letter with multiplicity $s \geq 0$, then $x$ must occur as blue and red letters with the same multiplicities in $\lambda^{\prime}$.

To show $\{123,1322\} \stackrel{m}{\sim}\{132,1223\}$, we first make the following further definition. Given $1<i \leq r$, let $\ell_{i}^{*}$ denote the largest member of $\left[\ell_{i}+1, \ell_{i-1}\right]$ occurring to the right of the leftmost occurrence of $\ell_{i}$, with $\ell_{1}^{*}=k$ if $\ell_{1}<k$. Note that $\ell_{i}^{*}$ need not exist if $\ell_{i-1}=\ell_{i}+1$. If $\pi \in \operatorname{Av}(S ; 132)$, consider for each $\ell_{i}$ the set of elements in $\left[\ell_{i}+1, k\right]$ for which there is at least one letter occurring to the right of the leftmost $\ell_{i}$ in $\pi$. One may verify that $\pi$ avoiding the pattern 1223 is equivalent to the condition that the only red letters of $\pi$ that may occur more than once are those equal to $\ell_{j}^{*}$ for some $j \in[r]$. Note that the subset of $[k]$ comprising those members that correspond to $\ell_{j}^{*}$ for some $j$ is the same for $\pi$ as it is for $\pi^{\prime}$. To see this, consider cases on whether $\ell_{i}^{*}=\ell_{i-1}$ or $\ell_{i}^{*}<\ell_{i-1}$ for each $i>1$, with the letter $k$ always possibly repeated as a red letter assuming $k>1$. Since the multiplicity of each red and blue letter is preserved by the mapping $\pi \mapsto \pi^{\prime}$, the latter condition above is equivalent to $\pi^{\prime}$ avoiding 1322, which implies the result.
Remark: The proof of Theorem 2.21 may be generalized to show $\left\{123,132^{r}\right\} \stackrel{m}{\sim}\left\{132,12^{r} 3\right\}$ for all $r \geq 1$.

Theorem 2.22. We have $\{121,1233\} \stackrel{m}{\sim}\{121,2133\}$ and $\{112,1233\} \stackrel{m}{\sim}\{112,2133\}$.
Proof. We first define a bijection between $\operatorname{Av}(S ; 121,1233)$ and $\operatorname{Av}(S ; 121,2133)$, where $S=$ $1^{b_{1}} \cdots k^{b_{k}}$. Let $\pi \in \operatorname{Av}(S ; 121,1233)$. We represent $\pi=\pi_{1} \pi_{2} \cdots \pi_{m}$ as a $k$-ary word, where $m=b_{1}+\cdots+b_{k}$. Let $a_{1}$ denote the largest $i \in[k]$ for which $b_{i}>1$, assuming it exists, and let $i_{1}$ be the index $j \in[m]$ such that $\pi_{j}$ corresponds to the second rightmost occurrence of the letter $a_{1}$ in $\pi$. Consider the decomposition of the initial section

$$
S_{1}=\pi_{1} \pi_{2} \cdots \pi_{i_{1}-1}=\alpha_{1} \beta_{1} \cdots \alpha_{r} \beta_{r},
$$

where $r \geq 1$ and $\alpha_{i}$ and $\beta_{j}$ denote runs of letters in $\left[a_{1}-1\right]$ and $\left[a_{1}, k\right]$, respectively, with all $\alpha_{i}$ and $\beta_{j}$ nonempty except for possibly $\alpha_{1}$ and $\beta_{r}$. Note that all letters in $S_{1}$ belonging to [ $\left.a_{1}-1\right]$ are (weakly) decreasing since $\pi$ avoids 1233 . We replace $S_{1}$ with $S_{1}^{\prime}$ within $\pi_{1}=\pi$ to obtain $\pi_{2}$, where

$$
S_{1}^{\prime}= \begin{cases}\operatorname{rev}\left(\alpha_{r}\right) \beta_{1} \operatorname{rev}\left(\alpha_{r-1}\right) \beta_{2} \cdots \operatorname{rev}\left(\alpha_{1}\right) \beta_{r}, & \text { if } \alpha_{1} \neq \varnothing ; \\ \beta_{1} \operatorname{rev}\left(\alpha_{r}\right) \beta_{2} \operatorname{rev}\left(\alpha_{r-1}\right) \cdots \beta_{r-1} \operatorname{rev}\left(\alpha_{2}\right) \beta_{r}, & \text { if } \alpha_{1}=\varnothing .\end{cases}
$$

Observe that $\pi_{2}$ contains no 2133 in which the role of 3 is played by $a_{1}$ since the entries in $\cup_{i=1}^{r} \alpha_{i}$ are decreasing. Note further that $\pi_{1}$ avoiding 121 implies that the $\alpha_{i}$ are pairwise disjoint and thus $\pi_{2}$ avoids 121 as well.
We now consider the largest letter in $\pi_{2}$ that can play the role of a 3 in a possible occurrence of 2133. Let $\pi_{2}=\pi_{1}^{(2)} \pi_{2}^{(2)} \cdots \pi_{m}^{(2)}$, as a word. Let $a_{2}$, if it exists, be the largest element of $\left[a_{1}-1\right]$ that is repeated whose second rightmost occurrence corresponds to $\pi_{\ell}^{(2)}$ for some $\ell>i_{1}$. We will denote this index $\ell$ by $i_{2}$. Note that $\pi_{2}$ avoiding 121 implies all letters $a_{2}$ occur to the right of the second rightmost $a_{1}$. Consider the decomposition of the initial section

$$
S_{2}=\pi_{1}^{(2)} \pi_{2}^{(2)} \cdots \pi_{i_{2}-1}^{(2)}=\alpha_{1}^{2} \beta_{1}^{(2)} \cdots \alpha_{s}^{(2)} \beta_{s}^{(2)},
$$

where $s \geq 1$ and $\alpha_{i}^{(2)}$ and $\beta_{j}^{(2)}$ denote runs of letters in $\left[a_{2}-1\right]$ and $\left[a_{2}, k\right]$, respectively, with only $\alpha_{1}^{(2)}$ and $\beta_{s}^{(2)}$ possibly empty. Suppose that the second rightmost $a_{1}$ occurs in $\beta_{t}^{(2)}$ for some $t \in[s]$. Then the entries in $\cup_{i=1}^{t} \alpha_{i}^{(2)}$ are (weakly) increasing via the first step of the algorithm above, whereas those in $\cup_{i=t+1}^{s} \alpha_{i}^{(2)}$ are decreasing, with either set of entries possibly
empty (which always is the case if $s=t$ or if $s=1$, with $\alpha_{1}^{(2)}$ empty). Note further that $\pi_{1}$ avoiding 1233 implies that the maximum of the set $\cup_{i=t+1}^{s} \alpha_{i}^{(2)}$ is less than or equal the minimum of $\cup_{i=1}^{t} \alpha_{i}^{(2)}$ assuming both sets are nonempty and indeed this inequality is strict since $\pi_{2}$ avoids 121 .
Define $S_{2}^{\prime}$ by

$$
S_{2}^{\prime}= \begin{cases}\operatorname{rev}\left(\alpha_{s}^{(2)}\right) \beta_{1}^{(2)} \cdots \operatorname{rev}\left(\alpha_{t+1}^{(2)}\right) \beta_{s-t}^{(2)} \alpha_{1}^{(2)} \beta_{s-t+1}^{(2)} \cdots \alpha_{t}^{(2)} \beta_{s}^{(2)}, & \text { if } t<s \text { and } \alpha_{1}^{(2)} \neq \varnothing ; \\ \beta_{1}^{(2)} \operatorname{rev}\left(\alpha_{s}^{(2)}\right) \cdots \beta_{s-t}^{(2)} \operatorname{rev}\left(\alpha_{t+1}^{(2)}\right) \beta_{s-t+1}^{(2)} \alpha_{2}^{(2)} \cdots \beta_{s-1}^{(2)} \alpha_{t}^{(2)} \beta_{s}^{(2)}, & \text { if } t<s \text { and } \alpha_{1}^{(2)}=\varnothing ; \\ S_{2}, & \text { if } t=s .\end{cases}
$$

We replace $S_{2}$ by $S_{2}^{\prime}$ in $\pi_{2}$, denoting the resulting multipermutation by $\pi_{3}$. Note that $\pi_{3}$ contains no 2133 in which the role of 3 is played by a member of $\left[a_{2}, k\right.$ ], by the maximality of $a_{1}$ and $a_{2}$. Furthermore, since complete runs of letters belonging to $\left[a_{2}-1\right]$ are shifted in the transition from $\pi_{2}$ to $\pi_{3}$, no occurrence of 121 is introduced.
Repeat the procedure above for $\pi_{3}=\pi_{1}^{(3)} \pi_{2}^{(3)} \cdots \pi_{m}^{(3)}$, considering for the largest member $a_{3}$ of $\left[a_{2}-1\right]$ for which there is a letter occurring at least twice to the right of $\pi_{i_{2}}^{(3)}$. Continue the procedure until one produces $\pi_{r}$ for which there is no letter in $\left[a_{r-1}-1\right]$ occurring to the right of the $\left(i_{r-1}\right)$-st entry that is repeated. Note that this process must terminate since $a_{i+1}<a_{i}$ for all $i$. One may verify $\pi_{r} \in \operatorname{Av}(S ; 121,2133)$.
Define $f: \operatorname{Av}(S ; 121,1233) \rightarrow \operatorname{Av}(S ; 121,2133)$ by setting $f(\pi)=\pi_{r}$ if $r>1$, with $f(\pi)=\pi$ if $\pi$ contains no repeated letters.
To show that $f$ is reversible, first let $a_{1}^{\prime}$ denote the largest repeated letter of $f(\pi)$. Then let $a_{i}^{\prime}$ be obtained from $a_{i-1}^{\prime}$ for $i>1$ by considering the largest member of $\left[a_{i-1}^{\prime}-1\right]$ for which there is a repeated letter occurring to the right of the second rightmost $a_{i-1}^{\prime}$ in $f(\pi)$. One can show that $a_{i}=a_{i}^{\prime}$ for $1 \leq i<r$, as the relative order of the penultimate occurrences of the $a_{i}$ is unchanged in each step of the process in the transformation from $\pi_{1}$ to $\pi_{r}$ (though the exact positions of these occurrences might change). Thus, $f$ may be reversed as follows. First undo the last step of the above procedure starting with $\pi_{r}$ upon considering the position of the penultimate $a_{r-1}$. Note that $a_{r-1}$ is actually the rightmost letter occurring at least twice in $\pi_{r}$. Then undo the second-to-last step considering $a_{r-2}$ in $\pi_{r-1}$ and so on, successively, until one obtains $\pi_{2}$. Inverting the mapping described in the first paragraph above then recovers $\pi_{1}=\pi$ from $\pi_{2}$.
The bijection $f$ also applies to $\{112,1233\} \stackrel{m}{\sim}\{112,2133\}$. Note that there is no 112 introduced in going from $\pi_{i-1}$ to $\pi_{i}$ for all $i$ since the letters to the left of $a_{i-1}$ belonging to [ $a_{i-1}-1$ ] are seen to be distinct, upon proceeding inductively (the $i=1$ case, by assumption). Also, one may obtain $\pi_{i}$ from $\pi_{i-1}$ in the proof of this equivalence simply by reversing the order of the terms in the subsequence of $\pi_{i-1}$ comprising all members of $\left[a_{i-1}-1\right]$ to the left of the penultimate $a_{i-1}$ letter since we need not avoid 121 in this case.

## 3. Concluding remarks

In Table 1 below is a list of the members of the non-singleton (3,4) Wilf-equivalence classes for compositions, up to symmetry (i.e., reversal), along with their respective theorem references. Note that $m$-equivalence of two sets of patterns clearly implies the equivalence of the sets with respect to avoidance in compositions.

| $\{\sigma, \tau\}$ | Reference | $\{\sigma, \tau\}$ | Reference |
| :---: | :---: | :---: | :---: |
| \{111, 1221\}, \{111, 2112\} | Thm. 2.10 | \{111, 1123\}, \{111, 1132\} | (c) Prop. 2.9 |
| $\{111,1223\},\{111,1232\}$, $\{111,1322\}$, $\{111,2132\},\{111,2123\}$, | Prop. 2.9 | \{111, 1233\}, \{111, 2133\} | Prop. 2.9 |
| $\{111,1234\},\{111,1243\}$, $\{111,1432\},\{111,2134\}$, $\{111,2143\},\{111,3214\}$ | Prop. 2.9 | \{112, 1111\}, \{121, 1111\} | (c) Prop. 2.9 |
| \{112, 1211\}, \{121, 1112\} | Thm. 2.10 | \{112, 2121\}, \{121, 1122\} | Thm. 2.13 |
| \{112, 1231\}, \{121, 1132\} | Thm. 2.16 | \{112, 2122\}, \{121, 1222\} | Thm. 2.12 |
| \{112, 2212\}, \{121, 2122\} | Thm. 2.11 | $\begin{aligned} & \{112,2312\},\{121,1223\}, \\ & \{121,2213\} \end{aligned}$ | Thm. 2.14 |
| $\begin{aligned} & \hline\{112,1232\},\{112,2132\}, \\ & \{121,1322\} \\ & \hline \end{aligned}$ | Thm. 2.15 | \{121, 1233\}, \{121, 2133\} | Thm. 2.22 |
| \{112, 1233\}, $\{112,2133\}$ | Thm. 2.22 | \{121, 1234\}, \{121, 2134\} | Thm. 2.19 |
| \{121,1243\}, $\{121,2143\}$ | Thm. 2.19 | \{112, 1243\}, \{112, 2143\} | Thm. 2.19 |
| \{121, 1342\}, \{121, 3142\} | Conj. 3.2 | \{112, 2314\}, \{112, 3124\} | Prop. 2.9 |
| $\begin{aligned} & \{112,1234\},\{112,2134\}, \\ & \{112,3214\} \end{aligned}$ | Prop. 2.9 | $\{122,1111\},\{212,1111\}$ | Prop. 2.9 |
| $\begin{aligned} & \{123,1111\},\{132,1111\}, \\ & \{213,1111\} \end{aligned}$ | Prop. 2.9 | \{122, 1121\}, \{212, 1112\} | (c) Thm. 2.12 |
| \{122, 1211\}, $\{212,1121\}$ | (c) Thm. 2.11 | \{123, 1211\}, \{213, 1211\} | Thm. 2.18 |
| $\begin{aligned} & \{123,1121\},\{132,1211\}, \\ & \{213,1121\} \end{aligned}$ | Thm. 2.17 | $\begin{aligned} & \{123,1112\},\{132,1121\}, \\ & \{213,1112\} \end{aligned}$ | Prop. 2.9, Thm. 2.18 |
| \{122, 2121\}, $\{212,1122\}$ | (c) Thm. 2.13 | \{122, 2212\}, \{212, 1222\} | (c) Thm. 2.10 |
| \{212, 1123\}, $\{212,1132\}$ | (c) Thm. 2.22 | \{122, 1123\}, \{122, 1132\} | (c) Thm. 2.22 |
| $\begin{aligned} & \{122,2312\},\{212,1223\}, \\ & \{212,1322\} \end{aligned}$ | (c) Thm. 2.14 | $\begin{aligned} & \{122,2123\},\{122,2132\}, \\ & \{212,2213\} \\ & \hline \end{aligned}$ | (c) Thm. 2.15 |
| \{122, 3123\}, \{212, 2133\} | (c) Thm. 2.16 | \{122, 2134\}, \{122, 2143\} | (c) Thm. 2.19 |
| \{212, 1234\}, $\{212,1243\}$ | (c) Thm. 2.19 | \{212, 3124\}, \{212, 3142\} | Conj. 3.2 |
| \{212,2134\}, \{212, 2143\} | (c) Thm. 2.19 | \{122, 1342\}, \{122, 1423\} | (c) Prop. 2.9 |
| $\begin{aligned} & \{122,1234\},\{122,1243\}, \\ & \{122,1432\} \end{aligned}$ | (c) Prop. 2.9 | \{123, 2212\}, \{132, 2212\} | (c) Thm. 2.18 |
| $\begin{aligned} & \{123,2122\},\{132,2122\}, \\ & \{213,2212\} \end{aligned}$ | (c) Thm. 2.17 | $\begin{aligned} & \{123,1222\},\{132,1222\}, \\ & \{213,2122\} \end{aligned}$ | $\begin{aligned} & \text { (c) Prop. 2.9, Thm. } \\ & 2.18 \end{aligned}$ |
| \{132, 2213\}, $\{213,1322\}$ | Thm. 2.20 | \{123, 1322\}, $\{132,1223\}$ | Thm. 2.21 |
| \{123, 2213\}, $\{213,1223\}$ | (c) Thm. 2.21 |  |  |

Table 1. Non-trivial Wilf classes for compositions, where (c) stands for complement.

Extending the arguments above (at times, treating separately the $r=1$ case) yields the following set of generalized equivalences, some of which have already been mentioned in prior remarks.

Theorem 3.1. The following equivalences hold, where it is assumed $r \geq 1$ unless stated otherwise:

$$
\begin{aligned}
& \text { (i) }\left\{1^{r+1}, \tau\right\} \stackrel{m}{\sim}\left\{1^{r+1}, \tau^{c}\right\}, \\
& \text { (ii) }\left\{112,121^{r-1}\right\} \stackrel{m}{\sim}\left\{121,1^{r} 2\right\}, \\
& \text { (iii) }\left\{112,1231^{r-1}\right\} \stackrel{\sim}{\sim}\left\{121,1^{r} 32\right\}, \\
& \text { (iv) }\left\{112,1232^{r-1}\right\} \stackrel{\sim}{\sim}\left\{112,2132^{r-1}\right\} \stackrel{m}{\sim}\left\{121,132^{r}\right\}, \\
& \text { (v) }\left\{112,123^{r}\right\} \stackrel{m}{\sim}\left\{112,213^{r}\right\}, \\
& \text { (vi) }\left\{112,212^{r}\right\} \stackrel{m}{\sim}\left\{121,12^{r+1}\right\}, \\
& \text { (vii) }\left\{112,2^{r} 12\right\} \stackrel{m}{\sim}\left\{121,212^{r}\right\}, \quad r \geq 2, \\
& \text { (viii) }\left\{112,2312^{r-1}\right\} \stackrel{m}{\sim}\left\{121,12^{r} 3\right\} \stackrel{m}{\sim}\left\{121,2^{r} 13\right\}, \quad r \geq 2, \\
& \text { (ix) }\left\{121,123^{r}\right\} \stackrel{m}{\sim}\left\{121,213^{r}\right\}, \\
& \text { (x) }\left\{123,1^{r} 2\right\} \stackrel{m}{\sim}\left\{132,1^{r-1} 21\right\}, \\
& \text { (xi) }\left\{123,1^{r} 21\right\} \stackrel{m}{\sim}\left\{213,1^{r} 21\right\}, \\
& \text { (xii) }\left\{123,121^{r}\right\} \stackrel{m}{\sim}\left\{213,121^{r}\right\}, \\
& \text { (xiii) }\left\{123,132^{r}\right\} \stackrel{m}{\sim}\left\{132,12^{r} 3\right\}, \\
& \text { (xiv) }\left\{132,2^{r} 13\right\} \stackrel{\sim}{\sim}\left\{213,132^{r}\right\},
\end{aligned}
$$

where $\tau$ in (i) denotes a permutation of the multiset $1^{r} \cdots k^{r}$ for some $k$.
Taking the appropriate value of $r$ in each part of Theorem 3.1 above yields the (3,4) equivalence which it generalizes.
Combining the $r=1$ cases above in Theorem 3.1, together with the strong equivalence of 123, 132 and 213 and also of 112 and 121, yields the following complete list of multiset equivalences for $(3,3)$ up to symmetry:

- $\{111,123\} \stackrel{m}{\sim}\{111,132\} \stackrel{m}{\sim}\{111,213\}$,
- $\{111,112\} \stackrel{m}{\sim}\{111,121\}$,
- $\{112,123\} \stackrel{m}{\sim}\{112,213\} \stackrel{m}{\sim}\{121,132\}$,
- $\{112,212\} \stackrel{m}{\sim}\{121,122\}$,
- $\{121,123\} \stackrel{m}{\sim}\{121,213\}$.

Let $C_{n}\left(\tau, \tau^{\prime}\right)$ denote the set of all compositions of $n$ that avoid the patterns $\tau$ and $\tau^{\prime}$ and let $c_{n}\left(\tau, \tau^{\prime}\right)=\left|C_{n}\left(\tau, \tau^{\prime}\right)\right|$. Our enumeration data suggest two possible compositional equivalences that we are unable to prove by our present methods, which we state here as a conjecture.

Conjecture 3.2. If $n \geq 1$, then

$$
c_{n}(121,1342)=c_{n}(121,3142) \quad \text { and } \quad c_{n}(212,3124)=c_{n}(212,3142) .
$$

Note that numerically we have confirmed both equivalences in Conjecture 3.2 for all $n$ up to $n=35$. However, the number of permutations of the multiset $1^{2} \cdots 5^{2}$ that avoid $\{121,1342\}$ is seen to differ from the number of permutations that avoid $\{121,3142\}$ ( 46566 vs. 45969). Taking reverse complements gives the same story for $\{212,3124\}$ and $\{212,3142\}$. Thus, if
demonstrated, the equivalences in Conjecture 3.2 would provide two examples of pattern sets that are Wilf-equivalent with respect to compositions, but are not multiset equivalent.

## 4. Appendix 1

Below we present, up to symmetry, the cardinalities of all $C_{n}\left(\tau, \tau^{\prime}\right)$ when $n=24$, where $\tau$ denotes a 3-letter and $\tau^{\prime}$ a 4-letter pattern. Note that in a few cases, some values for $n>24$ are given which are required to differentiate a specific class from another.

- $\{111,1212\}: 104335$,
- $\quad\{111,1221\},\{111,2112\}: 104873$,
- $\{111,1122\}: 105957$,
- $\{111,1312\}: 84928$,
- $\{111,1231\}: 86035$,
- $\{111,2113\}: 86215$,
- $\{111,1213\}: 86218$,
- $\quad\{111,1123\},\{111,1132\}: 86641$,

■ $\quad\{111,1223\},\{111,1232\},\{111,1322\}$, $\{111,2123\},\{111,2132\},\{111,2213\}:$ 95697,

- $\{111,2313\}: 109460$,
- $\{111,1332\}: 109787$,
- $\{111,1323\}: 109789$,
- $\{111,3123\}: 109781$,
- $\quad\{111,1233\},\{111,2133\}: 109932$,
- $\{111,2413\}: 94579,146512$,
- $\{111,1423\}: 94579,146775$,
- $\{111,1342\}$ : 94690 ,
- $\{111,2314\}: 94800$,
- $\{111,3124\}: 94816$,
- $\quad\{111,1234\},\{111,1243\},\{111,1432\}$, $\{111,2134\},\{111,2143\},\{111,3214\}:$ 94939,
- $\{111,1324\}: 94956$,
- $\{112,1111\},\{121,1111\}: 52825$,
- $\{112,2111\}: 37484$,
- $\{112,1211\},\{121,1112\}: 43592$,
- $\{112,2211\}: 46008$,
- $\{112,1221\}: 56883$,
- $\{112,2121\},\{121,1122\}: 58994$,
- $\{121,2112\}: 66160$,
- $\{112,3211\}: 28809$,
- $\{112,2311\}: 33701$,
- $\{112,2221\}: 62378$,
- $\{112,3121\}: 45347$,
- $\{112,1321\}: 49080$,
- $\{121,2113\}: 51680$,
- $\{121,1123\}: 51887$,
- $\{112,2131\}: 57398$,
- $\quad\{112,1231\},\{121,1132\}: 58001$,
- $\{112,1222\}: 71059$,
- $\{112,2122\},\{121,1222\}: 72586$,
- $\quad\{112,2212\},\{121,2122\}: 77327$,
- $\{112,3221\}: 45504$,
- $\{112,2321\}: 51758$,
- $\{112,3122\}: 60722$,
- $\{112,3212\}: 63624$,
- $\{112,1322\}: 63020$,
- $\boldsymbol{\square}\{112,2312\},\{121,1223\},\{121,2213\}:$

66262,

- $\{121,2123\}: 71678$,
- $\quad\{112,1232\},\{112,2132\},\{121,1322\}:$ 70572,
- $\{112,3321\}: 61826$,
- $\{112,2331\}: 67910$,
- $\{112,3231\}: 68588$,
- $\{112,3312\}: 75161$,
- $\{112,1332\}: 74353$,
- $\{112,3132\}: 74672$,
- $\quad\{121,1233\},\{121,2133\}: 77444$,
- $\{121,3123\}: 80111$,
- $\quad\{112,1233\},\{112,2133\}: 79160$,
- $\{112,2313\}: 79443$,
- $\{121,1332\}: 79444$,
- $\{112,1323\}: 79510$,
- $\{112,3123\}: 79638$,
- $\{112,3213\}: 79713$,
- $\{121,1323\}: \quad 80707,118366,172460$, 250527,362373,521428,
- $\{121,2313\}: \quad 80707,118366,172460$, 250527,362373,521429,
- $\{112,4321\}: 45291$,
- $\{112,3421\}: 50435$,
- $\{112,4231\}: 57822$,
- $\{112,2431\}: 59077$,
- \{112, 3241\}: 65123,
- $\{112,2341\}: 65182$,
- $\{112,4312\}: 65130$,
- $\{112,4132\}: 67293$,
- $\{112,1432\}: 68218$,
- $\{112,3412\}: 67262$,
- $\{112,3142\}: 72021$,
- $\{112,1342\}: 72102$,
- $\{121,3124\}: 74030$,
- $\{121,3214\}: 74040$,

■ \{121, 1234\}, $\{121,2134\}: 74041$,
■ \{121, 1243\}, $\{121,2143\}: 76542$,

- $\{112,4213\}: 75266$,
- $\{112,4123\}: 75303$,
- $\{112,2413\}: 75621$,
- $\{112,1423\}: 75647$,
- $\{112,1243\},\{112,2143\}: 76193$,
- $\{121,2314\}: 76770$,
- $\{121,1324\}: 76780$,

■ $\{121,1342\},\{121,3142\}: 78034,113633$, 164728,

- $\{121,1423\}: 78034,113633,164727$,
- $\{121,1432\}: 78044$,

■ \{112, 2314\}, $\{112,3124\}: 79157$,
■ $\{112,1234\},\{112,2134\},\{112,3214\}$ : 79198,115565,167895,

- $\{112,1324\}: 79198,115565,167896$,
- $\{122,1111\},\{212,1111\}: 138512$,

■ $\{123,1111\},\{132,1111\},\{213,1111\}$ : 83829,

- $\{122,2111\}: 97469$,
- $\{122,1112\}: 145591$,
- $\{122,1121\},\{212,1112\}: 149406$,
- $\{122,1211\},\{212,1121\}: 162336$,
- $\{123,2111\}: 45027$,
- $\{132,2111\}: 55190$,
- $\{213,2111\}: 90128$,
- $\{132,1112\}: 101522$,
- $\{123,1211\},\{213,1211\}: 108281$,

■ $\{123,1121\},\{132,1211\},\{213,1121\}$ : 120344,
■ $\{123,1112\},\{132,1121\},\{213,1112\}$ : 133919,

- $\{122,2211\}: 396868$,
- $\{122,2112\}: 444097$,
- $\{122,2121\},\{212,1122\}: 445294$,
- $\{212,1221\}: 448694$,
- $\{122,2221\}: 593122$,
- $\{122,2212\},\{212,1222\}: 607805$,
- $\{122,2311\}: 207967$,
- $\{122,3211\}: 217656$,
- $\{122,2131\}: 217373$,
- $\{122,2113\}: 218288$,
- $\{122,3112\}: 249010$,
- $\{122,3121\}: 251890$,
- $\{212,1123\},\{212,1132\}: 286877$,
- $\{212,1231\}: 306089$,
- \{212, 2113\}: 318167,
- $\{122,1312\}: 318616$,
- $\{122,1231\}: 322179$,
- $\{122,1123\},\{122,1132\}: 321098$,
- $\{122,1213\}: 326103$,
- $\{122,1321\}: 331513$,
- $\{212,1213\}: 326937,505465,776447$,
- $\{212,1312\}: 326937,505465,776446$,
- $\{122,2231\}: 472765$,
- $\{122,2213\}: 481694$,
- $\{122,2321\}: 506314$,
- $\{122,2312\},\{212,1223\},\{212,1322\}$ : 518302,
- $\{122,2123\},\{122,2132\},\{212,2213\}$ :

542447,

- $\{122,3221\}: 569643$,
- $\{122,3212\}: 586797$,
- $\{212,1232\}: 628352$,
- $\{122,3321\}: 639459$,
- $\{122,3231\}: 636726$,
- $\{122,3213\}: 642427$,
- $\{122,3312\}: 645365$,
- $\{212,1332\}: 658101$,
- $\{212,1233\}: 658147$,
- $\{122,3132\}: 668842$,
- $\{122,3123\},\{212,2133\}: 669649$,
- $\{122,3241\}: 502005$,
- $\{122,2431\}: 511743$,
- $\{122,2341\}: 512408$,
- $\{122,3214\}: 510131$,
- $\{122,2413\}: 516402$,
- $\{122,2314\}: 516912$,
- $\{122,2134\},\{122,2143\}: 529470$,
- $\{122,3421\}: 524533$,
- $\{122,4231\}: 525914$,
- $\{122,4321\}: 531152$,
- $\{122,4213\}: 531833$,
- $\{122,3412\}: 531977$,
- $\{122,4312\}: 539208$,
- $\{212,1423\}: 562177$,
- $\{212,1324\}: 562233$,
- $\{212,1342\}: 565100$,
- $\{212,1234\},\{212,1243\}: 565155,904528$,
- $\{212,1432\}: 565155,904529$,
- $\{122,3142\}: 550948$,
- $\{122,3124\}: 552176$,
- $\{122,4132\}: 565675$,
- $\{122,4123\}: 565838$,
- $\{212,2314\}: 583686$,
- $\{212,2413\},\{212,3124\}: 583691$,
- $\{212,3214\}: 583743$,
- $\{212,2134\},\{212,2143\}: 595438$,
- $\{122,1342\},\{122,1423\}: 601322$,

■ $\{122,1234\},\{122,1243\},\{122,1432\}$ : 602740,

- $\{122,1324\}: 602746$,
- \{132, 2211\}: 157727,
- $\{213,1122\}: 203230$,
- \{123, 2211\}: 186726,
- \{213, 1221\}: 214850,
- \{132, 2121\}: 228554,
- \{123, 2112\}: 262501,
- $\{213,2211\}: 224583$,
- \{123, 2121\}: 261305 ,
- \{132, 2112\}: 275288,
- $\{123,1122\}: 273983$,
- $\{213,1212\}: 279782$,
- $\{132,1221\}: 288149$,
- $\{132,1122\}: 312538$,
- $\{123,1221\}: 302929$,
- \{213, 2121\}: 305159,
- $\{132,1212\}: 338776$,
- \{213, 2112\}: 342635,
- $\{123,1212\}: 351544$,
- \{123, 3211\}: 195977,
- $\{213,3112\}: 242902$,
- $\{123,3112\}: 212143$,
- $\{213,3121\}: 246356$,
- $\{123,3121\}: 238941$,
- $\{213,3211\}: 234061$,
- $\{132,2311\}: 449163$,
- \{132, 2113\}: 354774,
- $\{213,1312\}: 391242$,
- $\{213,1321\}: 380778$,
- $\{132,1213\}: 399970$,
- $\{123,1312\}: 410195$,
- $\{132,1123\}: 414782$,
- $\{123,1321\}: 415510$,
- $\{132,2131\}: 494818$,
- $\{213,1132\}: 503790$,
- $\{123,1132\}: 529612$,
- $\{132,1231\}: 533009$,
- $\{132,2221\}: 339146$,
- $\{213,1222\}: 358327$,
- \{123, 2221\}: 378373,
- $\{213,2221\}: 390069$,
- $\{123,2212\},\{132,2212\}: 418354$,
- $\{123,2122\},\{132,2122\},\{213,2212\}$ :

440827,
■ $\{123,1222\},\{132,1222\},\{213,2122\}$ : 485327,

- $\{132,3211\}: 521377$,
- $\{123,2311\}: 524744$,
- $\{132,3121\}: 570656$,
- $\{213,2311\}: 599187$,
- $\{132,3112\}: 601383$,
- $\{123,2131\}: 641291$,
- $\{213,1231\}: 671515$,
- $\{123,2113\}: 744984$,
- $\{213,1123\}: 746159$,
- $\{213,3122\}: 342362$,
- $\{123,3221\}: 343971$,
- $\{213,3221\}: 340675$,
- $\{123,3212\}: 382554$,
- $\{123,3122\}: 381825$,
- $\{213,3212\}: 394852$,
- $\{132,2231\}: 526791$,
- $\{132,2213\},\{213,1322\}: 458220$,
- $\{132,2123\}: 506438$,
- $\{123,1322\},\{132,1223\}: 533568$,
- $\{132,3221\}: 603342$,
- $\{123,2321\}: 686218$,
- $\{132,3212\}: 653346$,
- $\{123,2312\}: 694424$,
- $\{213,2321\}: 697009$,
- $\{132,3122\}: 681004$,
- $\{213,2312\}: 732527$,
- $\{213,1232\}: 753461$,
- \{123, 2132\}: 771116,
- $\{123,2231\}: 854373$,
- $\{132,2321\}: 859462$,
- $\{213,2231\}: 875649$,
- $\{132,2312\}: 881784$,

■ $\{123,2213\},\{213,1223\}: 902785$,

- $\{213,3312\}: 708649$,
- $\{123,3321\}: 718470$,
- $\{123,3312\}: 714199$,
- $\{213,3321\}: 713750$,
- $\{132,2331\}: 815573$,
- $\{213,1332\}: 759017$,
- $\{123,3132\}: 801324$,
- \{132, 2313\}: 850475 ,
- $\{213,3132\}: 835356$,
- \{132, 2133\}: 836368 ,
- $\{123,1332\}: 844730$,
- $\{132,1233\}: 852658$,
- $\{123,2331\}: 881288$,
- $\{213,3231\}: 901887$,
- $\{132,3231\}: 896638$,
- $\{132,3213\}: 903969$,
- \{132, 3123\}: 915698,
- \{213, 1323\}: 920937,
- \{123, 2313\}: 930903,
- \{123, 2133\}: 935140,
- \{213, 3123\}: 954148,
- \{132,3321\}: 940340,
- $\{123,3231\}: 954767$,
- $\{132,3312\}: 955697$,
- \{213, 2331\}: 959257,
- $\{213,1233\}: 972460$,
- $\{123,3213\}: 979479$,
- $\{213,4312\}: 694835$,
- $\{123,4321\}: 705999$,
- \{123, 4312\}: 702556,
- $\{213,4321\}: 700152$,
- \{132, 2341\}: 850077,
- \{213, 1432\}: 787114,
- $\{123,4132\}: 779792$,
- $\{132,2314\}: 833222$,
- $\{213,4132\}: 795511$,
- $\{132,2134\}: 834511$,
- $\{132,1234\}: 851461$,
- $\{123,1432\}: 857921$,
- $\{213,4231\}: 873579$,
- $\{123,2431\}: 882598$,
- $\{132,3241\}: 922782$,
- $\{213,3412\}: 902041$,
- $\{123,3421\}: 901200$,
- $\{132,3214\}: 897047$,
- $\{123,3412\}: 900841$,
- $\{213,3421\}: 906726$,
- $\{213,1423\}: 918238$,
- $\{132,3124\}: 915830$,
- $\{123,2413\}: 922421$,
- $\{132,4231\}: 930808$,
- $\{213,4123\}: 922049$,
- $\{132,4213\}: 936394$,
- $\{213,1342\}: 944445$,
- \{132, 4123\}: 946103,
- $\{123,3142\}: 958210$,
- \{123, 2143\}: 958787,
- $\{132,3421\}: 960869$,
- $\{123,4231\}: 943423$,
- $\{132,3412\}: 970430$,
- $\{213,2431\}: 982832$,
- $\{132,4321\}: 965760$,
- $\{123,4213\}: 967564$,
- $\{123,3241\}: 980259$,
- $\{132,4312\}: 978603$,
- $\{213,1243\}: 1002198$,
- $\{213,2341\}: 1004697$,
- $\{123,3214\}: 1012078$,
- $\{213,1234\}: 1019326$.


## 5. Appendix 2

Below we present, up to symmetry, the cardinalities of all $C_{n}\left(\tau, \tau^{\prime}\right)$ when $n=25$, where $\tau$ and $\tau^{\prime}$ denote patterns of length three.

- $\{112,321\}: 9421$,
- $\{112,231\}: 17355$,
- $\{112,211\}: 17481$,
- $\{112,121\}: 26731$,
- $\{112,312\}: 32058$,

■ $\{111,123\}, \quad\{111,132\}, \quad\{111,213\}:$ 34552,

- $\{112,132\}: 38507$,
- $\{112,221\}: 40883$,

■ $\{111,112\},\{111,121\}: 44452$,


## References

[1] J. Backelin, J. West and G. Xin, Wilf-equivalence for singleton classes, Adv. in Appl. Mathematics 38(2) (2007), 133-148.
[2] R. L. Graham, D. E. Knuth and O. Patashnik, Concrete Mathematics: A Foundation for Computer Science, Addison-Wesley, Boston (1994).
[3] T. Guo, C. Krattenthaler and Y. Zhang, On (shape-)Wilf-equivalence for words, Adv. in Appl. Mathematics 100 (2018), 87-100.
[4] S. Heubach, T. Mansour and A. Munagi, Avoiding permutation patterns of type $(2,1)$ in compositions, Online J. Anal. Comb. 4 (2009), 1-9.
[5] V. Jelínek and T. Mansour, On pattern-avoiding partitions, Electron. J. Combin. 15:1 (2008), \#R39.
[6] V. Jelínek and T. Mansour, Wilf-equivalence on $k$-ary words, compositions, and parking functions, Electron. J. Combin. 16 (2009), \#R58.
[7] C. Krattenthaler, Growth diagrams, and increasing and decreasing chains in fillings of Ferrers shapes, Adv. in Appl. Math. 37(3) (2006), 404-431.
[8] C. D. Savage and H. S. Wilf, Pattern avoidance in compositions and multiset permutations, Adv. in Appl. Math. 36(2) (2006), 194-201.
[9] R. Simion and F. W. Schmidt, Restricted permutations, European J. Combin. 6 (1985), 383-406.
[10] Z. Stankova and J. West, A new class of Wilf-equivalent permutations, J. Algebraic Combin. 15(3) (2002), 271-290.
[11] J. West, Permutations with forbidden subsequences and stack-sortable permutations, PhD thesis, MIT, Cambridge, MA (1990).
[12] J. West, Generating trees and the Catalan and Schröder numbers, Discrete Math. 146 (1995), 247-262.

## Computer Science Institute, Charles University, Prague, Czechia

Email address: jelinek@iuuk.mff.cuni.cz
Department of Mathematics, University of Haifa, 3498838 Haifa, Israel
Email address: tmansour@univ.haifa.ac.il
Departamento de Matemáticas, Universidad Nacional de Colombia, Bogotá, Colombia
Email address: jlramirezr@unal.edu.co
Department of Mathematics, University of Tennessee, 37996 Knoxville, TN, USA
Email address: shattuck@math.utk.edu


[^0]:    Date: November 12, 2021.
    2010 Mathematics Subject Classification. 05A05, 05A15.
    Key words and phrases. pattern avoidance, composition, multiset, Wilf-equivalence.
    V. Jelínek is supported by project 18-19158S of the Czech Science Foundation.

