# A semi-algorithm to find elementary first order invariants of rational second order ordinary differential equations 

${ }^{a, b}$ J. Avellar, ${ }^{a}$ L.G.S. Duarte, ${ }^{c}$ S.E.S. Duarte and ${ }^{a}$ L.A.C.P. da Mota ${ }^{1}$<br>${ }^{a}$ Universidade do Estado do Rio de Janeiro, Instituto de Física, Depto. de Física Teórica, 20559-900 Rio de Janeiro - RJ, Brazil<br>${ }^{b}$ Fundação de Apoio à Escola Técnica, E.T.E. Juscelino Kubitschek, 21311-280 Rio de Janeiro - RJ, Brazil<br>${ }^{\text {c }}$ Fundação de Apoio à Escola Técnica, E.T.E. Visconde de Mauá, 20537-200 Rio de Janeiro - RJ, Brazil


#### Abstract

Here we present a method to find elementary first integrals of rational second order ordinary differential equations (SOODEs) based on a Darboux type procedure [16, 12, 13]. Apart from practical computational considerations, the method will be capable of telling us (up to a certain polynomial degree) if the SOODE has an elementary first integral and, in positive case, finds it via quadratures.


Keyword: Elementary first integrals, semi-algorithm, Darboux, Lie symmetry
PACS: 02.30.Hq

[^0]
## 1 Introduction

The differential equations (DEs) are the most widespread way to formulate the evolution of any given system in many scientific areas. Therefore, for the last three centuries, much effort has been made in trying to solve them.

Broadly speaking, we may divide the approaches to solving ODEs in the ones that classify the ODE and the ones that do not (classificatory and non-classificatory methods). Up to the end of the nineteenth century, we only had many (unconnected) classificatory methods to try to deal with the solving of ODEs. Sophus Lie then introduced his method $[1,2,3]$ that was meant to be general and try to solve any ODE, i.e., non-classificatory. Despite this appeal, the Lie approach had a shortcoming: namely, in order to deal with the ODE, one has to know the symmetries of the given ODE. Unfortunately, this part of the procedure was not algorithmic (mind you that the classificatory approach is algorithmic by nature). So, for many decades, the Lie method was not put to much "practical" use since to "guess" the symmetries was considered to be as hard as guessing the solution to the ODE itself. In $[4,5]$, an attempt was made to make this searching for the symmetries to the ODE practical and, consequently, make the Lie method more used.

Even thought the attempt mentioned above was very successful, the procedures applied to find the symmetries were heuristic. So, a non- classificatory algorithmic approach was still missing. The first algorithmic approach applicable to solving first order ordinary differential equations (FOODEs) was made by M. Prelle and M. Singer [6]. The attractiveness of the PS method lies not only in its basis on a totally different theoretical point of view but, also in the fact that, if the given FOODE has a solution in terms of elementary functions, the method guarantees that this solution will be found (though, in principle it can admittedly take an infinite amount of time to do so). The original PS method was built around a system of two autonomous FOODEs of the form $\dot{x}=P(x, y), \dot{y}=\mathcal{P}(x, y)$ with $P$ and $\mathcal{P}$ in $C[x, y]$ or, equivalently, the form $y^{\prime}=R(x, y)$, with $R(x, y)$ a rational function of its arguments.

The PS approach has its limitations, for instance, it deals only with rational FOODEs. But, since it is so powerful in many respects, it has generated many extensions $[7,8,9,10,11,12,13,14]$

Nevertheless, all these extensions deal only with FOODEs. In particular, the second order ordinary differential equations (SOODEs) play a very important role, for instance, in the physical sciences. So, with this in mind, we have produced [17] a PS-type approach to deal with SOODEs. This approach dealt with SOODEs that presented elementary ${ }^{2}$ solutions (with two elementary first order invariants).

Here, we present a different approach that, besides dealing with a much broader class of SOODEs (those with at least one elementary first order invariant), does not depend on a conjecture about the general structure of the first order invariants.

In section 2, we present the state of the art up to the present paper. In the following section, we introduce some important theoretical results for the building of the algorithm to find the integrating factor. In section 4, we present the algorithm for finding the integrating factor with examples of its application. Finally, we present our conclusions and point out some directions to further our work.

[^1]
## 2 Earlier Results

In the paper [6], one can find an important result that, translated to the case of SOODEs of the form

$$
\begin{equation*}
y^{\prime \prime}=\frac{M\left(x, y, y^{\prime}\right)}{N\left(x, y, y^{\prime}\right)}=\phi\left(x, y, y^{\prime}\right) \tag{1}
\end{equation*}
$$

where $M$ and $N$ are polynomials in $\left(x, y, y^{\prime}\right)$, can be stated as:
Theorem 1: If the SOODE (1) has a first order invariant that can be written in terms of elementary functions, then it has one of the form:

$$
\begin{equation*}
I=w_{0}+\sum_{i}^{m} c_{i} \ln \left(w_{i}\right), \tag{2}
\end{equation*}
$$

where $m$ is an integer and the $w^{\prime}$ s are algebraic functions ${ }^{3}$ of $\left(x, y, y^{\prime}\right)$.
The integrating factor for a SOODE of the form (1) is defined by:

$$
\begin{equation*}
R\left(\phi-y^{\prime \prime}\right)=\frac{d I\left(x, y, y^{\prime}\right)}{d x} \tag{3}
\end{equation*}
$$

where $\frac{d}{d x}$ represents the total derivative with respect to $x$.
Bellow we will present some results and definitions (previously presented on [17]) that we will need. First let us remember that, on the solutions, $d I=I_{x} d x+I_{y} d y+$ $I_{y^{\prime}} d y^{\prime}=0$. So, from equation (3), we have:

$$
\begin{equation*}
R\left(\phi d x-d y^{\prime}\right)=I_{x} d x+I_{y} d y+I_{y^{\prime}} d y^{\prime}=d I=0 . \tag{4}
\end{equation*}
$$

Since $y^{\prime} d x=d y$, we have

$$
\begin{equation*}
R\left[\left(\phi+S y^{\prime}\right) d x-S d y-d y^{\prime}\right]=d I=0, \tag{5}
\end{equation*}
$$

adding the null term $S y^{\prime} d x-S d y$, where $S$ is a function of $\left(x, y, y^{\prime}\right)$. From equation (5), we have:

$$
\begin{align*}
I_{x} & =R\left(\phi+S y^{\prime}\right) \\
I_{y} & =-R S  \tag{6}\\
I_{y^{\prime}} & =-R,
\end{align*}
$$

that must satisfy the compatibility conditions. Thus, defining the differential operator $D$ :

$$
\begin{equation*}
D \equiv \partial_{x}+y^{\prime} \partial_{y}+\phi \partial_{y^{\prime}} \tag{7}
\end{equation*}
$$

after a little algebra, that can be shown to be equivalent to:

$$
\begin{align*}
D[R] & =-R\left(S+\phi_{y^{\prime}}\right),  \tag{8}\\
D[R S] & =-R \phi_{y} . \tag{9}
\end{align*}
$$

[^2]
## 3 New theoretical results concerning the function $S$

Let us start this section by stating a corollary to theorem 1 concerning $S$ and $R$.
Corollary 1: If a SOODE of the form (1) has a first order elementary invariant then the integrating factor $R$ for such an SOODE and the function $S$ defined in the previous section can be written as algebraic functions of $\left(x, y, y^{\prime}\right)$.

Proof: Using the above mentioned result by Prelle and Singer, there is always a first order invariant $I=w_{0}+\sum_{i}^{m} c_{i} \ln \left(w_{i}\right)$ for the SOODE. So we have, using equation (3),

$$
\begin{equation*}
R\left(\frac{M}{N}-y^{\prime \prime}\right)=I_{x}+y^{\prime} I_{y}+y^{\prime \prime} I_{y^{\prime}} \Rightarrow R=-I_{y^{\prime}} \tag{10}
\end{equation*}
$$

where $I_{u} \equiv \partial_{u} I$. From equation (2), we have:

$$
\begin{equation*}
I_{y^{\prime}}=w_{0 y^{\prime}}+\sum_{i}^{m} c_{i} \frac{w_{i y^{\prime}}}{w_{i}} . \tag{11}
\end{equation*}
$$

Then $I_{y^{\prime}}$ is an algebraic function of $\left(x, y, y^{\prime}\right)$ and, by equation (10), so is $R$.
From equations (6), one can see that:

$$
\begin{equation*}
S=\frac{I_{y}}{I_{y^{\prime}}}=\frac{w_{0 y}+\sum_{i}^{m} c_{i} \frac{w_{i y}}{w_{i}}}{w_{0 y^{\prime}}+\sum_{i}^{m} c_{i} \frac{w_{i y^{\prime}}}{w_{i}}} . \tag{12}
\end{equation*}
$$

Therefore, $S$ is also an algebraic function of $\left(x, y, y^{\prime}\right)$.

Besides that, working on equations (8) and (9), we get[17]:

$$
\begin{equation*}
D[S]=S^{2}+\phi_{y^{\prime}} S-\phi_{y}=\frac{\mathcal{M}}{\mathcal{N}} \tag{13}
\end{equation*}
$$

where $\mathcal{M}$ and $\mathcal{N}$ are given by

$$
\begin{align*}
\mathcal{M} & \equiv(N S)^{2}+\left(N M_{y^{\prime}}-M N_{y^{\prime}}\right) S-\left(N M_{y}-M N_{y}\right),  \tag{14}\\
\mathcal{N} & \equiv N^{2} \tag{15}
\end{align*}
$$

Concerning eq.(13) we can demonstrate the following theorem:
Theorem 2: Consider the operator defined by $D_{S} \equiv \mathcal{M} \partial_{S}+\mathcal{N} D$. If $P$ is an eigenpolynomial of $D_{S}$ (i.e., $D_{S}[P]=\lambda P$, where $\lambda$ is a polynomial) that contains $S$, then $P=0$ defines a particular solution of eq.(13). Conversely, If $P$ is a polynomial that contains $S$, such that $P=0$ defines a particular solution of eq.(13), then $P$ is either an eigenpolynomial of $D_{S}$ or $P$ is an absolute invariant of the Lie transformation group defined by $D_{S}$.

Proof: In order to demonstrate theorem 2 we will, first, prove the following lema:

Lema 1: If eq.(13) $(D[S]=\mathcal{M} / \mathcal{N})$ has an algebraic solution defined by $\sum_{i} a_{i} S^{i}=$ 0 , where the $a_{i}$ are polynomials in $\left(x, y, y^{\prime}\right)$, then

$$
\begin{equation*}
\sum_{i}\left(\mathcal{N} D\left[a_{i}\right] S^{i}+\mathcal{M} a_{i} i S^{i-1}\right)=0 \tag{16}
\end{equation*}
$$

Conversely, if $\sum_{i}\left(\mathcal{N} D\left[a_{i}\right] S^{i}+\mathcal{M} a_{i} i S^{i-1}\right)=0$ then $\sum_{i} a_{i} S^{i}=0$ defines an algebraic function $(S)$ as a particular solution of eq.(13).

Proof of Lema 1: We begin by proving the first part of the Lema. Let $\sum_{i} a_{i} S^{i}=0$ define an algebraic particular solution of eq.(13). Applying the operator $D$ on $\sum_{i} a_{i} S^{i}=0$, one gets:

$$
\sum_{i}\left(D\left[a_{i}\right] S^{i}+a_{i} i S^{i-1} D[S]\right)=0 \Rightarrow \sum_{i}\left(D\left[a_{i}\right] S^{i}+a_{i} i S^{i-1} \frac{\mathcal{M}}{\mathcal{N}}\right)=0
$$

Multiplying this by $\mathcal{N}$ we get eq.(16).
Let us now prove the converse. Consider that eq.(16) applies. Multiplying it by $\mathcal{N}$ and remembering that $S$ obeys $D[S]=\mathcal{M} / \mathcal{N}$, one gets:

$$
\begin{equation*}
\sum_{i}\left(D\left[a_{i}\right] S^{i}+a_{i} i S^{i-1} \frac{\mathcal{M}}{\mathcal{N}}\right)=0 \Rightarrow-\sum_{i}\left(D\left[a_{i}\right] S^{i}\right) / \sum_{i}\left(a_{i} i S^{i-1}\right)=\frac{\mathcal{M}}{\mathcal{N}} \tag{17}
\end{equation*}
$$

However, note that, if we have an algebraic function $(B)$ defined by $\sum_{i} b_{i} B^{i}=0$, then $D[B]$ can be found as follows:

$$
\begin{gathered}
D\left[\sum_{i} b_{i} B^{i}\right]=0 \Rightarrow \sum_{i}\left(D\left[b_{i}\right] B^{i}+b_{i} i B^{i-1} D[B]\right)=0 \Rightarrow \\
D[B]=-\sum_{i}\left(D\left[b_{i}\right] B^{i}\right) / \sum_{i}\left(b_{i} i B^{i-1}\right)
\end{gathered}
$$

Therefore, one can see that eq.(17) can be put in the form:

$$
\begin{equation*}
D[S]=\frac{\mathcal{M}}{\mathcal{N}} \tag{18}
\end{equation*}
$$

where $S$ is an algebraic function defined by $\sum_{i} a_{i} S^{i}=0$.

Now, using Lema 1, we are going to demonstrate Theorem 2.
Consider that $P$ is an eigenpolynomial of $D_{S}$ that contains $S$. By definition, $D_{S}[P]=\lambda P$, where $\lambda$ is a polynomial. So, writing $P=\sum_{i} a_{i} S^{i}$, where the $a_{i}$ 's are polynomials in $\left(x, y, y^{\prime}\right)$, we have:

$$
\begin{equation*}
\sum_{i}\left(\mathcal{N} D\left[a_{i}\right] S^{i}+\mathcal{M} a_{i} i S^{i-1}\right)=\lambda P \tag{19}
\end{equation*}
$$

Over the algebraic function defined by $P=0$, we get $\sum_{i}\left(\mathcal{N} D\left[a_{i}\right] S^{i}+\mathcal{M} a_{i} i S^{i-1}\right)=$ 0. This, using Lema 1 , implies that $\sum_{i} a_{i} S^{i}=0$ defines a particular solution to eq.(13).

Conversely, consider that $P$ is a polynomial that contains $S$, such that $P=$ $\sum_{i} a_{i} S^{i}=0$ defines a particular solution of eq.(13). Again, via Lema 1, we have that:

$$
\begin{equation*}
\sum_{i}\left(\mathcal{N} D\left[a_{i}\right] S^{i}+\mathcal{M} a_{i} i S^{i-1}\right)=0 \tag{20}
\end{equation*}
$$

Consider now the following operator:

$$
\begin{equation*}
\mathcal{O}=N^{2}\left(x_{1}, x_{2}, x_{3}\right)\left(\partial_{x_{1}}+x_{3} \partial_{x_{2}}+\frac{M\left(x_{1}, x_{2}, x_{3}\right)}{N\left(x_{1}, x_{2}, x_{3}\right)} \partial_{x_{3}}\right)+\mathcal{M}\left(x_{1}, x_{2}, x_{3}, x_{4}\right) \partial_{x_{4}} \tag{21}
\end{equation*}
$$

Applying O to a generic polynomial $\mathcal{P}=\sum_{i} c_{i} x_{4}^{i}$, where the $c_{i}$ 's are polynomials in ( $x 1, x 2, x 3$ ), one obtains:

$$
\begin{equation*}
\mathcal{O}[\mathcal{P}]=\sum_{i}\left(N^{2}\left(x_{1}, x_{2}, x_{3}\right) \mathcal{D}\left[c_{i}\right] x_{4}^{i}+\mathcal{M}\left(x_{1}, x_{2}, x_{3}, x_{4}\right) c_{i} i x_{4}^{i-1}\right) \tag{22}
\end{equation*}
$$

where $\mathcal{D} \equiv\left(\partial_{x_{1}}+x_{3} \partial_{x_{2}}+\frac{M\left(x_{1}, x_{2}, x_{3}\right)}{N\left(x_{1}, x_{2}, x_{3}\right)} \partial_{x_{3}}\right)$.
Since the terms multiplying the partial derivatives are polynomials, applying $\mathcal{O}$ to a polynomial will generate a polynomial. So, from eq.(22), we have:

$$
\begin{equation*}
\sum_{i}\left(N^{2}\left(x_{1}, x_{2}, x_{3}\right) \mathcal{D}\left[c_{i}\right] x_{4}^{i}+\mathcal{M}\left(x_{1}, x_{2}, x_{3}, x_{4}\right) c_{i} i x_{4}^{i-1}\right)=Q \tag{23}
\end{equation*}
$$

where $Q$ is a polynomial in $\left(x_{1}, x_{2}, x_{3}, x_{4}\right)$.
Note that the left-hand side of equations (23) and (20) are formally equivalent. Consider that the hypothesis of the theorem 2 apply, i.e., $\mathcal{P}=\sum_{i} c_{i} x_{4}^{i}=0$ defines a algebraic function $x_{4}\left(x_{1}, x_{2}, x_{3}\right)$ that is a particular solution of $D\left[x_{4}\right]=$ $\mathcal{M}\left(x_{1}, x_{2}, x_{3}, x_{4}\right) / \mathcal{N}\left(x_{1}, x_{2}, x_{3}\right)$. Lema 1 implies then that, if we substitute the function $x_{4}=\operatorname{Root} O f\left(\sum_{i} c_{i} x_{4}^{i}\right)$ into eq.(23), we get $0=Q$. Therefore, $\mathcal{P}=0 \Rightarrow Q=$ 0 . Since both $\mathcal{P}$ and $Q$ are polynomials we have two possibilities:

- $Q$ is identically null - in that case, $\mathcal{P}$ is an absolute invariant of the Lie transformation group defined by the operator $\mathcal{O}$.
- $\mathcal{P}$ is a factor of $Q$ - in that case, $Q=\lambda \mathcal{P}$, where $\lambda$ is a polynomial in $\left(x_{1}, x_{2}, x_{3}, x_{4}\right)$ and $Q$ is a relative invariant of the Lie transformation group defined by the operator $\mathcal{O}$.

This completes the proof of theorem $2 . \square$

From the results above we may finally conclude:
Corollary 2: If the SOODE (1) has an elementary first integral, then there is a polynomial $P$ (containing $S$ ) that is either an eigenpolynomial of $D_{S}$ or is an absolute invariant of the Lie transformation group defined by $D_{S}$.

Proof: From corollary 1, since eq.(1) has a first order elementary invariant, there is an algebraic function $S$ that satisfies eq.(13). So, by definition, there is a polynomial $P$ (that contains $S$ ) such that $P=0$ defines $S$. From theorem 2, this
implies that the polynomial $P$ is either an eigenpolynomial of $D_{S}$ or is an absolute invariant of the Lie transformation group defined by $D_{S}$.

These theoretical results provide us an algorithm to find $S$. Briefly, in words, what we have to do is the following:

- Find the eigenpolynomials of the $D_{S}$ operator containing $S$. Let us call them $P_{i}$.
- In order to find $S$, we have to choose one of those $P_{i}$ (let us call it $P$ ) that contains $S$ and solve the equation $P=0$ for $S$.
- If you succeed in doing the above steps, you would have found $S$ for the SOODE you are dealing with.

The importance of these results and the above sketched method is that, as we shall briefly see, the finding of the algebraic function $S$ will allow us to produce a semi-algorithmic method to find the integrating factor (and, consequently, the first order invariant) for SOODEs of the type described in eq.(1).

## 4 Finding the integrating factor and the first integral

In what follows, we are going to demonstrate a result concerning the general structure of $R$, Let us state that result as a theorem.

Theorem 3: Consider a SOODE of the form (1), where $M$ and $N$ are polynomials in $\left(x, y, y^{\prime}\right)$, that presents an elementary first order invariant $I$. Then the integrating factor $R$ for this SOODE can be written as:

$$
\begin{equation*}
R=\prod_{i} f_{i}^{n_{i}} \tag{24}
\end{equation*}
$$

where $f_{i}$ are irreducible polynomials in $\left(x, y, y^{\prime}, S\right)$, which are eigenpolynomials of the operator $\mathcal{D} \equiv N D$ or are factors of $N$ and $n_{i}$ are non-zero rational numbers.

Proof:
Suppose that the hypothesis of the theorem is satisfied. Re-writing equation (4):

$$
\begin{equation*}
\frac{R}{N}\left[\left(M+S N y^{\prime}\right) d x-S N d y-N d y^{\prime}\right]=d I=0 \tag{25}
\end{equation*}
$$

For the sake of simplicity, let us establish some notation: $\bar{M} \equiv\left(M+y^{\prime} N S\right)$, $\bar{N} \equiv-(N S)$ and $\bar{R} \equiv\left(\frac{R}{N}\right)$. We can write $I_{x}=\overline{R M}$ and $I_{y}=-\overline{R N}$ and imposing the compatibility condition $I_{x y}=I_{y x}$, we have $\partial_{y}(\overline{R M})=-\partial_{x}(\overline{R N})$. Expanding this

$$
\begin{equation*}
\bar{R}_{y} \bar{M}+\bar{R}_{x} \bar{N}=-\bar{R}\left(\bar{M}_{y}+\bar{N}_{x}\right) \quad \Rightarrow \quad \frac{\bar{D}[\bar{R}]}{\bar{R}}=-\left(\bar{M}_{y}+\bar{N}_{x}\right) \tag{26}
\end{equation*}
$$

where $\bar{D} \equiv \bar{N} \partial_{x}+\bar{M} \partial_{y}$.
Since $\bar{M}$ and $\bar{N}$ are polynomials in $(x, y, S)^{4}$, equation (26) is formally equivalent to the equation which establish the conditions for the theorem by Prelle and Singer

[^3][6] with the extension by Shtokhamer $[7]^{5}$. More formally, consider the FOODE defined by:
\[

$$
\begin{equation*}
y^{\prime}=\frac{\widetilde{M}}{\widetilde{N}} \tag{27}
\end{equation*}
$$

\]

where $\widetilde{M}$ and $\widetilde{N}$ are the polynomials obtained by replacing $y^{\prime}$ by a constant $k$ in $\bar{M}$ and $\bar{N}$ respectively. Then we can write the solution for the FOODE (27) as $\widetilde{I}=C$, where $\widetilde{I}$ is defined by replacing $y^{\prime}$ (by the constant $k$ ) on $I$. By hypothesis, $I$ is elementary and, consequently, $\widetilde{I}$ is also elementary. Therefore, by the Theorem due to Prelle and Singer [6] and the Shtokhamer's extension [7], the integrating factor $\widetilde{R}$ (defined by replacing $y^{\prime}$ (by the constant $k$ ) on $R$ ) for equation (27) can be written as $\prod_{i} \widetilde{f}_{i}^{n_{i}}$, where $\widetilde{f}_{i}$ are irreducible polynomials in $(x, y, S)$. We could proceed analogously for the other two possible pairings of variables (i.e., $\left(x, y^{\prime}\right)$ and $\left.\left(y, y^{\prime}\right)\right)$. More explicitly, in the reasoning leading to equation (26), we could have imposed the compatibility condition $I_{x y^{\prime}}=I_{y^{\prime} x}$ and, in the same fashion we did above for the pairing $(x, y)$, conclude that the integrating factor can be written as a product of irreducible polynomials in $\left(x, y^{\prime}, S\right)$. The same can be done for the pairing $\left(y, y^{\prime}\right)$. So, we can conclude that $\bar{R}$ can be written as a product of irreducible polynomials in $\left(x, y, y^{\prime}, S\right)$. Since $\bar{R} \equiv\left(\frac{R}{N}\right)$, and $N$ is a polynomial in $\left(x, y, y^{\prime}\right)$, finally we have that $R=\prod_{i} f_{i}^{n_{i}}$, where $f_{i}$ are irreducible polynomials in $\left(x, y, y^{\prime}, S\right)$ and $n_{i}$ are non-zero rational numbers.

Let us now prove that the $f_{i}$ are eigenpolynomials of the operator $\mathcal{D}$ or factors of $N$. From equation (8) and remembering that $\bar{R} \equiv\left(\frac{R}{N}\right)$, we have:

$$
\begin{equation*}
\frac{D[\bar{R} N]}{\bar{R} N}=-S-\frac{N M_{y^{\prime}}-M N_{y^{\prime}}}{N^{2}} \tag{28}
\end{equation*}
$$

multiplying both sides of (28) by $N^{2}$, we get

$$
\begin{equation*}
N^{2} \frac{D[\bar{R}]}{\bar{R}}+N D[N]=-N^{2} S-\left(N M_{y^{\prime}}-M N_{y^{\prime}}\right) \tag{29}
\end{equation*}
$$

and finally,

$$
\begin{equation*}
N \frac{\mathcal{D}[\bar{R}]}{\bar{R}}=-N D[N]-N^{2} S-\left(N M_{y^{\prime}}-M N_{y^{\prime}}\right) \tag{30}
\end{equation*}
$$

Since the right-hand side of (30) is a polynomial in $\left(x, y, y^{\prime}, S\right)$, so is $N \frac{\mathcal{D}[\bar{R}]}{\bar{R}}$. Using (24) this implies that

$$
\begin{equation*}
N \frac{\mathcal{D}[\bar{R}]}{\bar{R}}=\sum_{i} n_{i} N \frac{\mathcal{D}\left[f_{i}\right]}{f_{i}} . \tag{31}
\end{equation*}
$$

is a polynomial. Since $f_{i}$ are irreducible, independent polynomials, we may conclude that: either $f_{i} \mid \mathcal{D}\left[f_{i}\right]$ or $f_{i}$ is a factor of $N$.

In order to obtain the integrating factor $R$, we are going to use eq.(8). Dividing it by $R$, we get:

$$
\begin{equation*}
D[R] / R=-\left(S+\phi_{y^{\prime}}\right)=-\left(S+\frac{N M_{y^{\prime}}-M N_{y^{\prime}}}{N^{2}}\right) \tag{32}
\end{equation*}
$$

[^4]Multiplying both sides of the equation above by $N^{2}$, one has:
$N^{2} D[R] / R=-\left(S N^{2}+N M_{y^{\prime}}-M N_{y^{\prime}}\right) \Rightarrow N \mathcal{D}[R] / R=-\left(S N^{2}+N M_{y^{\prime}}-M N_{y^{\prime}}\right)$
Due to the results of theorem 3, the equation (33) above it can be solved in the same manner as in the methods inspired on the Prelle- Singer Procedure. Once $R$ is found as a function of $\left(x, y, y^{\prime}, S\right)$, we can substitute the known $S$ (see section 3) and, finally, from eqs.(6), find the first order invariant via quadratures:

$$
\begin{gather*}
I\left(x, y, y^{\prime}\right)=\int R\left(\phi+S y^{\prime}\right) d x-\int\left[R S+\frac{\partial}{\partial y} \int R\left(\phi+S y^{\prime}\right) d x\right] d y- \\
\int\left[R+\frac{\partial}{\partial y^{\prime}}\left(\int R\left(\phi+S y^{\prime}\right) d x-\int\left[R S+\frac{\partial}{\partial y} \int R\left(\phi+S y^{\prime}\right) d x\right] d y\right)\right] d y^{\prime} . \tag{34}
\end{gather*}
$$

That concludes the presentation of our proposed approach which is a semialgorithm procedure to reduce soodes of the form of eq.(1).

### 4.1 Example

Here we are going to briefly introduce an example of a non-trivial ${ }^{6}$ SOODE that illustrates the results displayed above.

Consider the SOODE:

$$
\begin{equation*}
y^{\prime \prime}=-\frac{\left(y^{\prime}\right)^{2}+2\left(y^{\prime}\right)^{3}-1-2 y^{\prime}}{-x-\left(y^{\prime}\right) x-\left(y^{\prime}\right) y-1+\left(y^{\prime}\right)^{2}} \tag{35}
\end{equation*}
$$

For this SOODE, the operator $D$ becomes:

$$
\begin{align*}
D= & \left(y^{\prime 2}+2 y^{\prime 3}-1-2 y^{\prime}\right) \partial_{y^{\prime}}+\left(x+y^{\prime} x+y^{\prime} y+1-y^{\prime 2}\right) y^{\prime} \partial_{y}+ \\
& \left(x+y^{\prime} x+y^{\prime} y+1-y^{\prime 2}\right) \partial_{x} \tag{36}
\end{align*}
$$

and the operator $D_{S}$ is:

$$
\begin{equation*}
D_{S}=N D+\mathcal{M} \partial_{s} \tag{37}
\end{equation*}
$$

where $\mathcal{M}=2 s^{2} x y^{\prime} y+2 s^{2} y^{\prime 2} x y+7 s y^{\prime 2} x+4 s y^{\prime 3} x+s^{2}+s^{2} x^{2}+2 s^{2} x-2 s^{2} y^{\prime 2}+$ $s^{2} y^{\prime 4}-s x+4 s y^{\prime 2}-2 s y^{\prime 4}+s y+4 s y^{\prime 3} y-2 s^{2} y^{\prime 2} x+s^{2} y^{\prime 2} y^{2}-2 s^{2} y^{\prime 3} y+2 s^{2} y^{\prime} x^{2}+$ $2 s y^{\prime} x-2 s-y^{\prime}+s^{2} y^{\prime 2} x^{2}-2 s^{2} y^{\prime 3} x+2 s^{2} y^{\prime} y-2 y^{\prime 2}+y^{\prime 3}+2 y^{\prime 4}+s y^{\prime 2} y+2 s^{2} y^{\prime} x$.

The polynomial $P=\left(x+y^{\prime} x+y^{\prime} y+1-y^{\prime 2}\right) S+1-y^{\prime 2}$ is a eigenpolynomial of the $D_{S}$ operator. So, $P=0$ defines $S$ as:

$$
\begin{equation*}
S=-\frac{-1+y^{\prime 2}}{x+y^{\prime} x+y^{\prime} y+1-y^{\prime 2}} \tag{38}
\end{equation*}
$$

In order to calculate the integrating factor $R$, we need (see equations 31 e 32) to find the eigenpolynomials of the $D$ operator. These are found to be (up to the first degree):

$$
\begin{aligned}
& f_{1}=1+y^{\prime} \\
& f_{2}=1-y^{\prime} \\
& f_{3}=1+2 y^{\prime}
\end{aligned}
$$

[^5]and, for this particular case, these are sufficient (together with $N$ ) to build the integrating factor $R$. Combining equations 31 and 32 and solving for the $n_{i}$ 's we have that $\left\{n_{1}=-3 / 2, n_{2}=-3 / 2, n_{3}=0\right\}$. So, $R$ is found to be ${ }^{7}$ :
\[

$$
\begin{equation*}
R=\frac{x+y^{\prime} x+y^{\prime} y+1-y^{\prime 2}}{\left(\left(y^{\prime}-1\right)\left(1+y^{\prime}\right)\right)^{3 / 2}} \tag{39}
\end{equation*}
$$

\]

This leads to the first order invariant given by:

$$
\begin{equation*}
I=\frac{x+y+y^{\prime} x}{\sqrt{-1+y^{\prime 2}}}+\ln \left(y^{\prime}+\sqrt{-1+y^{\prime 2}}\right) . \tag{40}
\end{equation*}
$$

## 5 Computational considerations

Here, we are going to introduce an alternative way of calculating the integrating factor for a given SOODE. Why do we call this section computational considerations? The reason is that the material contained in this section is not applicable to all SOODEs ${ }^{8}$ of the form of eq.(1) and, indeed, we do not have a criteria of applicability. But, on the other hand, the method we are about to present is "less costly" computationally, therefore could be useful in practical applications. In the following, we will be using some results from the Lie theory.

Let us show a relation between the function $S$ and a symmetry of the SOODE (1). Making the following transformation

$$
\begin{equation*}
S=-\frac{D[\eta]}{\eta} \tag{41}
\end{equation*}
$$

eq.(13) becomes

$$
\begin{equation*}
D^{2}[\eta]=\phi_{y^{\prime}} D[\eta]+\phi_{y} \eta . \tag{42}
\end{equation*}
$$

From Lie theory we can see that eq.(42) represents the condition for a SOODE (1) to have a symmetry $[0, \eta]$. So, from (41) we can find a symmetry given by

$$
\begin{equation*}
\eta=e^{-\int S d x} . \tag{43}
\end{equation*}
$$

Looking at eq.(5) we can infer that $R$ is also an integrating factor for the auxiliary FOODE defined by

$$
\begin{equation*}
\frac{d y^{\prime}}{d y}=S \tag{44}
\end{equation*}
$$

where $x$ is regarded as a parameter. Besides, for this FOODE, $[\eta, D[\eta]]$ is a point symmetry. So, $R$ is given by

$$
\begin{equation*}
R=\frac{1}{\eta S+D[\eta]} . \tag{45}
\end{equation*}
$$

If in (45) we use $\eta$ defined by (43) we would get a singular $R$. However, eq.(42) has another solution independent from $\eta$ given by

[^6]\[

$$
\begin{equation*}
\bar{\eta}=\eta \int \frac{e^{\int \phi_{y^{\prime}} d x}}{\eta^{2}} d x \tag{46}
\end{equation*}
$$

\]

Using this $\bar{\eta}$ in (45) we get $R$ and, once this is done, we can calculate the first integral $I$ by using simple quadratures.

Actually, trying to solve eq.(43) could be complex. The operator $\int S d x$ that appears on equation (43) is meant to be the inverse of operator $D$ (defined on $(13))^{9}$. As one might expect, actually finding $\eta$ from (43) could be trick. Actually, in general, it will be impossible to integrate (43) and that is the reason why this shortcut is not applicable in general.

### 5.1 Example

Consider the SOODE

$$
\begin{equation*}
y^{\prime \prime}=-\frac{2 y^{\prime 2} x^{2}-2{y^{\prime 2}}^{2}-2 x y^{\prime} y-y^{2} x^{4}+2 y^{2} x^{2}-y^{2}}{2 y\left(x^{2}-1\right)} . \tag{47}
\end{equation*}
$$

Constructing the $D_{S}$ operator we get that

$$
\begin{equation*}
y^{2}\left(x^{2}-1\right)-S^{2} y^{2}-y^{\prime 2}+2 S y^{\prime} y \tag{48}
\end{equation*}
$$

is an eigenpolynomial of it. Then $S$ is given by $S=y^{\prime} / y+\sqrt{x^{2}-1}$ and $\eta, \bar{\eta}$ are respectively

$$
\begin{equation*}
\eta=\frac{\sqrt{x+\sqrt{x^{2}-1}}}{y e^{\frac{x \sqrt{x^{2}-1}}{2}}}, \quad \bar{\eta}=\frac{\sqrt{x+\sqrt{x^{2}-1}} \int \frac{\sqrt{x-1} \sqrt{x+1} e^{x} \sqrt{x^{2}-1}}{x+\sqrt{x^{2}-1}} d x}{y e^{\frac{x \sqrt{x^{2}-1}}{2}}} . \tag{49}
\end{equation*}
$$

$R$ can be written as

$$
\begin{equation*}
R=\frac{y \sqrt{x+\sqrt{x^{2}-1}}}{\sqrt{x^{2}-1} e^{\frac{x \sqrt{x^{2}-1}}{2}}} \tag{50}
\end{equation*}
$$

leading to the following first integral:

$$
\begin{equation*}
I=\frac{\sqrt{x+\sqrt{x^{2}-1}}\left(2 y^{\prime} y+\sqrt{x^{2}-1} y^{2}\right)}{\sqrt{x^{2}-1} e^{\frac{x \sqrt{x^{2}-1}}{2}}} . \tag{51}
\end{equation*}
$$

## 6 Conclusion

In [17], we have developed a method, based on a conjecture, to deal with SOODEs that presented an elementary solution (possessing two elementary first order invariants).

In that same paper, we have introduced a function $S$ to transform the Pfaffian equation related to the particular SOODE under consideration into a 1 -form proportional to differential of the first order differential invariant. That function $S$ was instrumental in finding the integrating factor for the SOODE.

[^7]Here, in the present paper, we introduce many theoretical results concerning that function $S$ and present an way to calculated it via a Darboux-type procedure. Here also the function $S$ is used to produce the integrating factor.

We then demonstrate general results about the structure of the integrating factor $R$ and that we can calculate it using a procedure very similar to the one inspired by the original work by Prelle-Singer [6] (applicable for FOODEs). These results assure that we have a semi- algorithmic procedure to deal with rational SOODEs, presenting at least one elementary first order invariant, i.e., there is no need to posses two such invariants and, consequently, we can cover a much broader class of SOODEs than before [17].

In the above section, we introduce an alternative way to calculate $R$ from the knowledge of $S$ for restricted cases. The motivation for that method is that, the general method, sometimes, can be computationally very demanding and it may be worth to have a go in the "shortcut" before embarking on long calculations (despite the general case being the sure think).

We are searching for better algorithms for the many steps of the method presented here in order to make it computationally more efficient. We are also working on a full computational implementation of the method as it stands today.

## References

[1] Stephani, H. Differential equations: their solution using symmetries, ed. M.A.H. MacCallum, Cambridge University Press, New York and London (1989).
[2] G.W. Bluman and S. Kumei, Symmetries and Differential Equations, Applied Mathematical Sciences 81, Springer-Verlag, (1989).
[3] P.J. Olver, Applications of Lie Groups to Differential Equations, SpringerVerlag, (1986).
[4] E.S. Cheb-Terrab, L.G.S. Duarte and L.A.C.P. da Mota, Computer Algebra Solving of First Order ODEs Using Symmetry Methods. Comput.Phys.Commun., 101, (1997), 254.
[5] E.S. Cheb-Terrab, L.G.S. Duarte and L.A.C.P. da Mota, Computer Algebra Solving of Second Order ODEs Using Symmetry Methods. Comput.Phys.Commun., 108, (1998), 90.
[6] M Prelle and M Singer, Elementary first integral of differential equations. Trans. Amer. Math. Soc., 279215 (1983).
[7] R. Shtokhamer, Solving first order differential equations using the Prelle-Singer algorithm, Technical report 88-09, Center for Mathematical Computation, University of Delaware (1988).
[8] C B Collins, Algebraic Invariants Curves of Polynomial Vector Fields in the Plane, Preprint. Canada: University of Waterloo (1993); C B Collins, Quadratic Vector Fields Possessing a Centre, Preprint. Canada: University of Waterloo (1993).
[9] C. Christopher, Liouvillian first integrals of second order polynomial differential equations. Electron. J. Differential Equations 1999, No. 49, 7 pp. (electronic).
[10] C. Christopher and J. Llibre, Integrability via invariant algebraic curves for Planar polynomial differential systems Ann.Differential Equations 16 (2000), no. 1, 5-19.
[11] J. Llibre, Integrability of polynomial differential systems, Handbook of Differential equations, Ordinary Differential Equations, volume 1, Chapter 5, pages 437-531. Edited by A. Cañada, P. Drábek and A. Fonda. Elsevier B.V. 2004.
[12] L.G.S. Duarte, S.E.S.Duarte and L.A.C.P. da Mota, A method to tackle first order ordinary differential equations with Liouvillian functions in the solution, in J. Phys. A: Math. Gen. 35 3899-3910 (2002).
[13] L.G.S. Duarte, S.E.S.Duarte and L.A.C.P. da Mota, Analyzing the Structure of the Integrating Factors for First Order Ordinary Differential Equations with Liouvillian Functions in the Solution, J. Phys. A: Math. Gen. 35 1001-1006 (2002)
[14] L.G.S. Duarte, S.E.S.Duarte, L.A.C.P. da Mota and J.F.E. Skea, Extension of the Prelle-Singer Method and a MAPLE implementation, Computer Physics Communications, Holanda, v. 144, n. 1, p. 46-62, 2002
[15] Davenport J.H., Siret Y. and Tournier E. Computer Algebra: Systems and Algorithms for Algebraic Computation. Academic Press, Great Britain (1993).
[16] Y K Man and M A H MacCallum, A Rational Approach to the Prelle-Singer Algorithm. J. Symbolic Computation, 11 1-11 (1996), and refferences therein.
[17] L G S Duarte, S E S Duarte, L A C P da Mota and J E F Skea, Solving second order ordinary differential equations by extending the Prelle-Singer method, $J$. Phys. A: Math.Gen., 34 3015-3024 (2001).


[^0]:    ${ }^{1}$ E-mails: javellar@dft.if.uerj.br, lduarte@dft.if.uerj.br, sduarte@dft.if.uerj.br and damota@dft.if.uerj.br

[^1]:    ${ }^{2}$ For a formal definition of elementary function, see [15].

[^2]:    ${ }^{3}$ For a formal definition of algebraic function, see [15].

[^3]:    ${ }^{4}$ Note that $\bar{M}$ and $\bar{N}$ are polynomials in $y^{\prime}$ as well.

[^4]:    ${ }^{5}$ Notice that, since $S$ is an algebraic function of $\left(x, y, y^{\prime}\right)$, it is a root of a polynomial equation of the form $\sum_{i} p_{i} S^{i}=0$, where $p_{i}$ are polynomials in $\left(x, y, y^{\prime}\right)$. Therefore, its derivatives can be expressed in terms of rational powers of itself. So, one may consider that would be dealing with a Shtokhamer extension with $U=S$.

[^5]:    ${ }^{6}$ For instance, the MAPLE package, release 7, could not reduce it.

[^6]:    ${ }^{7}$ Note that $R$ is formed by the product $f_{1}^{n_{1}} f_{2}^{n_{2}} f_{3}^{n_{3}} N^{1}=f_{1}^{-3 / 2} f_{2}^{-3 / 2} N^{1}$.
    ${ }^{8}$ Note that the method presented on the previous section is a general semi-algorithmic approach to deal with SOODEs of the form (1) that present an elementary first order invariant.

[^7]:    ${ }^{9}$ Actually, this meaning for the operator $\int$ is true for the whole of the present section.

