Self – similar solutions of the Burgers hierarchy

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Abstract

Self — similar solutions of the equations for the Burgers hierarchy are presented.

1 Introduction

The Burgers hierarchy can be written in the form [1-4]

$$u_t + \frac{\partial}{\partial x} \left(\frac{\partial}{\partial x} + u \right)^n u = 0, \quad n = 0, 1, 2, \dots$$
 (1)

Assuming n = 1 in Eq. (1) we have the Burgers equation

$$u_t + 2 \, u \, u_x + \, u_{xx} = 0. \tag{2}$$

Eq. (2) was firstly introduced in [5]. It is well known that this equation can be linearized by means of the Cole-Hopf transformation [6-8]. Exact solutions of Eq.(2) were considered in many papers (see, for example, [9-12]).

Assuming n = 2 in Eq. (1) we obtain the Sharma - Tasso - Olver equation

$$u_t + u_{xxx} + 3 u_x^2 + 3 u u_{xx} + 3 u^2 u_x = 0.$$
(3)

The Sharma - Tasso - Olver equation was derived in [1, 13]. Some exact solutions of this equation were presented in [14-21].

At n = 3 and n = 4 we obtain the following fourth and fifth order partial differential equations

$$u_t + u_{xxxx} + 10 u_x u_{xx} + 4 u u_{xxx} + 12 u u_x^2 + + 6 u^2 u_{xx} + 4 u^3 u_x = 0,$$
(4)

$$u_t + u_{xxxxx} + 10 u_{xx}^2 + 15 u_x u_{xxx} + 5 u u_{xxxx} + 15 u_x^3 + +50 u u_x u_{xx} + 10 u^2 u_{xxx} + 30 u^2 u_x^2 + 10 u^3 u_{xx} + 5 u^4 u_x = 0.$$
(5)

Assuming

$$x = L x', \quad u = C_0 u', \quad t = T t',$$
 (6)

we have that Eq.(1) is invariant under the dilation group in the case

$$C_0 L = 1, \quad T = L^{n+1}.$$
 (7)

Assuming $C_0 = e^{-a}$ in (7), we obtain the delation group for the Burgers hierarchy (1) in the form

$$u' = e^{-a} u, \quad x' = e^{a} x, \quad t' = e^{a(n+1)} t.$$
 (8)

From transformations (8) we have two invariants for Eq.(1)

$$I_1 = u t^{\frac{1}{n+1}} = u'(t')^{\frac{1}{n+1}}, \qquad I_2 = \frac{x}{t^{\frac{1}{n+1}}} = \frac{x'}{(t')^{\frac{1}{n+1}}}.$$
(9)

Therefore we look for the solutions of the Burgers hierarchy taking into account the variables

$$u(x,t) = \frac{A}{t^{\frac{1}{n+1}}} f(z), \quad z = \frac{Bx}{t^{\frac{1}{n+1}}}.$$
(10)

Substituting (10) into (1) we obtain the equation for f(z) at

$$A = B = \frac{1}{(n+1)^{\frac{1}{n+1}}}.$$
(11)

in the form

$$\left(\frac{d}{dz}+f\right)^n f - z f + \beta = 0, \tag{12}$$

where β is the constant of integration.

Solving Eq.(12) we obtain solutions of the Burgers hierarchy in the form

$$u(x,t) = \frac{1}{(n\,t+t)^{\frac{1}{n+1}}}f(z), \quad z = \frac{x}{(n\,t+t)^{\frac{1}{n+1}}}.$$
(13)

Let us study the solutions of nonlinear ordinary differential equation (12).

2 Exact solutions of equation(12)

First of all let us prove the following lemma.

Lemma 1. Equation (12) can be transformed to the linear equation of (n + 1) - th order by means of transformation

$$f = \frac{\psi_z}{\psi}.$$
(14)

Proof. The proof of this lemma can be given by means of the mathematical induction method.

Using the transformation (14) we have

$$\left(\frac{d}{dz}+f\right)f = \frac{\psi_{zz}}{\psi}, \quad \left(\frac{d}{dz}+f\right)^2 f = \frac{\psi_{zzz}}{\psi}$$
 (15)

Assuming that there is equality

$$\left(\frac{d}{dz}+f\right)^{k} f = \frac{\psi_{k+1,z}}{\psi}, \quad \psi_{k+1,z} = \frac{d^{k+1}\psi}{d\,z^{k+1}}.$$
 (16)

Differentiating Eq.(16) with respect to in z we have

$$\frac{d}{dz}\left(\frac{d}{dz}+f\right)^k f = \frac{\psi_{k+2,z}}{\psi} - \frac{\psi_z \,\psi_{k+1,z}}{\psi^2}.$$
(17)

From Eq.(17) we obtain the equality

$$\left(\frac{d}{dz}+f\right)^{k+1}f = \frac{\psi_{k+2,z}}{\psi}.$$
(18)

Therefore we obtain the formula

$$\left(\frac{d}{dz}+f\right)^n f = \frac{\psi_{n+1,z}}{\psi}.$$
(19)

Taking this formula into account we have the equality

$$\left(\frac{d}{dz}+f\right)^n f - zf + \beta = \frac{1}{\psi}\left(\psi_{n+1,z} - z\psi_z + \beta\psi\right).$$
(20)

As result of this lemma we obtain that solutions of Eq. (12) can be found by the formula (14), where $\psi(z)$ is the solution of the linear equation

$$\psi_{n+1,z} - z\,\psi_z + \beta\,\psi = 0,\tag{21}$$

Let us consider the partial cases. Assuming $\beta = 0$ in Eq.(21) we have

$$\psi_{n+1,z} - z\,\psi_z = 0. \tag{22}$$

Denoting $\psi_z = y$ we obtain

$$y_{n,z} - z \, y = 0. \tag{23}$$

In the case n = 1 we get solution of Eq.(23) in the form

$$y(z) = C_2 \, e^{-\frac{z^2}{2}}.\tag{24}$$

The general solution of Eq.(23) can be written as

$$\psi(z) = C_3 + C_2 \int_0^z e^{-\frac{\xi^2}{2}} d\xi, \qquad (25)$$

where C_2 and C_3 are arbitrary constants. In the case n = 2 we obtain the general solution of Eq.(23) in the form

$$y(z) = C_4 \sqrt{z} J_{\frac{1}{3}} \left(\frac{2}{3} z^{\frac{3}{2}}\right) + C_5 \sqrt{z} Y_{\frac{1}{3}} \left(\frac{2}{3} z^{\frac{3}{2}}\right),$$
(26)

where $J_{\frac{1}{3}}$ and $Y_{\frac{1}{3}}$ are the Bessel functions.

 $\psi(z=0) = b_0,$

In the case n > 2 solution of Eq.(23) has n solutions

$$y_j(z) = z^{j-1} E_{n,1+\frac{1}{n},1+\frac{j}{n}}(z^{n+1}), \quad j = 1, 2, \dots, n,$$
 (27)

where $E_{n,m,l}$ is a Mittag - Leffler type special function defined by [22];

$$E_{n,m,l}(z) = 1 + \sum_{k=1}^{\infty} b_k \, z^k, \quad b_k = \prod_{s=0}^{k-1} \frac{\Gamma\left(n(ms+l)+1\right)}{\Gamma(n(ms+l+1)+1)}$$
(28)

In the case $\beta \neq 0$ solutions of Eq.(23) can be referred to the type of the Laplace equations [23]. There are partial solutions $\psi(z) = -z^m$ of Eq.(21) at $\beta = m$, where $0 < m \leq n$ is integer. In the general case solutions of equations (21) can be found using the Laplace transformation or taking the expansions in the power series into account.

For a example let us solve the Cauchy problem for linear ordinary differential equation (21) at $\beta = -1$. We have the following problem

$$\psi_{n+1,z} - z \,\psi_z - \psi = 0,$$
(29)

$$\psi_z(z=0) = b_1, \dots, \psi_{n-2,z} = b_{n-2} \quad \psi_{n-1,z} = b_{n-1}.$$

Substituting

$$\psi(z) = \sum_{m=0}^{\infty} a_m \, z^m \tag{30}$$

into Eq.(29), we obtain the solution in the form

$$\psi(z) = a_0 \sum_{k=0}^{\infty} \frac{z^{nk} \prod_{j=0}^k (n \, j+1)}{(nk+1)!} + a_1 \sum_{k=0}^{\infty} \frac{z^{nk+1} \prod_{j=0}^k (n \, j+2)}{(nk+2)!} + + 2 a_2 \sum_{k=0}^{\infty} \frac{z^{nk+2} \prod_{j=0}^k (n \, j+3)}{(nk+3)!} + \dots + + (n-2)! a_{n-2} \sum_{k=0}^{\infty} \frac{z^{nk+n-2} \prod_{j=0}^k (n \, j+n-1)}{(nk+n-1)!} + + (n-1)! a_{n-1} \sum_{k=0}^{\infty} \frac{z^{nk+n-1} \prod_{j=0}^k (n \, j+n)}{(nk+n)!}.$$
(31)

The value of coefficients $a_0, a_1, a_2, \ldots, a_{n-2}$ and a_{n-1} are determined by the initial values $b_0, b_1, b_2, \ldots, b_{n-2}$ and b_{n-1} . We have

$$a_0 = b_0, \quad a_1 = b_1, \quad a_2 = \frac{b_2}{(2!)^2}, \dots, a_{n-1} = \frac{b_{n-1}}{((n-1)!)^2}.$$
 (32)

Let us present the partial cases of solution for equation (29). In the case n = 3 we have solution in the form

$$\psi(z) = a_0 \sum_{k=0}^{\infty} \frac{z^{3k} \prod_{j=0}^k (3j+1)}{(3k+1)!} + a_1 \sum_{k=0}^{\infty} \frac{z^{3k+1} \prod_{j=0}^k (3j+2)}{(3k+2)!} + 2a_2 \sum_{k=0}^{\infty} \frac{z^{3k+2} \prod_{j=0}^k (3j+3)}{(3k+3)!}.$$
(33)

Assuming n = 4 we obtain

$$\psi(z) = a_0 \sum_{k=0}^{\infty} \frac{z^{4k} \prod_{j=0}^k (4j+1)}{(4k+1)!} + a_1 \sum_{k=0}^{\infty} \frac{z^{4k+1} \prod_{j=0}^k (4j+2)}{(4k+2)!} + 2a_2 \sum_{k=0}^{\infty} \frac{z^{4k+2} \prod_{j=0}^k (4j+3)}{(4k+3)!} + 6a_3 \sum_{k=0}^{\infty} \frac{z^{4k+3} \prod_{j=0}^k (4j+4)}{(4k+4)!}.$$
(34)

One can show that these power series are conversed for any values z. Therefore self-similar solutions of equations for the Burgers hierarchy are found after substitution (34) into formula (14).

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