Asymptotic linearity of binomial random hypergraphs via cluster expansion under graph-dependence

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Abstract

Let integer $n \ge 3$ and integer $r = r(n) \ge 3$. Define the binomial random r-uniform hypergraph $H_r(n,p)$ to be the r-uniform graph on the vertex set [n] such that each r-set is an edge independently with probability p. A hypergraph is linear if every pair of hyperedges intersects in at most one vertex. We study the probability of linearity of random hypergraphs $H_r(n,p)$ via cluster expansion and give more precise asymptotics of the probability in question, improving the asymptotic probability of linearity obtained by McKay and Tian, in particular, when r = 3 and $p = o(n^{-7/5})$.

1 Introduction

For all positive integer $n \geq 3$, let [n] denote the integer set $\{1, 2, ..., n\}$. Let $[n]_t = n(n-1) \cdots (n-t+1)$ denote the t-th falling factorial for every positive integer $t \leq n$. Define the binomial random r-uniform hypergraph $H_r(n,p)$ to be the r-uniform hypergraph (r-graph for short) on the vertex set [n] such that each r-element subset (r-set for short) is an edge independently with probability p. A hypergraph is linear if every pair of hyperedges intersects in at most one vertex. Let $\mathcal{L}_r(n)$ be the set of all linear r-uniform hypergraphs with n vertices. All asymptotics in this note are with respect to $n \to \infty$. We study the probability of linearity of random hypergraphs $H_r(n,p)$, and improve the following result by giving more precise asymptotics of the probability.

Theorem 1.1 ([13, Theorem 1.2]). Let $r = r(n) \ge 3$. If $p\binom{n}{r} = O(r^{-2}n)$, then

$$\mathbb{P}(H_r(n,p) \in \mathcal{L}_r(n)) = \exp\left(-\frac{[r]_2^2}{4n^2} \binom{n}{r}^2 p^2 + O\left(\frac{r^6}{n^3} \binom{n}{r}^2 p^2\right)\right).$$

If
$$r^{-2}n \leq p\binom{n}{r} = o(r^{-3}n^{3/2})$$
, then

$$\mathbb{P}\left(H_r(n,p) \in \mathcal{L}_r(n)\right) = \exp\left(-\frac{[r]_2^2}{4n^2} \binom{n}{r}^2 p^2 + \frac{(3r-5)[r]_2^3}{6n^4} \binom{n}{r}^3 p^3 + O\left(\frac{\log^3(r^{-2}n)}{\sqrt{\binom{n}{r}p}} + \frac{r^6}{n^3} \binom{n}{r}^2 p^2\right)\right).$$

For random 3-uniform hypergraphs, the above theorem gives that if $p = o(n^{-3/2})$, then

(1)
$$\mathbb{P}(H_3(n,p) \in \mathcal{L}_3(n)) = \exp\left(-\frac{1}{4}n^4p^2 + \frac{2}{3}n^5p^3 + o(1)\right).$$

1.1 Cluster expansion and dependency graphs

Cluster expansion is a powerful tool in the rigorous study of statistical mechanics. It was pioneered by Mayer in the 1930's and remains widely used nowadays, see, for example, [4]. The cluster expansion allows us to express the logarithm of partition function as a sum over clusters. Here we introduce the standard cluster expansion setting, which is formulated in a way that is convenient for our application.

Definition 1.2. Given an undirected graph G = (V(G), E(G)).

- (d1) A connected component of G is a maximal set of vertices such that every pair of vertices is connected by a path. The number of connected components of G is denoted by c(G).
- (d2) The set of polymers $\mathcal{C}(G)$ of G is the vertex sets of all connected induced subgraphs of G, namely,

$$C(G) = \{ C \subseteq V(G) : c(G[C]) = 1 \},$$

where G[C] denotes the subgraph of G induced by the vertex set C. For any two distinct polymers $C_i, C_j \in \mathcal{C}(G)$, we write $C_i \sim C_j$ if $C_i \cup C_j \in \mathcal{C}(G)$; otherwise, $C_i \not\sim C_j$. Equivalently, $C_i \not\sim C_j$ if $d_G(C_i, C_j) > 1$ and otherwise, $C_i \sim C_j$, where $d_G(\cdot, \cdot)$ denotes the graph distance in G, that is, the number of edges in the shortest path among two subsets of vertices. Note that if we have that $C_i \sim C_j$ and $C_i \cap C_j = \emptyset$, then C_i and C_j are adjacent in G, that is, there exists an edge in E(G), with one endpoint in C_i and the other in C_j . The size of a polymer, denoted by |C|, is the number of vertices in it.

- (d3) For every non-empty ordered multiset of polymers $(C_1, \ldots, C_n) \in \mathcal{C}(G)^n$, let $\mathbb{G}(C_1, \ldots, C_n)$ be the graph on [n] with $\{i, j\} \in E(\mathbb{G})$ if $C_i \sim C_j$. For instance, fix a polymer $C \in \mathcal{C}(G)$. For a multiset of n copies of C, we have $\mathbb{G}(C, \ldots, C) = K_n$, where K_n denotes the complete graph on [n].
- (d4) A cluster γ is a non-empty ordered multiset of polymers $(C_1, \ldots, C_{|\gamma|})$ such that $\mathbb{G}(\gamma) = \mathbb{G}(C_1, \ldots, C_{|\gamma|})$ is connected. The size of a cluster γ , denoted by $|\gamma|$, is the number of polymers in it, and the number of vertices of a cluster γ , denoted by $||\gamma||$, is the sum of size of polymers it contains, that is, $||\gamma|| = \sum_{C \in \gamma} |C|$.
- (d5) The set of all clusters of G is denoted by $\Gamma(G)$. The set of all clusters of G with pairwise disjoint polymers is denoted by

$$\Gamma_{\emptyset}(G) = \{ \gamma \in \Gamma(G) : C_i \cap C_j = \emptyset \text{ for any distinct } C_i, C_j \in \gamma \}.$$

Note that each element in $\Gamma_{\emptyset}(G)$ is a cluster whose elements form a partition of a polymer, since for every $\gamma \in \Gamma_{\emptyset}(G)$, polymers $\{C : C \in \gamma\}$ are disjoint and their union $\cup_{C \in \gamma} C \in \mathcal{C}(G)$.

(d6) The set of all connected spanning subgraphs of G is

$$CSpan(G) = \{ (V(G), E) : E \subseteq E(G), c((V(G), E)) = 1 \}.$$

Here for every graph $H \in \mathrm{CSpan}(G)$, we have V(H) = V(G), $E(H) \subseteq E(G)$, and C(H) = 1.

The cluster expansion method can be naturally combined with dependency graphs. The dependency graph models have been widely used in probability and statistics to establish normal or Poisson approximation via the Stein's method, cumulants, etc. (see, for example, [7, 8]). They are also heavily used in probabilistic combinatorics, such as Lovász local lemma [3], Janson's inequality [9], concentration inequalities [21], etc.

Given a graph G = (V, E), we say that random variables $\{X_i\}_{i \in V}$ are G-dependent if for any disjoint $S, T \subset V$ such that $d_G(S, T) > 1$, random variables $\{X_i\}_{i \in S}$ and $\{X_j\}_{j \in T}$ are independent. Or equivalently, random variables $\{X_i\}_{i \in C_1}$ and $\{X_j\}_{j \in C_2}$ are independent for any two distinct polymers C_1 and C_2 of the graph G such that $C_1 \not\sim C_2$.

Note that the dependency graph for a set of random variables may not be necessarily unique and the sparser ones are the more interesting ones. Since no two disjoint $S, T \subset [n]$ are non-adjacent in K_n , then the trivial dependency graph K_n is a valid dependency graph for any set of variables $\{X_i\}_{i\in[n]}$.

Given G-dependent random variables $\{X_i\}_{i\in V(G)}$, for every set of vertices $S\subseteq V(G)$, the joint moment of random variables $\{X_i\}_{i\in S}$ is defined by

$$\mu(S) = \mathbb{E}\left[\prod_{i \in S} X_i\right],$$

with $\mu(\emptyset) := 1$. Let $\{C_i\}_{i \in [n]}$ be a set of pairwise non-adjacent disjoint polymers of G, in other words, for all distinct $i, j \in [n]$, we have $C_i \not\sim C_j$, or equivalently, $d_G(C_i, C_j) > 1$. Then one important factorisation

property for G-dependent variables, following from the definition of dependency graph, is that

(2)
$$\mu\left(\bigcup_{i\in[n]}C_i\right) = \prod_{i\in[n]}\mu\left(C_i\right).$$

Let $\{X_v\}_{v\in V(G)}$ be G-dependent random indicators and $X=\sum_{v\in V(G)}X_v$. In our application, each indicator indicates the occurrence of some combinatorial structure, and indicators are dependent with a dependency graph. By writing the probability of the non-existence of some combinatorial structure $\mathbb{P}(X=0)$ as a partition function, the cluster expansion then gives the formal expansion formula as a sum over clusters, whose truncation approximates the asymptotic probability. This is inspired by [16], in which they also treat $\mathbb{P}(X=0)$ as a partition function and investigate the connections between cluster expansion and the Lovász local lemma, giving a lower bound for $\mathbb{P}(X=0)$.

The standard cluster expansion gives the formal cluster expansion

(3)
$$\log \mathbb{P}\left(X=0\right) = \sum_{\gamma \in \Gamma(G)} \frac{\phi(\gamma)}{|\gamma|!} (-1)^{\|\gamma\|} \prod_{C \in \gamma} \mu\left(C\right),$$

with Ursell function $\phi: \Gamma(G) \to \mathbb{R}$ defined by

(4)
$$\phi(\gamma) = \sum_{H \in \mathrm{CSpan}(\mathbb{G}(\gamma))} (-1)^{e_H},$$

where e_H denotes the number of edges of the graph H. Note that if the cluster γ contains one single polymer $C \in \mathcal{C}(G)$, then $\phi(\gamma) = 1$, because $\mathbb{G}(C) = K_1$.

For completeness, we include a simple derivation of equation (3), following the routine cluster expansion derivation procedure (see, for example, [16, Section 2.2] or [4, Proposition 5.3.]). First, the inclusion-exclusion formula gives

(5)
$$\mathbb{P}(X=0) = \sum_{S \subseteq V(G)} (-1)^{|S|} \mu(S).$$

Let \mathcal{G}_c be a graph on vertex set $\mathcal{C}(G)$ such that for all distinct $C_i, C_j \in \mathcal{C}(G)$, if $C_i \sim C_j$, then $\{C_i, C_j\} \in E(\mathcal{G}_c)$. Next we utilise the factorisation property as shown in (2) to prove that the right hand side of equation (5) can be written as some partition function of hard-core model, more specifically, as a summation over independent sets of graph \mathcal{G}_c ,

(6)
$$\mathbb{P}\left(X=0\right) = \sum_{U \in \mathcal{I}(\mathcal{G}_c)} \prod_{C \in U} (-1)^{|C|} \mu\left(C\right),$$

where $\mathcal{I}(G)$ denotes the set of all independent sets for every graph G.

For every $S \subseteq V(G)$ such that $S \in \mathcal{C}(G)$, we have $\{S\} \in \mathcal{I}(\mathcal{G}_c)$. For every $S \subseteq V(G)$ such that $S \notin \mathcal{C}(G)$, we have S induces a union of pairwise non-adjacent maximal connected subgraphs, that is, there exists a unique set of polymers $U \in \mathcal{I}(\mathcal{G}_c)$ such that $S = \bigcup_{C \in U} C$, and $C_i \not\sim C_j$ for all pairs of distinct $C_i, C_j \in U$. The factorisation property (2) then gives

(7)
$$(-1)^{|S|}\mu(S) = \prod_{C \in S} (-1)^{|C|}\mu(C).$$

Conversely, for every $U \in \mathcal{I}(\mathcal{G}_c)$, we have $\bigcup_{C \in U} C \subseteq V(G)$, thus U determines S uniquely; combining with equation (7), it follows that equations (5) and (6) are equivalent.

Now we derive the formal cluster expansion. Let $\binom{S}{i}$ denote the family of *i*-sets of S for every set S and every positive integer $i \leq |S|$. From equation (6), we have

$$\mathbb{P}(X = 0) = \sum_{U \subseteq \mathcal{C}(G)} \prod_{C \in U} (-1)^{|C|} \mu(C) \prod_{\{C_i, C_j\} \in \binom{U}{2}} \mathbf{1}_{\{C_i \not\sim C_j\}}
= 1 + \sum_{n \geqslant 1} \frac{1}{n!} \sum_{(C_1, \dots, C_n) \in \mathcal{C}(G)^n} \prod_{i \in [n]} (-1)^{|C_i|} \mu(C_i) \prod_{1 \leqslant i < j \leqslant n} \mathbf{1}_{\{C_i \not\sim C_j\}}.$$

Note that by a simple expansion,

$$\prod_{1 \leqslant i < j \leqslant n} \mathbf{1}_{\{C_i \not\sim C_j\}} = \prod_{1 \leqslant i < j \leqslant n} \left(1 - \mathbf{1}_{\{C_i \sim C_j\}}\right) = \sum_{H \in \mathfrak{G}_n} (-1)^{e_H} \prod_{\{i,j\} \in E(H)} \mathbf{1}_{\{C_i \sim C_j\}},$$

where \mathfrak{G}_n denotes the set of all graphs on n vertices. Then formally, we have

$$\mathbb{P}(X=0) = 1 + \sum_{n \ge 1} \frac{1}{n!} \sum_{H \in \mathfrak{G}_n} W(H),$$

where

$$W(H) = \sum_{(C_1, \dots, C_{v_H}) \in \mathcal{C}(G)^{v_H}} (-1)^{e_H} \prod_{\{i, j\} \in E(H)} \mathbf{1}_{\{C_i \sim C_j\}} \prod_{k \in [v_H]} (-1)^{|C_k|} \mu(C_k),$$

and W(H) satisfies

- (a1) W(H) = W(H') whenever H and H' are isomorphic $H \cong H'$, that is, differ only by vertices relabelling;
- (a2) $W(H) = W(H_1)W(H_2)$ whenever H is isomorphic to the disjoint union of H_1 and H_2 .

Let \mathfrak{C}_n be the set of all connected graphs on n vertices. Via the exponential formula [19, Corollary 5.1.6], we reduce the sum over the set of all graphs to the set of all connected graphs

$$\log \mathbb{P}\left(X=0\right) = \sum_{n\geqslant 1} \frac{1}{n!} \sum_{H\in\mathfrak{C}_n} W(H) = \sum_{\gamma\in\Gamma(G)} \frac{1}{|\gamma|!} \sum_{H\in\operatorname{CSpan}\left(\mathbb{G}(\gamma)\right)} (-1)^{e_H+\|\gamma\|} \prod_{C\in\gamma} \mu\left(C\right),$$

where $\Gamma(G)$ denotes the set of all clusters of G. A similar derivation of the cluster expansion utilizing the exponential formula also appears in [4, Proposition 5.3]. Then equation (3) follows.

Remark 1.3. (r1) In probability theory and statistical physics, given a graph H and a vector $\mathbf{p} = \{p_v\}_{v \in V(H)}$, the partition function of the hard-core model (also the independence polynomial) on H is defined by

$$\mathbf{I}_{H}(\boldsymbol{p}) = \sum_{U \in \mathcal{I}(H)} \prod_{i \in U} p_{i}.$$

The cluster expansion is essentially the multivariate Taylor series for $\log \mathbf{I}_H(\mathbf{p})$ in variables $\{p_v\}_{v \in V(H)}$ around $\mathbf{0}$. Let $\boldsymbol{\mu} = ((-1)^{|C|}\boldsymbol{\mu}(C))_{C \in \mathcal{C}(G)}$. Then equation (6) can be regarded as the partition function $\mathbf{I}_{\mathcal{G}_c}(\boldsymbol{\mu})$ of the hard-core model on \mathcal{G}_c .

(r2) For independent indicators $\{X_i\}_{i\in[n]}$, if $0 \leq \mathbb{E}[X_i] < 1$ for all $i \in [n]$, then we have the Taylor series of logarithmic function

(8)
$$\log \mathbb{P}(X = 0) = \sum_{i \in [n]} \log (1 - \mathbb{E}[X_i]) = -\sum_{i \in [n]} \sum_{j \ge 1} \frac{1}{j} \mathbb{E}[X_i]^j.$$

The empty graph $\overline{K}_n := ([n], \emptyset)$ is a valid dependency graph for this independent case. Since the polymers of \overline{K}_n are all of size one containing a single vertex, and the clusters of \overline{K}_n are all multisets containing multiple copies of the same vertex, then for independent indicators $\{X_v\}_{v \in [n]}$, expansion in equation (3) becomes

(9)
$$\log \mathbb{P}(X = 0) = \sum_{\gamma \in \Gamma[\overline{K}_n]} \frac{\phi(\gamma)}{|\gamma|!} (-1)^{||\gamma||} \prod_{C \in \gamma} \mu(C) = \sum_{i \in [n]} \sum_{j \ge 1} \frac{1}{j!} \sum_{H \in CSpan(K_j)} (-1)^{e_H} (-1)^j \mathbb{E}[X_i]^j.$$

Comparing (8) and (9), it follows that

(10)
$$\sum_{H \in CSpan(K_n)} (-1)^{e_H} = (-1)^{n-1} (n-1)!,$$

which is well-known, see, for example, [16, Eq. (2.13)] or [20, Eq. (3.37)].

2 Linearity of binomial random hypergraphs

The probability of random hypergraphs being linear is equal to the probability of the non-existence of hyperedge pairs intersecting in more than one vertex. Then we accordingly define a set \mathcal{F} of 'forbidden' hypergraphs, containing all r-graphs $(e_1 \cup e_2, \{e_1, e_2\})$ on vertex set $e_1 \cup e_2$ such that $2 \leq |e_1 \cap e_2| < r$:

(11)
$$\mathcal{F} = \bigcup_{2 \le t \le r-1} \Big\{ (e_1 \cup e_2, \{e_1, e_2\}) : |e_1| = |e_2| = r, |e_1 \cap e_2| = t \Big\}.$$

Note that removing isomorphic duplicates from \mathcal{F} does not affect the probability that we are interested in, we thus assume that the r-graphs in \mathcal{F} are pairwise non-isomorphic. We also can assume that no hypergraphs in \mathcal{F} have isolated vertices.

The complete r-graph on n vertices, denoted by K_n^r , is the hypergraph consisting of n vertices and all possible edges of size r, that is, $K_n^r = ([n], \{S \subseteq [n] : |S| = r\})$. For every $F \in \mathcal{F}$, let A^F be the set of all subgraphs of K_n^r that are isomorphic to F. There are $[n]_{v_F}/\operatorname{Aut}(F)$ such subgraphs, where $\operatorname{Aut}(F)$ denotes the number of automorphisms of F. Let $A^{\mathcal{F}} = \bigcup_{F \in \mathcal{F}} A^F$. Then the random variable

$$X = \sum_{F \in A^{\mathcal{F}}} \mathbf{1}_{\{F \subset H_r(n,p)\}}$$

counts the number of all copies of all forbidden r-graphs of \mathcal{F} in $H_r(n,p)$. Hence the probability of random hypergraphs being linear $\mathbb{P}(H_r(n,p) \in \mathcal{L}_r(n))$ equals the probability $\mathbb{P}(X=0)$ such that $H_r(n,p)$ avoids all copies of all r-graphs of \mathcal{F} .

Next we define a dependency graph D on the vertex set $A^{\mathcal{F}}$ for random indicators $\{\mathbf{1}_{\{F\subset H_r(n,p)\}}\}_{F\in A^{\mathcal{F}}}$ with two indicators being dependent if the corresponding forbidden r-graphs share edges,

(12)
$$D = \left(A^{\mathcal{F}}, \left\{ \{F_1, F_2\} \in \binom{A^{\mathcal{F}}}{2} : E(F_1) \cap E(F_2) \neq \emptyset \right\} \right).$$

Using the above dependency graph for random indicators of the forbidden structures, we then can utilize the truncated cluster expansion series to approximate the probability of a binomial random r-uniform hypergraph being linear and to obtain more precise asymptotic probability of linearity. In this setting, a polymer is a set of forbidden subgraphs whose induced subgraph in D is connected. We will use the truncation of cluster expansion series involving clusters with certain restricted number of forbidden subgraphs. For every integer k > 0, denote the k-th term of the cluster expansion and the k-th truncated expansion with disjoint polymers as

$$L_{D,k}^{\emptyset} := \sum_{\gamma \in \Gamma_{\emptyset}(D): \|\gamma\| = k} \frac{\phi(\gamma)}{|\gamma|!} (-1)^{\|\gamma\|} \prod_{C \in \gamma} \mu\left(C\right) \quad \text{ and } \quad T_{D,k}^{\emptyset} := \sum_{i \in [k-1]} L_{D,i}^{\emptyset}.$$

Now we are ready to state our main result utilizing the truncated cluster expansion.

Theorem 2.1. Let $r = r(n) \ge 3$. If $p = o(n^{2-r})$, then for every integer k > 0,

(13)
$$\mathbb{P}\left(H_r(n,p) \in \mathcal{L}_r(n)\right) = \exp\left(T_{D,k}^{\emptyset} + O\left(\Delta_{k+1}(D)\right) + o\left(1\right)\right),$$

where D is the dependency graph for the indicators of forbidden r-graphs defined by (12) and $\Delta_i(D)$ denotes the sum of joint moments over polymers of size i in the graph D,

$$\Delta_i(D) = \sum_{C \in \mathcal{C}(D): |C| = i} \mu(C).$$

Moreover, for any $\varepsilon > 0$, if $p = o(n^{2-r-\varepsilon})$, then there exists an integer $k = k(\varepsilon) > 0$ such that

(14)
$$\mathbb{P}\left(H_r(n,p) \in \mathcal{L}_r(n)\right) = \exp\left(T_{D,k}^{\emptyset} + o(1)\right).$$

Theorem 2.1 gives the more precise asymptotics of the probability of random hypergraphs being linear. We next consider a specific example, by restricting to the 3-uniform hypergraphs case, and computing only the first few terms of the series explicitly for illustration purpose. This extends the asymptotic probability of linearity for $H_3(n, p)$ given by McKay and Tian in (1).

Corollary 2.2. If $p = o(n^{-7/5})$, then

(15)
$$\mathbb{P}\left(H_3(n,p) \in \mathcal{L}_3(n)\right) = \exp\left(-\frac{1}{4}n^4p^2 + \frac{2}{3}n^5p^3 - \frac{55}{24}n^6p^4 + \frac{3}{2}n^3p^2 + o\left(1\right)\right).$$

3 Proofs of main results

The density of a graph G is defined by $d(G) = e_G/v_G$, where v_G and e_G are the numbers of vertices and edges of G respectively. Another commonly used (see, for example, [15, 14]) density measure $m_{\star}(G)$ is defined by

(16)
$$m_{\star}(G) = \min_{H \subseteq G, e_H \geqslant 1} \frac{e_G - e_H}{v_G - v_H}.$$

Next we introduce joint cumulant, which is a fundamental tool in probability theory. Given G-dependent random variables $\{X_v\}_{v\in V(G)}$, for every set of vertices $S\subseteq V(G)$, the joint cumulant of random variables $\{X_i\}_{i\in S}$ is defined by

(17)
$$\kappa(S) = \sum_{\pi \in \Pi(S)} (-1)^{|\pi|-1} (|\pi|-1)! \prod_{P \in \pi} \mu(P),$$

where $\Pi(S)$ denotes the set of all partitions of S. The joint cumulant $\kappa(S)$ can be regarded as a measure of the mutual dependences of the variables in S. An important property of the joint cumulant $\kappa(S)$ is that if S can be partitioned into two subsets S_1 and S_2 such that the variables in S_1 are independent of the variables in S_2 , then $\kappa(S) = 0$. In other words, if $S \notin \mathcal{C}(G)$, then $\kappa(S) = 0$ (see, for example, [18]).

Given a family \mathcal{F} of r-graphs, we consider the probability that $H_r(n, p)$ is \mathcal{F} -free, that is, it simultaneously avoids all copies of all r-graphs in \mathcal{F} . Let i > 0 be an integer, define $\kappa_i(D)$ to be the sum of joint cumulants over polymers of size i in the dependency graph D, namely,

$$\kappa_i(D) = \sum_{C \in \mathcal{C}(D): |C| = i} \kappa(C).$$

Lemma 3.1 ([14, Corollary 12]). Let \mathcal{F} be a finite family of r-graphs and $p = p(n) \in (0,1)$ satisfy

(18)
$$np^{m_{\star}(\mathcal{F})} = o(1) \qquad and \qquad np^{2d(\mathcal{F})} = o(1),$$

where

$$m_{\star}(\mathcal{F}) = \min_{G \in \mathcal{F}} m_{\star}(G)$$
 and $d(\mathcal{F}) = \min_{G \in \mathcal{F}} d(G)$.

Then, for every integer k > 0, we have

(19)
$$\left| \log \mathbb{P} \left(H_r(n, p) \text{ is } \mathcal{F}\text{-}free \right) - \sum_{i \in [k]} (-1)^i \kappa_i(D) \right| = O\left(\Delta_{k+1}(D) \right) + o\left(1 \right).$$

Moreover, if $np^{m_*(\mathcal{F})} = n^{-\varepsilon}$ for some $\varepsilon > 0$, then there exists an integer $k = k(\varepsilon, \mathcal{F}) > 0$ such that $\Delta_{k+1}(D) = o(1)$.

Here we introduce another lemma relating the cluster expansion and cumulant series in equation (19).

Lemma 3.2 (Cumulant-cluster lemma). Let $\{X_v\}_{v\in V(G)}$ be G-dependent random indicators and k>0 be an integer. Then

(20)
$$T_{G,k+1}^{\emptyset} = \sum_{C \in \mathcal{C}_k(G)} (-1)^{|C|} \kappa(C),$$

where $C_k(G) = \{C \in C(G) : |C| \in [k]\}$ denotes the set of polymers with size at most k.

Now we are ready to prove the main result.

Proof of Theorem 2.1. Let \mathcal{F} defined by equation (11) be the set of forbidden r-graphs. Since each r-set of [n] is an edge independently with probability p in $H_r(n,p)$, then for distinct subgraphs $F_1, F_2 \in A^{\mathcal{F}}$, indicators $\mathbf{1}_{\{F_1 \subset H_r(n,p)\}}$ and $\mathbf{1}_{\{F_2 \subset H_r(n,p)\}}$ are dependent if $E(F_1) \cap E(F_2) \neq \emptyset$. Hence graph D defined by equation (12) is a dependency graph for random indicators $\mathbf{1}_{\{F \subset H_r(n,p)\}}$ for $F \in A^{\mathcal{F}}$.

Next we verify conditions in equation (18). Since

$$m_{\star}(\mathcal{F}) = \min_{G \in \mathcal{F}} \min_{H \subseteq G, e_H \geqslant 1} \frac{e_G - e_H}{v_G - v_H} = \frac{1}{\max_{G \in \mathcal{F}} \max_{H \subseteq G, e_H \geqslant 1} (v_G - v_H)} = \frac{1}{r - 2}$$

and

$$d(\mathcal{F}) = \min_{G \in \mathcal{F}} d(G) = \min_{G \in \mathcal{F}} \frac{e_G}{v_G} = \frac{2}{\max_{G \in \mathcal{F}} v_G} = \frac{1}{r - 1},$$

then $2d(\mathcal{F}) \geqslant m_{\star}(\mathcal{F})$ for all $r \geqslant 3$. Thus if $np^{1/(r-2)} = o(1)$, then

(21)
$$\left|\log \mathbb{P}\left(H_r(n,p) \in \mathcal{L}_r(n)\right) - T_{D,k}^{\emptyset}\right| = O\left(\Delta_{k+1}\right) + o\left(1\right).$$

Moreover, if $p = o(n^{-(r-2)-\varepsilon})$ for some $\varepsilon > 0$, then combining with Lemma 3.2, we complete the proof.

What remains is to show Lemma 3.2. We introduce an auxiliary lemma for its proof.

Lemma 3.3. For all connected graph H, we have

(22)
$$\sum_{\pi \in \Pi(V(H))} \sum_{G \in CSpan(K_{|\pi|})} (-1)^{e_G} \prod_{P \in \pi} \mathbf{1}_{\{P \in \mathcal{I}(H)\}} = \sum_{G \in CSpan(H)} (-1)^{e_G}.$$

We first introduce the chromatic polynomial. Given a graph H and a positive integer λ , a (proper) λ -colouring of H is a map $\Phi: V(H) \to [\lambda]$ such that $\Phi(u) \neq \Phi(v)$ for all $\{u, v\} \in E(H)$. The *chromatic polynomial* $P_H(\lambda)$ of H is the number of λ -colourings of H.

Given a graph H and a positive integer k, a partition containing k subsets $\{V_1, \ldots, V_k\}$ of V(H) is called a k-independent partition of H if for every $i \in [k]$, we have $V_i \neq \emptyset$ and $V_i \in \mathcal{I}(H)$. Let $\alpha(H, k)$ count the k-independent partition of H. Then we have the chromatic polynomial in factorial form

(23)
$$P_H(\lambda) = \sum_{k=1}^{v_H} \alpha(H, k) [\lambda]_k,$$

(see, for example, [2, Theorem 1.4.1]). An equivalent formula for $P_H(\lambda)$ written as a polynomial in λ , known as the Whitney-Tutte-Fortuin-Kasteleyn representation (see, for example, [12, Eq. (A.11)] or [17, Eq. (1.2)]) is

(24)
$$P_H(\lambda) = \sum_{E \subseteq E(H)} (-1)^{|E|} \lambda^{c(E)},$$

where c(E) = c(V(H), E) counts the number of the connected components of subgraph (V(H), E) for every edge set $E \subseteq E(H)$.

Proof of Lemma 3.3. By inspecting equations (24) and (4), one observes that the Ursell function is the linear term of the chromatic polynomial (this is also a well-known fact, see, for exmaple, [1]). Then we have the right hand side of equation (22)

$$(25) \qquad \sum_{G \in \mathrm{CSpan}(H)} (-1)^{e_G} = \frac{\mathrm{d}P_H(\lambda)}{\mathrm{d}\lambda} \Big|_{\lambda=0} = \frac{\mathrm{d}}{\mathrm{d}\lambda} \left(\sum_{k=1}^{v_H} \alpha(H,k)[\lambda]_k \right) \Big|_{\lambda=0} = \sum_{k=1}^{v_H} \alpha(H,k)(-1)^{k-1}(k-1)!.$$

Using the combinatorial identity obtained before in (10), the left hand side of equation (22) can be rewritten as

(26)
$$\sum_{\pi \in \Pi(V(H))} \sum_{G \in \text{CSpan}(K_{|\pi|})} (-1)^{e_G} \prod_{P \in \pi} \mathbf{1}_{\{P \in \mathcal{I}(H)\}} = \sum_{\pi \in \Pi(V(H))} (-1)^{|\pi|-1} (|\pi|-1)! \prod_{P \in \pi} \mathbf{1}_{\{P \in \mathcal{I}(H)\}}.$$

Notice that the right hand side of equation (26) is a sum of $|\pi|$ -independent partition for any $\pi \in \Pi(V(H))$. Thus we have

$$\sum_{\pi \in \Pi(V(H))} (-1)^{|\pi|-1} (|\pi|-1)! \prod_{P \in \pi} \mathbf{1}_{\{P \in \mathcal{I}(H)\}} = \sum_{k=1}^{v_H} \sum_{\pi \in \Pi(V(H)): |\pi|=k} (-1)^{k-1} (k-1)! \prod_{P \in \pi} \mathbf{1}_{\{P \in \mathcal{I}(H)\}}$$

$$= \sum_{k=1}^{v_H} \alpha(H,k) (-1)^{k-1} (k-1)!.$$
(27)

Then combining (25) and (27), we complete the proof.

Proof of Lemma 3.2. From the cluster expansion, we have

$$T_{G,k+1}^{\emptyset} = \sum_{i \in [k]} L_{G,i}^{\emptyset} = \sum_{\gamma \in \Gamma_{\emptyset}(G): \|\gamma\| \in [k]} \frac{\phi(\gamma)}{|\gamma|!} \prod_{C \in \gamma} (-1)^{|C|} \mu(C)$$

$$= \sum_{\substack{(C_1, \dots, C_n) \in \Gamma_{\emptyset}(G) \\ \sum_{i \in [n]} |C_i| \in [k]}} \frac{1}{n!} \sum_{H \in \mathrm{CSpan}(\mathbb{G}(C_1, \dots, C_n))} (-1)^{e_H} \prod_{i \in [n]} (-1)^{|C_i|} \mu(C_i)$$

$$= \sum_{\substack{\{C_1, \dots, C_n\} \in \Gamma_{\emptyset}(G) \\ \sum_{i \in [n]} |C_i| \in [k]}} \sum_{H \in \mathrm{CSpan}(\mathbb{G}(C_1, \dots, C_n))} (-1)^{e_H} \prod_{i \in [n]} (-1)^{|C_i|} \mu(C_i),$$

$$\sum_{i \in [n]} |C_i| \in [k]} (28)$$

where the first summation in the last line is an abuse of notation, and denotes the summation over (unordered) sets of polymers. From the definition of joint cumulants, we get

$$\begin{split} \sum_{C \in \mathcal{C}_k(G)} (-1)^{|C|} \kappa(C) &= \sum_{C \in \mathcal{C}_k(G)} (-1)^{|C|} \sum_{\pi \in \Pi(C)} (-1)^{|\pi|-1} (|\pi|-1)! \prod_{P \in \pi} \mu\left(P\right) \\ &= \sum_{C \in \mathcal{C}_k(G)} \sum_{\pi \in \Pi(C)} (-1)^{|\pi|-1} (|\pi|-1)! \prod_{P \in \pi} (-1)^{|P|} \mu\left(P\right). \end{split}$$

Combining with identity in (10), it follows that

$$\sum_{C \in \mathcal{C}_k(G)} (-1)^{|C|} \kappa(C) = \sum_{C \in \mathcal{C}_k(G)} \sum_{\pi \in \Pi(C)} \sum_{H \in \operatorname{CSpan}(K_{|\pi|})} (-1)^{e_H} \prod_{P \in \pi} (-1)^{|P|} \mu(P).$$

Fix an arbitrary polymer $C \in \mathcal{C}_k(G)$, for every partition $\pi = \{P_1, \dots, P_m\} \in \Pi(C)$, by factorising into pairwise non-adjacent maximal connected subgraphs, with each induced by C_1, \dots, C_n respectively, there exists a unique finest partition consisting of only polymers $\pi' = \pi'(\pi) = \{C_1, \dots, C_n\} \in \Pi(C)$ such that

- $(p1) n \geqslant m,$
- (p2) for all $i \in [n]$, we have $C_i \in \mathcal{C}(G)$,
- (p3) $(C_1, \ldots, C_n) \in \Gamma_{\emptyset}(G)$, and
- $(p4) \prod_{P \in \pi} \mu(P) = \prod_{C \in \pi'} \mu(C).$

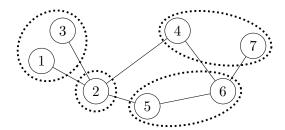


FIGURE 1. A polymer of size seven with a partition $\{\{1,3\},\{2\},\{4,7\},\{5,6\}\}\}$ and the corresponding polymer partition $\{\{1\},\{3\},\{2\},\{4\},\{7\},\{5,6\}\}\}$ such that $\mu(1,3) \mu(2) \mu(4,7) \mu(5,6) = \mu(1) \mu(2) \mu(3) \mu(4) \mu(5,6) \mu(7)$.

Then, we have

$$\sum_{C \in \mathcal{C}_k(G)} (-1)^{|C|} \kappa(C) = \sum_{C \in \mathcal{C}_k(G)} \sum_{\pi \in \Pi(C)} \sum_{H \in \text{CSpan}(K_{|\pi|})} (-1)^{e_H} \prod_{C \in \pi'(\pi)} (-1)^{|C|} \mu(C)$$

$$= \sum_{C \in \mathcal{C}_k(G)} \sum_{\substack{\pi \in \Pi(C) \\ \pi'(\pi) = \{C_1, \dots, C_n\}}} \sum_{H \in \text{CSpan}(K_{|\pi|})} (-1)^{e_H} \prod_{i \in [n]} (-1)^{|C_i|} \mu(C_i).$$

Since $\Gamma_{\emptyset}(G)$ is the set of all clusters of G with pairwise disjoint polymers, we then rearrange the partitions according to their corresponding polymer partitions and have that

$$\{\pi \in \Pi(C) : C \in \mathcal{C}_k(G)\} = \left\{\pi \in \Pi' : \{C_1, \dots, C_n\} \in \Gamma_{\emptyset}(G) : \sum_{i \in [n]} |C_i| \in [k]\right\},\,$$

where

$$\Pi' := \{ \pi \in \Pi(\cup_{i \in [n]} C_i) : \pi'(\pi) = (C_1, \dots, C_n) \}$$

denotes the set of partitions of $\bigcup_{i\in[n]}C_i$ for a given set of polymers $\{C_1,\ldots,C_n\}\in\Gamma_\emptyset(G)$. Hence

(29)
$$\sum_{C \in \mathcal{C}_{k}(G)} (-1)^{|C|} \kappa(C) = \sum_{\substack{\{C_{1}, \dots, C_{n}\} \in \Gamma_{\emptyset}(G) \\ \sum_{i \in [n]} |C_{i}| \in [k]}} \sum_{\pi \in \Pi'} \sum_{H \in \mathrm{CSpan}(K_{|\pi|})} (-1)^{e_{H}} \prod_{i \in [n]} (-1)^{|C_{i}|} \mu(C_{i}).$$

Note that we have

$$\prod_{P \in \pi} (-1)^{|P|} \mu(P) = \prod_{i \in [n]} (-1)^{|C_i|} \mu(C_i)$$

if and only if every element of the partition $\pi \in \Pi'$ is an independent set of $\mathbb{G}(C_1, \ldots, C_n)$. Then by comparing equations (28) and (29), it suffices to show that for all $(C_1, \ldots, C_n) \in \Gamma_{\emptyset}(G)$,

$$\sum_{\pi \in \Pi'} \sum_{G \in \operatorname{CSpan}(K_{|\pi|})} (-1)^{e_G} \prod_{P \in \pi} \mathbf{1}_{\{P \in \mathcal{I}(\mathbb{G}(C_1, \dots, C_n))\}} = \sum_{H \in \operatorname{CSpan}(\mathbb{G}(C_1, \dots, C_n))} (-1)^{e_H},$$

which follows from Lemma 3.3.

4 Computations for $H_3(n,p)$

The goal of this section is to compute the terms in Theorem 2.1 explicitly to prove Corollary 2.2.

Proof of Corollary 2.2. For 3-uniform hypergraphs, the forbidden hypergraph is on four vertices with a pair of 3-sets sharing two vertices, we call it a link. Then for random indicators of links, we construct the dependency graph D following equation (12), such that two links are adjacent if and only if they share one hyperedge. A polymer $C \in \mathcal{C}(D)$ of size k is a set of links $\{F_1, \ldots, F_k\}$ whose induced subgraph in D is connected.

We first enumerate all contributing non-isomorphic types of clusters, and compute value $\phi(\gamma)(-1)^{\|\gamma\|}$ $\prod_{C\in\gamma}\mu(C)/|\gamma|!$ for each cluster type γ . Then we multiply each value with the size of the respective isomorphism class. More precisely, noting a cluster is a set of link sets, an isomorphism between two clusters γ_1, γ_2 is a bijection between their vertices (the union of vertices in all links): $\bigcup_{C\in\gamma_1}\bigcup_{F\in C}V(F)\to \bigcup_{C\in\gamma_2}\bigcup_{F\in C}V(F)$, which induces a bijection from the hyperedges of γ_1 to the hyperedges of γ_2 , and a bijection from the polymers of γ_1 to the polymers of γ_2 . An automorphism of a cluster is an isomorphism to itself. For each cluster $\gamma\in\Gamma(D)$, we consider the distinct copies of γ in the complete r-graph on r vertices by choosing all the vertices in $\bigcup_{C\in\gamma_1}\bigcup_{F\in C}V(F)$ from [n] (ordered selections without repetition), and every element of $\Gamma(D)$ isomorphic to γ is counted once for every automorphism of γ .

Now, we compute the terms in (14) for $p = o(n^{-7/5})$ explicitly.

(c1) Clusters γ such that $||\gamma|| = 1$.

There is only one cluster type, a single forbidden link, namely, a hypergraph with two hyperedges intersecting in two vertices.

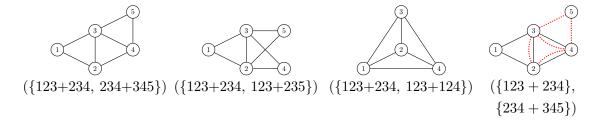


Thus, we have that

$$L_{D,1}^{\emptyset} = -\sum_{C \in \mathcal{C}(D): |C| = 1} \mu(C) = -\frac{[n]_4 p^2}{4} = -\frac{1}{4} n^4 p^2 + \frac{3}{2} n^3 p^2 + o(1).$$

(c2) Clusters γ such that $||\gamma|| = 2$.

There are two cluster types: one polymer of size two, and two polymers of size one, namely, one polymer consisting of two edge-sharing forbidden links, or two edge-sharing polymers with each being a single forbidden link. Also note that for any two (not necessarily distinct) polymers $(C_i, C_j) \in \mathcal{C}(D)^2$ such that $C_i \sim C_j$, we have that $\mathbb{G}(C_i, C_j) = K_2$, thus $\phi(C_i, C_j) = -1$.

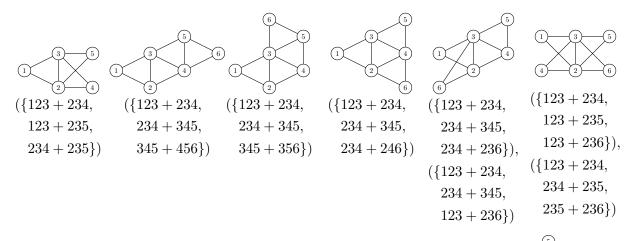


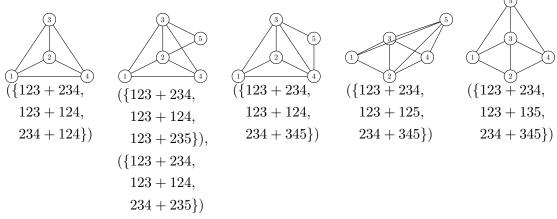
Therefore, we get

$$\begin{split} L_{D,2}^{\emptyset} &= \sum_{\gamma \in \Gamma_{\emptyset}(D): \|\gamma\| = 2} \frac{\phi(\gamma)}{|\gamma|!} (-1)^{\|\gamma\|} \prod_{C \in \gamma} \mu\left(C\right) \\ &= \sum_{C \in \mathcal{C}(D): |C| = 2} (-1)^{|C|} \mu\left(C\right) + \sum_{\substack{(C_1, C_2) \in \Gamma_{\emptyset}(D) \\ |C_1| = |C_2| = 1}} -\frac{1}{2} (-1)^2 \mu\left(C_1\right) \mu\left(C_2\right) \\ &= \frac{[n]_5 p^3}{2} + \frac{[n]_5 p^3}{4} + \frac{[n]_4 p^3}{2} - \frac{[n]_5 p^4}{4} = \frac{3}{4} n^5 p^3 + o\left(1\right). \end{split}$$

(c3) Clusters γ such that $\|\gamma\| = 3$.

We only focus on one cluster type: one polymer of size three, namely, one polymer consisting of three edge-sharing forbidden links, since if the cluster is formed by more then one polymer, then it must be extended from clusters γ such that $\|\gamma\| = 2$ and more than one polymer, which are already asymptotically negligible.





Hence, we have that

$$\begin{split} L_{D,3}^{\emptyset} &= \sum_{\gamma \in \Gamma_{\emptyset}(D): \|\gamma\| = 3} \frac{\phi(\gamma)}{|\gamma|!} (-1)^{\|\gamma\|} \prod_{C \in \gamma} \mu\left(C\right) \\ &= \sum_{C \in \mathcal{C}(D): |C| = 3} (-1)^{|C|} \mu\left(C\right) + O\left(n^4 p^3\right) + O\left(n^5 p^4\right) \\ &= -\frac{[n]_5 p^3}{2 \times 3!} - \frac{[n]_6 p^4}{2} - \frac{[n]_6 p^4}{2} - \frac{[n]_6 p^4}{3!} - \frac{[n]_6 p^4}{2} - [n]_6 p^4 - \frac{[n]_6 p^4}{2 \times 3!} - \frac{[n]_6 p^4}{2 \times 2} + o(1), \end{split}$$

where the last row of the types of polymers are of contribution $O(n^4p^3) = o(1)$.

(c4) Clusters γ such that $||\gamma|| = 4$.

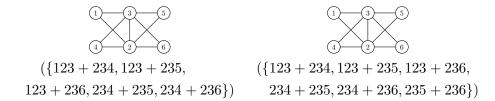
As before, we only focus on one cluster type: one polymer of size four, since clusters with more than one polymer contribute negligibly.

$$(\{123+234,123+235,123+236,234+235\}), \quad (\{123+234,123+235,\\(\{123+234,234+235,235+236,123+236\}) \quad 234+235,123+136\})$$

We then have

$$L_{D,4}^{\emptyset} = \sum_{\gamma \in \Gamma_{\emptyset}(D): \|\gamma\| = 4} \frac{\phi(\gamma)}{|\gamma|!} (-1)^{\|\gamma\|} \prod_{C \in \gamma} \mu\left(C\right) = \frac{[n]_6 p^4}{2 \times 2} + \frac{[n]_6 p^4}{2 \times 8} + \frac{[n]_6 p^4}{2} + o(1).$$

(c5) Clusters γ such that $\|\gamma\| \in \{5, 6\}$.



Then we have

$$L_{D,5}^{\emptyset} + L_{D,6}^{\emptyset} = \sum_{\gamma \in \Gamma_{\emptyset}(D): \|\gamma\| \in \{5.6\}} \frac{\phi(\gamma)}{|\gamma|!} (-1)^{\|\gamma\|} \prod_{C \in \gamma} \mu(C) = -\frac{[n]_6 p^4}{2 \times 2 \times 2} + \frac{[n]_6 p^4}{2 \times 4!} + o(1).$$

Since there is a finite number of types of polymers with size seven, we thus have $\Delta_7(D) = o(1)$. Hence, we ignore the remaining terms by equation (13). Adding up the contributing terms for $p = o(n^{-7/5})$ gives the asymptotic probability of $H_3(n,p)$ being linear in Corollary 2.2.

5 Concluding remarks

We have shown that the truncation of the cluster expansion series gives the asymptotic linearity of binomial random hypergraphs. The analysis of the truncation utilised results in [14], which exploit the correlation among random variables and rely heavily on FKG inequality. It would be interesting to investigate whether this is necessary for the truncation. Alternative ways of handling truncation that are commonly used include establishing the absolute convergence of the series via the Koteckỳ-Preiss criterion [11], for example, see [5, 10].

It also would be interesting to investigate whether the cluster expansion series also gives the linearity of random hypergraphs with given number of edges. However, in that case, all graph-dependent indicators are dependent, and the only valid dependency graph for them is the complete graph. In this case, we may need to modify the method by incorporating the notion of weak dependence, see, for example, [6].

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