

# REPRESENTATIONS OF DEGENERATE HERMITE POLYNOMIALS

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**ABSTRACT.** We introduce degenerate Hermite polynomials as a degenerate version of the ordinary Hermite polynomials. Then, among other things, by using the formula about representing one  $\lambda$ -Sheffer polynomial in terms of other  $\lambda$ -Sheffer polynomials we represent the degenerate Hermite polynomials in terms of the higher-order degenerate Bernoulli, Euler, and Frobenius-Euler polynomials and vice versa.

## 1. INTRODUCTION AND PRELIMINARIES

Carlitz initiated the study of degenerate Bernoulli and Euler polynomials and numbers and obtained some interesting arithmetical and combinatorial results in [3,4]. In recent years, some mathematicians began to investigate degenerate versions of quite a few special numbers and polynomials, which include the degenerate Bernoulli polynomials of the second kind, degenerate Stirling numbers of the first and second kinds, degenerate Bell polynomials, degenerate Frobenius-Euler polynomials, and so on (see [8,9,11-14] and the references therein). It is remarkable that studying degenerate versions is not only limited to polynomials but also extended to transcendental functions. Indeed, the degenerate gamma functions were introduced in connection with degenerate Laplace transforms in [10].

Gian-Carlo Rota began to construct a completely rigorous foundation for umbral calculus in the 1970s. The Rota's theory is based on the linear functionals in (11) and (12) and differential operators in (15) and (16). The Sheffer sequences, which are defined by (20), occupy the central position in the theory and are characterized by the generating functions as in (21), where the usual exponential function enters. The motivation for [8] started from the question that what if the usual exponential function in (21) was replaced by the degenerate exponential functions in (4). It turns out that it corresponds to replacing the linear functional in (11) and (12) by the family of  $\lambda$ -linear functionals in (9) and (10) and the differential operators in (15) and (16) by the family of  $\lambda$ -differential operators in (13) and (14). Indeed, with these replacements we were led to define  $\lambda$ -Sheffer polynomials, which are characterized by the desired generating functions (see (19)).

The aim of this paper is to introduce the degenerate Hermite polynomials as a degenerate version of the ordinary Hermite polynomials and to study their properties. Specifically, by using the formula (24) about representing one  $\lambda$ -Sheffer polynomial by other  $\lambda$ -Sheffer polynomials we will represent the degenerate Hermite polynomial in terms of three other degenerate polynomials, namely the higher-order degenerate Bernoulli, Euler and Frobenius-Euler polynomials and vice versa.

The outline of this paper is as follows. In Section 1, we will briefly go over very basics about umbral calculus including  $\lambda$ -linear functionals,  $\lambda$ -differential operators,  $\lambda$ -Sheffer sequences and the important formula in (24). For further details on these, we let the reader refer to [8]. In addition, we will recall the definitions for the Hermite polynomials, the degenerate Stirling numbers of the first and second kinds, higher-order degenerate Bernoulli polynomials, higher-order degenerate Euler polynomials and higher-order degenerate Frobenius-Euler polynomials. In Section 2, we will

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introduce the degenerate Hermite polynomials in a natural way and derive an explicit expression for them. Then we will represent the higher-order degenerate Bernoulli, Euler and Frobenius-Euler polynomials in terms of the degenerate Hermite polynomials and vice versa.

From now on, unless otherwise stated,  $\lambda$  is arbitrary but is a fixed non-zero real number. It is well known that the Hermite polynomials,  $H_n(x)$ , ( $n \geq 0$ ), are defined by the exponential generating function as

$$(1) \quad e^{2xt-t^2} = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}, \quad (\text{see [18]}).$$

From (1), we note that

$$(2) \quad H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}, \quad (n \geq 0).$$

In particular,

$$H_n(x) = \frac{n!}{2\pi i} \oint e^{-t^2+2xt} t^{-n-1} dt, \quad (n \geq 0),$$

where the contour encloses the origin and is traversed in a counterclockwise direction.

Carlitz introduced the degenerate Bernoulli polynomials of order  $r \in \mathbb{N}$  given by

$$(3) \quad \left( \frac{t}{e_\lambda(t) - 1} \right)^r e_\lambda^x(t) = \sum_{n=0}^{\infty} \beta_{n,\lambda}^{(r)}(x) \frac{t^n}{n!}, \quad (\text{see [3,4]}).$$

Here  $e_\lambda^x(t)$  are the degenerate exponential functions given by

$$(4) \quad e_\lambda^x(t) = (1 + \lambda t)^{\frac{x}{\lambda}}, \quad e_\lambda(t) = e_\lambda^1(t) = (1 + \lambda t)^{\frac{1}{\lambda}}, \quad (\text{see [10, 14, 15, 16]}).$$

For  $x = 0$ ,  $\beta_{n,\lambda}^{(r)} = \beta_{n,\lambda}^{(r)}(0)$  are called the degenerate Bernoulli numbers of order  $r$ .

Note that  $\lim_{\lambda \rightarrow 0} \beta_{n,\lambda}^{(r)}(x) = B_n^{(r)}(x)$ , where  $B_n^{(r)}(x)$  are the ordinary Bernoulli polynomials of order  $r$ , (see [1-20]).

Also, he considered the degenerate Euler polynomials of order  $r$  given by

$$(5) \quad \left( \frac{2}{e_\lambda(t) + 1} \right)^r e_\lambda^x(t) = \sum_{n=0}^{\infty} \mathcal{E}_{n,\lambda}^{(r)}(x) \frac{t^n}{n!}, \quad (\text{see [3,4]}),$$

where  $\mathcal{E}_{n,\lambda}^{(r)} = \mathcal{E}_{n,\lambda}^{(r)}(0)$  are called the degenerate Euler numbers of order  $r$ .

The degenerate logarithmic functions are defined by Kim-Kim as

$$(6) \quad \log_\lambda(1+t) = \sum_{n=0}^{\infty} \lambda^{n-1} (1)_{n,1/\lambda} \frac{t^n}{n!} = \frac{1}{\lambda} ((1+t)^\lambda - 1), \quad (\text{see [9]}),$$

where  $(x)_{0,\lambda} = 1$ ,  $(x)_{n,\lambda} = x(x-\lambda) \cdots (x-(n-1)\lambda)$ , ( $n \geq 1$ ).

Note that

$$\log_\lambda(e_\lambda(1+t)) = e_\lambda(\log_\lambda(1+t)) = 1+t.$$

The degenerate Stirling numbers of the first kind are defined by Kim-Kim as

$$(7) \quad (x)_n = \sum_{l=0}^n S_{1,\lambda}(n,l)(x)_{l,\lambda}, \quad (n \geq 0), \quad (\text{see [9, 11, 12, 13]}),$$

where  $(x)_0 = 1$ ,  $(x)_n = x(x-1) \cdots (x-n+1)$ , ( $n \geq 1$ ).

As an inversion formula of (7), the degenerate Stirling numbers of the second kind are defined as

$$(x)_{n,\lambda} = \sum_{l=0}^n S_{2,\lambda}(n,l)(x)_l, \quad (n \geq 0), \quad (\text{see [9, 13, 14, 16]}).$$

In [14], the degenerate Frobenius-Euler polynomials of order  $r$  are defined by

$$(8) \quad \left( \frac{1-u}{e_\lambda(t)-u} \right)^r e_\lambda^x(t) = \sum_{n=0}^{\infty} h_{n,\lambda}^{(r)}(x|u) \frac{t^n}{n!},$$

where  $u \in \mathbb{C}$  with  $u \neq 1$ , and  $h_{n,\lambda}^{(r)}(u) = h_{n,\lambda}^{(r)}(0|u)$  are called the Frobenius-Euler numbers of order  $r$ .

Let  $\mathbb{C}$  be the field of complex numbers. Let  $\mathcal{F}$  be the algebra of formal power series in  $t$  over  $\mathbb{C}$  given by

$$\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \mid a_k \in \mathbb{C} \right\},$$

and let  $\mathbb{P} = \mathbb{C}[x]$  be the algebra of polynomials in  $x$  over  $\mathbb{C}$ .

For  $f(t) \in \mathcal{F}$ , with  $f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!}$ , we define the  $\lambda$ -linear functional  $\langle f(t)|\cdot \rangle_\lambda$  on  $\mathbb{P}$  by

$$(9) \quad \langle f(t)|(x)_{k,\lambda} \rangle_\lambda = a_k, \quad (k \geq 0), \quad (\text{see [8]}).$$

In other words, the  $\lambda$ -linear functional  $\langle f(t)|\cdot \rangle_\lambda$  is the unique linear functional on  $\mathbb{P}$  mapping  $(x)_{k,\lambda}$  onto  $a_k$ . From (9), we note that

$$(10) \quad \langle t^k|(x)_{n,\lambda} \rangle_\lambda = n! \delta_{n,k}, \quad (n, k \geq 0), \quad (\text{see [8]}),$$

where  $\delta_{n,k}$  is the Kronecker's symbol.

If  $\lambda = 0$ , then the symbol  $\langle \cdot | \cdot \rangle_0$  is simply denoted by  $\langle \cdot | \cdot \rangle$ . With this notation, (9) and (10) respectively become the linear functionals used in Rota's theory (see [18]):

$$(11) \quad \langle f(t)|x^k \rangle = a_k, \quad (k \geq 0),$$

and

$$(12) \quad \langle t^k|x^n \rangle = n! \delta_{n,k}, \quad (n, k \geq 0).$$

The order  $o(f(t))$  of the formal power series  $f(t) (\neq 0)$  is the smallest integer  $k$  for which  $a_k$  does not vanish. If  $o(f(t)) = 1$ , then  $f(t)$  is called a delta series; if  $o(f(t)) = 0$ , then  $f(t)$  is called an invertible series, (see [8,18]).

For each non-negative integer  $k$ , we define the  $\lambda$ -differential operator on  $\mathbb{P}$  by

$$(13) \quad (t^k)_\lambda(x)_{n,\lambda} = \begin{cases} (n)_k(x)_{n-k,\lambda}, & \text{if } k \leq n, \\ 0, & \text{if } k > n, \end{cases} \quad (\text{see [8]}),$$

and, extending this linearly, any power series

$$f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \in \mathcal{F}$$

yields the  $\lambda$ -differential operator on  $\mathbb{P}$  given by

$$(14) \quad (f(t))_\lambda(x)_{n,\lambda} = \sum_{k=0}^n \binom{n}{k} a_k (x)_{n-k,\lambda}, \quad (n \geq 0).$$

In other words, we have

$$(f(t))_\lambda = \sum_{k=0}^{\infty} \frac{a_k}{k!} (t^k)_\lambda.$$

If  $\lambda = 0$ , then the differential operator  $(t^k)_0$  is simply denoted by  $t^k$ . Now, (13) and (14) respectively become the differential operators used in Rota's theory:

$$(15) \quad t^k x^n = \begin{cases} (n)_k x^{n-k}, & \text{if } k \leq n, \\ 0, & \text{if } k > n, \end{cases} \quad (\text{see [18]}),$$

and, extending this linearly, any formal power series

$$f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \in \mathcal{F}$$

yields the differential operator on  $\mathbb{P}$  given by

$$(16) \quad f(t)x^n = \sum_{k=0}^n \binom{n}{k} a_k x^{n-k}, \quad (n \geq 0).$$

From (14), we note that

$$(17) \quad (e_\lambda^y(t))_\lambda(x)_{n,\lambda} = (x+y)_{n,\lambda}, \quad (e_\lambda^y(t))_\lambda P(x) = P(x+y), \quad P(x) \in \mathbb{P},$$

$$(e_\lambda^y(t) - 1)_\lambda P(x) = P(x+y) - P(x), \quad \langle e_\lambda^y(t) | P(x) \rangle_\lambda = P(y), \quad (\text{see [8]}).$$

For  $f(t), g(t) \in \mathcal{F}$  with  $o(f(t)) = 1$  and  $o(g(t)) = 0$ , there exists a unique sequences  $s_{n,\lambda}(x)$  ( $\deg s_{n,\lambda}(x) = n$ ) of polynomials such that (see [8])

$$(18) \quad \langle g(t)(f(t))^k | s_{n,\lambda}(x) \rangle_\lambda = n! \delta_{n,k}, \quad (n, k \geq 0).$$

The sequence  $s_{n,\lambda}(x)$  is called the  $\lambda$ -Sheffer sequence for  $(g(t), f(t))$ , which is denoted by  $s_{n,\lambda}(x) \sim (g(t), f(t))_\lambda$ , (see [8]).

For  $s_{n,\lambda}(x) \sim (g(t), f(t))_\lambda$ , we have

$$(19) \quad \frac{1}{g(\bar{f}(t))} e_\lambda^y(\bar{f}(t)) = \sum_{k=0}^{\infty} s_{k,\lambda}(y) \frac{t^k}{k!}, \quad (\text{see [8]}),$$

for all  $y \in \mathbb{C}$ , where  $\bar{f}(t)$  is the compositional inverse of  $f(t)$  with  $\bar{f}(f(t)) = f(\bar{f}(t)) = t$ .

For  $\lambda = 0$ , and with  $f(t), g(t) \in \mathcal{F}$  as before, there exists a unique sequence  $s_n(x)$  ( $\deg s_n(x) = n$ ) of polynomials such that (see [18])

$$(20) \quad \langle g(t)(f(t))^k | s_n(x) \rangle = n! \delta_{n,k}, \quad (n, k \geq 0).$$

The sequence  $s_n(x)$  is called the Sheffer sequence for  $(g(t), f(t))$ , which is denoted by  $s_n(x) \sim (g(t), f(t))$ , (see [18]).

For  $s_n(x) \sim (g(t), f(t))$ , we have

$$(21) \quad \frac{1}{g(\bar{f}(t))} e^{y\bar{f}(t)} = \sum_{k=0}^{\infty} s_k(y) \frac{t^k}{k!}, \quad (\text{see [18]}),$$

for all  $y \in \mathbb{C}$ , where  $\bar{f}(t)$  is the compositional inverse of  $f(t)$  with  $\bar{f}(f(t)) = f(\bar{f}(t)) = t$ .

Assume that, for each  $\lambda \in \mathbb{R}^*$  of the set of nonzero real numbers,  $s_{n,\lambda}(x)$  is  $\lambda$ -Sheffer for  $(g_\lambda(t), f_\lambda(t))$ . Assume also that  $\lim_{\lambda \rightarrow 0} f_\lambda(t) = f(t)$ ,  $\lim_{\lambda \rightarrow 0} g_\lambda(t) = g(t)$ , for some delta series  $f(t)$  and an invertible series  $g(t)$ . Then we see that  $\lim_{\lambda \rightarrow 0} \bar{f}_\lambda(t) = \bar{f}(t)$ . Moreover, by (19), for each  $\lambda \in \mathbb{R}^*$  we have:

$$(22) \quad \frac{1}{g_\lambda(\bar{f}_\lambda(t))} e_\lambda^x(\bar{f}_\lambda(t)) = \sum_{k=0}^{\infty} s_{k,\lambda}(x) \frac{t^k}{k!}.$$

If  $\lim_{\lambda \rightarrow 0} s_{k,\lambda}(x) = s_k(x)$ , then, by (22), we have

$$\frac{1}{g(\bar{f}(t))} e^{x\bar{f}(t)} = \sum_{k=0}^{\infty} s_k(x) \frac{t^k}{k!}.$$

Hence  $s_n(x)$  is Sheffer for  $(g(t), f(t))$ , and

$$\langle g(t)f(t)^k | s_n(x) \rangle = n! \delta_{n,k}, \quad (n, k \geq 0).$$

In this case, we may say that the family  $\{s_{n,\lambda}(x)\}_{\lambda \in \mathbb{R}^*}$  of  $\lambda$ -Sheffer sequences  $s_{n,\lambda}(x)$  are the degenerate sequences for the Sheffer polynomial  $s_n(x)$ . For example,  $\{\beta_{n,\lambda}(x)\}_{\lambda \in \mathbb{R}^*}$  (see (3), with  $r = 1$ ) are the degenerate sequences for the Bernoulli polynomials  $B_n(x)$ , where  $B_n(x) = \lim_{\lambda \rightarrow 0} \beta_{n,\lambda}(x)$  are given by

$$\frac{t}{e^t - 1} e^{xt} = \sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!}.$$

However,  $s_{n,\lambda}(x)$  itself is oftentimes called the degenerate Sheffer polynomial for the Sheffer polynomial  $s_n(x)$ .

For  $f(t), g(t) \in \mathcal{F}$  and  $P(x) \in \mathbb{P}$ , it is easy to show that

$$\langle f(t)g(t)|P(x) \rangle_{\lambda} = \langle g(t)|(f(t))_{\lambda} P(x) \rangle_{\lambda} = \langle f(t)|(g(t))_{\lambda} P(x) \rangle_{\lambda}, \quad (\text{see [8]}).$$

For  $s_{n,\lambda}(x) \sim (g(t), f(t))_{\lambda}$ ,  $r_{n,\lambda}(x) \sim (h(t), l(t))_{\lambda}$ , we have

$$(23) \quad s_{n,\lambda}(x) = \sum_{k=0}^n c_{n,k} r_{k,\lambda}(x), \quad (\text{see [8]}),$$

where

$$(24) \quad c_{n,k} = \frac{1}{k!} \left\langle \frac{h(\bar{f}(t))}{g(\bar{f}(t))} (l(\bar{f}(t)))^k \middle| (x)_{n,\lambda} \right\rangle_{\lambda}.$$

## 2. REPRESENTATIONS OF DEGENERATE HERMITE POLYNOMIALS IN TERMS OF OTHER DEGENERATE SHEFFER POLYNOMIALS AND VICE VERSA

In light of (1), we may consider the degenerate Hermite polynomials which are given by

$$(25) \quad e_{\lambda}^{-1}(t^2) \cdot e_{\lambda}^x(2t) = \sum_{n=0}^{\infty} H_{n,\lambda}(x) \frac{t^n}{n!}.$$

Note that

$$\sum_{n=0}^{\infty} \lim_{\lambda \rightarrow 0} H_{n,\lambda}(x) \frac{t^n}{n!} = e^{-t^2+2xt} = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}.$$

By (25), we get

$$(26) \quad \begin{aligned} \sum_{n=0}^{\infty} H_{n,\lambda}(x) \frac{t^n}{n!} &= e_{\lambda}^{-1}(t^2) \cdot e_{\lambda}^x(2t) = \sum_{l=0}^{\infty} (-1)_{l,\lambda} \frac{t^{2l}}{l!} \sum_{m=0}^{\infty} (x)_{m,\lambda} 2^m \frac{t^m}{m!} \\ &= \sum_{n=0}^{\infty} \left( n! \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)_{l,\lambda} 2^{n-2l} (x)_{n-2l,\lambda}}{l!(n-2l)!} \right) \frac{t^n}{n!}. \end{aligned}$$

By comparing the coefficients on both sides of (26), we obtain the following theorem.

**Theorem 1.** For  $n \geq 0$ , we have the expression given by

$$H_{n,\lambda}(x) = n! \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} \frac{2^{n-2l} (-1)_{l,\lambda}}{l!(n-2l)!} (x)_{n-2l,\lambda},$$

where  $\lfloor x \rfloor$  denotes the greatest integer not exceeding  $x$ .

Note that  $H_{n,\lambda}(x)$  is the  $\lambda$ -Sheffer sequence for  $(e_{\lambda}(\frac{1}{4}t^2), \frac{t}{2})$ . Now, we consider the following two  $\lambda$ -Sheffer sequences:

$$(27) \quad \mathcal{E}_{n,\lambda}^{(r)}(x) \sim \left( \left( \frac{e_{\lambda}(t)+1}{2} \right)^r, t \right)_{\lambda}, \quad H_{n,\lambda}(x) \sim \left( e_{\lambda} \left( \frac{1}{4}t^2 \right), \frac{1}{2}t \right)_{\lambda}.$$

By (23), (24) and (27), we get

$$(28) \quad \mathcal{E}_{n,\lambda}^{(r)}(x) = \sum_{k=0}^n C_{n,k} H_{n,\lambda}(x),$$

where

$$(29) \quad \begin{aligned} C_{n,k} &= \frac{1}{k!} \left\langle \frac{e_\lambda \left(\frac{1}{4}t^2\right)}{\left(\frac{e_\lambda(t)+1}{2}\right)^r} \left(\frac{t}{2}\right)^k \middle| (x)_{n,\lambda} \right\rangle_\lambda \\ &= \frac{1}{2^k k!} \left\langle \left(\frac{2}{e_\lambda(t)+1}\right)^r e_\lambda \left(\frac{1}{4}t^2\right) \middle| (t^k)_\lambda (x)_{n,\lambda} \right\rangle_\lambda \\ &= \frac{\binom{n}{k}}{2^k} \left\langle \left(\frac{2}{e_\lambda(t)+1}\right)^r \middle| \left(e_\lambda \left(\frac{1}{4}t^2\right)\right)_\lambda (x)_{n-k,\lambda} \right\rangle_\lambda \\ &= 2^{-k} \binom{n}{k} \sum_{l=0}^{\lfloor \frac{n-k}{2} \rfloor} \frac{(1)_{l,\lambda}}{4^l l!} (n-k)_{2l} \left\langle \left(\frac{2}{e_\lambda(t)+1}\right)^r \middle| (x)_{n-k-2l,\lambda} \right\rangle_\lambda \\ &= 2^{-k} \binom{n}{k} \sum_{l=0}^{\lfloor \frac{n-k}{2} \rfloor} \frac{(1)_{l,\lambda}}{2^{2l} l!} (n-k)_{2l} \mathcal{E}_{n-k-2l,\lambda}^{(r)} \\ &= n! \sum_{\substack{0 \leq l \leq n-k \\ l: \text{even}}} \frac{(1)_{\frac{l}{2},\lambda}}{2^{l+k} \left(\frac{l}{2}\right)! (n-k-l)! k!} \mathcal{E}_{n-k-l,\lambda}^{(r)}. \end{aligned}$$

Therefore, by (28) and (29), we obtain the following theorem.

**Theorem 2.** For  $n \geq 0$ , we have the representation given by

$$\mathcal{E}_{n,\lambda}^{(r)}(x) = n! \sum_{k=0}^n \left\{ \sum_{\substack{0 \leq l \leq n-k \\ l: \text{even}}} \frac{(1)_{\frac{l}{2},\lambda}}{k! (n-k-l)! 2^{l+k} \left(\frac{l}{2}\right)!} \mathcal{E}_{n-k-l,\lambda}^{(r)} \right\} H_{n,\lambda}(x).$$

For

$$(30) \quad \beta_{n,\lambda}^{(r)}(x) \sim \left( \left( \frac{e_\lambda(t)-1}{t} \right)^r, t \right)_\lambda, \quad H_{n,\lambda}(x) \sim \left( e_\lambda \left( \frac{1}{4}t^2 \right), \frac{t}{2} \right)_\lambda,$$

we have

$$(31) \quad \beta_{n,\lambda}^{(r)}(x) = \sum_{k=0}^n C_{n,k} H_{k,\lambda}(x),$$

where

$$(32) \quad \begin{aligned} C_{n,k} &= \frac{1}{k!} \left\langle \frac{e_\lambda \left(\frac{1}{4}t^2\right)}{\left(\frac{e_\lambda(t)-1}{t}\right)^r} \left(\frac{t}{2}\right)^k \middle| (x)_{n,\lambda} \right\rangle_\lambda \\ &= \frac{1}{k! 2^k} \left\langle \left(\frac{t}{e_\lambda(t)-1}\right)^r e_\lambda \left(\frac{1}{4}t^2\right) \middle| (t^k)_\lambda (x)_{n,\lambda} \right\rangle_\lambda \\ &= 2^{-k} \binom{n}{k} \sum_{l=0}^{\lfloor \frac{n-k}{2} \rfloor} \frac{(1)_{l,\lambda}}{l! 4^l} (n-k)_{2l} \left\langle \left(\frac{t}{e_\lambda(t)-1}\right)^r \middle| (x)_{n-k-2l,\lambda} \right\rangle_\lambda \end{aligned}$$

$$\begin{aligned}
 &= 2^{-k} \binom{n}{k} \sum_{l=0}^{\lfloor \frac{n-k}{2} \rfloor} \frac{(1)_{l,\lambda}}{l!4^l} (n-k)_{2l} \beta_{n-k-2l,\lambda}^{(r)} \\
 &= n! \sum_{\substack{0 \leq l \leq n-k \\ l: \text{ even}}} \frac{(1)_{\frac{l}{2},\lambda}}{k!(n-k-l)!2^{k+l}(\frac{l}{2})!} \beta_{n-k-l,\lambda}^{(r)}.
 \end{aligned}$$

Therefore, by (31) and (32), we obtain the following theorem.

**Theorem 3.** For  $n \geq 0$ , we have the representation given by

$$\beta_{n,\lambda}^{(r)}(x) = n! \sum_{k=0}^n \left\{ \sum_{\substack{0 \leq l \leq n-k \\ l: \text{ even}}} \frac{(1)_{\frac{l}{2},\lambda} \beta_{n-k-l,\lambda}^{(r)}}{k!(n-k-l)!(\frac{l}{2})!2^{k+l}} \right\} H_{k,\lambda}(x).$$

From (8), (19) and (25), we note that

$$(33) \quad h_{n,\lambda}^{(r)}(x|u) \sim \left( \left( \frac{e_\lambda(t) - u}{1-u} \right)^r, t \right)_\lambda, \quad H_{n,\lambda}(x) \sim \left( e_\lambda \left( \frac{1}{4}t^2 \right), \frac{t}{2} \right)_\lambda.$$

By (23), (24) and (33), we get

$$(34) \quad h_{n,\lambda}^{(r)}(x|u) = \sum_{k=0}^n C_{n,k} H_{k,\lambda}(x),$$

where

$$\begin{aligned}
 (35) \quad C_{n,k} &= \frac{1}{k!} \left\langle \frac{e_\lambda \left( \frac{1}{4}t^2 \right)}{\left( \frac{e_\lambda(t) - u}{1-u} \right)^r} \left( \frac{t}{2} \right)^k \middle| (x)_{n,\lambda} \right\rangle_\lambda \\
 &= \frac{1}{2^k k!} \left\langle \left( \frac{1-u}{e_\lambda(t) - u} \right)^r e_\lambda \left( \frac{1}{4}t^2 \right) \middle| (t^k)_\lambda (x)_{n,\lambda} \right\rangle_\lambda \\
 &= \frac{\binom{n}{k}}{2^k} \left\langle \left( \frac{1-u}{e_\lambda(t) - u} \right)^r \left( e_\lambda \left( \frac{1}{4}t^2 \right) \right) \middle| (x)_{n-k,\lambda} \right\rangle_\lambda \\
 &= \frac{\binom{n}{k}}{2^k} \sum_{l=0}^{\lfloor \frac{n-k}{2} \rfloor} \frac{(n-k)_{2l} (1)_{l,\lambda}}{l!2^{2l}} \left\langle \left( \frac{1-u}{e_\lambda(t) - u} \right)^r \middle| (x)_{n-k-2l,\lambda} \right\rangle_\lambda \\
 &= \frac{\binom{n}{k}}{2^k} \sum_{l=0}^{\lfloor \frac{n-k}{2} \rfloor} \frac{(n-k)_{2l} (1)_{l,\lambda}}{l!2^{2l}} h_{n-k-2l,\lambda}^{(r)}(u) \\
 &= n! \sum_{\substack{0 \leq l \leq n-k \\ l: \text{ even}}} \frac{(1)_{\frac{l}{2},\lambda} h_{n-k-l,\lambda}^{(r)}(u)}{\left( \frac{l}{2} \right)! 2^{l+k} (n-k-l)! k!}.
 \end{aligned}$$

Therefore, by (34) and (35), we obtain the following theorem.

**Theorem 4.** For  $n \geq 0$ , we have the representation given by

$$h_{n,\lambda}^{(r)}(x|u) = n! \sum_{k=0}^n \left\{ \sum_{\substack{0 \leq l \leq n-k \\ l: \text{ even}}} \frac{(1)_{\frac{l}{2},\lambda} h_{n-k-l,\lambda}^{(r)}(u)}{\left( \frac{l}{2} \right)! 2^{l+k} (n-k-l)! k!} \right\} H_{k,\lambda}(x).$$

For  $s_{n,\lambda}(x) \sim (g(t), f(t))_\lambda$ ,  $p_{n,\lambda}(x) \sim (1, f(t))_\lambda$ , we have

$$\begin{aligned} n! \delta_{n,k} &= \langle g(t)(f(t))^k | s_{n,\lambda}(x) \rangle_\lambda = \langle (f(t))^k | (g(t))_\lambda s_{n,\lambda}(x) \rangle_\lambda \\ (36) \quad &= \langle (f(t))^k | p_{n,\lambda}(x) \rangle_\lambda \end{aligned}$$

By (36), we get

$$(37) \quad (g(t))_\lambda s_{n,\lambda}(x) \sim (1, f(t))_\lambda, \quad (n \geq 0).$$

From (25), we have

$$(38) \quad H_{n,\lambda}(x) \sim \left( e_\lambda \left( \frac{1}{4} t^2 \right), \frac{1}{2} t \right)_\lambda \iff \left( e_\lambda \left( \frac{1}{4} t^2 \right) \right)_\lambda H_{n,\lambda}(x) \sim \left( 1, \frac{1}{2} t \right)_\lambda.$$

Since  $2^n(x)_{n,\lambda} \sim (1, \frac{1}{2}t)_\lambda$ ,  $(e_\lambda(\frac{1}{4}t^2))_\lambda H_{n,\lambda}(x) = 2^n(x)_{n,\lambda}$ .

Hence,

$$\begin{aligned} H_{n,\lambda}(x) &= 2^n \left( e_\lambda^{-1} \left( \frac{1}{4} t^2 \right) \right)_\lambda (x)_{n,\lambda} \\ &= 2^n \sum_{l=0}^{\infty} \frac{(-1)_{l,\lambda}}{l! 4^l} (t^{2l})_\lambda (x)_{n,\lambda} \\ &= 2^n \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)_{l,\lambda} (n)_{2l}}{l! 4^l} (x)_{n-2l,\lambda} \\ &= n! \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)_{l,\lambda} 2^{n-2l}}{l! (n-2l)!} (x)_{n-2l,\lambda}, \end{aligned}$$

which gives another proof for Theorem 1. Assume that

$$(39) \quad H_{n,\lambda}(x) = \sum_{k=0}^n C_{n,k} \mathcal{E}_{k,\lambda}^{(r)}(x).$$

Then, by (23), (24) and (27), we get

$$\begin{aligned} (40) \quad C_{n,k} &= \frac{1}{k!} \left\langle \left( \frac{e_\lambda(2t)+1}{2} \right)^r (2t)^k \middle| (x)_{n,\lambda} \right\rangle_\lambda \\ &= \frac{2^k}{k!} \left\langle \left( \frac{e_\lambda(2t)+1}{2} \right)^r e_\lambda^{-1}(t^2) \middle| (t^k)_\lambda (x)_{n,\lambda} \right\rangle_\lambda \\ &= 2^k \binom{n}{k} \left\langle \left( \frac{e_\lambda(2t)+1}{2} \right)^r \middle| (e_\lambda^{-1}(t^2))_\lambda (x)_{n-k,\lambda} \right\rangle_\lambda \\ &= 2^k \binom{n}{k} \sum_{l=0}^{\lfloor \frac{n-k}{2} \rfloor} \frac{(-1)_{l,\lambda} (n-k)_{2l}}{l!} \left\langle \left( \frac{e_\lambda(2t)+1}{2} \right)^r \middle| (x)_{n-k-2l,\lambda} \right\rangle_\lambda \end{aligned}$$

$$\begin{aligned}
 &= \frac{2^k}{2^r} \binom{n}{k} \sum_{j=0}^r \sum_{l=0}^{\lfloor \frac{n-k}{2} \rfloor} \frac{(-1)_{l,\lambda} (n-k)_{2l} \binom{r}{j}}{l!} \langle e_\lambda^j(2t) | (x)_{n-k-2l,\lambda} \rangle_\lambda \\
 &= \frac{1}{2^r} \sum_{j=0}^r \sum_{l=0}^{\lfloor \frac{n-k}{2} \rfloor} \frac{(-1)_{l,\lambda} (n-k)_{2l} \binom{r}{j} \binom{n}{k} 2^k 2^{n-k-2l} (j)_{n-k-2l,\lambda}}{l!} \\
 &= \frac{1}{2^r} \sum_{j=0}^r \sum_{l=0}^{\lfloor \frac{n-k}{2} \rfloor} \frac{(-1)_{l,\lambda} (n-k)! \binom{r}{j} \binom{n}{k} 2^{n-2l}}{l!(n-k-2l)!} (j)_{n-k-2l,\lambda}.
 \end{aligned}$$

Therefore, by (39) and (40), we obtain the following theorem.

**Theorem 5.** For  $n \geq 0$ , we have the representation given by

$$H_{n,\lambda}(x) = \frac{1}{2^r} \sum_{k=0}^n \left\{ \sum_{j=0}^r \sum_{l=0}^{\lfloor \frac{n-k}{2} \rfloor} \frac{(-1)_{l,\lambda} (n-k)! \binom{r}{j} \binom{n}{k} 2^{n-2l}}{l!(n-k-2l)!} (j)_{n-k-2l,\lambda} \right\} \mathcal{E}_{k,\lambda}^{(r)}(x).$$

Next, we would like to find another representation of  $H_{n,\lambda}(x)$  in terms of  $\mathcal{E}_{k,\lambda}^{(r)}(x)$ . From (40), we observe that

$$\begin{aligned}
 (41) \quad C_{n,k} &= \frac{1}{k!} \left\langle \left( \frac{e_\lambda(2t) + 1}{2} \right)^r e_\lambda^{-1} \left( \frac{1}{4} (2t)^2 \right) (2t)^k \middle| (x)_{n,\lambda} \right\rangle_\lambda \\
 &= \frac{1}{k!} \left\langle \left( \frac{e_\lambda(t) + 1}{2} \right)^r e_\lambda^{-1} \left( \frac{1}{4} t^2 \right) t^k \middle| 2^n (x)_{n,\lambda} \right\rangle_\lambda \\
 &= \frac{1}{k!} \left\langle \left( \frac{e_\lambda(t) + 1}{2} \right)^r t^k \middle| 2^n \left( e_\lambda^{-1} \left( \frac{1}{4} t^2 \right) \right)_\lambda (x)_{n,\lambda} \right\rangle_\lambda \\
 &= \frac{1}{k!} \left\langle \left( \frac{e_\lambda(t) + 1}{2} \right)^r t^k \middle| H_{n,\lambda}(x) \right\rangle_\lambda \\
 &= \frac{1}{k!} \left\langle \left( \frac{e_\lambda(t) + 1}{2} \right)^r \middle| (t^k)_\lambda H_{n,\lambda}(x) \right\rangle_\lambda.
 \end{aligned}$$

To proceed further, we note that

$$\begin{aligned}
 (42) \quad (t^k)_\lambda H_{n,\lambda}(x) &= n! \sum_{l=0}^{\lfloor \frac{n-k}{2} \rfloor} \frac{2^{n-2l} (-1)_{l,\lambda}}{l!(n-2l)!} (t^k)_\lambda (x)_{n-2l,\lambda} \\
 &= n! \sum_{l=0}^{\lfloor \frac{n-k}{2} \rfloor} \frac{2^{n-2l} (-1)_{l,\lambda}}{l!(n-2l-k)!} (x)_{n-2l-k,\lambda} \\
 &= 2^k (n)_k (n-k)! \sum_{l=0}^{\lfloor \frac{n-k}{2} \rfloor} \frac{2^{n-k-2l} (-1)_{l,\lambda}}{l!(n-k-2l)!} (x)_{n-2l-k,\lambda} \\
 &= 2^k (n)_k H_{n-k,\lambda}(x).
 \end{aligned}$$

By (41) and (42), we get

$$\begin{aligned}
(43) \quad C_{n,k} &= \frac{2^k (n)_k}{k!} \left\langle \left( \frac{e_\lambda(t) + 1}{2} \right)^r \middle| H_{n-k,\lambda}(x) \right\rangle_\lambda \\
&= \frac{2^k \binom{n}{k}}{2^r} \sum_{j=0}^r \binom{r}{j} \langle e_\lambda^j(t) | H_{n-k,\lambda}(x) \rangle_\lambda \\
&= \frac{2^k}{2^r} \binom{n}{k} \sum_{j=0}^r \binom{r}{j} H_{n-k,\lambda}(j).
\end{aligned}$$

Therefore, by (39) and (43), we obtain the following theorem.

**Theorem 6.** For  $n \geq 0$ , we have the representation given by

$$H_{n,\lambda}(x) = \frac{1}{2^r} \sum_{k=0}^n \binom{n}{k} 2^k \left[ \sum_{j=0}^r \binom{r}{j} H_{n-k,\lambda}(j) \right] \mathcal{E}_{k,\lambda}^{(r)}(x).$$

Let us assume that

$$(44) \quad H_{n,\lambda}(x) = \sum_{k=0}^n C_{n,k} \beta_{n,\lambda}^{(r)}(x).$$

Then, by (23), (24) and (30), we have

$$\begin{aligned}
(45) \quad C_{n,k} &= \frac{1}{k!} \left\langle \frac{(e_\lambda(2t)-1)^r}{e_\lambda\left(\frac{1}{4}(2t)^2\right)} (2t)^k \middle| (x)_{n,\lambda} \right\rangle_\lambda \\
&= \frac{1}{k!} \left\langle \frac{(e_\lambda(t)-1)^r}{e_\lambda\left(\frac{1}{4}t^2\right)} t^k \middle| 2^n (x)_{n,\lambda} \right\rangle_\lambda \\
&= \frac{1}{k!} \left\langle \left( \frac{e_\lambda(t)-1}{t} \right)^r t^k \middle| \left( e_\lambda^{-1}\left(\frac{1}{4}t^2\right) \right)_\lambda 2^n (x)_{n,\lambda} \right\rangle_\lambda \\
&= \frac{1}{k!} \left\langle \left( \frac{e_\lambda(t)-1}{t} \right)^r t^k \middle| H_{n,\lambda}(x) \right\rangle_\lambda.
\end{aligned}$$

For  $r > k$ , we have

$$\begin{aligned}
(46) \quad C_{n,k} &= \frac{1}{k!} \left\langle (e_\lambda(t)-1)^k \middle| \left( \left( \frac{e_\lambda(t)-1}{t} \right)^{r-k} \right)_\lambda H_{n,\lambda}(x) \right\rangle_\lambda \\
&= \frac{1}{k!} \sum_{l=0}^n \frac{S_{2,\lambda}(l+r-k, r-k)}{\binom{l+r-k}{l} l!} \langle (e_\lambda(t)-1)^k | (t^l)_\lambda H_{n,\lambda}(x) \rangle_\lambda \\
&= \frac{1}{k!} \sum_{l=0}^n \frac{S_{2,\lambda}(l+r-k, r-k)}{\binom{l+r-k}{l} l!} \langle (e_\lambda(t)-1)^k | 2^l (n)_l H_{n-l,\lambda}(x) \rangle_\lambda \\
&= \frac{1}{k!} \sum_{l=0}^n \sum_{j=0}^k \frac{2^l \binom{n}{l} \binom{k}{j}}{\binom{l+r-k}{l}} S_{2,\lambda}(l+r-k, r-k) (-1)^{k-j} \langle e_\lambda^j(t) | H_{n-l,\lambda}(x) \rangle_\lambda \\
&= \frac{1}{k!} \sum_{l=0}^n \sum_{j=0}^k \frac{2^l \binom{n}{l} \binom{k}{j}}{\binom{l+r-k}{l}} S_{2,\lambda}(l+r-k, r-k) (-1)^{k-j} H_{n-l,\lambda}(j).
\end{aligned}$$

The next theorem now follows from (44) and (46).

**Theorem 7.** For  $r > n \geq 0$ , we have the representation given by

$$H_{n,\lambda}(x) = \sum_{k=0}^n \frac{1}{k!} \left\{ \sum_{j=0}^k \sum_{l=0}^n \frac{2^l \binom{n}{l} \binom{k}{j}}{\binom{l+r-k}{l}} S_{2,\lambda}(l+r-k, r-k) (-1)^{k-j} H_{n-l,\lambda}(j) \right\} \beta_{k,\lambda}^{(r)}(x).$$

For  $r \leq k \leq n$  in (45), we have

$$\begin{aligned} (47) \quad C_{n,k} &= \frac{1}{k!} \sum_{j=0}^r \binom{r}{j} (-1)^{r-j} \langle e_\lambda^j(t) t^{k-r} | H_{n,\lambda}(x) \rangle_\lambda \\ &= \frac{1}{k!} \sum_{j=0}^r \binom{r}{j} (-1)^{r-j} \langle e_\lambda^j(t) | (t^{k-r})_\lambda H_{n,\lambda}(x) \rangle_\lambda \\ &= \frac{2^{k-r} (n)_{k-r}}{k!} \sum_{j=0}^r \binom{r}{j} (-1)^{r-j} \langle e_\lambda^j(t) | H_{n-k+r,\lambda}(x) \rangle_\lambda \\ &= \frac{2^{k-r} (n)_{k-r}}{k!} \sum_{j=0}^r \binom{r}{j} (-1)^{r-j} H_{n-k+r,\lambda}(j). \end{aligned}$$

Therefore, by (44) and (47), we obtain the following theorem.

**Theorem 8.** For  $n \geq r$ , we have the representation given by

$$\begin{aligned} H_{n,\lambda}(x) &= \sum_{k=0}^{r-1} \frac{1}{k!} \left\{ \sum_{j=0}^k \sum_{l=0}^n \frac{2^l \binom{n}{l} \binom{k}{j}}{\binom{l+r-k}{l}} S_{2,\lambda}(l+r-k, r-k) (-1)^{k-j} H_{n-l,\lambda}(j) \right\} \beta_{k,\lambda}^{(r)}(x) \\ &\quad + n! \sum_{k=r}^n \left\{ \sum_{j=0}^r \frac{(-1)^{r-j} \binom{r}{j} 2^{k-r}}{k!(n-k+r)!} H_{n-k+r,\lambda}(j) \right\} \beta_{k,\lambda}^{(j)}(x). \end{aligned}$$

Let

$$(48) \quad H_{n,\lambda}(x) = \sum_{k=0}^n C_{n,k} h_{k,\lambda}^{(r)}(x|u).$$

Then, by (23), (24) and (33), we have

$$\begin{aligned} (49) \quad C_{n,k} &= \frac{1}{k!} \left\langle \frac{\left( \frac{e_\lambda(2t)-u}{1-u} \right)^r}{e_\lambda\left(\frac{1}{4}(2t)^2\right)} (2t)^k \middle| (x)_{n,\lambda} \right\rangle_\lambda \\ &= \frac{1}{k!} \left\langle \frac{\left( \frac{e_\lambda(t)-u}{1-u} \right)^r}{e_\lambda\left(\frac{1}{4}t^2\right)} t^k \middle| 2^n (x)_{n,\lambda} \right\rangle_\lambda \\ &= \frac{1}{k!} \left\langle \left( \frac{e_\lambda(t)-u}{1-u} \right)^r t^k \middle| \left( e_\lambda^{-1}\left(\frac{1}{4}t^2\right) \right)_\lambda 2^n (x)_{n,\lambda} \right\rangle_\lambda \\ &= \frac{1}{k!(1-u)^r} \langle (e_\lambda(t)-u)^r | (t^k)_\lambda H_{n,\lambda}(x) \rangle_\lambda \\ &= \frac{1}{(1-u)^r k!} \langle (e_\lambda(t)-u)^r | 2^k (n)_k H_{n-k,\lambda}(x) \rangle_\lambda \\ &= \frac{\binom{n}{k} 2^k}{(1-u)^r} \sum_{j=0}^r \binom{r}{j} (-u)^{r-j} \langle e_\lambda^j(t) | H_{n-k,\lambda}(x) \rangle_\lambda \\ &= \frac{\binom{n}{k} 2^k}{(1-u)^r} \sum_{j=0}^r \binom{r}{j} (-u)^{r-j} H_{n-k,\lambda}(j). \end{aligned}$$

Therefore, by (48) and (49), we obtain the following theorem.

**Theorem 9.** For  $n \geq 0$ , we have the representation given by

$$H_{n,\lambda}(x) = \frac{1}{(1-u)^r} \sum_{k=0}^n \binom{n}{k} 2^k \left[ \sum_{j=0}^r \binom{r}{j} (-u)^{r-j} H_{n-k,\lambda}(j) \right] h_{k,\lambda}^{(r)}(x|u).$$

Finally, we would like to obtain another expression of  $H_{n,\lambda}(x)$  in terms of  $h_{k,\lambda}^{(r)}(x|u)$ . With  $C_{n,k}$  as in (48), we have

$$\begin{aligned} C_{n,k} &= \frac{1}{k!} \left\langle \frac{(e_\lambda(2t)-u)^r}{e_\lambda(\frac{1}{4}(2t)^2)} (2t)^k \middle| (x)_{n,\lambda} \right\rangle_\lambda \\ &= \frac{2^k (n)_k}{k! (1-u)^r} \left\langle \frac{(e_\lambda(2t)-u)^r}{e_\lambda(t^2)} \middle| (x)_{n-k,\lambda} \right\rangle_\lambda \\ &= \frac{\binom{n}{k} 2^k}{(1-u)^r} \sum_{l=0}^{\lfloor \frac{n-k}{2} \rfloor} \frac{(-1)_{l,\lambda}}{l!} \langle (e_\lambda(2t)-u)^r | (t^{2l})_\lambda (x)_{n-k,\lambda} \rangle_\lambda \\ &= \frac{\binom{n}{k} 2^k}{(1-u)^r} \sum_{l=0}^{\lfloor \frac{n-k}{2} \rfloor} \frac{(-1)_{l,\lambda} (n-k)_{2l}}{l!} \langle (e_\lambda(2t)-u)^r | (x)_{n-k-2l,\lambda} \rangle_\lambda \\ &= \frac{\binom{n}{k} 2^k}{(1-u)^r} \sum_{l=0}^{\lfloor \frac{n-k}{2} \rfloor} \frac{(-1)_{l,\lambda} (n-k)!}{l! (n-k-2l)!} \sum_{j=0}^r \binom{r}{j} (-u)^{r-j} \langle e_\lambda^j(2t) | (x)_{n-k-2l,\lambda} \rangle_\lambda \\ &= \frac{1}{(1-u)^r} \sum_{j=0}^r \sum_{l=0}^{\lfloor \frac{n-k}{2} \rfloor} \frac{\binom{n}{k} \binom{r}{j} 2^k (-1)_{l,\lambda} (n-k)!}{l! (n-k-2l)!} (-u)^{r-j} 2^{n-k-2l} (j)_{n-k-2l,\lambda} \end{aligned}$$

Hence, we have the following theorem.

**Theorem 10.** For  $n \geq 0$ , we have the representation given by

$$H_{n,\lambda}(x) = \frac{1}{(1-u)^r} \sum_{k=0}^n \left[ \sum_{j=0}^r \sum_{l=0}^{\lfloor \frac{n-k}{2} \rfloor} \frac{\binom{n}{k} \binom{r}{j} (-1)_{l,\lambda} (n-k)!}{l! (n-k-2l)!} (-u)^{r-j} 2^{n-2l} (j)_{n-k-2l,\lambda} \right] h_{k,\lambda}^{(r)}(x).$$

### 3. CONCLUSION

Special polynomials and numbers can be studied by several different methods, which include generating functions, combinatorial methods, umbral calculus,  $p$ -adic analysis, differential equations, probability, orthogonal polynomials and special functions. These various means of investigating special polynomials and numbers can be applied also to degenerate special polynomials and numbers. Indeed, in recent years, degenerate versions of many special polynomials and numbers were explored with such methods, and some of their arithmetical and combinatorial properties were discovered. Moreover, those degenerate versions of some special polynomials found some applications to other areas of mathematics such as differential equations, identities of symmetry and probability theory.

Hermite polynomials are orthogonal polynomials that arise in such diverse areas as combinatorics, numerical analysis, probability, physics, random matrix theory and systems theory. In light of the regained recent interests in degenerate special numbers and polynomials, the introduction of the degenerate Hermite polynomials followed naturally. In [8], the  $\lambda$ -linear functionals and the  $\lambda$ -differential operators were introduced in order to effectively treat  $\lambda$ -Sheffer polynomials (see (18), (19)). In particular, the important formula in (23) and (24), which expresses one  $\lambda$ -Sheffer

polynomial in terms of other  $\lambda$ -Sheffer polynomials, were derived in [8]. In this paper, we applied this formula and expressed the higher-order degenerate Bernoulli, Euler and Frobenius-Euler polynomials respectively in terms of the degenerate Hermite polynomials and vice versa.

As one of our future research projects, we would like to continue to study ' $\lambda$ -umbral calculus' and their applications to physics, science and engineering as well as to mathematics.

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### Authors' contributions

TK and DSK conceived of the framework and structured the whole paper; TK wrote the paper; L-CJ, HL and HYK checked the results of the paper; DSK completed the revision of the article. All authors read and approved the final manuscript.

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