

Leitmann's direct method for fractional optimization problems*

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Abstract

Based on a method introduced by Leitmann [Internat. J. Non-Linear Mech. **2** (1967), 55–59], we exhibit exact solutions for some fractional optimization problems of the calculus of variations and optimal control.

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1 Introduction

A direct method to determine solutions for some problems of the calculus of variations, avoiding the variational approach, was introduced by George Leitmann in 1967 [19]. Leitmann's method is based on the following two steps: (i) by an appropriate coordinate transformation, we rewrite the initial problem as an equivalent simpler one; (ii) knowing the solution for the new equivalent problem, and since there exists an one-to-one correspondence between the minimizers (or maximizers) of the new problem with the ones of the original problem, we determine the desired solution. We also note [28, 29], where this method is extended to optimal control. The idea behind the direct method of Leitmann is clever and simple, providing a “principle” — a general assertion that can be formulated in precise terms, as a theorem, in many different contexts, under many different assumptions. Leitmann's principle reflects the fact that optimality does not depend on the particular coordinates used, being preserved under invariance transformations up to an exact differential. For more on the subject we refer the reader to [8, 20, 21, 29].

Fractional calculus is a branch of mathematics where one allows integrals and derivatives to be of an arbitrary real or complex order [24, 25, 27]. It has found many applications in pure and applied mathematics, physics, and engineering [10, 18]. An area of fractional calculus that is receiving remarkable attention consists in developing a theory of the calculus

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of variations and optimal control in the presence of fractional (non-integer) derivatives, and investigating its applications — see [1, 6, 9, 13, 16] and references therein.

The literature on necessary optimality conditions for fractional optimal control problems is vast (cf., e.g., [1, 4, 5, 6, 12, 15]). Applications of such necessary conditions involve, however, the solution of fractional differential equations, which is a difficult task to accomplish analytically. Often, numerical methods or truncated Taylor series are used [2, 6, 11]. In this note, motivated by the results of [17, 28, 29], we give the first examples of fractional problems of the calculus of variations and optimal control with a known exact expression for the global minimizers. Such minimizers are found utilizing Leitmann's principle.

Although simple and easy to understand, Leitmann's principle is not sufficiently well-known and used. This is particularly true in the present context of fractional calculus, where Leitmann's direct method went completely unnoticed while necessary optimality conditions, extremely difficult to apply and often proved under (too) strong hypotheses, are widely explored. Recently there has been an increase of interest in Leitmann's method with several papers devoted to various aspects of the subject, including discrete-time problems of the calculus of variations and, more generally, variational problems on time scales [22]. There is, however, no direct link between the time scale calculus and the fractional calculus: they are different areas of research, with different communities. Moreover, there are profound and interesting implications of Leitmann's method for fractional problems (cf. e.g. Remark 2) that have no counterpart in other problems of the calculus of variations, such as the ones studied in [14, 22, 23]. This is due to the fact that Euler-Lagrange type conditions proved in the literature of the fractional calculus of variations are often deduced imposing that admissible functions have continuous left and right fractional derivatives on the closed interval where the problem is defined — see [1] and discussions in [3, 4]. This is different from the calculus of variations in different contexts, e.g., different from what happens with time scale variational problems [22].

The text is organized as follows. In Section 2 we present a short introduction to fractional calculus, providing the necessary concepts and results needed in the sequel. A simple version of Leitmann's method for solving fractional variational problems is given in Section 3, while in Section 4 we illustrate the effectiveness of the method by obtaining exact global minimizers for fractional variational problems depending on fractional derivatives and integrals, as well as fractional optimal control problems. We end with conclusions in Section 5.

2 Preliminaries

Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function, $\alpha > 0$ a given real, and $n := [\alpha] + 1$, where $[\alpha]$ denotes the integer part of α . Let $\Gamma(\cdot)$ be the Gamma function, i.e.,

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \quad z > 0.$$

The Riemann-Liouville fractional integral of order α is defined by

$${}_a I_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt,$$

while the Riemann-Liouville fractional derivative of order α is given by

$${}_a D_x^\alpha f(x) = \frac{d^n}{dx^n} {}_a I_x^{n-\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x (x-t)^{n-\alpha-1} f(t) dt.$$

In particular,

$${}_aD_x^0 f(x) = f(x) \quad \text{and} \quad {}_aD_x^m f(x) = f^{(m)}(x), \quad m \in \mathbb{N}.$$

There exist composition formulas between fractional integrals and fractional derivatives, similar to the standard ones (proofs may be found, e.g., in [18]):

Theorem 1. *If $f \in L_p(a, b)$ ($1 \leq p \leq \infty$), $\alpha > 0$ and $\beta > 0$, then*

$${}_aI_x^\alpha {}_aI_x^\beta f(x) = {}_aI_x^{\alpha+\beta} f(x)$$

almost everywhere on $[a, b]$.

Theorem 2. *If $f \in L_p(a, b)$ ($1 \leq p \leq \infty$), then*

$${}_aD_x^\alpha {}_aI_x^\alpha f(x) = f(x)$$

almost everywhere on $[a, b]$.

Theorem 3. *Let $f_{n-\alpha}(x) = {}_aI_x^{n-\alpha} f(x)$. If $f \in L_1(a, b)$ and $f_{n-\alpha} \in AC^n[a, b]$, then*

$${}_aI_x^\alpha {}_aD_x^\alpha f(x) = f(x) - \sum_{j=1}^n \frac{f_{n-\alpha}^{(n-j)}(a)}{\Gamma(\alpha - j + 1)} (x - a)^{\alpha-j}$$

almost everywhere on $[a, b]$.

In particular, if ${}_aI_a^{1-\alpha} f(a) = 0$ and $\alpha \in (0, 1)$, then ${}_aI_x^\alpha {}_aD_x^\alpha f(x) = f(x)$.

3 Fractional version of Leitmann's approach

Consider the following fractional problem of the calculus of variations: to minimize

$$\mathfrak{J}(y) := \int_a^b F(x, y, {}_aD_x^\alpha y(x)) dx \tag{1}$$

under the constraint

$${}_aI_b^{1-\alpha} y(b) = y_b, \tag{2}$$

where y_b is a fixed real. Here we consider continuous functions $y : [a, b] \rightarrow \mathbb{R}$ such that ${}_aD_x^\alpha y(x)$ exists on $[a, b]$. We also emphasize that the condition ${}_aI_a^{1-\alpha} y(a) = 0$ appears implicitly, since function y is continuous (cf. [24, pag. 46]). If we allow $\alpha = 1$, functional (1) becomes the standard one,

$$\mathfrak{J}(y) := \int_a^b F(x, y, y'(x)) dx, \tag{3}$$

and equality (2) reads as $y(b) = y_b$, that is, we obtain a classical problem of the calculus of variations: to minimize (3) under given boundary conditions $y(a) = 0$ and $y(b) = y_b$.

The next result is obtained by following the method presented in [19].

Theorem 4. Let $y(x) = z(x, \tilde{y}(x))$ be a continuous transformation having a unique inverse $\tilde{y}(x) = \tilde{z}(x, y(x))$, such that there exists an one-to-one correspondence

$$y(x) \Leftrightarrow \tilde{y}(x),$$

for every function y satisfying

$${}_a I_b^{1-\alpha} y(b) = y_b \quad (4)$$

and for every function \tilde{y} satisfying

$${}_a I_b^{1-\alpha} \tilde{y}(b) = {}_a I_b^{1-\alpha} \tilde{z}(x, y(x))|_{x=b}. \quad (5)$$

In addition assume that there exists a C^1 function $H : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ such that the relation

$$F(x, y, {}_a D_x^\alpha y(x)) - F(x, \tilde{y}, {}_a D_x^\alpha \tilde{y}(x)) = \frac{d}{dx} H(x, {}_a I_x^{1-\alpha} \tilde{y}(x))$$

holds. Then there exists a one-to-one correspondence between the minimizers $y^*(x)$ of \mathfrak{J} verifying (4) and the minimizers $\tilde{y}^*(x) = \tilde{z}(x, y^*(x))$ of \mathfrak{J} verifying (5).

Proof. This is obvious from

$$\begin{aligned} \mathfrak{J}(y^*) - \mathfrak{J}(\tilde{y}^*) &= \int_a^b F(x, y^*, {}_a D_x^\alpha y^*(x)) - F(x, \tilde{y}^*, {}_a D_x^\alpha \tilde{y}^*(x)) dx \\ &= \int_a^b \frac{d}{dx} H(x, {}_a I_x^{1-\alpha} \tilde{y}^*(x)) \\ &= H(b, {}_a I_b^{1-\alpha} \tilde{y}^*(b)) - H(a, 0) \end{aligned}$$

and the fact that the right-hand side is a constant. \square

In the next section we show that Theorem 4 can be applied with success not only to fractional variational problems of form (1)–(2) but also to fractional optimal control problems [2, 16] and problems depending on both fractional derivatives and fractional integrals [4].

4 Applications

We illustrate the application of Leitmann's method with examples from three different classes of problems: Proposition 1 gives an exact solution for a standard fractional variational problem (1)–(2); Proposition 2 gives the global minimizer for a family of problems whose Lagrangian depends simultaneously on a fractional derivative and a fractional integral of y ; Proposition 3 gives the exact solution of a fractional optimal control problem.

Proposition 1. Let α be an arbitrary real between 0 and 1. The global minimizer of the fractional problem

$$\mathfrak{J}(y) = \int_0^1 ({}_0 D_x^\alpha y(x))^2 dx \longrightarrow \min \quad (6)$$

under the constraint

$${}_0 I_1^{1-\alpha} y(1) = c \quad (7)$$

is

$$y(x) = \frac{c}{\alpha \Gamma(\alpha)} x^\alpha. \quad (8)$$

Remark 1. We note that for $\alpha = 1$ problem (6)–(7) reduces to the standard problem of the calculus of variations

$$\begin{aligned} \int_0^1 (y'(x))^2 dx &\longrightarrow \min \\ y(0) = 0, \quad y(1) &= c. \end{aligned} \quad (9)$$

The global minimizer of problem (9) is $y(x) = cx$ (cf. e.g. [28, pag. 509]), which coincides with our solution (8) for $\alpha = 1$.

Remark 2. Problem (6)–(7) is trivial when $c = 0$: since $\mathfrak{J}(y) \geq 0$ for any function y and $\mathfrak{J}(0)$ is zero, it is obvious that the null function is the solution of (6)–(7). We are thus interested in the nontrivial case $c \neq 0$. In [1] a fractional Euler-Lagrange necessary optimality condition is proved, under the hypothesis that admissible functions y have continuous left and right fractional derivatives on the closed interval $[a, b]$. Such an assumption implies that $y(a) = y(b) = 0$ (cf. [26]). This means that Agrawal's fractional Euler-Lagrange equation given in [1] can only be applied to (6)–(7) when we restrict ourselves to the trivial case $c = 0$. Indeed, if $c \neq 0$, then $y(1) = \frac{c}{\alpha\Gamma(\alpha)} \neq 0$ and therefore the right fractional derivative is not defined everywhere on interval $[0, 1]$. It is easy to check that for $c \neq 0$ the solution (8) of problem (6)–(7) does not satisfy the Euler-Lagrange equation derived in [1]. The trivial situation $c = 0$ is the only case when the methods of [1] succeed in finding the solution of (6)–(7).

Proof. (of Proposition 1) Let $y = \tilde{y} + f$, where f is a function such that ${}_0D_x^\alpha f(x) = K$, for all $x \in [0, 1]$ (K to be specified later). We consider functions \tilde{y} satisfying the equation

$${}_0I_1^{1-\alpha} \tilde{y}(x) = 0. \quad (10)$$

Since ${}_0D_x^\alpha f = K$, then

$${}_0I_x^\alpha {}_0D_x^\alpha f(x) = {}_0I_x^\alpha K$$

and the continuity on f implies that ${}_aI_a^{1-\alpha} f(a) = 0$. Consequently

$$f(x) = \frac{K}{\alpha\Gamma(\alpha)} x^\alpha.$$

Moreover,

$$\begin{aligned} \int_0^1 ({}_0D_x^\alpha y)^2 dx &= \int_0^1 ({}_0D_x^\alpha \tilde{y} + {}_0D_x^\alpha f)^2 dx \\ &= \int_0^1 ({}_0D_x^\alpha \tilde{y})^2 dx + \int_0^1 (2K {}_0D_x^\alpha \tilde{y} + K^2) dx \\ &= \int_0^1 ({}_0D_x^\alpha \tilde{y})^2 dx + \int_0^1 \frac{d}{dx} (2K {}_0I_x^{1-\alpha} \tilde{y} + K^2 x) dx. \end{aligned}$$

Observe that $\int_0^1 ({}_0D_x^\alpha \tilde{y})^2 dx \geq 0$ and $\int_0^1 {}_0D_x^\alpha \tilde{y} dx = 0$ if $\tilde{y}(x) \equiv 0$. Therefore $\tilde{y}(x) \equiv 0$ is a minimizer of \mathfrak{J} , and satisfies equation (10). Since the second integral of the last sum is constant (actually, is equal to K^2), it follows that

$$y(x) = \frac{K}{\alpha\Gamma(\alpha)} x^\alpha$$

is a solution of problem (6)–(7). Since

$${}_0I_x^{1-\alpha} \frac{K}{\alpha\Gamma(\alpha)} x^\alpha = Kx$$

it follows that ${}_0I_1^{1-\alpha}y(1) = K$ and so $K = c$. \square

We now study a variational problem where not only a fractional derivative of y but also a fractional integral of y appears in the Lagrangian. This new class of fractional functionals of the calculus of variations, that depend not only on fractional derivatives but also on fractional integrals, has been recently studied in [4]. In that paper necessary and sufficient conditions of optimality for the fundamental problem of the calculus of variations and for problems subject to integral constraints (isoperimetric problems) are studied. However, similarly to [1], the conditions are difficult to apply, and no example has been solved analytically. We now show how Leitmann's direct method can also be used to solve such problems.

Proposition 2. *Let g be a given function of class C^1 with $g(x) \neq 0$ on $[0, 1]$, and α and ξ real numbers with α between zero and one. The global minimizer of the fractional variational problem*

$$\int_0^1 [{}_0D_x^\alpha y(x) \cdot g(x) + ({}_0I_x^{1-\alpha} y(x) + 1)g'(x)]^2 dx \longrightarrow \min \quad (11)$$

$${}_0I_1^{1-\alpha} y(1) = \xi$$

is given by the function

$$y(x) = {}_0D_x^{1-\alpha} \left(\frac{[g(1)(\xi + 1) - g(0)]x + g(0)}{g(x)} - 1 \right). \quad (12)$$

Remark 3. *For $\alpha = 1$ problem (11) coincides with the classical problem of the calculus variations*

$$\int_0^1 [y'(x) \cdot g(x) + (y(x) + 1)g'(x)]^2 dx \longrightarrow \min$$

$$y(0) = 0, \quad y(1) = \xi$$

that has been studied by Leitmann in his seminal paper of 1967 [19]. For the integer-order case $\alpha = 1$, our function (12) reduces to $y(x) = \frac{[g(1)(\xi+1)-g(0)]x+g(0)}{g(x)} - 1$, which coincides with the results of Leitmann [19].

Proof. (of Proposition 2) To begin, observe that

$$\frac{d}{dx} [({}_0I_x^{1-\alpha} y(x) + 1)g(x)] = {}_0D_x^\alpha y(x) \cdot g(x) + ({}_0I_x^{1-\alpha} y(x) + 1)g'(x).$$

Let y be an admissible function to problem (11), and consider the transformation $y(x) = \tilde{y}(x) + f(x)$, with f to be determined later. Then

$$\begin{aligned} & \left(\frac{d}{dx} [({}_0I_x^{1-\alpha} \tilde{y}(x) + {}_0I_x^{1-\alpha} f(x) + 1)g(x)] \right)^2 - \left(\frac{d}{dx} [({}_0I_x^{1-\alpha} \tilde{y}(x) + 1)g(x)] \right)^2 \\ &= 2 \frac{d}{dx} [({}_0I_x^{1-\alpha} \tilde{y}(x) + 1)g(x)] \cdot \frac{d}{dx} [{}_0I_x^{1-\alpha} f(x) \cdot g(x)] + \left(\frac{d}{dx} [{}_0I_x^{1-\alpha} f(x) \cdot g(x)] \right)^2 \\ &= \frac{d}{dx} [{}_0I_x^{1-\alpha} f(x) \cdot g(x)] \cdot \frac{d}{dx} [(2({}_0I_x^{1-\alpha} \tilde{y}(x) + 1) + {}_0I_x^{1-\alpha} f(x)) g(x)]. \end{aligned}$$

Let us determine f in such a way that

$$\frac{d}{dx} [{}_0I_x^{1-\alpha} f(x) \cdot g(x)] = \text{const.}$$

Integrating, we deduce that

$${}_0I_x^{1-\alpha} f(x) \cdot g(x) = Ax + B,$$

i.e.,

$$f(x) = {}_0D_x^{1-\alpha} \left(\frac{Ax + B}{g(x)} \right).$$

Consider now the new problem

$$\begin{aligned} \int_0^1 ({}_0D_x^\alpha \tilde{y}(x) \cdot g(x) + ({}_0I_x^{1-\alpha} \tilde{y}(x) + 1)g'(x))^2 dx &\longrightarrow \min \\ {}_0I_1^{1-\alpha} \tilde{y}(1) &= \frac{1}{g(1)} - 1. \end{aligned} \quad (13)$$

It is easy to see, and a trivial exercise to check, that

$$\tilde{y}(x) = {}_0D_x^{1-\alpha} \left(\frac{1}{g(x)} - 1 \right)$$

is a solution of problem (13). Therefore

$$y(x) = {}_0D_x^{1-\alpha} \left(\frac{Ax + C}{g(x)} - 1 \right), \quad \text{with } C = B + 1,$$

is a solution of (11). Using the boundary conditions

$${}_0I_0^{1-\alpha} y(0) = 0 \quad \text{and} \quad {}_0I_1^{1-\alpha} y(1) = \xi$$

we obtain the values for the constants A and C : $A = g(1)(\xi + 1) - g(0)$ and $C = g(0)$. \square

Proposition 3 deals with a fractional optimal control problem. Using Leitmann's method we prove it. For the classical (non-fractional) approach, we refer the reader to [28, 29].

Proposition 3. *Let $\alpha \in (0, 1)$ be a real number. Consider the following fractional optimal control problem:*

$$\mathfrak{J}(u_1, u_2) = \int_0^1 [(u_1(x))^2 + (u_2(x))^2] dx \longrightarrow \min \quad (14)$$

subject to the fractional control system

$$\begin{cases} {}_0D_x^\alpha y_1(x) = \exp(u_1(x)) + u_1(x) + u_2(x) \\ {}_0D_x^\alpha y_2(x) = u_2(x) \end{cases} \quad (15)$$

and fractional boundary conditions

$$\begin{cases} {}_0I_1^{1-\alpha} y_1(1) = 2 \\ {}_0I_1^{1-\alpha} y_2(1) = 1. \end{cases} \quad (16)$$

The global minimizer of problem (14)–(16) is

$$(u_1(x), u_2(x)) \equiv (0, 1), \quad (y_1(x), y_2(x)) = \left(\frac{2x^\alpha}{\alpha\Gamma(\alpha)}, \frac{x^\alpha}{\alpha\Gamma(\alpha)} \right). \quad (17)$$

Proof. We use the following change of variables:

$$\begin{cases} \tilde{y}_1(x) = y_1(x) - {}_0I_x^\alpha 1 \\ \tilde{y}_2(x) = y_2(x) - {}_0I_x^\alpha 1 \\ \tilde{u}_1(x) = u_1(x) \\ \tilde{u}_2(x) = u_2(x) - 1. \end{cases}$$

For the new variables, we compute

$$\begin{aligned} {}_0D_x^\alpha \tilde{y}_1 &= {}_0D_x^\alpha y_1 - {}_0D_x^\alpha {}_0I_x^\alpha 1 \\ &= \exp(u_1) + u_1 + u_2 - 1 \\ &= \exp(\tilde{u}_1) + \tilde{u}_1 + \tilde{u}_2 \end{aligned}$$

and

$$\begin{aligned} {}_0D_x^\alpha \tilde{y}_2 &= {}_0D_x^\alpha y_2 - {}_0D_x^\alpha {}_0I_x^\alpha 1 \\ &= \tilde{u}_2. \end{aligned}$$

Therefore, $(\tilde{y}_1, \tilde{y}_2, \tilde{u}_1, \tilde{u}_2)$ satisfies (15). System (16) becomes

$${}_0I_1^{1-\alpha} \tilde{y}_1(1) = 1 \quad \text{and} \quad {}_0I_1^{1-\alpha} \tilde{y}_2(1) = 0 \quad (18)$$

since ${}_0I_x^{1-\alpha} {}_0I_x^\alpha 1 = {}_0I_x^1 1 = x$.

Consider a new problem, which we label as (\tilde{P}) : minimize $\mathfrak{J}(\cdot, \cdot)$ subject to system (15) and conditions (18). Evaluating the functional $\mathfrak{J}(\cdot, \cdot)$ at $(\tilde{u}_1, \tilde{u}_2)$, we obtain

$$\begin{aligned} \mathfrak{J}(\tilde{u}_1, \tilde{u}_2) &= \int_0^1 [(\tilde{u}_1(x))^2 + (\tilde{u}_2(x))^2] dx \\ &= \int_0^1 [(u_1(x))^2 + (u_2(x))^2 - 2u_2(x) + 1] dx \\ &= \mathfrak{J}(u_1, u_2) + \int_0^1 (-2{}_0D_x^\alpha y_2(x) + 1) dx \\ &= \mathfrak{J}(u_1, u_2) + \int_0^1 \frac{d}{dx} (-2{}_0I_x^{1-\alpha} y_2(x) + x) dx. \end{aligned}$$

Observe that the second term of the last expression is constant. Because $\tilde{u}_1(x) \equiv \tilde{u}_2(x) \equiv 0$ is a solution of the problem (\tilde{P}) , we get

$$\begin{cases} {}_0D_x^\alpha \tilde{y}_1 = 1 \\ {}_0D_x^\alpha \tilde{y}_2 = 0, \end{cases}$$

i.e.,

$$\begin{cases} \tilde{y}_1(x) = {}_0I_x^\alpha 1 = \frac{x^\alpha}{\alpha\Gamma(\alpha)} \\ \tilde{y}_2(x) = {}_0I_x^\alpha 0 = 0. \end{cases}$$

Therefore, the solution of problem (14)–(16) is given by

$$(u_1(x), u_2(x)) = (0, 1)$$

and

$$(y_1(x), y_2(x)) = \left(\frac{2x^\alpha}{\alpha\Gamma(\alpha)}, \frac{x^\alpha}{\alpha\Gamma(\alpha)} \right).$$

□

For $\alpha = 1$ the problem (14)–(16) was solved in [29, pag. 56]: the solution coincides with (17) taking $\alpha = 1$. It is worth noticing that it is far from trivial to obtain Proposition 3 using the techniques of [2, 16].

5 Conclusions

In this paper we illustrate the direct approach of George Leitmann within the fractional variational context. Although we only consider the continuous fractional calculus of variations, the techniques remain valid for the discrete variational calculus recently introduced in [7].

To the best of our knowledge, no direct methods have ever been applied before to fractional optimal control or variational problems. In our opinion there is an urgent need to promote such direct methods among the community of fractional variational problems. Indeed, as illustrated in Section 4, and notwithstanding the simplicity of the statement of Theorem 4, in some cases a difficult problem can be simplified and easily solved analytically without numerical solution of complicated fractional Euler-Lagrange type conditions. More than that, Leitmann's approach allows one to solve problems for which available methods of the literature of fractional calculus fail to apply (cf. Remark 2).

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