# Self - similar solutions of the Burgers hierarchy 

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#### Abstract

Self — similar solutions of the equations for the Burgers hierarchy are presented.


## 1 Introduction

The Burgers hierarchy can be written in the form [1-4]

$$
\begin{equation*}
u_{t}+\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}+u\right)^{n} u=0, \quad n=0,1,2, \ldots \tag{1}
\end{equation*}
$$

Assuming $n=1$ in Eq. (11) we have the Burgers equation

$$
\begin{equation*}
u_{t}+2 u u_{x}+u_{x x}=0 . \tag{2}
\end{equation*}
$$

Eq. (2) was firstly introduced in [5]. It is well known that this equation can be linearized by means of the Cole-Hopf transformation [6-8]. Exact solutions of Eq.(2) were considered in many papers (see, for example, [9-12]).

Assuming $n=2$ in Eq. (1) we obtain the Sharma - Tasso - Olver equation

$$
\begin{equation*}
u_{t}+u_{x x x}+3 u_{x}^{2}+3 u u_{x x}+3 u^{2} u_{x}=0 \tag{3}
\end{equation*}
$$

The Sharma - Tasso - Olver equation was derived in [1, 13]. Some exact solutions of this equation were presented in [14-21].

At $n=3$ and $n=4$ we obtain the following fourth and fifth order partial differential equations

$$
\begin{gather*}
u_{t}+u_{x x x x}+10 u_{x} u_{x x}+4 u u_{x x x}+12 u u_{x}^{2}+ \\
+6 u^{2} u_{x x}+4 u^{3} u_{x}=0 \tag{4}
\end{gather*}
$$

$$
\begin{gather*}
u_{t}+u_{x x x x x}+10 u_{x x}^{2}+15 u_{x} u_{x x x}+5 u u_{x x x x}+15 u_{x}^{3}+ \\
+50 u u_{x} u_{x x}+10 u^{2} u_{x x x}+30 u^{2} u_{x}^{2}+10 u^{3} u_{x x}+5 u^{4} u_{x}=0 . \tag{5}
\end{gather*}
$$

Assuming

$$
\begin{equation*}
x=L x^{\prime}, \quad u=C_{0} u^{\prime}, \quad t=T t^{\prime}, \tag{6}
\end{equation*}
$$

we have that Eq.(1) is invariant under the dilation group in the case

$$
\begin{equation*}
C_{0} L=1, \quad T=L^{n+1} \tag{7}
\end{equation*}
$$

Assuming $C_{0}=e^{-a}$ in (7), we obtain the delation group for the Burgers hierarchy (1) in the form

$$
\begin{equation*}
u^{\prime}=e^{-a} u, \quad x^{\prime}=e^{a} x, \quad t^{\prime}=e^{a(n+1)} t . \tag{8}
\end{equation*}
$$

From transformations (8) we have two invariants for Eq.(1)

$$
\begin{equation*}
I_{1}=u t^{\frac{1}{n+1}}=u^{\prime}\left(t^{\prime}\right)^{\frac{1}{n+1}}, \quad I_{2}=\frac{x}{t^{\frac{1}{n+1}}}=\frac{x^{\prime}}{\left(t^{\prime}\right)^{\frac{1}{n+1}}} . \tag{9}
\end{equation*}
$$

Therefore we look for the solutions of the Burgers hierarchy taking into account the variables

$$
\begin{equation*}
u(x, t)=\frac{A}{t^{\frac{1}{n+1}}} f(z), \quad z=\frac{B x}{t^{\frac{1}{n+1}}} . \tag{10}
\end{equation*}
$$

Substituting (10) into (1) we obtain the equation for $f(z)$ at

$$
\begin{equation*}
A=B=\frac{1}{(n+1)^{\frac{1}{n+1}}} . \tag{11}
\end{equation*}
$$

in the form

$$
\begin{equation*}
\left(\frac{d}{d z}+f\right)^{n} f-z f+\beta=0 \tag{12}
\end{equation*}
$$

where $\beta$ is the constant of integration.
Solving Eq.(12) we obtain solutions of the Burgers hierarchy in the form

$$
\begin{equation*}
u(x, t)=\frac{1}{(n t+t)^{\frac{1}{n+1}}} f(z), \quad z=\frac{x}{(n t+t)^{\frac{1}{n+1}}} . \tag{13}
\end{equation*}
$$

Let us study the solutions of nonlinear ordinary differential equation (12).

## 2 Exact solutions of equation (12)

First of all let us prove the following lemma.
Lemma 1. Equation (12) can be transformed to the linear equation of $(n+1)$ - th order by means of transformation

$$
\begin{equation*}
f=\frac{\psi_{z}}{\psi} . \tag{14}
\end{equation*}
$$

Proof. The proof of this lemma can be given by means of the mathematical induction method.

Using the transformation (14) we have

$$
\begin{equation*}
\left(\frac{d}{d z}+f\right) f=\frac{\psi_{z z}}{\psi}, \quad\left(\frac{d}{d z}+f\right)^{2} f=\frac{\psi_{z z z}}{\psi} \tag{15}
\end{equation*}
$$

Assuming that there is equality

$$
\begin{equation*}
\left(\frac{d}{d z}+f\right)^{k} f=\frac{\psi_{k+1, z}}{\psi}, \quad \psi_{k+1, z}=\frac{d^{k+1} \psi}{d z^{k+1}} . \tag{16}
\end{equation*}
$$

Differentiating Eq.(16) with respect to in $z$ we have

$$
\begin{equation*}
\frac{d}{d z}\left(\frac{d}{d z}+f\right)^{k} f=\frac{\psi_{k+2, z}}{\psi}-\frac{\psi_{z} \psi_{k+1, z}}{\psi^{2}} \tag{17}
\end{equation*}
$$

From Eq.(17) we obtain the equality

$$
\begin{equation*}
\left(\frac{d}{d z}+f\right)^{k+1} f=\frac{\psi_{k+2, z}}{\psi} \tag{18}
\end{equation*}
$$

Therefore we obtain the formula

$$
\begin{equation*}
\left(\frac{d}{d z}+f\right)^{n} f=\frac{\psi_{n+1, z}}{\psi} \tag{19}
\end{equation*}
$$

Taking this formula into account we have the equality

$$
\begin{equation*}
\left(\frac{d}{d z}+f\right)^{n} f-z f+\beta=\frac{1}{\psi}\left(\psi_{n+1, z}-z \psi_{z}+\beta \psi\right) \tag{20}
\end{equation*}
$$

As result of this lemma we obtain that solutions of Eq. (12) can be found by the formula (14), where $\psi(z)$ is the solution of the linear equation

$$
\begin{equation*}
\psi_{n+1, z}-z \psi_{z}+\beta \psi=0 \tag{21}
\end{equation*}
$$

Let us consider the partial cases. Assuming $\beta=0$ in Eq.(21) we have

$$
\begin{equation*}
\psi_{n+1, z}-z \psi_{z}=0 . \tag{22}
\end{equation*}
$$

Denoting $\psi_{z}=y$ we obtain

$$
\begin{equation*}
y_{n, z}-z y=0 . \tag{23}
\end{equation*}
$$

In the case $n=1$ we get solution of Eq.(23) in the form

$$
\begin{equation*}
y(z)=C_{2} e^{-\frac{z^{2}}{2}} \tag{24}
\end{equation*}
$$

The general solution of Eq.(23) can be written as

$$
\begin{equation*}
\psi(z)=C_{3}+C_{2} \int_{0}^{z} e^{-\frac{\xi^{2}}{2}} d \xi \tag{25}
\end{equation*}
$$

where $C_{2}$ and $C_{3}$ are arbitrary constants. In the case $n=2$ we obtain the general solution of Eq.(23) in the form

$$
\begin{equation*}
y(z)=C_{4} \sqrt{z} J_{\frac{1}{3}}\left(\frac{2}{3} z^{\frac{3}{2}}\right)+C_{5} \sqrt{z} Y_{\frac{1}{3}}\left(\frac{2}{3} z^{\frac{3}{2}}\right), \tag{26}
\end{equation*}
$$

where $J_{\frac{1}{3}}$ and $Y_{\frac{1}{3}}$ are the Bessel functions.
In the case $n>2$ solution of Eq.(23) has $n$ solutions

$$
\begin{equation*}
y_{j}(z)=z^{j-1} E_{n, 1+\frac{1}{n}, 1+\frac{j}{n}}\left(z^{n+1}\right), \quad j=1,2, \ldots, n, \tag{27}
\end{equation*}
$$

where $E_{n, m, l}$ is a Mittag - Leffler type special function defined by [22];

$$
\begin{equation*}
E_{n, m, l}(z)=1+\sum_{k=1}^{\infty} b_{k} z^{k}, \quad b_{k}=\prod_{s=0}^{k-1} \frac{\Gamma(n(m s+l)+1)}{\Gamma(n(m s+l+1)+1)} \tag{28}
\end{equation*}
$$

In the case $\beta \neq 0$ solutions of Eq.(23) can be referred to the type of the Laplace equations [23]. There are partial solutions $\psi(z)=-z^{m}$ of Eq.(21) at $\beta=m$, where $0<m \leq n$ is integer. In the general case solutions of equations (21) can be found using the Laplace transformation or taking the expansions in the power series into account.

For a example let us solve the Cauchy problem for linear ordinary differential equation (21) at $\beta=-1$. We have the following problem

$$
\begin{gather*}
\psi_{n+1, z}-z \psi_{z}-\psi=0 \\
\psi(z=0)=b_{0}, \quad \psi_{z}(z=0)=b_{1}, \ldots, \psi_{n-2, z}=b_{n-2} \quad \psi_{n-1, z}=b_{n-1} . \tag{29}
\end{gather*}
$$

Substituting

$$
\begin{equation*}
\psi(z)=\sum_{m=0}^{\infty} a_{m} z^{m} \tag{30}
\end{equation*}
$$

into Eq.(29), we obtain the solution in the form

$$
\begin{align*}
& \psi(z)=a_{0} \sum_{k=0}^{\infty} \frac{z^{n k} \prod_{j=0}^{k}(n j+1)}{(n k+1)!}+a_{1} \sum_{k=0}^{\infty} \frac{z^{n k+1} \prod_{j=0}^{k}(n j+2)}{(n k+2)!}+ \\
&+2 a_{2} \sum_{k=0}^{\infty} \frac{z^{n k+2} \prod_{j=0}^{k}(n j+3)}{(n k+3)!}+\ldots+ \\
&+(n-2)!a_{n-2} \sum_{k=0}^{\infty} \frac{z^{n k+n-2} \prod_{j=0}^{k}(n j+n-1)}{(n k+n-1)!}+  \tag{31}\\
&+(n-1)!a_{n-1} \sum_{k=0}^{\infty} \frac{z^{n k+n-1} \prod_{j=0}^{k}(n j+n)}{(n k+n)!} .
\end{align*}
$$

The value of coefficients $a_{0}, a_{1}, a_{2}, \ldots, a_{n-2}$ and $a_{n-1}$ are determined by the initial values $b_{0}, b_{1}, b_{2}, \ldots, b_{n-2}$ and $b_{n-1}$. We have

$$
\begin{equation*}
a_{0}=b_{0}, \quad a_{1}=b_{1}, \quad a_{2}=\frac{b_{2}}{(2!)^{2}}, \ldots, a_{n-1}=\frac{b_{n-1}}{((n-1)!)^{2}} . \tag{32}
\end{equation*}
$$

Let us present the partial cases of solution for equation (29). In the case $n=3$ we have solution in the form

$$
\begin{gather*}
\psi(z)=a_{0} \sum_{k=0}^{\infty} \frac{z^{3 k} \prod_{j=0}^{k}(3 j+1)}{(3 k+1)!}+a_{1} \sum_{k=0}^{\infty} \frac{z^{3 k+1} \prod_{j=0}^{k}(3 j+2)}{(3 k+2)!}+ \\
+2 a_{2} \sum_{k=0}^{\infty} \frac{z^{3 k+2} \prod_{j=0}^{k}(3 j+3)}{(3 k+3)!} . \tag{33}
\end{gather*}
$$

Assuming $n=4$ we obtain

$$
\begin{align*}
& \psi(z)=a_{0} \sum_{k=0}^{\infty} \frac{z^{4 k} \prod_{j=0}^{k}(4 j+1)}{(4 k+1)!}+a_{1} \sum_{k=0}^{\infty} \frac{z^{4 k+1} \prod_{j=0}^{k}(4 j+2)}{(4 k+2)!}+ \\
& +2 a_{2} \sum_{k=0}^{\infty} \frac{z^{4 k+2} \prod_{j=0}^{k}(4 j+3)}{(4 k+3)!}+6 a_{3} \sum_{k=0}^{\infty} \frac{z^{4 k+3} \prod_{j=0}^{k}(4 j+4)}{(4 k+4)!} . \tag{34}
\end{align*}
$$

One can show that these power series are conversed for any values $z$. Therefore self-similar solutions of equations for the Burgers hierarchy are found after substitution (34) into formula (14).

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