Fractional Noether's theorem in the Riesz-Caputo sense*

Gastão S. F. Frederico gfrederico@mat.ua.pt Department of Science and Technology University of Cape Verde Praia, Santiago, Cape Verde

Delfim F. M. Torres[†]
delfim@ua.pt
Department of Mathematics
University of Aveiro
3810-193 Aveiro, Portugal

Abstract

We prove a Noether's theorem for fractional variational problems with Riesz-Caputo derivatives. Both Lagrangian and Hamiltonian formulations are obtained. Illustrative examples in the fractional context of the calculus of variations and optimal control are given.

Keywords: calculus of variations, optimal control, fractional derivatives, invariance, Noether's theorem, Leitmann's direct method.

2000 Mathematics Subject Classification: 49K05, 26A33.

1 Introduction

Variational symmetries are defined by parameter transformations that keep a problem of the calculus of variations or optimal control invariant [28, 35, 51]. Their importance, as recognized by Noether in 1918, is connected with the existence of conservation laws that can be used to reduce the order of the Euler-Lagrange differential equations [21, 27, 50]. Noether's symmetry theorem is

^{*}Accepted (25/Jan/2010) for publication in Applied Mathematics and Computation.

 $^{^{\}dagger}$ Corresponding author. Partially supported by the *Centre for Research on Optimization and Control* (CEOC) of the University of Aveiro, cofinanced by the European Community fund FEDER/POCI 2010.

nowadays recognized as one of the most beautiful results of the calculus of variations and optimal control [14, 49, 52].

In 1967 a direct method for the problems of the calculus of variations, which allow to obtain absolute extremizers directly, without using the Euler-Lagrange equations, was introduced by George Leitmann [34, 36, 37]. Time as shown that Leitmann's method is a general and fruitful principle that can be applied with success to a myriad of different classes of problems [8, 9, 10, 11, 38, 39, 40]. Interestingly, it turns out that Leitmann's and Noether's principles are closely connected [48, 53].

The fractional calculus is an area of current strong research with many different and important applications [31, 43, 45, 47]. In the last years its importance in the calculus of variations and optimal control has been perceived, and a fractional variational theory began to be developed by several different authors [1, 7, 12, 18, 19, 29, 44, 46]. Most part of the results in this direction make use of fractional derivatives in the sense of Riemann-Liouville [7, 18, 23, 25, 44], FALVA [19, 20, 22], or Caputo [1, 5, 24]. In 2007 generalized Euler-Lagrange fractional equations and transversality conditions were studied for variational problems defined in terms of Riesz fractional derivatives [2]. In this paper we develop further the theory by obtaining a fractional version of Noether's symmetry theorem for variational problems with Riesz-Caputo derivatives (Theorems 24 and 35). Both fractional problems of the calculus of variations and optimal control are considered.

We finish this introduction comparing in some details the results here obtained with the ones of references [18, 19, 20].

In [18] a (α, β) fractional derivative is considered, which involves a right fractional derivative of order α and a left fractional derivative of order β combined using a complex number γ . The (α, β) fractional derivative is useful when one needs to deal with complex valued functions. In our paper we consider real valued functions only. Moreover, we consider left and right derivatives in the sense of Caputo, while the (α, β) derivative in [18] is defined via left and right Riemann-Liouville derivatives. The advantage of using the Riesz symmetrized Caputo fractional derivative instead of the (α, β) derivative in [18] is that Caputo derivatives allow us to use the standard boundary conditions of the calculus of variations, which explains why they are more popular in engineer and physics.

In our paper we consider problems of the calculus of variations for functions with one independent variable. Paper [19] initiates a new area of fractional variational calculus by proposing a fractional variational theory involving multiple integrals. Some important consequences of such theory in mechanical problems involving dissipative systems with infinitely many degrees of freedom are given in [19], but a formal theory for that is missing. Generalization of our present results to multiple fractional variational integrals is an interesting and challenging open question. The recent results proved in [3] may be useful to that objective.

The results of [20] are for fractional Riemann-Liouville cost integrals that depend on a parameter α but not on fractional-order derivatives of order α as we do here: the variational problems of [20] are defined for Lagrangians that depend on the classical derivative, while here we deal with fractional derivatives.

2 Preliminaries on Fractional Calculus

In this section we fix notations by collecting the definitions of fractional derivatives in the sense of Riemann-Liouville, Caputo, and Riesz [2, 43, 45, 47].

Definition 1 (Riemann-Liouville fractional integrals). Let f be a continuous function in the interval [a,b]. For $t \in [a,b]$, the left Riemann-Liouville fractional integral ${}_aI^{\alpha}_tf(t)$ and the right Riemann-Liouville fractional integral ${}_tI^{\alpha}_bf(t)$ of order α , $\alpha > 0$, are defined by

$${}_{a}I_{t}^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)} \int_{a}^{t} (t - \theta)^{\alpha - 1} f(\theta) d\theta, \qquad (1)$$

$${}_{t}I_{b}^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)} \int_{t}^{b} (\theta - t)^{\alpha - 1} f(\theta) d\theta, \qquad (2)$$

where Γ is the Euler gamma function.

Definition 2 (Riesz fractional integral). Let f be a continuous function in the interval [a,b]. For $t \in [a,b]$, the Riesz fractional integral ${}_a^R I_b^{\alpha} f(t)$ of order α , $\alpha > 0$, is defined by

$${}_a^R I_b^{\alpha} f(t) = \frac{1}{2\Gamma(\alpha)} \int_a^b |t - \theta|^{\alpha - 1} f(\theta) d\theta. \tag{3}$$

Remark 3. From equations (1)–(3) it follows that

$${}_{a}^{R}I_{b}^{\alpha}f(t) = \frac{1}{2} \left[{}_{a}I_{t}^{\alpha}f(t) + {}_{t}I_{b}^{\alpha}f(t) \right]. \tag{4}$$

Definition 4 (fractional derivative in the sense of Riemann-Liouville). Let f be a continuous function in the interval [a,b]. For $t \in [a,b]$, the left Riemann-Liouville fractional derivative ${}_aD_t^{\alpha}f(t)$ and the right Riemann-Liouville fractional derivative ${}_tD_b^{\alpha}f(t)$ of order α are defined by

$${}_{a}D_{t}^{\alpha}f(t) = D^{n}{}_{a}I_{t}^{n-\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^{n} \int_{a}^{t} (t-\theta)^{n-\alpha-1}f(\theta)d\theta, \quad (5)$$

$${}_tD_b^{\alpha}f(t) = (-D)^n{}_tI_b^{n-\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d}{dt}\right)^n \int_t^b (\theta-t)^{n-\alpha-1} f(\theta) d\theta \,, \tag{6}$$

where $n \in \mathbb{N}$ is such that $n-1 \leq \alpha < n$, and D is the usual derivative.

Definition 5 (fractional derivative in the sense of Caputo). Let f be a continuous function in [a,b]. For $t \in [a,b]$, the left Caputo fractional derivative ${}_{t}^{C}D_{b}^{\alpha}f(t)$ and the right Caputo fractional derivative ${}_{t}^{C}D_{b}^{\alpha}f(t)$ of order α are defined in the following way:

$${}_{a}^{C}D_{t}^{\alpha}f(t) = {}_{a}I_{t}^{n-\alpha}D^{n}f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{a}^{t} (t-\theta)^{n-\alpha-1} \left(\frac{d}{d\theta}\right)^{n} f(\theta)d\theta , \quad (7)$$

$${}_{t}^{C}D_{b}^{\alpha}f(t) = {}_{t}I_{b}^{n-\alpha}(-D)^{n}f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t}^{b} (\theta-t)^{n-\alpha-1} \left(-\frac{d}{d\theta}\right)^{n} f(\theta)d\theta,$$
(8)

where $n \in \mathbb{N}$ is such that $n - 1 \le \alpha < n$.

Definition 6 (fractional derivatives in the sense of Riesz and Riesz-Caputo). Let f be a continuous function in [a,b]. For $t \in [a,b]$, the Riesz fractional derivative ${}_a^R D_b^{\alpha} f(t)$ and the Riesz-Caputo fractional derivative ${}_a^{RC} D_b^{\alpha} f(t)$ of order α are defined by

$${}_{a}^{R}D_{b}^{\alpha}f(t) = D_{a}^{nR}I_{t}^{n-\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^{n} \int_{a}^{b} |t-\theta|^{n-\alpha-1}f(\theta)d\theta, \quad (9)$$

$${}_{a}^{RC}D_{b}^{\alpha}f(t) = {}_{a}^{R}I_{t}^{n-\alpha}D^{n}f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{a}^{b} |t-\theta|^{n-\alpha-1} \left(\frac{d}{d\theta}\right)^{n} f(\theta)d\theta, \quad (10)$$

where $n \in \mathbb{N}$ is such that $n - 1 \le \alpha < n$.

Remark 7. Using equations (4) and (5)–(10) it follows that

$$_{a}^{R}D_{b}^{\alpha}f(t)=\frac{1}{2}\left[{}_{a}D_{t}^{\alpha}f(t)+(-1)^{n}\,{}_{t}D_{b}^{\alpha}f(t)\right] \label{eq:equation:equation:equation}$$

and

$$_{a}^{RC}D_{b}^{\alpha}f(t)=\frac{1}{2}\left[_{a}^{C}D_{t}^{\alpha}f(t)+\left(-1\right)^{n}{}_{t}^{C}D_{b}^{\alpha}f(t)\right]\,.\label{eq:eq:energy_energy}$$

In the particular case $0 < \alpha < 1$, we have:

$${}_{a}^{R}D_{b}^{\alpha}f(t) = \frac{1}{2} \left[{}_{a}D_{t}^{\alpha}f(t) - {}_{t}D_{b}^{\alpha}f(t) \right]$$
 (11)

and

$${}_{a}^{RC}D_{b}^{\alpha}f(t) = \frac{1}{2} \left[{}_{a}^{C}D_{t}^{\alpha}f(t) - {}_{t}^{C}D_{b}^{\alpha}f(t) \right]. \tag{12}$$

Remark 8. If $\alpha = 1$, equalities (5)–(8) give the classical derivatives:

$$_{a}D_{t}^{1}f(t) = {}_{a}^{C}D_{t}^{1}f(t) = \frac{d}{dt}f(t), \quad _{t}D_{b}^{1}f(t) = {}_{t}^{C}D_{b}^{1}f(t) = -\frac{d}{dt}f(t).$$

Substituting these quantities into (11) and (12), we obtain that

$${}_{a}^{R}D_{b}^{1}f(t) = {}_{a}^{RC}D_{b}^{1}f(t) = \frac{d}{dt}f(t)$$
.

3 Main Results

In 2007 a formulation of the Euler-Lagrange equations was given for problems of the calculus of variations with fractional derivatives in the sense of Riesz-Caputo [2]. Here we prove a fractional version of Noether's theorem valid along the Riesz-Caputo fractional Euler-Lagrange extremals [2]. For that we introduce an appropriate fractional operator that allow us to generalize the classical concept of conservation law. Under the extended fractional notion of conservation law we begin by proving in §3.1 a fractional Noether theorem without changing

the time variable t, i.e., without transformation of the independent variable (Theorem 21). In §3.2 we proceed with a time-reparameterization to obtain the fractional Noether's theorem in its general form (Theorem 24). Finally, in §3.3 we consider more general fractional optimal control problems in the sense of Riesz-Caputo, obtaining the corresponding fractional Noether's theorem in Hamiltonian form (Theorem 35).

3.1 On the Riesz-Caputo conservation of momentum

We begin by defining the fractional functional under consideration.

Problem 9 (The fractional problem of the calculus of variations in the sense of Riesz-Caputo). The fractional problem of the calculus of variations in the sense of Riesz-Caputo consists to find the stationary functions of the functional

$$I[q(\cdot)] = \int_{a}^{b} L\left(t, q(t), {}_{a}^{RC} D_{b}^{\alpha} q(t)\right) dt, \qquad (13)$$

where $[a,b] \subset \mathbb{R}$, a < b, $0 < \alpha < 1$, and the admissible functions $q: t \mapsto q(t)$ and the Lagrangian $L: (t,q,v_l) \mapsto L(t,q,v_l)$ are assumed to be functions of class C^2 :

$$q(\cdot) \in C^{2}([a, b]; \mathbb{R}^{n});$$

$$L(\cdot, \cdot, \cdot) \in C^{2}([a, b] \times \mathbb{R}^{n} \times \mathbb{R}^{n}; \mathbb{R}).$$

Along the work, we denote by $\partial_i L$ the partial derivative of L with respect to its i-th argument, i = 1, 2, 3.

Remark 10. When $\alpha = 1$ the functional (13) is reduced to the classical functional of the calculus of variations:

$$I[q(\cdot)] = \int_{a}^{b} L(t, q(t), \dot{q}(t)) dt.$$
 (14)

The next theorem summarizes the main result of [2].

Theorem 11 ([2]). If $q(\cdot)$ is an extremizer of (13), then it satisfies the following fractional Euler-Lagrange equation in the sense of Riesz-Caputo:

$$\partial_2 L\left(t, q(t), {}^{RC}_a D^\alpha_b q(t)\right) - {}^R_a D^\alpha_b \partial_3 L\left(t, q(t), {}^{RC}_a D^\alpha_b q(t)\right) = 0 \tag{15}$$

for all $t \in [a, b]$.

Remark 12. The functional (13) involves Riesz-Caputo fractional derivatives only. However, both Riesz-Caputo and Riesz fractional derivatives appear in the fractional Euler-Lagrange equation (15).

Remark 13. Let $\alpha = 1$. Then the fractional Euler-Lagrange equation in the sense of Riesz-Caputo (15) is reduced to the classical Euler-Lagrange equation:

$$\partial_2 L(t, q(t), \dot{q}(t)) - \frac{d}{dt} \partial_3 L(t, q(t), \dot{q}(t)) = 0.$$

Theorem 11 leads to the concept of fractional extremal in the sense of Riesz-Caputo.

Definition 14 (fractional extremal in the sense of Riesz-Caputo). A function $q(\cdot)$ that is a solution of (15) is said to be a fractional Riesz-Caputo extremal for functional (13).

In order to prove a fractional Noether's theorem we adopt a technique used in [23, 30]. For that, we begin by introducing the notion of variational invariance and by formulating a necessary condition of invariance without transformation of the independent variable t.

Definition 15 (invariance of (13) without transformation of the independent variable). Functional (13) is said to be invariant under an ε -parameter group of infinitesimal transformations $\bar{q}(t) = q(t) + \varepsilon \xi(t, q(t)) + o(\varepsilon)$ if

$$\int_{t_a}^{t_b} L\left(t, q(t), {}_a^{RC} D_b^{\alpha} q(t)\right) dt = \int_{t_a}^{t_b} L\left(t, \bar{q}(t), {}_a^{RC} D_b^{\alpha} \bar{q}(t)\right) dt \tag{16}$$

for any subinterval $[t_a, t_b] \subseteq [a, b]$.

The next theorem establishes a necessary condition of invariance.

Theorem 16 (necessary condition of invariance). If functional (13) is invariant in the sense of Definition 15, then

$$\partial_2 L\left(t,q(t),{}^{RC}_aD^\alpha_bq(t)\right)\cdot\xi(t,q(t)) + \partial_3 L\left(t,q(t),{}^{RC}_aD^\alpha_bq(t)\right)\cdot{}^{RC}_aD^\alpha_b\xi(t,q(t)) = 0\,. \eqno(17)$$

Remark 17. Let $\alpha = 1$. From (17) we obtain the classical condition of invariance of the calculus of variations without transformation of the independent variable t (cf., e.g., $\lfloor 41 \rfloor$):

$$\partial_2 L(t,q,\dot{q}) \cdot \xi(t,q) + \partial_3 L(t,q,\dot{q}) \cdot \dot{\xi}(t,q) = 0$$
.

Proof. Having in mind that condition (16) is valid for any subinterval $[t_a, t_b] \subseteq [a, b]$, we can get rid off the integral signs in (16). Differentiating this condition with respect to ε , then substituting $\varepsilon = 0$, and using the definitions and properties of the fractional derivatives given in Section 2, we arrive to the intended

conclusion:

$$\begin{split} 0 &= \partial_2 L\left(t,q(t),{}^{RC}_aD^\alpha_bq(t)\right) \cdot \xi(t,q) \\ &+ \partial_3 L\left(t,q(t),{}^{RC}_aD^\alpha_bq(t)\right) \cdot \frac{d}{d\varepsilon} \left[\frac{1}{\Gamma(n-\alpha)}\int_a^b |t-\theta|^{n-\alpha-1}\left(\frac{d}{d\theta}\right)^n \bar{q}(\theta)d\theta\right]_{\varepsilon=0} \\ &= \partial_2 L\left(t,q,{}^{RC}_aD^\alpha_bq\right) \cdot \xi(t,q) \\ &+ \partial_3 L\left(t,q,{}^{RC}_aD^\alpha_bq\right) \cdot \frac{d}{d\varepsilon} \left[\frac{1}{\Gamma(n-\alpha)}\int_a^b |t-\theta|^{n-\alpha-1}\left(\frac{d}{d\theta}\right)^n q(\theta)d\theta\right. \\ &+ \frac{\varepsilon}{\Gamma(n-\alpha)}\int_a^b |t-\theta|^{n-\alpha-1}\left(\frac{d}{d\theta}\right)^n \xi(\theta,q)d\theta\right]_{\varepsilon=0} \\ &= \partial_2 L\left(t,q,{}^{RC}_aD^\alpha_bq\right) \cdot \xi(t,q) \\ &+ \partial_3 L\left(t,q,{}^{RC}_aD^\alpha_bq\right) \cdot \frac{1}{\Gamma(n-\alpha)}\int_a^b |t-\theta|^{n-\alpha-1}\left(\frac{d}{d\theta}\right)^n \xi(\theta,q)d\theta \\ &= \partial_2 L\left(t,q,{}^{RC}_aD^\alpha_bq\right) \cdot \xi(t,q) + \partial_3 L\left(t,q,{}^{RC}_aD^\alpha_bq\right) \cdot {}^{RC}_aD^\alpha_b\xi(t,q) \,. \end{split}$$

The following definition is useful in order to introduce an appropriate concept of fractional conserved quantity in the sense of Riesz-Caputo.

Definition 18. Given two functions f and g of class C^1 in the interval [a, b], we introduce the following operator:

$$\mathcal{D}_t^{\gamma}\left(f,g\right) = g \cdot {}_a^R D_b^{\gamma} f + f \cdot {}_a^{RC} D_b^{\gamma} g \,,$$

where $t \in [a, b]$ and $\gamma \in \mathbb{R}_0^+$.

Remark 19. Similar operators were used in [23, Definition 19] but involving Riemann-Liouville fractional derivatives. We note that the new operator \mathcal{D}_t^{γ} proposed here involves both Riesz and Riesz-Caputo fractional derivatives.

Remark 20. In the classical context one has $\gamma = 1$ and

$$\mathcal{D}_t^1(f,g) = f' \cdot g + f \cdot g' = \frac{d}{dt}(f \cdot g) = \mathcal{D}_t^1(g,f) .$$

Roughly speaking, $\mathcal{D}_t^{\gamma}(f,g)$ is a fractional version of the derivative of the product of f with g. Differently from the classical context, in the fractional case one has, in general, $\mathcal{D}_t^{\gamma}(f,g) \neq \mathcal{D}_t^{\gamma}(g,f)$.

We now prove the fractional Noether's theorem in the sense of Riesz-Caputo without transformation of the independent variable t.

Theorem 21 (Noether's theorem in the sense of Riesz-Caputo without transformation of time). If functional (13) is invariant in the sense of Definition 15, then

$$\mathcal{D}_{t}^{\alpha} \left[\partial_{3} L \left(t, q(t), {}_{a}^{RC} D_{b}^{\alpha} q(t) \right), \xi(t, q(t)) \right] = 0 \tag{18}$$

along any fractional Riesz-Caputo extremal q(t), $t \in [a, b]$ (Definition 14).

Remark 22. In the particular case when $\alpha = 1$ we get from the fractional conservation law in the sense of Riesz-Caputo (18) the classical Noether's conservation law of momentum (cf., e.g., [30, 41]):

$$\frac{d}{dt} \left[\partial_3 L \left(t, q(t), \dot{q}(t) \right) \cdot \xi(t, q(t)) \right] = 0$$

along any Euler-Lagrange extremal $q(\cdot)$ of (14). For this reason, we call the fractional law (18) the fractional Riesz-Caputo conservation of momentum.

Proof. Using the fractional Euler-Lagrange equation (15), we have:

$$\partial_2 L\left(t, q, {}^{RC}_a D^{\alpha}_b q\right) = {}^{R}_a D^{\alpha}_b \partial_3 L\left(t, q, {}^{RC}_a D^{\alpha}_b q\right). \tag{19}$$

Replacing (19) in the necessary condition of invariance (17), we get:

$${}_{a}^{R}D_{b}^{\alpha}\partial_{3}L\left(t,q,{}_{a}^{RC}D_{b}^{\alpha}q\right)\cdot\xi(t,q)+\partial_{3}L\left(t,q,{}_{a}^{RC}D_{b}^{\alpha}q\right)\cdot{}_{a}^{RC}D_{t}^{\alpha}\xi(t,q)=0\,. \tag{20}$$

By definition of the operator $\mathcal{D}_{t}^{\gamma}(f,g)$ it results from (20) that

$$\mathcal{D}_{t}^{\alpha}\left[\partial_{3}L\left(t,q,_{a}^{RC}D_{b}^{\alpha}q\right),\xi(t,q)\right]=0.$$

3.2 The Noether theorem in the sense of Riesz-Caputo

The next definition gives a more general notion of invariance for the integral functional (13). The main result of this section, the Theorem 24, is formulated with the help of this definition.

Definition 23 (invariance of (13)). The integral functional (13) is said to be invariant under the one-parameter group of infinitesimal transformations

$$\begin{cases} \bar{t} = t + \varepsilon \tau(t, q(t)) + o(\varepsilon), \\ \bar{q}(t) = q(t) + \varepsilon \xi(t, q(t)) + o(\varepsilon), \end{cases}$$

if

$$\int_{t_a}^{t_b} L\left(t,q(t),{}^{RC}_aD^\alpha_bq(t)\right)dt = \int_{\bar{t}(t_a)}^{\bar{t}(t_b)} L\left(\bar{t},\bar{q}(\bar{t}),{}^{RC}_aD^\alpha_b\bar{q}(\bar{t})\right)d\bar{t}$$

for any subinterval $[t_a, t_b] \subseteq [a, b]$.

Our next theorem gives a generalization of Noether's theorem for fractional problems of the calculus of variations in the sense of Riesz-Caputo.

Theorem 24 (Noether's fractional theorem in the sense of Riesz-Caputo). If the integral functional (13) is invariant in the sense of Definition 23, then

$$\mathcal{D}_{t}^{\alpha} \left[\partial_{3}L \left(t, q, {_{a}^{RC}} D_{t}^{\alpha} q \right), \xi(t, q) \right] + \mathcal{D}_{t}^{\alpha} \left[L \left(t, q, {_{a}^{RC}} D_{t}^{\alpha} q \right) - \alpha \partial_{3}L \left(t, q, {_{a}^{RC}} D_{t}^{\alpha} q \right) \cdot {_{a}^{RC}} D_{b}^{\alpha} q, \tau(t, q) \right] = 0 \quad (21)$$

along any fractional Riesz-Caputo extremal $q(\cdot)$.

Remark 25. In the particular case $\alpha = 1$ we obtain from (21) the classical Noether's conservation law (cf., e.g., [30, 41]):

$$\frac{d}{dt}\left[\partial_{3}L\left(t,q,\dot{q}\right)\cdot\xi(t,q)+\left(L(t,q,\dot{q})-\partial_{3}L\left(t,q,\dot{q}\right)\cdot\dot{q}\right)\tau(t,q)\right]=0$$

along any Euler-Lagrange extremal $q(\cdot)$ of (14).

Proof. Our proof is an extension of the method used in [30] to prove the classical Noether's theorem. For that we reparameterize the time (the independent variable t) with a Lipschitzian transformation

$$[a,b] \ni t \longmapsto \sigma f(\lambda) \in [\sigma_a, \sigma_b]$$

that satisfies

$$t'_{\sigma} = \frac{dt(\sigma)}{d\sigma} = f(\lambda) = 1 \text{ if } \lambda = 0.$$
 (22)

In this way one reduces (13) to an autonomous integral functional

$$\bar{I}[t(\cdot), q(t(\cdot))] = \int_{\sigma_a}^{\sigma_b} L\left(t(\sigma), q(t(\sigma)), {}^{RC}_{\sigma_a} D^{\alpha}_{\sigma_b} q(t(\sigma))\right) t'_{\sigma} d\sigma, \tag{23}$$

where $t(\sigma_a) = a$ and $t(\sigma_b) = b$. Using the definitions and properties of fractional derivatives given in Section 2, we get successively that

$$\begin{split} R^{C}_{\sigma_{a}}D^{\alpha}_{\sigma_{b}}q(t(\sigma)) &= \frac{1}{\Gamma(n-\alpha)} \int_{\frac{a}{f(\lambda)}}^{\frac{b}{f(\lambda)}} |\sigma f(\lambda) - \theta|^{n-\alpha-1} \left(\frac{d}{d\theta(\sigma)}\right)^{n} q\left(\theta f^{-1}(\lambda)\right) d\theta \\ &= \frac{(t'_{\sigma})^{-\alpha}}{\Gamma(n-\alpha)} \int_{\frac{a}{(t'_{\sigma})^{2}}}^{\frac{b}{(t'_{\sigma})^{2}}} |\sigma - s|^{n-\alpha-1} \left(\frac{d}{ds}\right)^{n} q(s) ds \\ &= (t'_{\sigma})^{-\alpha} {}_{\chi}^{RC} D^{\alpha}_{\omega} q(\sigma) \quad \left(\chi = \frac{a}{(t'_{\sigma})^{2}}, \ \omega = \frac{b}{(t'_{\sigma})^{2}}\right). \end{split}$$

We then have

$$\begin{split} \bar{I}[t(\cdot),q(t(\cdot))] &= \int_{\sigma_a}^{\sigma_b} L\left(t(\sigma),q(t(\sigma)),(t_\sigma^{'})^{-\alpha} \underset{\chi}{R^C} D_\omega^\alpha \, q(\sigma) \, q(\sigma)\right) t_\sigma^{'} d\sigma \\ &\doteq \int_{\sigma_a}^{\sigma_b} \bar{L}_f\left(t(\sigma),q(t(\sigma)),t_\sigma^{'},\underset{\chi}{R^C} D_\omega^\alpha \, q(\sigma)\right) d\sigma \\ &= \int_a^b L\left(t,q(t),\underset{a}{R^C} D_b^\alpha q(t)\right) dt \\ &= I[q(\cdot)] \,. \end{split}$$

If the integral functional (13) is invariant in the sense of Definition 23, then the integral functional (23) is invariant in the sense of Definition 15. It follows from Theorem 21 that

$$\mathcal{D}_{t}^{\alpha} \left[\partial_{4} \bar{L}_{f}, \xi \right] + \mathcal{D}_{t}^{\alpha} \left[\frac{\partial}{\partial t_{\sigma}'} \bar{L}_{f}, \tau \right] = 0 \tag{24}$$

is a fractional conserved law in the sense of Riesz-Caputo. For $\lambda=0$ the condition (22) allows us to write that

$${}^{RC}_{\chi}D^{\alpha}_{\omega}q(\sigma) = {}^{RC}_{a}D^{\alpha}_{b}q(t),$$

and therefore we get that

$$\partial_4 \bar{L}_f = \partial_3 L \tag{25}$$

and

$$\frac{\partial}{\partial t'_{\sigma}} \bar{L}_{f} = \partial_{4} \bar{L}_{f} \cdot \frac{\partial}{\partial t'_{\sigma}} \left[\frac{(t'_{\sigma})^{-\alpha}}{\Gamma(n-\alpha)} \int_{\chi}^{\omega} |\sigma - s|^{n-\alpha-1} \left(\frac{d}{ds} \right)^{n} q(s) \, ds \right] t'_{\sigma} + L$$

$$= \partial_{4} \bar{L}_{f} \cdot \left[\frac{-\alpha(t'_{\sigma})^{-\alpha-1}}{\Gamma(n-\alpha)} \int_{\chi}^{\omega} |\sigma - s|^{n-\alpha-1} \left(\frac{d}{ds} \right)^{n} q(s) \, ds \right] t'_{\sigma} + L$$

$$= -\alpha \partial_{3} L \cdot {}_{\sigma}^{RC} D_{h}^{\alpha} q + L.$$
(26)

Substituting the quantities (25) and (26) into (24), we obtain the fractional conservation law in the sense of Riesz-Caputo (21).

3.3 Optimal control of Riesz-Caputo fractional systems

We now adopt the Hamiltonian formalism in order to generalize the Noether type results found in [14, 49] for the more general context of fractional optimal control in the sense of Riesz-Caputo. For this, we make use of our Noether's Theorem 24 and the standard Lagrange multiplier technique (cf. [14]). The fractional optimal control problem in the sense of Riesz-Caputo is introduced, without loss of generality, in Lagrange form:

$$I[q(\cdot), u(\cdot)] = \int_{a}^{b} L(t, q(t), u(t)) dt \longrightarrow \min, \qquad (27)$$

subject to the fractional differential system

$${}^{RC}_{a}D^{\alpha}_{a}q(t) = \varphi\left(t, q(t), u(t)\right) \tag{28}$$

and initial condition

$$q(a) = q_a. (29)$$

The Lagrangian $L:[a,b]\times\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}$ and the fractional velocity vector $\varphi:[a,b]\times\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^n$ are assumed to be functions of class C^1 with respect to all their arguments. We also assume, without loss of generality, that $0<\alpha\leq 1$. In conformity with the calculus of variations, we are considering that the control functions $u(\cdot)$ take values on an open set of \mathbb{R}^m .

Definition 26. The fractional differential system (28) is said to be a fractional control system in the sense of Riesz-Caputo.

Remark 27. In the particular case $\alpha = 1$ the problem (27)–(29) is reduced to the classical optimal control problem

$$I[q(\cdot), u(\cdot)] = \int_{a}^{b} L(t, q(t), u(t)) dt \longrightarrow \min,$$
(30)

$$\dot{q}(t) = \varphi(t, q(t), u(t)), \quad q(a) = q_a.$$
 (31)

Remark 28. The fractional functional of the calculus of variations in the sense of Riesz-Caputo (13) is obtained from (27)–(28) choosing $\varphi(t, q, u) = u$.

Definition 29 (fractional process in the sense of Riesz-Caputo). An admissible pair $(q(\cdot), u(\cdot))$ which satisfies the fractional control system (28) of the fractional optimal control problem (27)–(29), $t \in [a,b]$, is said to be a fractional process in the sense of Riesz-Caputo.

Theorem 30 ([2]). If $(q(\cdot), u(\cdot))$ is a fractional process of problem (27)–(29) in the sense of Riesz-Caputo, then there exists a co-vector function $p(\cdot) \in PC^1([a,b];\mathbb{R}^n)$ such that for all $t \in [a,b]$ the triple $(q(\cdot), u(\cdot), p(\cdot))$ satisfy the following conditions:

• the Hamiltonian system

$$\begin{cases} {}^{RC}_{a}D^{\alpha}_{b}q(t) = \partial_{4}\mathcal{H}(t,q(t),u(t),p(t)), \\ {}^{R}_{a}D^{\alpha}_{b}p(t) = -\partial_{2}\mathcal{H}(t,q(t),u(t),p(t)); \end{cases}$$

• the stationary condition

$$\partial_3 \mathcal{H}(t, q(t), u(t), p(t)) = 0$$
;

where the Hamiltonian \mathcal{H} is given by

$$\mathcal{H}(t,q,u,p) = L(t,q,u) + p \cdot \varphi(t,q,u) . \tag{32}$$

Definition 31 (fractional Pontryagin extremal in the sense of Riesz-Caputo). A triple $(q(\cdot), u(\cdot), p(\cdot))$ satisfying Theorem 30 will be called a fractional Pontryagin extremal in the sense of Riesz-Caputo.

Remark 32. In the case of the fractional calculus of variations in the sense of Riesz-Caputo one has $\varphi(t,q,u)=u$ (Remark 28) and $\mathcal{H}=L+p\cdot u$. From Theorem 30 we get ${}_a^{RC}D_a^{\alpha}q=u$ and ${}_a^RD_b^{\alpha}p=-\partial_2 L$ from the Hamiltonian system, and from the stationary condition $\partial_3\mathcal{H}=0$ it follows that $p=-\partial_3 L$, thus ${}_a^RD_b^{\alpha}p=-{}_a^RD_b^{\alpha}\partial_3 L$. Comparing both expressions for ${}_a^RD_b^{\alpha}p$, we arrive to the fractional Euler-Lagrange equations (15): $\partial_2 L={}_a^RD_b^{\alpha}\partial_3 L$.

Minimizing (27) subject to (28) is equivalent, by the Lagrange multiplier rule, to minimize

$$J[q(\cdot), u(\cdot), p(\cdot)] = \int_{a}^{b} \left[\mathcal{H}\left(t, q(t), u(t), p(t)\right) - p(t) \cdot {}_{a}^{RC} D_{a}^{\alpha} q(t) \right] dt \tag{33}$$

with \mathcal{H} given by (32).

Remark 33. Theorem 30 is easily proved applying the optimality condition (15) to the equivalent functional (33).

The notion of variational invariance for (27)–(28) is defined with the help of the augmented functional (33).

Definition 34 (variational invariance of (27)–(28)). We say that the integral functional (33) is invariant under the one-parameter family of infinitesimal transformations

$$\begin{cases}
\bar{t} = t + \varepsilon \tau(t, q(t), u(t), p(t)) + o(\varepsilon), \\
\bar{q}(t) = q(t) + \varepsilon \xi(t, q(t), u(t), p(t)) + o(\varepsilon), \\
\bar{u}(t) = u(t) + \varepsilon \varrho(t, q(t), u(t), p(t)) + o(\varepsilon), \\
\bar{p}(t) = p(t) + \varepsilon \varsigma(t, q(t), u(t), p(t)) + o(\varepsilon),
\end{cases}$$
(34)

if

$$\left[\mathcal{H}(\bar{t}, \bar{q}(\bar{t}), \bar{u}(\bar{t}), \bar{p}(\bar{t})) - \bar{p}(\bar{t}) \cdot \frac{RC}{\bar{a}} D_{\bar{b}}^{\alpha} \bar{q}(\bar{t}) \right] d\bar{t}
= \left[\mathcal{H}(t, q(t), u(t), p(t)) - p(t) \cdot \frac{RC}{\bar{a}} D_{\bar{b}}^{\alpha} q(t) \right] dt . \quad (35)$$

The next theorem provides us with an extension of Noether's theorem to the wider fractional context of optimal control in the sense of Riesz-Caputo.

Theorem 35 (Noether's fractional theorem in Hamiltonian form). If (27)–(28) is variationally invariant, in the sense of Definition 34, then

$$\mathcal{D}_{t}^{\alpha}\left[\mathcal{H}(t,q(t),u(t),p(t))-\left(1-\alpha\right)p(t)\cdot_{a}^{RC}D_{b}^{\alpha}q(t),\tau(t,q(t))\right]\\ -\mathcal{D}_{t}^{\alpha}\left[p(t),\xi(t,q(t))\right]=0\quad(36)$$

along any fractional Pontryagin extremal $(q(\cdot), u(\cdot), p(\cdot))$ of problem (27)-(29).

Proof. The fractional conservation law (36) in the sense of Riesz-Caputo is obtained by applying Theorem 24 to the equivalent functional (33).

Remark 36. In the particular case $\alpha = 1$ the fractional optimal control problem (27)–(29) is reduced to the standard optimal control problem (30)–(31). In this situation one gets from Theorem 35 the Noether-type theorem associated with the classical optimal control problem [14, 49]: invariance under a one-parameter family of infinitesimal transformations (34) implies that

$$\mathcal{H}(t, q(t), u(t), p(t))\tau(t, q(t)) - p(t) \cdot \xi(t, q(t)) = constant$$
 (37)

along all the Pontryagin extremals (we obtain the conservation law (37) by choosing $\alpha = 1$ in (36)).

Theorem 35 gives a new and interesting result for autonomous fractional variational problems. Let us consider an autonomous fractional optimal control

problem, i.e., (27) and (28) with the Lagrangian L and the fractional velocity vector φ not depending explicitly on the independent variable t:

$$I[q(\cdot), u(\cdot)] = \int_{a}^{b} L(q(t), u(t)) dt \longrightarrow \min,$$
(38)

$${}_{a}^{RC}D_{b}^{\alpha}q(t) = \varphi\left(q(t), u(t)\right). \tag{39}$$

Corollary 37. For the autonomous fractional problem (38)–(39)

$$_{a}^{R}D_{b}^{\alpha}\left[\mathcal{H}(t,q(t),u(t),p(t))+\left(\alpha-1\right)p(t)\cdot_{a}^{RC}D_{b}^{\alpha}q(t)\right]=0$$

along any fractional Pontryagin extremal $(q(\cdot), u(\cdot), p(\cdot))$.

Proof. As the Hamiltonian \mathcal{H} does not depend explicitly on the independent variable t, we can easily see that (38)–(39) is invariant under translation of the time variable: the condition of invariance (35) is satisfied with $\bar{t}(t)=t+\varepsilon$, $\bar{q}(t)=q(t), \ \bar{u}(t)=u(t), \ \text{and} \ \bar{p}(t)=p(t).$ Indeed, given that $d\bar{t}=dt$, the invariance condition (35) is verified if $\frac{RC}{\bar{a}}D_{\bar{b}}^{\alpha}\bar{q}(\bar{t})=\frac{RC}{a}D_{\bar{b}}^{\alpha}q(t)$. This is true because

$$\begin{split} {}^{RC}D^{\alpha}_{\bar{b}}\bar{q}(\bar{t}) &= \frac{1}{\Gamma(n-\alpha)}\int_{\bar{a}}^{\bar{b}}|\bar{t}-\theta|^{n-\alpha-1}\left(\frac{d}{d\theta}\right)^{n}\bar{q}(\theta)d\theta \\ &= \frac{1}{\Gamma(n-\alpha)}\int_{a+\varepsilon}^{b+\varepsilon}|t+\varepsilon-\theta|^{n-\alpha-1}\left(\frac{d}{d\theta}\right)^{n}\bar{q}(\theta)d\theta \\ &= \frac{1}{\Gamma(n-\alpha)}\int_{a}^{b}|t-s|^{n-\alpha-1}\left(\frac{d}{ds}\right)^{n}\bar{q}(t+\varepsilon)ds \\ &= \frac{R^{C}}{a}D^{\alpha}_{b}\bar{q}(t+\varepsilon) = \frac{R^{C}}{a}D^{\alpha}_{b}\bar{q}(\bar{t}) \\ &= \frac{R^{C}}{a}D^{\alpha}_{b}q(t) \,. \end{split}$$

Using the notation in (34) we have $\tau = 1$ and $\xi = \varrho = \varsigma = 0$. From Theorem 35 we arrive to the intended conclusion.

Corollary 37 asserts that unlike the classical autonomous problem of optimal control, for (38)–(39) the fractional Hamiltonian \mathcal{H} is not conserved. Instead of $\frac{d}{dt}(H) = 0$ we have

$${}_{a}^{R}D_{b}^{\alpha}\left[\mathcal{H}+\left(\alpha-1\right)p(t)\cdot{}_{a}^{RC}D_{b}^{\alpha}q(t)\right]=0\,,\tag{40}$$

i.e., fractional conservation of the Hamiltonian \mathcal{H} plus a quantity that depends on the fractional order α of differentiation. This seems to be explained by violation of the homogeneity of space-time caused by the fractional derivatives, $\alpha \neq 1$. In the particular $\alpha = 1$ we obtain from (40) the classical result: the Hamiltonian \mathcal{H} is preserved along all the Pontryagin extremals of (30)–(31).

4 Examples

To illustrate our results, we consider in this section two examples where the fractional Lagrangian does not depend explicitly on the independent variable t (autonomous case). In both examples we use our Corollary 37 to establish the fractional conservation laws.

Example 38. Let us consider the following fractional problem of the calculus of variations:

$$I[q(\cdot)] = \frac{1}{2} \int_0^1 \left({}_0^{RC} D_1^{\alpha} q(t) \right)^2 dt \longrightarrow \min.$$

The Hamiltonian (32) takes the form $\mathcal{H} = -\frac{1}{2}p^2$. From Corollary 37 we conclude that

$${}_{0}^{R}D_{1}^{\alpha}\left[\frac{p^{2}(t)}{2}(1-2\alpha)\right] = 0.$$
 (41)

Example 39. Consider now the fractional optimal control problem

$$I[q(\cdot)] = \frac{1}{2} \int_0^1 \left[q^2(t) + u^2(t) \right] dt \longrightarrow \min,$$

$${}_0^{RC} D_1^{\alpha} q(t) = -q(t) + u(t),$$

under the initial condition q(0) = 1. The Hamiltonian \mathcal{H} defined by (32) takes the following form:

$$\mathcal{H} = \frac{1}{2} (q^2 + u^2) + p(-q + u).$$

It follows from our Corollary 37 that

$${}_{0}^{R}D_{1}^{\alpha} \left[\frac{1}{2} \left(q^{2} + u^{2} \right) + \alpha p(-q + u) \right] = 0$$
 (42)

along any fractional Pontryagin extremal $(q(\cdot), u(\cdot), p(\cdot))$ of the problem.

For $\alpha = 1$ the conservation laws (41) and (42) give the well known results of conservation of energy.

5 Conclusions and Possible Extensions

The standard approach to solve fractional problems of the calculus of variations is to make use of the necessary optimality conditions given by the fractional Euler-Lagrange equations. These are, in general, nonlinear fractional differential equations, very hard to be solved. One way to address the problem is to find conservation laws, i.e., quantities which are preserved along the Euler-Lagrange extremals, and that can be used to simplify the problem at hands. The main questions addressed in this paper are: (i) What should be a conservation law in the fractional setting? (ii) How to determine such fractional conservation laws? Answer to (i) is given by substituting the classical operator $\frac{d}{dt}$ by a new

fractional operator \mathcal{D}_t^{γ} (see Definition 18 and Remarks 20, 22, and 25); answer to (ii) is given by a fractional Noether's theorem (Theorem 24) that establishes a relation between the existence of variational symmetries and the existence of fractional conservation laws.

Our results can also bring insight to the issue of nonlinear fractional differential equations and fractional calculus of variations from a different perspective. Indeed, the present results together with [42, 48, 53] suggest that Leitmann's direct method can also be adapted to cope with fractional variational problems. Having in mind that the fractional Euler-Lagrange equations are in general difficult or even impossible to be solved analytically, this is an interesting open question, with some preliminary results already obtained in [4].

We believe that there is no approach to fractional calculus better than all the others. All fractional derivatives have some advantages and disadvantages. Several fractional variational problems have been proposed in the literature. This means that for a given classical Lagrangian we have at our disposal several different methods to obtain the fractional Euler-Lagrange equations and corresponding Hamiltonians. The fractional dynamics depend on the fractional derivatives used to construct the Lagrangian, thus the existence of several options can be used to treat different physical systems with different specifications and characteristics. We trust that application of our results to the fractional dynamics with Riesz-Caputo derivatives will bring new opportunities.

The main aim of our paper was to prove a Noether theorem – one of the most beautiful and useful results of the calculus of variations - for fractional variational problems via Riesz-Caputo fractional derivatives. A Noether theorem is valid along the Euler-Lagrange extremals of the problems. Since previous Euler-Lagrange equations obtained in the literature via Riesz-Caputo fractional derivatives (cf. Theorem 11) assume both left and right derivatives to be of the same order α , we have restricted ourselves to this case. However, in principle it is possible to consider in the definition of Riesz-Caputo fractional derivative a different order for the left and right derivative and prove a corresponding Euler-Lagrange equation. With such two-order Euler-Lagrange equation one can then try to generalize our results to this more general setting. Even more general, one can try to extend Theorems 11 and 30 and Theorems 24 and 35 by using generalized Erdélyi-Kober fractional integrals and derivatives [6, 13, 32, 33]. This is highly interesting because of recent applications within the framework of micro structure [15, 16, 17, 26]. These are some interesting possibilities for future work.

We are grateful to two anonymous referees for several good ideas of future research and encouragement words.

References

O. P. Agrawal, Generalized Euler-Lagrange equations and transversality conditions for FVPs in terms of the Caputo derivative, J. Vib. Control 13 (2007), no. 9-10, 1217–1237.

- [2] O. P. Agrawal, Fractional variational calculus in terms of Riesz fractional derivatives, J. Phys. A 40 (2007), no. 24, 6287-6303.
- [3] R. Almeida, A. B. Malinowska and D. F. M. Torres, A fractional calculus of variations for multiple integrals with application to vibrating string, Journal of Mathematical Physics, in press. arXiv:1001.2722
- [4] R. Almeida and D. F. M. Torres, Leitmann's direct method for fractional optimization problems, submitted.
- [5] R. Almeida and D. F. M. Torres, Necessary and sufficient conditions for the fractional calculus of variations with Caputo derivatives, submitted.
- [6] B. Al-Saqabi and V. S. Kiryakova, Explicit solutions of fractional integral and differential equations involving Erdélyi-Kober operators, Appl. Math. Comput. 95 (1998), no. 1, 1–13.
- [7] D. Baleanu and T. Avkar, Lagrangians with linear velocities within Riemann-Liouville fractional derivatives, Nuovo Cimento, 119 (2004), 73–79.
- [8] D. A. Carlson and G. Leitmann, An extension of the coordinate transformation method for open-loop Nash equilibria, J. Optim. Theory Appl. 123 (2004), no. 1, 27–47.
- [9] D. A. Carlson and G. Leitmann, Coordinate transformation method for the extremization of multiple integrals, J. Optim. Theory Appl. 127 (2005), no. 3, 523-533.
- [10] D. A. Carlson and G. Leitmann, A direct method for open-loop dynamic games for affine control systems, in *Dynamic games: theory and applications*, 37–55, Springer, New York, 2005.
- [11] P. Cartigny and C. Deissenberg, An extension of Leitmann's direct method to inequality constraints, Int. Game Theory Rev. 6 (2004), no. 1, 15–20.
- [12] J. Cresson, Fractional embedding of differential operators and Lagrangian systems, J. Math. Phys. 48 (2007), no. 3, 033504, 34 pp.
- [13] D. Cvijovic and J. Klinowski, Integrals involving complete elliptic integrals, J. Comput. Appl. Math. 106 (1999), no. 1, 169–175.
- [14] D. S. Đukić, Noether's theorem for optimum control systems, Internat. J. Control (1) 18 (1973), 667–672.
- [15] A. R. El-Nabulsi, The fractional calculus of variations from extended Erdélyi-Kober operator, Int. J. Mod. Phys. B 23 (2009), no. 16, 3349–3361.
- [16] A. R. El-Nabulsi, Complexified fractional heat kernel and physics beyond the spectral triplet action in noncommutative geometry, Int. J. Geom. Meth. Mod. Phys. 6 (2009), no. 6, 941–963.
- [17] A. R. El-Nabulsi, Complexified quantum field theory and "mass without mass" from multidimensional fractional actionlike variational approach with dynamical fractional exponents, Chaos Solitons Fractals **42** (2009), no. 4, 2384–2398.
- [18] R. A. El-Nabulsi and D. F. M. Torres, Necessary optimality conditions for fractional action-like integrals of variational calculus with Riemann-Liouville derivatives of order (α, β) , Math. Methods Appl. Sci. **30** (2007), no. 15, 1931–1939. arXiv:math-ph/0702099
- [19] R. A. El-Nabulsi and D. F. M. Torres, Fractional actionlike variational problems, J. Math. Phys. 49 (2008), no. 5, 053521, 7 pp. arXiv:0804.4500

- [20] G. S. F. Frederico and D. F. M. Torres, Constants of motion for fractional action-like variational problems, Int. J. Appl. Math. 19 (2006), no. 1, 97–104. arXiv:math.0C/0607472
- [21] G. S. F. Frederico and D. F. M. Torres, Conservation laws for invariant functionals containing compositions, Appl. Anal. 86 (2007), no. 9, 1117–1126. arXiv:0704.0949
- [22] G. S. F. Frederico and D. F. M. Torres, Non-conservative Noether's theorem for fractional action-like variational problems with intrinsic and observer times, Int. J. Ecol. Econ. Stat. 9 (2007), no. F07, 74–82. arXiv:0711.0645
- [23] G. S. F. Frederico and D. F. M. Torres, A formulation of Noether's theorem for fractional problems of the calculus of variations, J. Math. Anal. Appl. 334 (2007), no. 2, 834–846. arXiv:math.0C/0701187
- [24] G. S. F. Frederico and D. F. M. Torres, Fractional optimal control in the sense of Caputo and the fractional Noether's theorem, Int. Math. Forum 3 (2008), no. 9-12, 479-493. arXiv:0712.1844
- [25] G. S. F. Frederico and D. F. M. Torres, Fractional conservation laws in optimal control theory, Nonlinear Dynam. 53 (2008), no. 3, 215–222. arXiv:0711.0609
- [26] L. Galué, S. L. Kalla and V. K. Tuan, Composition of Erdélyi-Kober fractional operators, Integral Transform. Spec. Funct. 9 (2000), no. 3, 185–196.
- [27] P. D. F. Gouveia and D. F. M. Torres, Automatic computation of conservation laws in the calculus of variations and optimal control, Comput. Methods Appl. Math. 5 (2005), no. 4, 387–409. arXiv:math/0509140
- [28] P. D. F. Gouveia, D. F. M. Torres and E. A. M. Rocha, Symbolic computation of variational symmetries in optimal control, Control Cybernet. 35 (2006), no. 4, 831–849. arXiv:math.0C/0604072
- [29] R. Hilfer, Applications of fractional calculus in physics, World Sci. Publishing, River Edge, NJ, 2000.
- [30] J. Jost and X. Li-Jost, Calculus of variations, Cambridge Univ. Press, Cambridge, 1998.
- [31] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, Theory and applications of fractional differential equations, Elsevier, Amsterdam, 2006.
- [32] V. Kiryakova, Generalized fractional calculus and applications, Longman Sci. Tech., Harlow, 1994.
- [33] V. Kiryakova, A brief story about the operators of the generalized fractional calculus, Fract. Calc. Appl. Anal. 11 (2008), no. 2, 203–220.
- [34] G. Leitmann, A note on absolute extrema of certain integrals, Internat. J. Non-Linear Mech. 2 (1967), 55–59.
- [35] G. Leitmann, The calculus of variations and optimal control, Plenum, New York, 1981.
- [36] G. Leitmann, On a class of direct optimization problems, J. Optim. Theory Appl. 108 (2001), no. 3, 467–481.
- [37] G. Leitmann, Some extensions to a direct optimization method, J. Optim. Theory Appl. 111 (2001), no. 1, 1–6.

- [38] G. Leitmann, On a method of direct optimization, Vychisl. Tekhnol. 7 (2002), Special Issue, 63–67.
- [39] G. Leitmann, A direct method of optimization and its application to a class of differential games, Cubo Mat. Educ. 5 (2003), no. 3, 219–228.
- [40] G. Leitmann, A direct method of optimization and its application to a class of differential games, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. 11 (2004), no. 2-3, 191–204.
- [41] J. D. Logan, Applied mathematics, Wiley, New York, 1987.
- [42] A. B. Malinowska and D. F. M. Torres, Leitmann's direct method of optimization for absolute extrema of certain problems of the calculus of variations on time scales, Appl. Math. Comput., in press. DOI:10.1016/j.amc.2010.01.015 arXiv:1001.1455
- [43] K. S. Miller and B. Ross, An introduction to the fractional calculus and fractional differential equations, Wiley, New York, 1993.
- [44] S. I. Muslih and D. Baleanu, Hamiltonian formulation of systems with linear velocities within Riemann-Liouville fractional derivatives, J. Math. Anal. Appl. 304 (2005), no. 2, 599–606.
- [45] I. Podlubny, Fractional differential equations, Academic Press, San Diego, CA, 1999.
- [46] F. Riewe, Mechanics with fractional derivatives, Phys. Rev. E (3) 55 (1997), no. 3, part B, 3581–3592.
- [47] S. G. Samko, A. A. Kilbas and O. I. Marichev, Fractional integrals and derivatives, Translated from the 1987 Russian original, Gordon and Breach, Yverdon, 1993.
- [48] C. J. Silva and D. F. M. Torres, Absolute extrema of invariant optimal control problems, Commun. Appl. Anal. 10 (2006), no. 4, 503–515. arXiv:math/0608381
- [49] D. F. M. Torres, On the Noether theorem for optimal control, Eur. J. Control 8 (2002), no. 1, 56–63.
- [50] D. F. M. Torres, Conservation laws in optimal control, in *Dynamics, bifurcations, and control (Kloster Irsee, 2001)*, 287–296, Lecture Notes in Control and Inform. Sci., 273, Springer, Berlin, 2002.
- [51] D. F. M. Torres, Quasi-invariant optimal control problems, Port. Math. (N.S.) 61 (2004), no. 1, 97–114. arXiv:math/0302264
- [52] D. F. M. Torres, Proper extensions of Noether's symmetry theorem for nonsmooth extremals of the calculus of variations, Commun. Pure Appl. Anal. 3 (2004), no. 3, 491–500.
- [53] D. F. M. Torres and G. Leitmann, Contrasting two transformation-based methods for obtaining absolute extrema, J. Optim. Theory Appl. 137 (2008), no. 1, 53–59. arXiv:0704.0473