# REFINED EULERIAN NUMBERS AND BALLOT PERMUTATIONS 

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#### Abstract

A ballot permutation is a permutation $\pi$ such that in any prefix of $\pi$ the descent number is not more than the ascent number. In this article, we obtained a formula in close form for the multivariate generating function of $\{A(n, d, j)\}_{n, d, j}$, which denote the number of permutations of length $n$ with $d$ descents and $j$ as the first letter. Besides, by a series of calculations with generatingfunctionology, we confirm a recent conjecture of Wang and Zhang for ballot permutations.


## 1. Introduction

Let $\mathcal{S}_{n}$ be the symmetric group of all permutations of $[n]=\{1,2, \ldots, n\}$. A position $i \in[n-1]$ in a permutation $\pi=\pi_{1} \pi_{2} \cdots \pi_{n} \in \mathcal{S}_{n}$ is a descent if $\pi_{i}>\pi_{i+1}$, and an ascent if $\pi_{i}<\pi_{i+1}$. Denote the number of descents of $\pi$ by $\operatorname{des}(\pi)$, and the number of ascents by $\operatorname{asc}(\pi)$. We call the number $h(\pi)=\operatorname{asc}(\pi)-\operatorname{des}(\pi)$ the height of $\pi$. The permutation $\pi$ is said to be a ballot permutation if $h\left(\pi_{1} \pi_{2} \cdots \pi_{i}\right) \geq 0$ for all $i \in[n]$. Let $\mathscr{B}_{n}$ denote the set of ballot permutations on $[n]$. Define $\mathscr{B}_{0}=\{\epsilon\}$, where $\epsilon$ is the empty permutation.

The problem of enumerating ballot permutations is closely related with that of enumerating ordinary permutations with a given up-down signature, see [4, 11, 25, 28]. A ballot permutation of length $2 n+1$ with $n$ descents is said to be a Dyck permutation, whose enumeration is the Eulerian-Catalan number, see Bidkhori and Sullivant [8]. Let $\mathscr{O}_{n}$ be the set of odd order permutations of [ $n$ ], viz., the set of permutations of $[n]$ which are the products of cycles with odd lengths. Bernardi, Duplantier and Nadeau [6] proved that the number of ballot permutations of length $n$ equals the number of odd order permutations of length $n$, by using compositions of bijections, and thus Theorem 1.1 follows. A short proof is given by Wang and Zhang [31].

Theorem 1.1 (Bernardi, Duplantier and Nadeau). The number of ballot permutations of length $n$ is

$$
b_{n}= \begin{cases}{[(n-1)!!]^{2},} & \text { if } n \text { is even } \\ n!!(n-2)!!, & \text { if } n \text { is odd }\end{cases}
$$

where $(-1)!!=1$.

[^0]In order to give a refinement for Theorem 1.1, Spiro [1] introduced a statistic $M(\pi)$ which is defined for a permutation $\pi$ such that

$$
M(\pi)=\sum_{c} \min (\operatorname{cdes}(c), \operatorname{casc}(c))
$$

where the sum runs over all cycles $c=\left(c_{1} c_{2} \cdots c_{k}\right)$ of $\pi$, with the cyclic descent

$$
\operatorname{cdes}(c)=\mid\left\{i \in[k]: c_{i}>c_{i+1} \text { where } c_{k+1}=c_{1}\right\} \mid
$$

and the cyclic ascent

$$
\operatorname{casc}(c)=\mid\left\{i \in[k]: c_{i}<c_{i+1} \text { where } c_{k+1}=c_{1}\right\}|=|c|-\operatorname{cdes}(c),
$$

where $|c|$ is the length of $c$. Spiro conjectured that the number of ballot permutations of length $n$ with $d$ descents equals the number of odd order permutations $\pi$ of length $n$ such that $M(\pi)=d$, which is confirmed by Wang and the first author [32], and thus Theorem 1.2 follows.
Theorem 1.2. Let $n \geq 1$ and $0 \leq d \leq\lfloor(n-1) / 2\rfloor$. The number of ballot permutations of length $n$ with $d$ descents equals the number of odd order permutations $\pi$ of length $n$ with $M(\pi)=d$.

Theorem 1.2 is proved by computing their bivariate generating functions in terms of the Eulerian number [31]. For $n \geq 1$ and $0 \leq d \leq n-1$, the Eulerian number, denoted as $A(n, d)$ or $\left\langle\begin{array}{l}n \\ d\end{array}\right\rangle$, is the number of permutations of $[n]$ with $d$ descents, see OEIS [29, A008292]. We adopt the convention $A(0,0)=1$ and

$$
A(n, d)=0, \quad \text { if } n<0, \text { or } d<0, \text { or } d=n \geq 1, \text { or } d>n
$$

The $n$th Eulerian polynomial is

$$
A_{n}(t)=\sum_{\pi \in \mathcal{S}_{n}} t^{\operatorname{des}(\pi)}=\sum_{d} A(n, d) t^{d} \quad \text { for } n \geq 1
$$

where the notation $\sum_{i}$ implies that the index $i$ runs over all nonnegative integers making the summation meaningful, and $A_{0}(t)=1$, see Kyle Petersen [22, §1.4]. The exponential generating function of the Eulerian polynomials is

$$
\begin{equation*}
\sum_{n} A_{n}(t) \frac{x^{n}}{n!}=\frac{t-1}{t-e^{(t-1) x}} \tag{1.1}
\end{equation*}
$$

see [22, Theorem 1.6] and [17, Formula (75)]. Then the bivariate generating function

$$
\begin{equation*}
A(t, x)=\sum_{n \geq 1} \sum_{d} \frac{A(n, d) t^{d} x^{n}}{n!}=\sum_{n \geq 1} A_{n}(t) \frac{x^{n}}{n!}=\frac{t-1}{t-e^{(t-1) x}}-1=\frac{e^{(1-t) x}-1}{1-t e^{(1-t) x}} \tag{1.2}
\end{equation*}
$$

Denote by $b(n, d)$ the number of ballot permutations of length $n$ with $d$ descents, and denote the bivariate generating function of $\{b(n, d)\}_{n, d}$ by

$$
B(t, x)=\sum_{n} \sum_{d \leq n-1} \frac{b(n, d) t^{d} x^{n}}{n!}
$$

Wang and the first author [32] give the following Theorem by calculating the joint distribution of the peak and descent statistics over ballot permutations.

Theorem 1.3 (Wang and Zhao). We have

$$
B(t, x)=\exp \left(x+2 \sum_{k \geq 1} \sum_{d \leq k-1} A(2 k, d) t^{d+1} \frac{x^{2 k+1}}{(2 k+1)!}\right)
$$

Wang and Zhang [31] gave another conjecture which refined Theorem 1.2 by tracking the neighbors of the largest letter in these permutations. They defined a word $u$ as a factor of a word $w$ if there exist words $x$ and $y$ such that $w=x u y$, and a word $u$ as a cyclic factor of a permutation $\pi \in \mathcal{S}_{n}$ if $u$ is a factor of some word $v$ such that $(v)$ is a cycle of $\pi$. This conjecture is confirmed in this paper by computing their multivariate generating functions respectively, and thus Theorem 1.4 follows.

Theorem 1.4 (Wang and Zhang). For all $n$, $d$, and $2 \leq j \leq n-1$, we have $b_{n, d}(1, j)+$ $b_{n, d}(j, 1)=2 p_{n, d}(1, j)$, where $b_{n, d}(i, j)$ is the number of ballot permutations of length $n$ with $d$ descents which have inj as a factor, and $p_{n, d}(i, j)$ is the number of odd order permutations of length $n$ with $M(\pi)=d$ which have inj as a cyclic factor.

With Theorem 1.4 and the Toeplitz property of $b_{n, d}(i, j)+b_{n, d}(j, i)$ and $p_{n, d}(i, j)$ showed in [31], we obtain in Theorem 4.1 the generating function which P. R. Stanley expressed interest in private communication, and is defined by

$$
P(t, x, y, z)=\sum_{n, d} \sum_{1 \leq i<j \leq n-1} \frac{2 p_{n, d}(i, j) t^{d} x^{n} y^{i} z^{j}}{(j-i-1)!(n-j+i-2)!}
$$

There are various generalizations of Eulerian numbers. Here we focus on a refinement $A(n, d, j)$ of $A(n, d)$ defined by Brenti and Welker [10], which is the number of permutations of length $n$ with $d$ descents and first letter $j$ for $n \geq 1,0 \leq d \leq n-1$ and $1 \leq j \leq n$. They show the real-rootedness of the following polynomials

$$
A_{n}^{\langle j\rangle}(t)=\sum_{d} A(n, d, j) t^{d} \quad \text { for } n \geq 1
$$

which refine the Eulerian polynomials. We adopt the convention $A(n, d, j)=0$ for other $n, d$ and $j$. Define

$$
A(t, x, y):=\sum_{n, d} \sum_{j=1}^{n} \frac{A(n, d, j) t^{d} x^{n} y^{j}}{(j-1)!(n-j)!}
$$

We would give a formula for $A(t, x, y)$ in close form, based what a series of formulas with generatingfunctionology could be established, and prove Theorem 1.4.

For this purpose, we give several definitions of enumerating sequences and calculate their multivariate generating functions in the following sections. For $n \geq 2,1 \leq j \leq n$ and $0 \leq d \leq n$, let $U(n, d, j)$ denote the number of permutations of length $n$ with $d-1$ descents or $n-d-1$ ascents and first letter $j$. Then

$$
\begin{equation*}
U(n, d, j)=A(n, n-d-1, j)+A(n, d-1, j) \tag{1.3}
\end{equation*}
$$

and

$$
\begin{equation*}
U(n, d, j)=U(n, n-d, j) \tag{1.4}
\end{equation*}
$$

We define the multivariate generating function for $\{U(n, d, j)\}_{n, d, j}$

$$
\hat{U}(t, x, y):=\sum_{n, d} \sum_{j=1}^{n} \frac{U(n, d, j) t^{d} x^{n} y^{j}}{(j-1)!(n-j)!},
$$

and

$$
U(t, x, y):=\sum_{n \text { is odd }, j} \sum_{d \leq \frac{n-1}{2}} \frac{U(n, d, j) t^{d} x^{n} y^{j}}{(j-1)!(n-j)!} .
$$

For $n \geq 3,2 \leq j \leq n-1$ and $0 \leq d \leq n-1$, we denote by $E(n, d, j)$ the number of permutations in $S_{n}$ of length $n$ with $d$ descents which have $1 n j$ or $j n 1$ as a factor, and define $b(n, d, j):=b_{n, d}(1, j)+b_{n, d}(j, 1)$. Define

$$
\begin{aligned}
& E(t, x, y):=\sum_{n, d} \sum_{j=2}^{n-1} \frac{E(n, d, j) t^{d} x^{n} y^{j}}{(n-j-1)!(j-2)!}, \\
& B(t, x, y):=\sum_{n, d} \sum_{j=2}^{n-1} \frac{b(n, d, j) t^{d} x^{n} y^{j}}{(n-j-1)!(j-2)!},
\end{aligned}
$$

and

$$
P(t, x, y):=\sum_{n, d} \sum_{j=2}^{n-1} \frac{p_{n, d}(1, j) t^{d} x^{n} y^{j}}{(j-2)!(n-j-1)!}
$$

The rest of this paper is organized as follows. In Section 2, we calculate the formulas for $A(t, x, y), \hat{U}(t, x, y)$ and $U(t, x, y)$. In Section 3, we calculate the formulas for $E(t, x, y), P(t, x, y)$ and a relation between $B(t, x), B(t, x, y)$ and $E(t, x, y)$. Finally, in Section 4 we confirm the conjecture of Wang and Zhang and calculate $P(t, x, y, z)$.

## 2. Formulas for $A(t, x, y), \hat{U}(t, x, y)$ and $U(t, x, y)$

This section is devoted to giving the generating functions of $\{A(n, d, j)\}_{n, d, j}$, which is the basis of section 3. One can get the following recursion for $A(n, d, j)$ by the values of the second letters in permutations, which could be found in [10].

Lemma 2.1. For $n \geq 2,1 \leq j \leq n$ and $0 \leq d \leq n-1$, we have

$$
\begin{equation*}
A(n, d, j)=\sum_{i=1}^{j-1} A(n-1, d-1, i)+\sum_{i=j+1}^{n} A(n-1, d, i-1) \tag{2.1}
\end{equation*}
$$

By Lemma 2.1, we can obtain the following Theorem by generating function calculation.

Theorem 2.2. We have

$$
A(t, x, y)=\frac{(t-1) x y e^{(t-1) x y}}{t-e^{(t-1) x(1+y)}}
$$

Proof. Substituting $j$ by $j+1$ in (2.1), we have

$$
\begin{equation*}
A(n, d, j+1)=\sum_{i=1}^{j} A(n-1, d-1, i)+\sum_{i=j+2}^{n} A(n-1, d, i-1) \tag{2.2}
\end{equation*}
$$

subtracting (2.1) from (2.2), we get

$$
\begin{equation*}
A(n, d, j+1)=A(n, d, j)+A(n-1, d-1, j)-A(n-1, d, j) \tag{2.3}
\end{equation*}
$$

where $n \geq 2,1 \leq j \leq n-1$ and $0 \leq d \leq n-1$.
Multiplying each term in (2.3) by $\frac{t^{d} x^{n} y^{j}}{(j-1)!(n-j)!}$ and summing over all integers $n, d$ and $1 \leq j \leq n-1$ with subscript transformation, we have

$$
\begin{align*}
\sum_{n, d} \sum_{1 \leq j \leq n-1} \frac{A(n, d, j+1) t^{d} x^{n} y^{j+1}}{(n-j-1)!(j-1)!} & =\sum_{n, d} \sum_{j \geq 1} \frac{(j-1) A(n, d, j) t^{d} x^{n} y^{j}}{(n-j)!(j-1)!} \\
& =y \sum_{n, d} \sum_{j \geq 1} \frac{A(n, d, j) j t^{d} x^{n} y^{j-1}}{(n-j)!(j-1)!}-A(t, x, y) \\
& =y \frac{\partial A(t, x, y)}{\partial y}-A(t, x, y) \tag{2.4}
\end{align*}
$$

$$
\begin{aligned}
\sum_{n, d} \sum_{1 \leq j \leq n-1} \frac{A(n, d, j) t^{d} x^{n} y^{j+1}}{(n-j-1)!(j-1)!}= & \sum_{n, d} \sum_{1 \leq j \leq n-1} A(n, d, j) \frac{(n-j) t^{d} x^{n} z^{j+1}}{(n-j)!(j-1)!} \\
= & x y \sum_{n, d} \sum_{1 \leq j \leq n-1} \frac{n A(n, d, j) t^{d} x^{n-1} y^{j}}{(n-j)!(j-1)!} \\
& -y^{2} \sum_{n, d} \sum_{1 \leq j \leq n-1} \frac{j A(n, d, j) t^{d} x^{n} y^{j-1}}{(n-j)!(j-1)!} \\
= & x y \frac{\partial A(t, x, y)}{\partial x}-y^{2} \frac{\partial A(t, x, y)}{\partial y}
\end{aligned}
$$

$$
\begin{equation*}
\sum_{n, d} \sum_{1 \leq j \leq n-1} \frac{A(n-1, d-1, j) t^{d} x^{n} y^{j+1}}{(n-j-1)!(j-1)!}=t x y A(t, x, y) \tag{2.6}
\end{equation*}
$$

$$
\begin{equation*}
\sum_{n, d} \sum_{1 \leq j \leq n-1} \frac{A(n-1, d, j) t^{d} x^{n} y^{j+1}}{(n-j-1)!(j-1)!}=x y A(t, x, y) \tag{2.7}
\end{equation*}
$$

Combining (2.4), (2.5), (2.6) and (2.7), we get

$$
y \frac{\partial A(t, x, y)}{\partial y}-A(t, x, y)=x y \frac{\partial A(t, x, y)}{\partial x}-y^{2} \frac{\partial A(t, x, y)}{\partial y}+t x y A(t, x, y)-x y A(t, x, y)
$$

By Maple, we have

$$
\begin{equation*}
A(t, x, y)=x y e^{(t-1) x y} F(t, x+x y) \tag{2.8}
\end{equation*}
$$

where $F(t, x)$ is an arbitrary differentiable function.
In the following we will determine $F(t, x)$. Take derivatives on both sides of (2.8), we have

$$
\frac{\partial A(t, x, y)}{\partial y}=x[(t-1) x y+1] e^{(t-1) x y} F(t, x+x y)+x^{2} y e^{(t-1) x y} F_{2}^{\prime}(t, x+x y)
$$

then

$$
\frac{\partial A}{\partial z}(t, x, 0)=x F(t, x)
$$

On the other hand, it is easy to see that $A(n, d, 1)=A(n-1, d)$, then

$$
\frac{\partial A}{\partial z}(t, x, 0)=\sum_{n, d} \frac{A(n, d, 1) t^{d} x^{n}}{(n-1)!}=x \sum_{n, d} \frac{A(n, d) x^{n} y^{d}}{n!}=x A(t, x)+x
$$

Thus

$$
\begin{equation*}
F(t, x)=A(t, x)+1=\frac{t-1}{t-e^{(t-1) x}} \tag{2.9}
\end{equation*}
$$

Plugging (2.9) into (2.8), one can complete the proof.
Remark 2.3. The close form of the generating function for $\{A(n, d, j)\}$ in (2.2) doesn't seem to be seen elsewhere. The conclusion could also be obtained by generating function calculation through Zhang's lemma in [36]: for $1 \leq j \leq n$,

$$
\frac{A_{n}^{\langle l\rangle}(x)}{(1-x)^{n}}=\delta_{l, 1}+\sum_{i=1}^{\infty} j^{l-1}(j+1)^{n-l} x^{j}
$$

It is direct to calculate $\hat{U}(t, x, y)$ according to the definition of (1.3) and Theorem 2.2.
Corollary 2.4. We have

$$
\hat{U}(t, x, y)=t A(t, x, y)+\frac{1}{t} A\left(\frac{1}{t}, x t, y\right)
$$

i.e.,

$$
\hat{U}(t, x, y)=\frac{(t-1) x y\left(t e^{(t-1) x y}+e^{(t-1) x}\right)}{t-e^{(t-1) x(y+1)}}
$$

With Corollary 2.4, we obtain the following corollary.

Proposition 2.5. we have

$$
U(t, x, y)+U\left(\frac{1}{t}, t x, y\right)=\frac{\hat{U}(t, x, y)-\hat{U}(t,-x, y)}{2}
$$

In other words,

$$
U(t, x, y)=\mathrm{D}^{t, x}\left(\frac{x y(t+1)(t-1)^{2} e^{(t-1) x}\left(e^{2(t-1) x y}+1\right)}{2\left(e^{(t-1) x(y+1)}\right)-1\left(t-e^{(t-1) x(y+1)}\right)}\right)
$$

Proof. According to (1.4) and using subscript transformation, we have

$$
\begin{aligned}
U(t, x, y)+U\left(\frac{1}{t}, t x, y\right) & =\sum_{n \text { is odd }, j} \sum_{d \leq \frac{n-1}{2}} \frac{U(n, d, j) t^{d} x^{n} y^{j}}{(j-1)!(n-j)!}+\sum_{n \text { is odd, } j} \sum_{d \leq \frac{n-1}{2}} \frac{U(n, d, j) t^{n-d} x^{n} y^{j}}{(j-1)!(n-j)!} \\
& =\sum_{n \text { is odd }, d, j} \frac{U(n, d, j) t^{d} x^{n} y^{j}}{(j-1)!(n-j)!}=\frac{\hat{U}(t, x, y)-\hat{U}(t,-x, y)}{2} \\
& =\frac{x y(t+1)(t-1)^{2} e^{(t-1) x}\left(e^{2(t-1) x y}+1\right)}{2\left(e^{(t-1) x(y+1)}\right)-1\left(t-e^{(t-1) x(y+1)}\right)} .
\end{aligned}
$$

Noting that $U(t, x, y)$ is a multivariate formal power series with terms of the form $t^{d} x^{n} y^{k}$ such that $d \leq(n-1) / 2$, and the terms of $U\left(\frac{1}{t}, t x, y\right)$ are of the form $t^{d} x^{n} y^{k}$ such that $d>(n-1) / 2$. According the definition of $\mathrm{D}^{t, x}$, we obtain the desired equation.

## 3. Formulas for $E(t, x, y), P(t, x, y)$ and $B(t, x, y)$

### 3.1. The generating function of $\{E(n, d, j)\}_{n, d, j}$.

Theorem 3.1. For $n \geq 3,2 \leq j \leq n-1$ and $0 \leq d \leq n-1$, We have

$$
\begin{array}{r}
E(n, d, j)=\sum_{l=1}^{n-2} \sum_{k=0}^{l-1} \sum_{u=0}^{j-2}\binom{j-2}{u}\binom{n-j-1}{l-1-u} A(n-l-2, d-k-1) U(l, k, u+1)  \tag{3.1}\\
+V(n-2, d-1, j-1)-V(n-2, d-2, j-1)
\end{array}
$$

where $U(l, k, u+1)=V(l, l-1-k, u+1)+V(l, k-1, u+1)$.

Proof. First we give a formula of $E(n, d, j)$ with which one can derive the expression of $E(t, x, y)$.

For any permutation $\pi=\pi_{1} \pi_{2} \cdots \pi_{n}$ of length $n$ with $d$ descents which has $1 n j$ or $j n 1$ as a factor, there are three cases of $\pi$ :

- $j n 1$ is in $\pi$. Suppose that $\pi_{l}=j$, $\operatorname{des}\left(\pi_{1} \pi_{2} \cdots \pi_{j}\right)=k$ and there are $u$ letters in $\left\{\pi_{1}, \pi_{2}, \cdots \pi_{l-1}\right\}$ which are less than $j$, where $1 \leq k+1 \leq l \leq n-2$ and
$0 \leq u \leq j-2$. Then $\pi_{l} \pi_{l-1} \cdots \pi_{1}$ is a permutation of length $l$ with the $(u+1)$ th largest letter as the first letter, and

$$
\operatorname{des}\left(\pi_{l} \pi_{l-1} \cdots \pi_{1}\right)=l-1-\operatorname{asc}\left(\pi_{l} \pi_{l-1} \cdots \pi_{1}\right)=l-1-k
$$

Considering $n$ is a descent and 1 is a ascent in $\pi, \pi_{l+3} \pi_{l+4} \cdots \pi_{n}$ is a permutation of length $n-l-2$ with

$$
\operatorname{des}\left(\pi_{l+3} \pi_{l+4} \cdots \pi_{n}\right)=d-1-\operatorname{des}\left(\pi_{1} \pi_{2} \cdots \pi_{j}\right)=d-k-1
$$

Note that as a subset of $[n]$, there $\operatorname{are}\binom{j-2}{u}\binom{n-j-1}{l-1-u}$ possibilities of $\left\{\pi_{1}, \pi_{2}, \cdots \pi_{l-1}\right\}$. Thus the number of such $\pi \mathrm{s}$ is

$$
\sum_{l=1}^{n-2} \sum_{k=0}^{l-1} \sum_{u=0}^{j-2}\binom{j-2}{u}\binom{n-j-1}{l-1-u} A(l, l-1-k, u+1) A(n-l-2, d-k-1)
$$

- $1 n j$ is in $\pi$ and $\pi_{1} \neq 1$. Suppose that $\pi_{n-l+1}=j$, $\operatorname{des}\left(\pi_{n-l+1} \pi_{n-l+2} \cdots \pi_{l-1}\right)=$ $k-1$ and there are $u$ letters in $\left\{\pi_{n-l+1}, \pi_{n-l+2}, \cdots \pi_{l-1}\right\}$ which are less than $j$, where $1 \leq k+1 \leq l \leq n-3$ and $0 \leq u \leq j-2$. Similar to case 1 , $\pi=\pi_{1} \pi_{2} \cdots \pi_{n-l-2}$ is a permutation of length $n-l-2$ with $d-1-k$ descents and $j \pi_{n-l+2} \pi_{n-l+3} \cdots \pi_{n}$ is a permutation of length $l$ with $k-1$ descents and the $(u+1)$ th largest letter as the first letter. Thus the number of such $\pi \mathrm{s}$ is

$$
\sum_{l=1}^{n-3} \sum_{k=0}^{l-1} \sum_{u=0}^{j-2}\binom{j-2}{u}\binom{n-j-1}{l-1-u} V(l, k-1, u+1) A(n-l-2, d-k-1)
$$

- $1 n j$ is in $\pi$ and $\pi_{1}=1$. So the $\pi_{3}=j$. Then $j \pi_{4} \pi_{5} \cdots \pi_{n}$ is a permutation of length $n-2$ with $d-1$ descents and the $(j-1)$ th largest letter as the first letter. The number of such $\pi \mathrm{s}$ is $V(n-2, d-1, j-1)$.

Combining the above three cases, we have

$$
\begin{gathered}
E(n, d, j)=\sum_{l=1}^{n-2} \sum_{k=0}^{l-1} \sum_{u=0}^{j-2}\binom{j-2}{u}\binom{n-j-1}{l-1-u} V(l, l-1-k, u+1) A(n-l-2, d-k-1) \\
+\sum_{l=1}^{n-3} \sum_{k=0}^{l-1} \sum_{u=0}^{j-2}\binom{j-2}{u}\binom{n-j-1}{l-1-u} V(l, k-1, u+1) A(n-l-2, d-k-1)+V(n-2, d-1, j-1) \\
=\sum_{l=1}^{n-2} \sum_{k=0}^{l-1} \sum_{u=0}^{j-2}\binom{j-2}{u}\binom{n-j-1}{l-1-u} A(n-l-2, d-k-1) U(l, k, u+1) \\
-\sum_{k=0}^{n-2} \sum_{u=0}^{j-2}\binom{j-2}{u}\binom{n-j-1}{l-1-u} V(l, k-1, u+1) A(n-l-2, d-k-1)+V(n-2, d-1, j-1)
\end{gathered}
$$

It is easy to check that when $l=n-2$,
$\sum_{k=0}^{l} \sum_{u=0}^{j-2}\binom{j-2}{u}\binom{n-j-1}{l-1-u} V(l, k-1, u+1) A(n-l-2, d-k-1)=V(n-2, d-2, j-1)$,
which complete the proof.

Theorem 3.1 is translated into the language of generating functions as follows.

## Theorem 3.2.

$$
E(t, x, y)=t x^{2} y[A(t, x+x y)+1] \hat{U}(t, x, y)+t(1-t) x^{2} y A(t, x, y)
$$

i.e.,

$$
E(t, x, y)=\frac{t(t-1)^{2} x^{3} y^{2} e^{(t-1) x}\left(e^{2(t-1) x y}+1\right)}{\left(t-e^{(t-1) x(y+1)}\right)^{2}}
$$

Proof. Multiplying each term in (3.1) by $\frac{t^{d} x^{n} y^{j}}{(j-2)!(n-j-1)!}$, summing over all integers $n, j$ and $d$ such that $n \geq 3,2 \leq j \leq n-1$ and $0 \leq d \leq n-1$, we deduce that

$$
\begin{aligned}
E(t, x, y)= & \sum_{n=3}^{\infty} \sum_{d=0}^{n-1} \sum_{j=2}^{n-1} \frac{E(n, d, j) t^{d} x^{n} y^{j}}{(j-2)!(n-j-1)!} \\
= & \sum_{n=3}^{\infty} \sum_{d=0}^{n-1} \sum_{j=2}^{n-1} \sum_{l=1}^{n-2} \sum_{k=0}^{l-1} \sum_{u=0}^{j-2} \frac{\left.\binom{j-2}{u}\binom{n-j-1}{l-1-u}\right) t^{d} x^{n} y^{j}}{(j-2)!(n-j-1)!} A(n-l-2, d-k-1) U(l, k, u+1) \\
& +t x^{2} y \sum_{n=3}^{\infty} \sum_{d=0}^{n-1} \sum_{j=2}^{n-1} \frac{A(n-2, d-1, j-1) t^{d-1} x^{n-2} y^{j-1}}{(j-2)!(n-j-1)!} \\
& -t^{2} x^{2} y \sum_{n=3}^{\infty} \sum_{d=0}^{n-1} \sum_{j=2}^{n-1} \frac{A(n-2, d-2, j-1) t^{d-2} x^{n-2} y^{j-1}}{(j-2)!(n-j-1)!} \\
= & t x^{2} \sum_{n=3}^{\infty} \sum_{d=0}^{n-1} \sum_{j=2}^{n-1} \sum_{l=1}^{n-2} \sum_{k=0}^{l-1} \sum_{u=0}^{j-2}\binom{n-l-2}{j-2-u} y^{j} \cdot \frac{A(n-l-2, d-k-1) x^{n-l-2} t^{d-k-1}}{(n-2-l)!} \\
& \cdot \frac{U(l, k, u+1) t^{k} x^{l}}{u!(l-1-u)!}+t x^{2} y A(t, x, y)-t^{2} x^{2} y A(t, x, y) .
\end{aligned}
$$

Exchanging the order of $u$ and $j$ in the summation, we have

$$
\begin{aligned}
\sum_{n=3}^{\infty} \sum_{d=0}^{n-1} \sum_{j=2}^{n-1} \sum_{l=1}^{n-2} \sum_{k=0}^{l-1} \sum_{u=0}^{j-2}\binom{n-l-2}{j-2-u} y^{j} \cdot \frac{A(n-l-2, d-k-1) x^{n-l-2} t^{d-k-1}}{(n-2-l)!} \\
\cdot \frac{U(l, k, u+1) t^{k} x^{l}}{u!(l-1-u)!}=\sum_{n=3}^{\infty} \sum_{d=0}^{n-1} \sum_{u=0}^{n-3} \sum_{l=1}^{n-2} \sum_{k=0}^{l-1} \sum_{j=u+2}^{n-1}\binom{n-l-2}{j-2-u} y^{j} \\
\cdot \frac{A(n-l-2, d-k-1) x^{n-l-2} t^{d-k-1}}{(n-2-l)!} \cdot \frac{U(l, k, u+1) t^{k} x^{l}}{u!(l-1-u)!}
\end{aligned}
$$

For fixed nonnegative integer $n, l$ and $u$ such that $n-2 \geq l \geq u+1$ (then $n-u-3 \geq$ $n-l-2$ ), we have

$$
\sum_{j=2}^{n-1}\binom{n-l-2}{j-2-u} y^{j}=y^{u+2} \sum_{i=-u}^{n-u-3}\binom{n-l-2}{i} y^{i}=(1+y)^{n-l-2} y^{u+2}
$$

Thus

$$
\begin{aligned}
E(t, x, y)= & t x^{2} \sum_{n=3}^{\infty} \sum_{d=0}^{n-1} \sum_{u=0}^{n-3} \sum_{l=1}^{n-2} \sum_{k=0}^{l-1} \sum_{j=u+2}^{n-1}\binom{n-l-2}{j-2-u} y^{j} \cdot \frac{A(n-l-2, d-k-1) x^{n-l-2} t^{d-k-1}}{(n-2-l)!} \\
& \cdot \frac{U(l, k, u+1) t^{k} x^{l}}{u!(l-1-u)!}+t x^{2} y A(t, x, y)-t^{2} x^{2} y A(t, x, y) . \\
= & t x^{2} \sum_{n=3}^{\infty} \sum_{d=0}^{n-1} \sum_{u=0}^{n-3} \sum_{l=1}^{n-2} \sum_{k=0}^{l-1}(1+y)^{n-l-2} y^{u+2} \cdot \frac{A(n-l-2, d-k-1) x^{n-l-2} t^{d-k-1}}{(n-2-l)!} \\
& \cdot \frac{U(l, k, u+1) t^{k} x^{l}}{u!(l-1-u)!}+t x^{2} y A(t, x, y)-t^{2} x^{2} y A(t, x, y) \\
= & t x^{2} y \sum_{n=3}^{\infty} \sum_{d=0}^{n-1} \sum_{u=0}^{n-3} \sum_{l=1}^{n-2} \sum_{k=0}^{l-1} \frac{A(n-l-2, d-k-1)[x(1+y)]^{n-l-2} t^{d-k-1}}{(n-2-l)!} \\
\cdot & \frac{U(l, k, u+1) t^{k} x^{l} y^{u+1}}{u!(l-1-u)!}+t(1-t) x^{2} y A(t, x, y) \\
= & t x^{2} y \sum_{n, d} \frac{A(n, d)[x(1+y)]^{n} t^{d}}{n!} \cdot \sum_{l, k} \sum_{u=1}^{k} \frac{U(l, k, u) t^{k} x^{l} y^{u}}{(u-1)!(l-u)!}+t(1-t) x^{2} y A(t, x, y) \\
= & t x^{2} y[A(t, x+x y)+1] \hat{U}(t, x, y)+t(1-t) x^{2} y A(t, x, y) .
\end{aligned}
$$

Thus completing the proof.
3.2. The generating function of $\{b(n, d, j)\}_{n, d, j}$. In this subsection we will give a relation between $B(t, x, y), B(t, x)$ and $E(t, x, y)$ by the relations between their coefficients. We give a bijection proof adopting the idea of the reversal-concatenation map in [32]. The concept of lowest point in [32] play a important role in the proof, which is defined to be the position $1 \leq k \leq n$ of a permutation $\pi=\pi_{1} \pi_{2} \cdots \pi_{n}$ satisfying

$$
h\left(\pi_{1} \pi_{2} \cdots \pi_{k}\right)=\min \left\{h\left(\pi_{1} \pi_{2} \cdots \pi_{i}\right) \mid 1 \leq i \leq n\right\} .
$$

From the definition, it is easy to see that if $l$ is the minimal lowest point of a permutation $\pi=\pi_{1} \pi_{2} \cdots \pi_{n}$, then $\pi_{l-1} \pi_{l-2} \cdots \pi_{1}$ and $\pi_{l} \pi_{l+1} \cdots \pi_{n}$ are both ballot permutations (empty permutation is also ballot permutation). For example, the first lowest point of permutation 143265 is 4 , then 341 and 265 are ballot permutations. Similarly, if $l$ is the maximal lowest point of a permutation $\pi=\pi_{1} \pi_{2} \cdots \pi_{n}$, then $\pi_{l} \pi_{l-1} \cdots \pi_{1}$ and $\pi_{l+1} \pi_{l+2} \cdots \pi_{n}$ are ballot permutations.

Theorem 3.3. For all integers $n, d$ and $j$ such that $n \geq 3,0 \leq d \leq n-1$ and $2 \leq j \leq n-1$, we have

$$
\begin{equation*}
E(n, d, j)+E(n, d-1, j)=\sum_{l, k, u}\binom{j-2}{u}\binom{n-j-1}{n-l-3-u} b(l, k) b(n-l, d-l+k, u+2) \tag{3.2}
\end{equation*}
$$

$$
+\sum_{l, k, u}\binom{j-2}{u}\binom{n-j-1}{l-3-u} b(n-l, d-l+k) b(l, k, u+2)
$$

Proof. Define

$$
\begin{gathered}
\mathcal{E}(n, d, j)=\left\{\pi \in \mathcal{S}_{n}: \operatorname{des}(\pi)=d \text { and } \pi \text { has } 1 n j \text { or } j n 1 \text { as a factor }\right\}, \\
\mathcal{B}(n, d)=\left\{\pi \in \mathcal{S}_{n}: \operatorname{des}(\pi)=d \text { and } \pi \text { is a ballot permutation }\right\}, \\
\mathcal{B}_{j}(n, d)=\{\pi \in \mathcal{B}(n, d): \pi \text { has } 1 n j \text { or } j n 1 \text { as a factor }\}, \\
\mathcal{B}^{1}(n, d, j)=\{(\rho, \tau): \exists l, k, u \geq 0 \text { such that } \\
\left.\qquad \rho \in \mathcal{B}(l, k), \tau \in \mathcal{B}_{u+2}(n-l, d-l+k), \text { and } \rho \tau \in \mathcal{S}_{n}\right\}, \text { and } \\
\mathcal{B}^{2}(n, d, j)=\{(\rho, \tau): \exists l, k, u \geq 0 \text { such that } \\
\\
\left.\quad \rho \in \mathcal{B}_{u+2}(l, k), \tau \in \mathcal{B}(n-l, d-l+k), \text { and } \rho \tau \in \mathcal{S}_{n}\right\} .
\end{gathered}
$$

Now we give a bijection between $\mathcal{E}(n, d, j) \cup \mathcal{E}(n, d-1, j)$ and $\mathcal{B}^{1}(n, d, j) \cup \mathcal{B}^{2}(n, d, j)$.
For any permutation $\pi \in \mathcal{E}(n, d, j)$, assume that $l+1$ is the minimal lowest point. Let $\rho=\pi_{l} \pi_{l-1} \cdots \pi_{1}$ and $\pi(2)=\pi_{l+1} \pi_{l+2} \cdots \pi_{n}$. Assume that $\operatorname{des}(\rho)=l-k-1$. Then $\pi_{l}<\pi_{l+1}$, and $\rho$ or $\tau$ has $1 n j$ or $j n 1$ as a factor:

- if $\rho$ has the factor $1 n j$ or $j n 1$. Assume that there are $u+1$ numbers less than $j$ in $\rho$, then $(\rho, \tau) \in \mathcal{B}^{2}(n, d, j)$.
- if $\tau$ has the factor $1 n j$ or $j n 1$. Assume that there are $u+1$ numbers less than $j$ in $\tau$, then $(\rho, \tau) \in \mathcal{B}^{1}(n, d, j)$.

Similarly, for any permutation $\pi \in \mathcal{E}(n, d-1, j)$, assume that $l$ is the maximal lowest point. Let $\rho=\pi_{l} \pi_{l-1} \cdots \pi_{1}$ and $\pi(2)=\pi_{l+1} \pi_{l+2} \cdots \pi_{n}$. Assume that $\operatorname{des}(\rho)=l-k-1$. Then $(\rho, \tau) \in \mathcal{B}^{1}(n, d, j) \bigcup \mathcal{B}^{2}(n, d, j)$.

It is not difficult to check that the map

$$
\begin{aligned}
\phi: \mathcal{E}(n, d, j) \cup \mathcal{E}(n, d-1, j) & \rightarrow \mathcal{B}^{1}(n, d, j) \bigcup \mathcal{B}^{2}(n, d, j) \\
\pi & \mapsto(\rho, \tau)
\end{aligned}
$$

is a bijection. Thus completing the proof.

With Theorem 3.3, we obtain the following relation between $B(t, x, y), B(t, x)$ and $E(t, x, y)$.

Theorem 3.4. We have that

$$
\begin{equation*}
B(t, x, y) B\left(\frac{1}{t}, t x(1+y)\right)+B\left(\frac{1}{t}, t x, y\right) B(t, x+x y)=(1+t) E(t, x, y) \tag{3.3}
\end{equation*}
$$

Proof. Multiplying each term in (3.2) by $\frac{t^{d} x^{n} y^{j}}{(n-j-1)!(j-2)!}$, we have

$$
\begin{align*}
& \frac{E(n, d, j) t^{d} x^{n} y^{j}}{(n-j-1)!(j-2)!}+t \cdot \frac{E(n, d-1, j) t^{d-1} x^{n} y^{j}}{(n-j-1)!(j-2)!}  \tag{3.4}\\
= & \sum_{l, k, u} \frac{b(l, k) t^{-k}(t x)^{l} y^{j-u-2}}{l!} \cdot \frac{b_{n-l, d-l+k}(1, u+2) t^{d-l+k} x^{n-l} y^{u+2}}{(n-l-3-u)!u!}\binom{l}{j-2-u} \tag{3.5}
\end{align*}
$$

$$
\begin{equation*}
+\sum_{l, k, u} \frac{b(n-l, d-l+k) t^{d-l+k} x^{n-l} y^{j-2-u}}{(n-l)!} \cdot \frac{b_{l, k}(1, u+2) t^{-k}(x y)^{l} z^{u+2}}{(l-3-u)!(n-l!)}\binom{n-l}{j-2-u} \tag{3.6}
\end{equation*}
$$

Summing over (3.4) for all $n, d$ and $j$ such that $n \geq 3,0 \leq d \leq n-1$ and $2 \leq$ $j \leq n-1$, the desired generating function can be obtained by using techniques in generatingfunctionology as that is used in the proof of Theorem 3.2.
3.3. The generating function of $\left\{p_{n, d}(1, j)\right\}_{n, d, j}$. To calculate $P(t, x, y)$, we first prove the following formula for $p_{n, d}(1, j)$.
Theorem 3.5. For $n \geq 3,2 \leq j \leq n-1$ and $0 \leq d \leq n-1$, We have

$$
\begin{align*}
p_{n, d}(1, j)= & \sum_{\substack{m_{1}+m_{2}+\lambda_{1} n_{1}+\cdots+\lambda_{s} n_{s}=n-3, d_{0} \leq \frac{m_{1}+m_{2}+2}{d_{0}}, d_{0}+\lambda_{1} d_{1}+\cdots+\lambda_{s} d_{s}=d, n_{i} \text { is odd, } m_{1}+m_{2} \text { is even }}}\binom{j-2}{m_{1}}\binom{n-j-1}{m_{2}} \frac{\left(n-3-m_{1}-m_{2}\right)!}{\prod_{i=1}^{s}\left[\left(n_{i}!\right)^{\lambda_{i}} \lambda_{i}!\right]}  \tag{3.7}\\
& \cdot \prod_{i=1}^{s}\left[l\left(n_{i}, d_{i}\right)\right]^{\lambda_{i}} U\left(m_{1}+m_{2}+1, d_{0}-1, m_{1}+1\right)
\end{align*}
$$

Proof. It is easy the see that, $p_{n, d}(1, j)$ is the ways of decomposition of $[n]$ satisfying the following conditions:

- (i) $s, m_{1}, m_{2}, d_{0}, d_{i}, n_{i}, \lambda_{i}(1 \leq i \leq s)$ are nonnegative integers such that $d_{0} \leq$ $\frac{m_{1}+m_{2}+2}{2}, n_{i}$ is odd $(1 \leq i \leq s), m_{1}+m_{2}$ is even and

$$
\left(m_{1}+m_{2}, d_{0}\right)+\sum_{i=1}^{s} \lambda_{i}\left(n_{i}, d_{i}\right)=(n-3, d)
$$

- (ii) $[n]$ is divided into $1+\sum_{i=1}^{s} \lambda_{i}$ odd order cycles, such that there are $\lambda_{i}$ cycles of length $n_{i}$ with $M(\cdot)=d_{i}(1 \leq i \leq s)$ and the remaining cycle (denoted by $c$ ) has length $m_{1}+m_{2}+3$ with $M(c)=d_{0}$.
- (iii) $c$ has $1 n j$ as a cyclic factor and $m_{1}$ numbers belonging to $\{2,3, \cdots, j-1\}$.

For odd integer $n$ and integer $0 \leq d \leq\left\lfloor\frac{n-1}{2}\right\rfloor$, let $l(n, d)$ denote the number of cyclic permutations over [ $n$ ] of length $n$ with $M(\cdot)=d$. When $n>1$, for any such cyclic
permutation $\left(n c_{1} c_{2} \cdots c_{n-1}\right)$, the permutation $c_{1} c_{2} \cdots c_{n-1}$ has length $n-1$ and $d-1$ or $n-d-1$ descents. Since $n$ is odd, $d-1 \neq n-d-1$. Noting that $A(n-1, d-1)=$ $A(n-1, n-1-1-(d-1))=A(n-1, n-d-1)$, we have

$$
l(n, d)=\left\{\begin{array}{lr}
2 A(n-1, d-1), & n>1, d \leq \frac{n-1}{2} \\
1, & n=1, d=0
\end{array}\right.
$$

According to Theorem 1.3, we have

$$
B(t, x)=\exp \left(x+2 \sum_{k \geq 1} \sum_{d \leq k-1} A(2 k, d) t^{d+1} \frac{x^{2 k+1}}{(2 k+1)!}\right)=\exp \left(\sum_{n \text { is odd }, d} l(n, d) \frac{t^{d} x^{n}}{n!}\right)
$$

Similarly, assume that a odd order cyclic permutation $c=\left(1 n j \pi_{1} \pi_{2} \cdots \pi_{n-3}\right)$ over $[n]$ with $M(\pi)=d$ has $m$ numbers belonging to $\{2,3, \cdots, j-1\}$. Then the permutation $j \pi_{1} \pi_{2} \cdots \pi_{n-3}$ is over $\{2,3, \cdots, n-1\}$ which have $d-2$ or $n-d-2$ descents. Then the number of such cyclic $c$ is

$$
A(n-2, d-2, j-1)+A(n-2, n-d-2, j-1)=U(n-2, d-1, j-1)
$$

Thus for fixed $s, m_{1}, m_{2}, d_{0}, d_{i}, n_{i}, \lambda_{i}(1 \leq i \leq s)$ satisfying condition (i), the ways of decomposition of $[n]$ satisfying condition (ii) and (iii) is

$$
\begin{aligned}
\binom{j-2}{m_{1}}\binom{n-j-1}{m_{2}} \frac{\left(n-3-m_{1}-m_{2}\right)!}{\Pi_{i=1}^{s}\left[\left(n_{i}!\right)^{\lambda_{i}} \lambda_{i}!\right]} \prod_{i=1}^{s} & {\left[l\left(n_{i}, d_{i}\right)\right]^{\lambda_{i}} } \\
& \cdot U\left(m_{1}+m_{2}+1, d_{0}-1, m_{1}+1\right) .
\end{aligned}
$$

Summing over the integers $s, m_{1}, m_{2}, d_{0}, d_{i}, n_{i}, \lambda_{i}(1 \leq i \leq s)$ satisfying condition (i) and thus completing the proof.

Theorem 3.6. We have

$$
\begin{equation*}
P(t, x, y)=t x^{2} y B(t, x+x y) U(t, x, y) \tag{3.8}
\end{equation*}
$$

Proof. Multiplying each term in (3.7) by $\frac{t^{d} x^{n} y^{j}}{(j-2)!(n-j-1)!}$, we obtain

$$
\begin{aligned}
& \frac{p_{n, d}(1, j) t^{d} x^{n} y^{j}}{(j-2)!(n-j-1)!}=t x^{2} y \sum_{\substack{m_{1}+m_{2}+\lambda_{1} n_{1}+\ldots+\lambda_{s} n_{s}=n-3, d_{0} \leq \frac{m_{1}+m_{2}+2}{} \\
d_{0}+\lambda_{1} d_{1}+\ldots+\lambda_{s} d_{s}=d, n_{i} \text { is odd, } m_{1}+m_{2} \text { is even }}} \prod_{i=1}^{s}\left[\frac{t^{d_{i}} x^{n_{i}} l\left(n_{i}, d_{i}\right)}{n_{i}!}\right]^{\lambda_{i}} \frac{1}{\lambda_{i}!} \\
& \cdot \frac{U\left(m_{1}+m_{2}+1, d_{0}-1, m_{1}+1\right) t^{d_{0}-1} x^{m_{1}+m_{2}+1} y^{m_{1}+1}}{m_{1}!m_{2}!}\binom{n-m_{1}-m_{2}-3}{j-m_{1}-2} y^{j-m_{1}-2}
\end{aligned}
$$

Summing over all integers $n, d, j$ and noting that

$$
\sum_{j}\binom{n-3-m_{1}-m_{2}}{j-2-m_{1}} y^{j-2-m_{1}}=(1+z)^{n-3-m_{1}-m_{2}}=\prod_{i=1}^{s}(1+y)^{\lambda_{i} n_{i}}
$$

we have

$$
\begin{align*}
& \sum_{n, d, j} \frac{t^{d} x^{n} y^{j} p_{n, d}(1, j)}{(j-2)!(n-j-1)!} \\
= & t x^{2} y \sum_{\substack{n, d \\
n, ~}} \sum_{\substack{m_{1}+m_{2}+\lambda_{1} n_{1}+\cdots+\lambda_{s} n_{s}=n-3, d_{0} \leq \underline{m_{1}+m_{2}+2} \\
d_{0}+\lambda_{1} d_{1}+\cdots+\lambda_{s} d_{s}=d_{n} n_{i} \text { is odd, } m_{1}+m_{2} \text { is even }}} \prod_{i=1}^{s}\left[\frac{t^{d_{i}} x^{n_{i}}(1+y)^{n_{i}} l\left(n_{i}, d_{i}\right)}{n_{i}!}\right]^{\lambda_{i}} \frac{1}{\lambda_{i}!} \\
& \cdot \frac{U\left(m_{1}+m_{2}+1, d_{0}-1, m_{1}+1\right) t^{t_{0}-1} x^{m_{1}+m_{2}+1} y^{m_{1}+1}}{m_{1}!m_{2}!} \\
= & t x^{2} y B(t, x(1+y)) \sum_{m_{1}+m_{2} \text { is even }} \sum_{d_{0} \leq \frac{m_{1}+m_{2}+2}{2}} U\left(m_{1}+m_{2}+1, d_{0}-1, m_{1}+1\right)  \tag{3.9}\\
& \cdot \frac{t^{d_{0}-1} x^{m_{1}+m_{2}+1} y^{m_{1}+1}}{m_{1}!m_{2}!} \\
= & t x^{2} y B(t, x(1+y)) \sum_{n \text { is odd }, j} \sum_{d \leq \frac{n-1}{2}} \frac{U(n, d, j) t^{d} x^{n} y^{j}}{(j-1)!(n-j)!} \\
= & x^{2} y B(t, x(1+y)) U(t, x, y) .
\end{align*}
$$

Thus completing the proof.

## 4. Proof of Theorem 1.4

Now we are in a position to prove Theorem 1.4.

Proof. The conclusion is equivalent to $B(t, x, y)=2 P(t, x, y)$. Since $B(t, x, y)$ is uniquely determined by (3.3), we only need to prove

$$
(1+t) E(t, x, y)=2 B\left(\frac{1}{t}, t x(1+y)\right) P(t, x, y)+2 B(t, x+x y) P\left(\frac{1}{t}, t x, y\right)
$$

According to (3.8), we just need to verify the following equation

$$
\begin{align*}
(1+t) E(t, x, y)=2 t x^{2} y B & \left.\left(\frac{1}{t}, t x(1+y)\right) B(t, x+x y)\right) U(t, x, y)  \tag{4.1}\\
& +\frac{2(t x)^{2} y}{t} B(t, x+x y) B\left(\frac{1}{t}, t x(1+y)\right) U\left(\frac{1}{t}, t x, y\right) .
\end{align*}
$$

According to [32, Theorem 3.7], we have

$$
B(t, x) B\left(\frac{1}{t}, t x\right)=1+(1+t) A(t, x)
$$

Then

$$
B(t, x+x y) B\left(\frac{1}{t}, t x(1+y)\right)=1+(1+t) A(t, x+x y)
$$

(4.1) is equivalent to

$$
(1+t) E(t, x, y)=2 t x^{2} y(1+(1+t) A(t, x+x y))\left(U(t, x, y)+U\left(\frac{1}{t}, t x, y\right)\right)
$$

According to Proposition 2.5, we only need to prove

$$
(1+t) E(t, x, y)=t x^{2} y(1+(1+t) A(t, x+x y))(\hat{U}(t, x, y)-\hat{U}(t,-x, y))
$$

Plugging the formulas of $A(t, x), \hat{U}(t, x, y)$ and $E(t, x, y)$ into (4), one can complete the proof.

With Theorem 1.4, we can prove the following theorem.
Theorem 4.1. We have

$$
P(t, x, y, z)=\frac{2 y}{1-y z}\left(P(t, x, z)-P\left(t, x y z, \frac{1}{y}\right)\right)
$$

Proof.

$$
\begin{aligned}
P(t, x, y, z) & =\sum_{n, d} \sum_{1 \leq i<j \leq n-1} \frac{2 p_{n, d}(i, j) t^{d} x^{n} y^{i} z^{j}}{(j-i-1)!(n-j+i-2)!}(\text { let } u=j-i+1) \\
& =\sum_{n, d} \sum_{u=2}^{n-1} \sum_{i=1}^{n-u} \frac{2 p_{n, d}(1, u) t^{d} x^{n} y^{i} z^{u+i-1}}{(u-2)!(n-u-1)!} \\
& =\frac{2 y}{1-y z} \sum_{n, d} \sum_{u=2}^{n-1} \frac{p_{n, d}(1, u)\left(t^{d} x^{n} z^{u}-t^{d}(x y z)^{n} y^{-u}\right)}{(u-2)!(n-u-1)!} \\
& =\frac{2 y}{1-y z}\left(P(t, x, z)-P\left(t, x y z, \frac{1}{y}\right)\right) .
\end{aligned}
$$

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