Global optimality conditions and optimization methods for quadratic assignment problems☆

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Abstract

In this paper some global optimality conditions for general quadratic {0, 1} programming problems with linear equality constraints are discussed and then some global optimality conditions for quadratic assignment problems (QAP) are presented. A local optimization method for (QAP) is derived according to the necessary global optimality conditions. A global optimization method for (QAP) is presented by combining the sufficient global optimality conditions, the local optimization method and some auxiliary functions. Some numerical examples are given to illustrate the efficiency of the given optimization methods.

Keywords:
Quadratic assignment program, global optimality condition, local optimization method, global optimization method, auxiliary function.

1. Introduction

The quadratic assignment problem (QAP) was introduced by Koopmans and Beckmann in 1957, as a mathematical model for the location of indivisible economical activities, see [1]. The (QAP) in Koopmans-Beckmann form can be written as

$$
\begin{aligned}
(QAP) \qquad \min \quad & \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} a_{ik}b_{jl}x_{ij}x_{kl} + \sum_{i,j=1}^{n} c_{ij}x_{ij} \\
\text{s.t.} \quad & \sum_{i=1}^{n} x_{ij} = 1, \quad \sum_{j=1}^{n} x_{ij} = 1 \\
& x_{ij} \in \{0,1\}, i, j = 1, 2, \ldots, n.
\end{aligned}
$$

It is astonishing how many real life applications can be modeled as (QAP)s. An early natural application in location theory was used by Dickey and Hopkins (see [2]) in a campus planning model. In addition to facility location, (QAP)s appear in a variety of applications such as computer manufacturing, scheduling, process communications and turbine balancing. The traveling salesman problem may be seen as a special case of (QAP) if one assumes that the flows connect all facilities only along a single ring, all flows have the same non-zero (constant) value. Many other problems of standard combinatorial optimization problems may be also written in this form , see [3].

Besides the wide range of practical applications of the problem (QAP), it is NP-hard. The proof that the (QAP) is indeed NP-complete was first shown by Sahni and Gonzalez [4] in 1976. Sahni and Gonzalez [4] also proved that any

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routine that finds even an ε-approximate solution is also NP-complete, thus making the (QAP) among the "hardest of the hard" of all combinatorial optimization problems. It is therefore not surprising that verifying optimality is also an NP-hard problem. In fact, even checking local optimality is a hard problem, see [5].

Because of their many real world applications and complexity, many authors have investigated this problem class, see [5, 6, 7, 8, 9]. When it comes to the global optimization methods, there are two classes of strategies: exact or heuristic [3]. In the first case, the different methods used to achieve a global optimum for the (QAP) include branch-and-bound, cutting plane methods [10, 11] or combinations of these methods, like branch-and-cut [12] and dynamic programming [13]. Heuristic procedures include constructive methods [14, 15, 16], limited enumeration methods [17], improvement methods [18], simulated annealing [19], genetic algorithms [20], scatter search [21], ant colony optimization [22], tabu search [23, 24], greedy randomized adaptive search procedures [25] and variable neighborhood search [26]. However, no dominant algorithm has emerged [27].

In this paper, we will first investigate some global optimality conditions for problem (QAP), including some sufficient global optimality conditions and some necessary global optimality conditions. We will then present a new local optimization method for problem (QAP) by using the presented necessary global optimality conditions. Finally we present a new global optimization method by combining the presented sufficient global optimality conditions, the local optimization methods and some auxiliary functions, which belongs to improvement methods.

The rest of the paper is organized as follows. In section 2, we discuss some global optimality conditions for general quadratic $\{0, 1\}$ programming problem with linear equality constraints. We provide in section 3 some global optimality conditions for problem (QAP). In section 4, we present some optimization methods for problem (QAP), including a local optimization method and a global optimization method based on the presented global optimality conditions. In section 5, we give some numerical examples to illustrate the efficiency of these optimization methods for problem (QAP).

2. Global Optimality Conditions for {0, 1} Quadratic Problems with Linear Equivalent Constraints

The real line is denoted by R and the n-dimensional Euclidean space is denoted by R^n. For vectors $x, y \in R^n$, $x \geq y$ means that $x_i \geq y_i$, for $i = 1, \ldots, n$. The notation $A \geq B$ means $A - B$ is a positive semidefinite matrix and $A \leq 0$ means $-A \geq 0$. A diagonal matrix with diagonal elements $\alpha_1, \ldots, \alpha_n$ is denoted by $\mathrm{diag}(\alpha_1, \ldots, \alpha_n)$ or $\mathrm{diag}(\alpha)$, where $\alpha = (\alpha_1, \ldots, \alpha_n)^T$. S^n denotes the set of all the $n \times n$ symmetric matrixes.

Firstly, consider the following unconstrained quadratic $\{0, 1\}$ programming problem: (UQP):

$$(UQP) \qquad \begin{array}{ll} \min & f(x) := x^T A_0 x + x^T a_0 \\ s.t. & x \in U = \{0, 1\}^n, \end{array}$$

where $A_0 \in S^n, a_0 = (a_1^0, \ldots, a_n^0)^T \in R^n$. For a given $\bar{x} \in U$, let $\bar{X} = \mathrm{diag}(\bar{x})$ and let $e := (1, \ldots, 1)^T$ and $I = \mathrm{diag}(e)$.

By Theorem 3.1 in reference [28], we can obtain the following sufficient global optimality condition for problem (UQP).

Proposition 1. [28] **(Sufficient Global Optimality Condition for** (UQP)**)** Let $\bar{x} \in U$. If

$$[SCU] \qquad \mathrm{Diag}((2\bar{X} - I)(a_0 + 2A_0\bar{x})) \leq A_0,$$

then $\bar{x}$ is a global minimizer of problem (UQP).

Proof. Let $y := 2x - e$. Then problem (UQP) is equivalent to the following problem:

$$(UQP)' \qquad \begin{array}{ll} \min & g(y) := 1/2 y^T A_0 y + y^T (A_0 e + a_0) \\ s.t. & y \in \{-1, 1\}^n. \end{array}$$

Thus $\bar{x}$ satisfies condition $[SCU]$ implied that $\bar{y} := 2\bar{x} - e$ satisfies that

$$\mathrm{Diag}(\bar{Y}(a_0 + A_0 e + A_0\bar{y})) \leq A_0,$$

which satisfies the sufficient condition for problem $(UQP)'$ given by Theorem 3.1 in reference [28]. Hence, $\bar{y}$ is a global minimizer of problem $(UQP)'$, which means that $\bar{x}$ is a global minimizer of problem (UQP).

Let

$$\Gamma = \{\gamma = \{i_1, \ldots, i_k\} \mid i_j \in \{1, \ldots, n\}, i_j \neq i_r, 1 \leq j, r \leq k, 1 \leq k \leq n\}. \tag{1}$$

Obviously, Γ is a finite set. For $A = (a_{ij})_{n\times n} \in S^n$, $a := (a_j)_{n\times 1} = (a_1, \ldots, a_n)^T \in R^n$ and $\gamma = \{i_1, \ldots, i_k\} \in \Gamma$, let

$$\begin{aligned}
\text{diag}(A) : &= \text{diag}(a_{11}, \ldots, a_{nn}) \\
[A]_\gamma : &= (a_{jr})_{j,r\in\gamma} = \begin{pmatrix} a_{i_1i_1} & a_{i_1i_2} & \cdots & a_{i_1i_k} \\ a_{i_2i_1} & a_{i_2i_2} & \cdots & a_{i_2i_k} \\ \cdots & \cdots & \cdots & \cdots \\ a_{i_ki_1} & a_{i_ki_2} & \cdots & a_{i_ki_k} \end{pmatrix}_{k\times k} &(2) \\
[a]_\gamma : &= (a_j)_{j\in\gamma} = (a_{i_1}, \ldots, a_{i_k})^T, &(3) \\
e_\gamma : &= (e^1, \ldots, e^n)^T, &(4)
\end{aligned}$$

where $e^j = \begin{cases} 1, & \text{if } j \in \gamma \\ 0, & \text{if } j \notin \gamma \end{cases}$.

Theorem 1. (Necessary and Sufficient Global Optimality Condition for Problem (UQP)) *Let $\bar{x} \in U$. Then $\bar{x}$ is a global minimizer of problem (UQP), if and only if*

$$[NSCU] \qquad [2\bar{x} - e]_\gamma^T[\text{diag}((2\bar{X} - I)(a_0 + 2A_0\bar{x})) - A_0]_\gamma[2\bar{x} - e]_\gamma \leq 0, \forall\gamma \in \Gamma.$$

Proof. By definition, $\bar{x}$ is a global minimizer of problem (UQP) if and only if

$$(x - \bar{x})^T A_0(x - \bar{x}) + (x - \bar{x})^T(a_0 + 2A_0\bar{x}) \geq 0, \forall x \in U. \tag{5}$$

For any $\gamma = \{i_1, \ldots, i_k\} \in \Gamma$, let

$$x^\gamma = (x_1, \ldots, x_n)^T, \text{ where } x_i := \begin{cases} 1 - \bar{x}_i, & \text{if } i \in \gamma \\ \bar{x}_i, & \text{if } i \notin \gamma \end{cases},$$

then $x^\gamma \in U$ and $x^\gamma - \bar{x} = (I - 2\bar{X})e_\gamma$. Furthermore, we can easily to verify that $x \in U$ if and only if there exists a $\gamma \in \Gamma$, such that $x = x^\gamma$. For any $\gamma \in \Gamma$, from (5), we can obtain that

$$((I - 2\bar{X})e_\gamma)^T A_0((I - 2\bar{X})e_\gamma) - ((2\bar{X} - I)e_\gamma)^T(a_0 + 2A_0\bar{x}) \geq 0,$$

which is equivalent to

$$[2\bar{x} - e]_\gamma^T[\text{diag}((2\bar{X} - I)(a_0 + 2A_0\bar{x})) - A_0]_\gamma[2\bar{x} - e]_\gamma \leq 0$$

by

$$\begin{aligned}
((2\bar{X} - I)e_\gamma)^T A_0((2\bar{X} - I)e_\gamma) &= [2\bar{x} - e]_\gamma^T[A_0]_\gamma[2\bar{x} - e]_\gamma \\
((2\bar{X} - I)e_\gamma)^T(a_0 + 2A_0\bar{x}) &= [2\bar{x} - e]_\gamma^T[a_0 + 2A_0\bar{x}]_\gamma,
\end{aligned}$$

and

$$[2\bar{x} - e]_\gamma^T[a_0 + 2A_0\bar{x}]_\gamma = [2\bar{x} - e]_\gamma^T[\text{diag}((2\bar{X} - I)(a_0 + 2A_0\bar{x})]_\gamma[2\bar{x} - e]_\gamma$$

since $(2\bar{x}_i - 1)^2 = 1, \forall i = 1, \ldots, n$. Thus, $\bar{x}$ is a global minimizer of problem (UQP) if and only if condition $[NSCU]$ holds.

Now consider the following quadratic $\{0, 1\}$ problem with linear equality constraints:

$$\begin{aligned}
(LQP) \qquad &\min \quad f(x) := x^T A_0 x + x^T a_0 \\
&s.t. \quad Bx + b = 0 \\
&\qquad\quad x \in U = \{0, 1\}^n,
\end{aligned}$$

where $A_0 \in S^n, a_0 = (a_1^0, \dots, a_n^0)^T \in R^n, B = (B_1, \dots, B_n) = (b_{ij})_{m\times n}$ is a $m \times n$ matrix, $b \in R^m$, where $B_i = (b_{1i}, \dots, b_{mi})^T, i = 1, \dots, n$. Let

$$\begin{aligned} U_L : &= \{x \in U \mid Bx + b = 0\}. && (6)\\ m_0 : &= \min_{x\in U\setminus U_L} (Bx+b)^T(Bx+b) && (7)\\ M_0 : &= \min_{x\in U} f(x) && (8) \end{aligned}$$

For convenience, let $\min_{\emptyset} = +\infty$. Let $x^* \in U_L$ be a global minimizer of problem (LQP), and let

$$q_0 : = \frac{f(x^*) - M_0}{m_0}. \tag{9}$$

Then $m_0 > 0$ and $q_0 \geq 0$. For a given $q \in R$, let

$$F_q(x) := x^T A_0 x + x^T a_0 + q(Bx+b)^T(Bx+b), \tag{10}$$

i.e.,

$$F_q(x) = x^T(A_0 + qB^TB)x + x^T(a_0 + 2qB^Tb) + qb^Tb.$$

For given $q \in R$, $\lambda \in R^m$, let

$$G_{q,\lambda}(x) := x^T A_0 x + x^T a_0 + q(Bx+b)^T(Bx+b) + \lambda^T(Bx+b), \tag{11}$$

i.e.,

$$G_{q,\lambda}(x) = x^T(A_0 + qB^TB)x + x^T(a_0 + 2qB^Tb + B^T\lambda) + qb^Tb + \lambda^Tb.$$

Consider the following problems:

$$(LQP)_q^F \qquad \begin{array}{ll} \min & F_q(x) \\ s.t. & x \in U \end{array}$$

and

$$(LQP)_{q,\lambda}^G \qquad \begin{array}{ll} \min & G_{q,\lambda}(x) \\ s.t. & x \in U. \end{array}$$

We have the following results:

Proposition 2. *Let $\bar{x} \in U$. Then when $q > q_0$, $\bar{x}$ is a global minimizer of problem (LQP) if and only if $\bar{x}$ is a global minimizer of problem $(LQP)_q^F$.*

Proof. Let $\bar{x}$ be a global minimizer of problem (LQP). Then, $\bar{x} \in U_L$. And for any $x \in U_L$, we have that $f(x) \geq f(\bar{x})$. By the definition of q_0, we know that $q_0 = \frac{f(\bar{x})-M_0}{m_0}$. Then, for any $q > q_0$ and $x \in U \setminus U_L$, we have that

$$\begin{aligned} F_q(x) &= f(x) + q(Bx+b)^T(Bx+b) \\ &\geq M_0 + qm_0 \\ &> M_0 + f(\bar{x}) - M_0 \\ &= F_q(\bar{x}). \end{aligned}$$

For any $x \in U_L$, we have that

$$F_q(x) = f(x) + q(Bx+b)^T(Bx+b) = f(x) \geq f(\bar{x}) = F_q(\bar{x}).$$

Hence, for any $x \in U$, we have that $F_q(x) \geq F_q(\bar{x})$. Thus, $\bar{x}$ is a global minimizer of problem $(LQP)_q^F$.

Conversely, if $\bar{x}$ is a global minimizer of problem $(LQP)_q^F$, then when $q > q_0$, we can prove that $\bar{x} \in U_L$. Indeed, if $\bar{x} \in U \setminus U_L$, then $(B\bar{x}+b)^T(B\bar{x}+b) \geq m_0$ and

$$\begin{array}{rcl} F_q(\bar{x}) & = & f(\bar{x}) + q(B\bar{x}+b)^T(B\bar{x}+b) \\ & > & f(\bar{x}) + q_0 m_0 = f(\bar{x}) + f(x^*) - M_0 \\ & \geq & f(x^*) = F_q(x^*), \end{array}$$

where x^* is a global minimizer of problem (LQP). This contradicts that $\bar{x}$ is a global minimizer of problem $(LQP)_q^F$. Hence $\bar{x} \in U_L$. Therefore, for any $x \in U_L$, we have that

$$f(x) = F_q(x) \geq F_q(\bar{x}) = f(\bar{x}),$$

which means that $\bar{x}$ is a global minimizer of problem (LQP).

Proposition 3. *Let $\bar{x} \in U_L$. If there exist $q \in R$ and $\lambda \in R^m$ such that $\bar{x}$ is a global minimizer of problem $(LQP)_{q,\lambda}^G$, then $\bar{x}$ is a global minimizer of problem (LQP).*

Proof. If there exist $q \in R$ and $\lambda \in R^m$ such that $\bar{x}$ is a global minimizer of problem $(LQP)_{q,\lambda}^G$, then for any $x \in U_L$, we have that

$$f(x) = G_{q,\lambda}(x) \geq G_{q,\lambda}(\bar{x}) = f(\bar{x}).$$

Hence, $\bar{x}$ is a global minimizer of problem (LQP).

By Propositions 1 and 3, we can obtain the following results.

Theorem 2. *(**Sufficient Condition for Problem** $[LQP]$) Let $\bar{x} \in U_L$, $\bar{X} = \mathrm{diag}(\bar{x})$. If there exist $q_1 \in R$ and $\lambda \in R^m$ such that*

$$[SCL] \qquad A_0 + q_1 B^T B \geq \mathrm{diag}((2\bar{X} - I)(a_0 + B^T\lambda + 2A_0\bar{x})),$$

then $\bar{x}$ is a global minimizer of problem (LQP).

Proof. By Proposition 1, we know that if

$$A_0 + q_1 B^T B \geq \mathrm{diag}((2\bar{X} - I)(a_0 + 2q_1 B^T b + B^T\lambda + 2(A_0 + q_1 B^T B)\bar{x})), \tag{12}$$

then $\bar{x}$ is a global minimizer of problem $(LQP)_{q_1,\lambda}^G$. By Proposition 3, we know that if $\bar{x} \in U_L$, then $\bar{x}$ is also a global minimizer of problem (LQP). We can verify that (12) is equivalent to $[SCL]$ since $b + B\bar{x} = 0$.

Corollary 1. *Let $\bar{x} \in U$, $\bar{X} = \mathrm{diag}(\bar{x})$. If there exists a $q_2 > q_0$ such that*

$$[SCL1] \qquad A_0 + q_2 B^T B \geq \mathrm{diag}((2\bar{X} - I)(a_0 + 2A_0\bar{x})),$$

then $\bar{x}$ is a global minimizer of problem (LQP).

Proof. If $\bar{x} \in U_L$, by Theorem 2, we know that if $[SCL1]$ holds, then $\bar{x}$ is a global minimizer of problem (LQP), where $\lambda = 0$. Here we just need to prove that if $[SCL1]$ holds, then $\bar{x} \in U_L$. By Proposition 1, we know that if

$$A_0 + q_2 B^T B \geq \mathrm{diag}((2\bar{X} - I)(a_0 + 2q_2 B^T b + 2(A_0 + q_2 B^T B)\bar{x})), \tag{13}$$

then $\bar{x}$ is a global minimizer of problem $(LQP)_{q_2}^F$. By Proposition 2, we know that if also $q_2 > q_0$, then $\bar{x} \in U_L$ and $\bar{x}$ is a global minimizer of problem (LQP). Moreover, (13) is equivalent to $[SCL1]$ since $b + B\bar{x} = 0$.

Remark 1. Theorem 2.1 in reference [29] gives the following sufficient condition $[SCL2]$ for problem (LQP):

$$[SCL2] \qquad A_0 \geq \mathrm{diag}((2\bar{X} - I)(a_0 + B^T\lambda + 2A_0\bar{x}),$$

where $\lambda \in R^m$. Obviously, condition $[SCL2]$ implies condition $[SCL]$, but the following numerical example illustrates that $[SCL]$ does not imply $[SCL2]$. Hence, sufficient condition $[SCL]$ strictly extends the sufficient condition $[SCL2]$ given in [29].

Example 1. Consider the problem

$$
\begin{array}{lll}
[EX] & \min & f(x) = 5x_1^2 + 2x_1x_2 + 6x_2^2 + 2x_1x_3 - x_3^2 - 8x_1 - 8x_2 - x_3 \\
 & s.t. & x_1 - x_2 - x_3 = 0 \\
 & & x_i \in \{0,1\}, i = 1,2,3.
\end{array}
$$

Let $A_0 = \begin{pmatrix} 5 & 1 & 1 \\ 1 & 6 & 0 \\ 1 & 0 & -1 \end{pmatrix}$, $B = (1,-1,-1)$ and $b = 0$, $a_0 = (-8,-8,-1)^T$. Let $U_L := \{x \in \{0,1\}^3 \mid x_1 - x_2 - x_3 = 0\}$ and $\bar{x} = (1,1,0)^T$. Then, $\bar{x} \in U_L$. We can verify that for $q = 1$ and $\lambda = 0$, condition $[SCL]$ holds, but for any $\lambda \in R$, condition $[SCL2]$ does not hold. Indeed, obviously,

$$A_0 - \mathrm{diag}((2\bar{X} - I)(a_0 + 2A_0\bar{x}) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

For $q = 1$, we have that

$$qB^TB = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}.$$

Thus, we have that

$$A_0 - \mathrm{diag}((2\bar{X} - I)(a_0 + 2A_0\bar{x}) + qB^TB = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \geq 0,$$

i.e., condition $[SCL]$ holds for $q = 1$ and $\lambda = 0$. Hence, $\bar{x}$ is a global minimizer of problem (EX).

But for any $\lambda \in R$, $\mathrm{diag}((2\bar{X} - I)B^T\lambda) = \mathrm{diag}(\lambda, -\lambda, \lambda)$. Thus,

$$A_0 - \mathrm{diag}((2\bar{X} - I)(a_0 + 2A_0\bar{x}) - \mathrm{diag}((2\bar{X} - I)B^T\lambda) = \begin{pmatrix} 1-\lambda & 1 & 1 \\ 1 & \lambda & 0 \\ 1 & 0 & -\lambda \end{pmatrix}.$$

Then, we can easily verify that for any $\lambda \in R$, $A_0 - \mathrm{diag}((2\bar{X} - I)(a_0 + B^T\lambda + 2A_0\bar{x})$ is not a positive semidefinite matrix, i.e., condition $[SCL2]$ does not hold for any $\lambda \in R$.

From necessary and sufficient condition $[NSCU]$ for problem (UQP), we can obtain the following necessary and sufficient condition for problem (LQP). For $\gamma = \{i_1, \ldots, i_p\} \in \Gamma$ and $B = (b_{ij})_{m\times n} = (B_1, \ldots, B_n)$, where $1 \leq p \leq n$ and $B_i = (b_{1i}, \ldots, b_{mi})^T$, let

$$B^\gamma := (B_{i_1}, \ldots, B_{i_p}), i_j \in \gamma, j = 1, \ldots, p. \tag{14}$$

Then B^γ is a $m \times p$ matrix.

Theorem 3. **(Necessary and Sufficient Condition for Problem** (LQP)**)** *Let $\bar{x} \in U_L$. Then $\bar{x}$ is a global minimizer of problem (LQP) if and only if*

$$[NSCL] \quad \begin{cases} \text{for any } \gamma \in \Gamma \text{ with } B^\gamma[2\bar{x} - e]_\gamma = 0, \\ [2\bar{x} - e]_\gamma^T \Big[\mathrm{diag}\Big((2\bar{X} - I)(a_0 + 2A_0\bar{x})\Big) - A_0\Big]_\gamma [2\bar{x} - e]_\gamma \leq 0. \end{cases}$$

Proof. By Proposition 2, we know that if $q > q_0$, then $\bar{x}$ is a global minimizer of problem (LQP) if and only if $\bar{x}$ is a global minimizer of problem $(LQP)_q^F$, where q_0 is given by (9). By Theorem 1, $\bar{x}$ is a global minimizer of problem $[LQP]_q^F$ if and only if

$$\begin{cases} \text{for any } \gamma \in \Gamma, \\ [2\bar{x} - e]_\gamma^T \Big[\mathrm{diag}\Big((2\bar{X} - I)\left(a_0 + 2qB^Tb + 2(A_0 + qB^TB)\bar{x}\right)\Big) - (A_0 + qB^TB)\Big]_\gamma [2\bar{x} - e]_\gamma \leq 0. \end{cases}$$

By $\bar{x} \in U_L$, i.e., $B\bar{x} + b = 0$, the above condition is equivalent to the following condition:

$$[NSCL]' \quad \left\{ \begin{array}{l} \text{for any } \gamma \in \Gamma, \\ [2\bar{x} - e]_\gamma^T \Big[\mathrm{diag}\Big((2\bar{X} - I)(a_0 + 2A_0\bar{x})\Big) - (A_0 + qB^T B)\Big]_\gamma [2\bar{x} - e]_\gamma \le 0. \end{array} \right.$$

Here we just need to prove that there exists a $q_1 \ge q_0$, such that condition $[NSCL]$ is equivalent to condition $[NSCL]'$ when $q \ge q_1$. In fact, $[NSCL]'$ is equivalent to the following condition:

$$[NSCL]'' \quad \left\{ \begin{array}{l} \text{for any } \gamma \in \Gamma, \\ [2\bar{x} - e]_\gamma^T [\mathrm{diag}((2\bar{X} - I)(a_0 + 2A_0\bar{x})) - A_0]_\gamma [2\bar{x} - e]_\gamma \le q[2\bar{x} - e]_\gamma^T [B^T B]_\gamma [2\bar{x} - e]_\gamma. \end{array} \right.$$

Furthermore, $[B^T B]_\gamma = B^{\gamma T} B^\gamma$. Hence

$$[2\bar{x} - e]_\gamma^T [B^T B]_\gamma [2\bar{x} - e]_\gamma = (B^\gamma [2\bar{x} - e]_\gamma)^T B^\gamma [2\bar{x} - e]_\gamma.$$

Thus, if $[NSCL]'$ holds, then for any $\gamma \in \Gamma$ with

$$B^\gamma [2\bar{x} - e]_\gamma = 0,$$

we must have that

$$[2\bar{x} - e]_\gamma^T [\mathrm{diag}((2\bar{X} - I)(a_0 + 2A_0\bar{x})) - A_0]_\gamma [2\bar{x} - e]_\gamma \le 0,$$

i.e., $[NSCL]$ holds.

Conversely, if $[NSCL]$ holds, then for any $q \ge 0$ and $\gamma \in \Gamma$ with

$$B^\gamma [2\bar{x} - e]_\gamma = 0,$$

we have that

$$[2\bar{x} - e]_\gamma^T [\mathrm{diag}((2\bar{X} - I)(a_0 + 2A_0\bar{x})) - A_0]_\gamma [2\bar{x} - e]_\gamma \le 0 = q[2\bar{x} - e]_\gamma^T [B^T B]_\gamma [2\bar{x} - e]_\gamma.$$

So here we just need to prove that there exists a $q_1 > 0$ such that when $q \ge q_1$, for any $\gamma \in \Gamma$ with $B^\gamma [2\bar{x} - e]_\gamma \ne 0$, we also have that

$$[2\bar{x} - e]_\gamma^T [\mathrm{diag}((2\bar{X} - I)(a_0 + 2A_0\bar{x})) - A_0]_\gamma [2\bar{x} - e]_\gamma \le q[2\bar{x} - e]_\gamma^T [B^T B]_\gamma [2\bar{x} - e]_\gamma.$$

For any $\gamma \in \Gamma$, if $B^\gamma [2\bar{x} - e]_\gamma \ne 0$, then

$$[2\bar{x} - e]_\gamma^T [B^T B]_\gamma [2\bar{x} - e]_\gamma > 0.$$

Hence, there must exist a $q_\gamma > 0$, such that when $q \ge q_\gamma$,

$$[2\bar{x} - e]_\gamma^T [\mathrm{diag}((2\bar{X} - I)(a_0 + 2A_0\bar{x})) - A_0]_\gamma [2\bar{x} - e]_\gamma \le q[2\bar{x} - e]_\gamma^T [B^T B]_\gamma [2\bar{x} - e]_\gamma.$$

For $\gamma \in \Gamma$ with $B^\gamma [2\bar{x} - e]_\gamma = 0$, let $q_\gamma = 0$. Let $q_1 = \max\{q_{\gamma|\gamma\in\Gamma}, q_0\}$. Then $q_1 \ge q_0$ is a finite nonnegative number since Γ is a finite set. When $q \ge q_1$, for any $\gamma \in \Gamma$, we have

$$[2\bar{x} - e]_\gamma^T [\mathrm{diag}((2\bar{X} - I)(a_0 + 2A_0\bar{x})) - A_0]_\gamma [2\bar{x} - e]_\gamma \le q[2\bar{x} - e]_\gamma^T [B^T B]_\gamma [2\bar{x} - e]_\gamma,$$

i.e., $[NSCL]'$ holds.

3. Global Optimality Conditions for Quadratic Assignment Problems

Consider the following quadratic assignment problem (QAP):

$$\begin{array}{lll}(QAP) & \min & \displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} a_{ik}b_{jl}x_{ij}x_{kl} + \sum_{i,j=1}^{n} c_{ij}x_{ij} \\ & \text{s.t.} & \displaystyle\sum_{i=1}^{n} x_{ij} = 1, \quad \sum_{j=1}^{n} x_{ij} = 1 \\ & & x_{ij} \in \{0,1\}, i, j = 1, 2, \ldots, n,\end{array}$$

which is a very famous combinatorial optimization problem suggested by Koopmans and Beckmann [1]. Let

$$\begin{array}{rcl}
x_i : & = & (x_{i1}, \ldots, x_{in}), i = 1, \ldots, n, \\
x : & = & (x_1, \ldots, x_n)^T, \\
c_i : & = & (c_{i1}, \ldots, c_{in}), \\
c : & = & (c_1, \ldots, c_n)^T, \\
U_A : & = & \left\{ x \left| \displaystyle\sum_{i=1}^{n} x_{ij} = 1, \quad \sum_{j=1}^{n} x_{ij} = 1, x_{ij} \in \{0,1\}, i, j = 1, 2, \ldots, n \right.\right\}, \\
Q : & = & (b_{ij})_{n\times n}, \\
R_{ij} : & = & a_{ij}Q, \\
R : & = & \begin{pmatrix} R_{11} & \ldots & R_{1n} \\ \vdots & \vdots & \vdots \\ R_{n1} & \ldots & R_{nn} \end{pmatrix}, \\
D : & = & \dfrac{R + R^T}{2}, \\
E : & = & \begin{pmatrix}
1 & 1 & \ldots & 1 & 0 & 0 & \ldots & 0 & \ldots & \ldots & 0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 & 1 & 1 & \ldots & 1 & \ldots & \ldots & 0 & 0 & \ldots & 0 \\
 & & \ldots & & & & \ldots & & \ldots & \ldots & & & \ldots & \\
0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & \ldots & \ldots & 1 & 1 & \ldots & 1 \\
1 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 & \ldots & \ldots & 1 & 0 & \ldots & 0 \\
0 & 1 & \ldots & 0 & 0 & 1 & \ldots & 0 & \ldots & \ldots & 0 & 1 & \ldots & 0 \\
 & & \ldots & & & & \ldots & & \ldots & \ldots & & & \ldots & \\
0 & 0 & \ldots & 1 & 0 & 0 & \ldots & 1 & \ldots & \ldots & 0 & 0 & \ldots & 1
\end{pmatrix}_{2n\times n^2}
\end{array}$$

So R_{ij} is a $n \times n$ matrix, R is a $n^2 \times n^2$ matrix, D is a $n^2 \times n^2$ symmetric matrix and $f(x) := x^T Dx + c^T x$. Problem (QAP) can be rewritten as the following equivalent problem, which is also noted as (QAP):

$$\begin{array}{lll}(QAP) & \min & f(x) := x^T Dx + c^T x \\ & s.t. & g(x) := Ex - e = 0 \\ & & x \in \{0,1\}^{n^2}.\end{array}$$

By Theorem 2, we can obtain the following sufficient condition for problem (QAP).

Theorem 4. *(***Sufficient Condition for Assignment Problem** *(QAP)) Let $\bar{x} \in U_A$, $\bar{X} = \operatorname{diag}(\bar{x})$. If there exist $\bar{q} \in R$ and $\lambda \in R^m$ such that*

$$[SCA] \qquad D + \bar{q}E^T E \geq \operatorname{diag}((2\bar{X} - I)(c + E^T\lambda + 2D\bar{x})),$$

then $\bar{x}$ is a global minimizer of problem (QAP).

Proof. It can be obtained easily from Theorem 2.

Remark 2. Condition $[SCA]$ extends the results given by Theorem 2.2 in reference [29], which is equivalent to the following condition:

$$[SCA1] \qquad D \geq \mathrm{diag}((2\bar{X} - I)(c + E^T\lambda + 2D\bar{x})).$$

Let

$$\Gamma_A := \{\gamma \mid \gamma = \{i_1 j_1, \ldots, i_p j_p\} \mid i_k, j_k \in \{1, \ldots, n\}, 1 \leq k \leq p, 1 \leq p \leq n\}. \tag{15}$$

For $D = (d_{i,j})_{n^2\times n^2} \in S^{n^2}$, $E = (e_{i,j})_{2n\times n^2} = (e_1, \ldots, e_{n^2})$, where $e_i \in R^{2n}$, $a = (a_1, \ldots, a_{n^2})^T \in R^{n^2}$, for any $\gamma = \{i_1 j_1, \ldots, i_p j_p\} \in \Gamma_A$, $1 \leq p \leq n$, let

$$[D]_\gamma : = (d_{i,j})_{p\times p}, \text{ where } i, j \in \{(i_k - 1)n + j_k, i_k j_k \in \gamma, k = 1, \ldots, p\}, \tag{16}$$
$$E^\gamma : = (e_{k_1}, \ldots, e_{k_p})_{2n\times p}, \ k_r = (i_r - 1)n + j_r, i_r j_r \in \gamma, r = 1, \ldots, p, \tag{17}$$
$$[a]_\gamma : = (a_{k_1}, \ldots, a_{k_p})^T \in R^p, \ k_r = (i_r - 1)n + j_r, i_r j_r \in \gamma, r = 1, \ldots, p, \tag{18}$$
$$e_\gamma : = (e_{11}, \ldots, e_{1n}, \ldots, e_{n1}, \ldots, e_{nn})^T \in R^{n^2}, \ e_{ij} = \begin{cases} 1, & \text{if } ij \in \gamma \\ 0, & \text{otherwise} \end{cases}. \tag{19}$$

By Theorem 3, we can obtain the following necessary and sufficient conditions for problem (QAP).

Theorem 5. (Necessary and Sufficient Condition for Assignment Problem (QAP)**)** *Let $\bar{x} \in U_A$ and let $\bar{X} = \mathrm{diag}(\bar{x})$. Then $\bar{x}$ is a global minimizer of problem (QAP), if and only if*

$$[NSCA] \quad \begin{cases} \text{for any } \gamma \in \Gamma_A \text{ such that } E^\gamma[2\bar{x} - e]_\gamma = 0, \\ [2\bar{x} - e]_\gamma^T\Big[\mathrm{diag}\Big((2\bar{X} - I)(c + 2D\bar{x})\Big) - D\Big]_\gamma [2\bar{x} - e]_\gamma \leq 0 \end{cases},$$

where $e = (1, \ldots, 1)^T \in R^{n^2}$.

Proof. It can be obtained easily from Theorem 3.

Proposition 4. *Let $\bar{x} \in U_A$. Then*

(1) *for any $i = 1, \ldots, n$, there exists one and only one $j_{i,\bar{x}} \in \{1, \ldots, n\}$ such that*

$$\begin{cases} \bar{x}_{ij_{i,\bar{x}}} = 1 \\ \bar{x}_{ij} = 0, \forall j \in \{1, \ldots, n\}, j \neq j_{i,\bar{x}} \\ \bar{x}_{rj_{i,\bar{x}}} = 0, \forall r = 1, \ldots, n, r \neq i; \end{cases}$$

(2) *for any $j = 1, \ldots, n$, there exists one and only one $i_{j,\bar{x}} \in \{1, \ldots, n\}$ such that*

$$\begin{cases} \bar{x}_{i_{j,\bar{x}}j} = 1 \\ \bar{x}_{ij} = 0, \forall i \in \{1, \ldots, n\}, i \neq i_{j,\bar{x}} \\ \bar{x}_{i_{j,\bar{x}},r} = 0, \forall r \in \{1, \ldots, n\}, r \neq j; \end{cases}$$

(3) $\Gamma_{U_A} \supset \Gamma_{\bar{x},U_A}$ *and $|\Gamma_{\bar{x},U_A}| = n(n-1)$, where*

$$\Gamma_{U_A} : = \{\gamma \in \Gamma_A \mid E^\gamma[2\bar{x} - e]_\gamma = 0\}, \tag{20}$$
$$\Gamma_{\bar{x},U_A} : = \{\{ij_{i,\bar{x}}, ij, i_{j,\bar{x}}j_{i,\bar{x}}, i_{j,\bar{x}}j\} \mid i, j \in \{1, \ldots, n\}, i \neq i_{j,\bar{x}}, j \neq j_{i,\bar{x}}\}. \tag{21}$$

Proof. (1) For any $i = 1, \ldots, n$, by $\sum_{j=1}^n \bar{x}_{ij} = 1$ and $\bar{x}_{ij} \in \{0, 1\}$, we know that there exists one and only one $j_{i,\bar{x}} \in \{1, \ldots, n\}$ such that $\bar{x}_{ij_{i,\bar{x}}} = 1$ and for the other $j \in \{1, \ldots, n\}$, $j \neq j_{i,\bar{x}}$, $\bar{x}_{ij} = 0$. By $\sum_{r=1}^n \bar{x}_{rj_{i,\bar{x}}} = 1$, we know that for any $r = 1, \ldots, n$ and $r \neq i$, $\bar{x}_{rj_{i,\bar{x}}} = 0$.

Similarly, (2) can be also obtained.

(3) For any $\gamma \in \{\{ij_{i,\bar{x}}, ij, i_{j,\bar{x}}j_{i,\bar{x}}, i_{j,\bar{x}}j\} \mid i, j \in \{1, \ldots, n\}, i \neq i_{j,\bar{x}}, j \neq j_{i,\bar{x}}\}$, we can obtain that

$$E^{\gamma}[2\bar{x} - e]_{\gamma} = 0$$

by $[2\bar{x} - e]_{\gamma} = (1, -1, -1, 1)^T$ and $E^{\gamma} = (e_1, , e_2, e_3, e_4)_{2n\times 4}$, where

$$e_1 = (e_{1,1}, \ldots, e_{1,2n})^T, e_{1,p} = \begin{cases} 1, & p = i, n + j_{i,\bar{x}} \\ 0, & \text{otherwise,} \end{cases}$$
$$e_2 = (e_{2,1}, \ldots, e_{2,2n})^T, e_{2,p} = \begin{cases} 1, & p = i, n + j \\ 0, & \text{otherwise,} \end{cases}$$
$$e_3 = (e_{3,1}, \ldots, e_{3,2n})^T, e_{3,p} = \begin{cases} 1, & p = i_{j,\bar{x}}, n + j_{i,\bar{x}} \\ 0, & \text{otherwise,} \end{cases}$$
$$e_4 = (e_{4,1}, \ldots, e_{4,2n})^T, e_{4,p} = \begin{cases} 1, & p = i_{j,\bar{x}}, n + j \\ 0, & \text{otherwise .} \end{cases}$$

Thus, $\{\{ij_{i,\bar{x}}, ij, i_{j,\bar{x}}j_{i,\bar{x}}, i_{j,\bar{x}}j\} \mid i, j \in \{1, \ldots, n\}, i \neq i_{j,\bar{x}}, j \neq j_{i,\bar{x}}\} \subset \Gamma_{\bar{x},U_A}$. Obviously $|\Gamma_{\bar{x},U_A}| = n(n-1)$.

By Proposition 4 and Theorem 5, we can obtain the following necessary condition.

Theorem 6. (Necessary Condition for Assignment Problem (QAP)**)** *Let $\bar{x} \in U_A$. If $\bar{x}$ is a global minimizer of problem* (QAP)*, then*

$$[NCA] \qquad [2\bar{x} - e]_{\gamma}^T[\operatorname{diag}((2\bar{X} - I)(c + 2D\bar{x})) - D]_{\gamma}[2\bar{x} - e]_{\gamma} \leq 0, \forall \gamma \in \Gamma_{\bar{x},U_A}.$$

For a given $\bar{x} \in U_A$, the following algorithm gives a method to obtain the set $\Gamma_{\bar{x},U_A}$:

Algorithm 1. *(Algorithm for Set $\Gamma_{\bar{x},U_A}$:)*

Step 0. Set $i := 1$ and $\Gamma = \emptyset$, goto Step 1;

Step 1. If $i > n$, goto Step 4; otherwise, let $p_i := \operatorname{argmax}\{\bar{x}_{ij}, j = 1, \ldots, n\}$, i.e., $\bar{x}_{ip_i} = 1$, and let $j := 1$, goto Step 2;

Step 2. If $j = p_i$, let $j := j + 1$, goto Step 3; otherwise let $q_j := \operatorname{argmax}\{\bar{x}_{ij}, i = 1, \ldots, n\}$, i.e., $\bar{x}_{q_j j} = 1$, and let $\Gamma := \Gamma \cup \{ip_i, ij, q_j p_i, q_j j\}$, $j := j + 1$, goto Step 3;

Step 3. If $j > n$, let $i := i + 1$, goto Step 1; otherwise, goto Step 2;

Step 4. Stop. Γ is the set of $\Gamma_{\bar{x},U_A}$.

4. Optimization Methods for Quadratic Assignment Problems

In this section, we will first derive a new local optimization method for quadratic assignment problem (QAP), then we will introduce an auxiliary function to derive a global optimization method for problem (QAP).

4.1. Local Optimization Method

Consider the following general problem (AGP).

$$(AGP) \qquad \min \; f(x) \quad x \in U_A,$$

where f is a general objective function,

$$\begin{aligned} U_A &= \left\{ x \,\middle|\, \sum_{i=1}^{n} x_{ij} = 1, \quad \sum_{j=1}^{n} x_{ij} = 1, x_{ij} \in \{0, 1\}, i, j = 1, 2, \ldots, n \right\} \\ &= \{x \mid Ex = 1, x \in \{0, 1\}^{n^2}\}. \end{aligned}$$

If $f(x) = x^T Dx + c^T x$, where D and c are given in problem (QAP), then (AGP) is the quadratic assignment problem (QAP). Let $\bar{x} \in U_A$ and let

$$D_{\bar{x}} : \;=\; \{d_\gamma = (I - 2\bar{X})e_\gamma \mid \gamma \in \Gamma_{\bar{x},U_A}\}, \tag{22}$$

$$N(\bar{x}) : \;=\; \{\bar{x}\} \cup \{\bar{x} + d \mid d \in D_{\bar{x}}\}, \tag{23}$$

where $\bar{X} = \mathrm{diag}(\bar{x})$ and e_γ is defined by (19). $N(\bar{x})$ is said to be the neighborhood of $\bar{x}$. Obviously, we have that $N(\bar{x}) \subset U_A$ and $|N(\bar{x})| = n(n-1) + 1$. Indeed, for any $d \in D(\bar{x})$, there exists a $\gamma = (ij_{i,\bar{x}}, ij, i_{j,\bar{x}}j_{i,\bar{x}}, i_{j,\bar{x}}j) \in \Gamma_{\bar{x},U_A}$ such that $d = (I - 2\bar{X})e_\gamma$. Hence,

$$x = \bar{x} + d = \bar{x} + (I - 2\bar{X})e_\gamma = (x_{11}, \ldots, x_{1n}, \ldots, x_{n1}, \ldots, x_{nn})^T,$$

where $x_{kr} = \begin{cases} 1 - \bar{x}_{kr} & kr \in \gamma \\ \bar{x}_{kr} & kr \notin \gamma \end{cases} \in \{0, 1\}$ and

$$\sum_{r=1}^{n} x_{kr} = \begin{cases} \sum_{r=1}^{n} \bar{x}_{kr} = 1, & \forall k \neq i, i_{j,\bar{x}} \\ \displaystyle\sum_{\substack{r=1 \\ r \neq j \\ r \neq j_{i,\bar{x}}}}^{n} \bar{x}_{ir} + (1 - \bar{x}_{ij_{i,\bar{x}}}) + (1 - \bar{x}_{ij}) = (1 - \bar{x}_{ij}) = 1, & \text{if } k = i \\ \displaystyle\sum_{\substack{r=1 \\ r \neq j \\ r \neq j_{i,\bar{x}}}}^{n} \bar{x}_{i_{j,\bar{x}}r} + (1 - \bar{x}_{i_{j,\bar{x}}j_{i,\bar{x}}}) + (1 - \bar{x}_{i_{j,\bar{x}}j}) = (1 - \bar{x}_{i_{j,\bar{x}}j}) = 1, & \text{if } k = i_{j,\bar{x}} \end{cases}$$

$$\sum_{k=1}^{n} x_{kr} = \begin{cases} \sum_{k=1}^{n} \bar{x}_{kr} = 1, & \forall r \neq j, j_{i,\bar{x}} \\ \displaystyle\sum_{\substack{k=1 \\ k \neq i \\ k \neq i_{j,\bar{x}}}}^{n} \bar{x}_{kj_{i,\bar{x}}} + (1 - \bar{x}_{ij_{i,\bar{x}}}) + (1 - \bar{x}_{i_{j,\bar{x}}j_{i,\bar{x}}}) = (1 - \bar{x}_{i_{j,\bar{x}}j_{i,\bar{x}}}) = 1, & \text{if } r = i_{j,\bar{x}} \\ \displaystyle\sum_{\substack{k=1 \\ k \neq i \\ k \neq i_{j,\bar{x}}}}^{n} \bar{x}_{kj} + (1 - \bar{x}_{i_{j,\bar{x}}j}) + (1 - \bar{x}_{ij}) = (1 - \bar{x}_{ij}) = 1, & \text{if } r = j \end{cases}.$$

Thus, $x = \bar{x} + d \in U_A$.

By $|\Gamma_{\bar{x},U_A}| = n(n-1)$, we have that $|N(\bar{x})| = n(n-1) + 1$.

Definition 1. Let $\bar{x} \in U_A$. $\bar{x}$ is said to be a local minimizer (maximizer) of problem (AGP) if for any $x \in N(\bar{x})$, $f(x) \geq f(\bar{x})(f(x) \leq f(\bar{x}))$. $\bar{x}$ is said to be a strict local minimizer (maximizer) of problem (AGP) if for any $x \in N(\bar{x}) \setminus \{\bar{x}\}$, $f(x) > f(\bar{x})(f(x) < f(\bar{x}))$.

Definition 2. Let $\bar{x} \in U_A$. $\bar{x}$ is said to be a global minimizer of problem (AGP) if for any $x \in U_A$, $f(x) \geq f(\bar{x})$.

Definition 3. $d \in D_{\bar{x}}$ is said to be a descent direction of problem (AGP) at point $\bar{x} \in U_A$ if $f(\bar{x} + d) < f(\bar{x})$.

Algorithm 2. *(Local Optimization Method for Problem (AGP):)*

Step 1. Take an initial point $x \in U_A$.

Step 2. If x is already a local minimizer of problem (AGP), i.e., $f(x + d) \geq f(x)$ for any $d \in D_x$, then stop; otherwise, let d_x be a descent direction of problem (AGP) at point x, i.e., $f(x + d_x) < f(x)$, go to Step 3.

Step 3. Set $x := x + d_x$, and go to Step 2.

Theorem 7. *Let $\bar{x} \in U_A$ and let $f(x) = x^T Dx + c^T x$. Then $\bar{x}$ is a local minimizer of problem* (QAP) *if and only if* $[NCA]$ *holds.*

Proof. By the definition of local minimizer of problem (QAP) given by Definition 1, we know that $\bar{x}$ is a local minimizer of problem (QAP) if and only if for any $\gamma \in \Gamma_{\bar{x},U_A}$, we have that

$$f(\bar{x} + d_\gamma) - f(\bar{x}) \geq 0,$$

where $d_\gamma = (I - 2\bar{X})e_\gamma$, i.e.

$$(\bar{x} + d_\gamma)^T D(\bar{x} + d_\gamma) + c^T(\bar{x} + d_\gamma) - \bar{x}^T D\bar{x} - c^T \bar{x} \geq 0. \tag{24}$$

We can easily to verify that (24) is equivalent to (NCA).

Note that here we design a local optimization method for problem (QAP) by using the necessary global optimality condition (NCA). In the following, we will derive a global optimization method for problem (QAP) by using the global optimality sufficient condition $[SCA]$, the local optimization method given by Algorithm 5.1 and some auxiliary functions.

4.2. Global Optimization Method for Quadratic Assignment Problem (QAP)

In order to derive the global minimization method, here we need to introduce the following auxiliary functions. Let x^* be a local minimizer of quadratic assignment problem (QAP) and let

$$\varphi_r(t) = \begin{cases} 0 & t \leq -r \\ \frac{t}{r} + 1 & -r < t \leq 0 \\ 1 & t > 0 \end{cases}, \tag{25}$$

$$\Phi_{r,x^*}(x) = \frac{1}{\|x - x^*\| + 1}\varphi_r\Big(f(x) - f(x^*)\Big), \tag{26}$$

where $f(x) = x^T Dx + c^T x$, D, c are given in problem (QAP), and $\|x - x^*\| = \sum_{i,j=1}^n |x_{ij} - x^*_{ij}|$. Consider the following problem:

$$(AQAP) \qquad \min \quad \Phi_{r,x^*}(x) \qquad x \in U_A,$$

where $r > 0$ is a parameter.

Theorem 8. *For any $r > 0$, x^* is a strict local maximizer of problem* $(AQAP)$*.*

Proof. Since x^* is a local minimizer of problem (QAP), for any $d \in D_{x^*}$, we have that

$$f(x^* + d) \geq f(x^*).$$

Hence, for any $r > 0$ and for any $d \in D_{x^*}$, we have that

$$\Phi_{r,x^*}(x^* + d) = \frac{1}{\|x^* + d - x^*\| + 1} = 1/5 < 1 = \Phi_{r,x^*}(x^*),$$

where we can easily verify that $\|d\| = 4$ for any $d \in D_{x^*}$. Thus, x^* is a strict local maximizer of problem $(AQAP)$.

Theorem 9. *Let $\bar{x}$ be a local minimizer of problem* $(AQAP)$ *obtained by Algorithm* 5.1 *with $x^1 := x^* + d^0$ being an initial point, where $d^0 \in D_{x^*}$. Then, $\bar{x} \in U_A$ and one of the following conditions holds:*

$$(1) \qquad f(\bar{x}) < f(x^*)$$

or

$$(2) \qquad \bar{x}^T x^* = 0.$$

Proof. Let $\bar{x}$ be a local minimizer of problem $(AQAP)$. By Algorithm 5.1, we know that $\bar{x} \in U_A$. Suppose that $f(\bar{x}) \geq f(x^*)$ and $\bar{x}^T x^* \neq 0$, then there exists an index pair ij such that $\bar{x}_{ij} = x^*_{ij} = 1$. Let $\gamma \in \Gamma_{\bar{x},U_A}$ such that $ij \in \gamma$. Then there exist $k, r \in \{1, \ldots, n\}$ and $k \neq i, r \neq j$ such that $\gamma = \{ij, ir, kj, kr\}$ and $\bar{x}_{ir} = \bar{x}_{kj} = 0, \bar{x}_{kr} = 1$. Then $\gamma \in \Gamma_{\bar{x},U_A}$. Obviously, we also have that $x^*_{ir} = x^*_{kj} = 0$ by $x^*_{ij} = 1$. Let $d = (I - 2\bar{X})e_\gamma$, where $\bar{X} = \mathrm{diag}(\bar{x})$. Then $d \in D_{\bar{x}}$ and

$$\begin{aligned}
\Phi_{r,x^*}(\bar{x} + d) &\leq \frac{1}{\|\bar{x} + d - x^*\| + 1} \\
&\leq \frac{1}{\sum_{pq \notin \gamma} |\bar{x}_{pq} - x^*_{pq}| + 4} \\
&< \frac{1}{\sum_{pq \notin \gamma} |\bar{x}_{pq} - x^*_{pq}| + 2} \\
&\leq \Phi_{r,x^*}(\bar{x}),
\end{aligned}$$

which contradicts that $\bar{x}$ is a local minimizer of problem $(AQAP)$.

Theorem 10. *If x^* is not a global minimizer of problem (QAP), then there exists a $r_0 > 0$ such that when $r \leq r_0$, any $x \in U_A$ with $f(x) < f(x^*)$ is a local and also a global minimizer of problem $(AQAP)$.*

Proof. Let $L_{x^*} := \{x \in U_A \mid f(x) < f(x^*)\}$ and $r_0 := \min\{f(x^*) - f(x) \mid x \in L_{x^*}\}$. Then $r_0 > 0$ since L_{x^*} is a finite set. And for any $r \leq r_0$, for any $x \in L_{x^*}$, we have that

$$f(x) - f(x^*) \leq -r_0 \leq -r.$$

Hence, we have that

$$\Phi_{r,x^*}(x) = 0 \leq \Phi_{r,x^*}(y), \forall y \in U_A.$$

Thus, $x \in L_{x^*}$ is a local and also a global minimizer of problem $(AQAP)$.

Theorem 11. *Let $\bar{x} \in U_A$. Then when $r \leq r_0$, $\bar{x}$ satisfies that $f(\bar{x}) < f(x^*)$ if and only if $\Phi_{r,x^*}(\bar{x}) = 0$, where r_0 is decided by Theorem 10.*

Proof. If $\bar{x}$ satisfies that $f(\bar{x}) < f(x^*)$, then $r_0 > 0$ and $f(\bar{x}) - f(x^*) \leq -r$ when $r \leq r_0$. Thus, $\Phi_{r,x^*}(\bar{x}) = 0$.

Obviously, for any $r > 0$, if $\Phi_{r,x^*}(\bar{x}) = 0$, then we must have that $f(\bar{x}) - f(x^*) \leq -r$.

In the following, we will give a global optimization method for problem (QAP) based on the given local optimization method, the global optimality sufficient condition $[SCA]$ and the filled function Φ_{r,x^*}.

Algorithm 3. *(Global Optimization Method for Quadratic Assignment Problem (QAP):)*
Step 0. Take an initial point $x_1 \in U_A$ (for example, in the following examples, we take $x_1 = (x^1_{ij})$, where $x^1{}_{ii} = 1, i = 1, \ldots, n$ and $x^1{}_{ij} = 0, i, j \in \{1, \ldots, n\}, i \neq j$), a sufficiently small positive number μ, and an initial $r_1 > 0$. Set $k := 1$ and $r := r_1$.

Step 1. Use the local optimization method (Algorithm 2) to solve problem (QAP) starting from x_k. Let x^*_k be the obtained local minimizer. If $k \geq 2$ and $f(x^*_k) \geq f(x^*_{k-1})$, go to Step 5; otherwise, go to Step 2.

Step 2. Verify whether x^*_k satisfies the following global optimality sufficient condition: there exist $\bar{q} \in R$ and $\lambda \in R^m$ such that

$$[SCA]_{x^*_k} \qquad D + qE^T E \geq \mathrm{diag}((2X^*_k - I)(c + E^T\lambda + 2Dx^*_k),$$

where $X^*_k = \mathrm{diag}(x^*_k)$. If $[SCA]_{x^*_k}$ holds, then go to Step 7; otherwise, go to Step 3.

Step 3. Let

$$\Phi_{r,x^*_k}(x) = \frac{1}{\|x - x^*_k\| + 1}\varphi_r(f(x) - f(x^*_k)).$$

Consider the following problem:

$$\begin{aligned}
\min \quad & \Phi_{r,x^*_k}(x) \\
s.t. \quad & x \in U_A.
\end{aligned} \tag{27}$$

Set $D_{x_k^*} = \{I - 2X_k^* e_\gamma \mid \gamma \in \Gamma_{x_k^*, U_A}\} = \{d_1(x_k^*), \dots, d_{n(n-1)}(x_k^*)\}$, set $i := 1$ and $(x_k)_i^* := x_k^* + d_i(x_k^*)$, go to Step 4.

Step 4. Use the local optimization method (Algorithm 2) to solve problem (27) starting from $(x_k)_i^*$. If in the process of minimization, at some point $y_k^* \in U_A$, the condition $\Phi_{r,x_k^*}(y_k^*) = 0$ holds, then set $x_{k+1} := y_k^*,\ k := k+1$, go to Step 1; otherwise continue the procedure. Let $\bar{(x_k)_i}^*$ be the local minimizer of problem (27) and let $x_{k+1} := \bar{(x_k)_i}^*, k := k+1$, go to Step 1.

Step 5. If $i \geq n(n-1)$, go to Step 6; otherwise, set $i := i+1$ and $(x_k)_i^* := x_k^* + d_i(x_k^*)$, go to Step 4.

Step 6. If $r \geq \mu$, decrease r, such as, let $r := r/10$, and let $i := 1$, go to Step 3; otherwise, go to Step 7.

Step 7. Stop x_k^* is the obtained global minimizer of problem (QAP).

4.3. Numerical Examples

In this subsection, we will give several numerical examples to illustrate the efficiency of the given optimization method **Algorithm 3** (note that **Algorithm 2** is used here for local optimization) to obtain a global minimizer of problem (QAP). In the following numerical examples, we use the following notations:

x_1 is the initial point taken arbitrarily;

$x_k, k \geq 2$ is the point obtained by solving the auxiliary function problem (27) with local optimization method Algorithm 2;

$x_k^*, k \geq 1$ is the local minimizer of problem (QAP) obtained by Algorithm 2 starting from x_k.

Example 2. [30] Consider the following quadratic assignment problem:

$$
\begin{aligned}
[EX1] \qquad \min \quad & f(x) := \sum_{i=1}^{10} \sum_{j=1}^{10} \sum_{k=1}^{10} \sum_{l=1}^{10} f_{ij} d_{kl} x_{ik} x_{jl} \\
s.t. \quad & \sum_{i=1}^{10} x_{ij} = 1, j = 1, \dots, 10, \\
& \sum_{j=1}^{10} x_{ij} = 1, i = 1, \dots, 10, \\
& x_{ij} \in \{0, 1\}, i, j = 1, \dots, 10,
\end{aligned}
$$

where f_{ij} and d_{kl} are given by the following matrices, respectively:

$$
F := (f_{ij})_{10\times 10} = \begin{bmatrix}
0 & 5 & 3 & 7 & 9 & 3 & 9 & 2 & 9 & 0 \\
5 & 0 & 7 & 8 & 3 & 2 & 3 & 3 & 5 & 7 \\
3 & 7 & 0 & 9 & 3 & 5 & 3 & 3 & 9 & 3 \\
7 & 8 & 9 & 0 & 8 & 4 & 1 & 8 & 0 & 4 \\
9 & 2 & 2 & 8 & 0 & 8 & 8 & 7 & 5 & 9 \\
3 & 2 & 4 & 4 & 9 & 0 & 4 & 8 & 0 & 3 \\
8 & 4 & 1 & 1 & 8 & 4 & 0 & 7 & 9 & 5 \\
3 & 2 & 2 & 8 & 6 & 8 & 6 & 0 & 5 & 5 \\
9 & 6 & 9 & 0 & 7 & 0 & 9 & 5 & 0 & 5 \\
0 & 7 & 2 & 4 & 9 & 1 & 4 & 7 & 4 & 0
\end{bmatrix}
$$

and

$$D := (d_{kl})_{10\times 10} = \begin{bmatrix} 0 & 7 & 4 & 6 & 8 & 8 & 8 & 6 & 6 & 5 \\ 7 & 0 & 8 & 2 & 6 & 5 & 6 & 8 & 3 & 6 \\ 4 & 8 & 0 & 10 & 4 & 4 & 7 & 2 & 6 & 7 \\ 6 & 2 & 10 & 0 & 6 & 6 & 9 & 3 & 2 & 6 \\ 8 & 6 & 4 & 6 & 0 & 6 & 4 & 8 & 8 & 6 \\ 8 & 5 & 4 & 6 & 6 & 0 & 3 & 8 & 3 & 2 \\ 8 & 6 & 7 & 9 & 4 & 3 & 0 & 6 & 7 & 8 \\ 6 & 8 & 2 & 3 & 8 & 8 & 6 & 0 & 8 & 8 \\ 6 & 3 & 6 & 2 & 8 & 3 & 7 & 8 & 0 & 9 \\ 5 & 6 & 7 & 6 & 6 & 2 & 8 & 8 & 9 & 0 \end{bmatrix}.$$

The optimal permutation given by [30] is $(9, 1, 8, 3, 6, 7, 2, 5, 4, 10)$ and the optimal objective value given by [30] is $f^* = 2227$. Table 1 records the numerical results of solving Example $[EX1]$ by Algorithm 3.

Table 1: Numerical results for Example $[EX1]$

k	x_k	$f(x_k)$	local minimizer x_k^*	$f(x_k^*)$
1	$(1, 2, 3, 4, 5, 6, 7, 8, 9, 10)^T$	2879	$(10, 2, 9, 4, 5, 1, 3, 8, 6, 7)^T$	2413
2	$(1, 3, 4, 9, 6, 10, 2, 7, 5, 8)^T$	2701	$(1, 9, 4, 2, 6, 10, 3, 5, 8, 7)^T$	2399
3	$(4, 5, 1, 3, 7, 8, 2, 9, 10, 6)^T$	2653	$(9, 1, 8, 3, 6, 7, 2, 5, 4, 10)^T$	2227

From Table 1, we see that the first local minimizer is not the global one, and then we use the filled function to obtain the second and the third one. The third local minimizer is the global one.

Example 3. [30] Consider the following quadratic assignment problem:

$$
\begin{aligned}
[EX2] \qquad & \min \quad f(x) := \sum_{i=1}^{10}\sum_{j=1}^{10}\sum_{k=1}^{10}\sum_{l=1}^{10} f_{ij}d_{kl}x_{ik}x_{jl} \\
& s.t. \quad \sum_{i=1}^{10} x_{ij} = 1, j = 1, \ldots, 10, \\
& \qquad\quad \sum_{j=1}^{10} x_{ij} = 1, i = 1, \ldots, 10, \\
& \qquad\quad x_{ij} \in \{0, 1\}, i, j = 1, \ldots, 10,
\end{aligned}
$$

where f_{ij} and d_{kl} are given by the following matrices, respectively:

$$
F := (f_{ij})_{10\times 10} = \begin{bmatrix}
0 & 9 & 4 & 2 & 2 & 9 & 7 & 4 & 4 & 3 \\
8 & 0 & 6 & 9 & 0 & 9 & 7 & 2 & 5 & 8 \\
1 & 7 & 0 & 1 & 9 & 9 & 7 & 9 & 5 & 3 \\
0 & 0 & 9 & 0 & 4 & 8 & 8 & 8 & 8 & 2 \\
5 & 4 & 6 & 2 & 0 & 8 & 8 & 7 & 2 & 7 \\
7 & 8 & 0 & 4 & 4 & 0 & 9 & 3 & 4 & 3 \\
8 & 5 & 5 & 9 & 8 & 3 & 0 & 9 & 5 & 2 \\
7 & 1 & 5 & 0 & 3 & 5 & 4 & 0 & 9 & 9 \\
7 & 5 & 3 & 2 & 6 & 3 & 3 & 4 & 0 & 0 \\
3 & 7 & 1 & 3 & 3 & 3 & 5 & 8 & 9 & 0
\end{bmatrix}
$$

and

$$
D := (d_{kl})_{10\times 10} = \begin{bmatrix}
0 & 8 & 7 & 6 & 8 & 8 & 6 & 106 & 5 & 9 \\
7 & 0 & 8 & 2 & 10 & 2 & 8 & 8 & 4 & 7 \\
1 & 7 & 0 & 10 & 7 & 7 & 8 & 1 & 5 & 10 \\
7 & 1 & 9 & 0 & 6 & 6 & 10 & 3 & 3 & 2 \\
8 & 2 & 1 & 5 & 0 & 5 & 1 & 7 & 10 & 10 \\
7 & 8 & 1 & 7 & 6 & 0 & 4 & 9 & 3 & 1 \\
9 & 4 & 6 & 8 & 6 & 2 & 0 & 4 & 6 & 9 \\
2 & 8 & 3 & 4 & 9 & 6 & 8 & 0 & 8 & 8 \\
7 & 2 & 7 & 2 & 7 & 4 & 9 & 8 & 0 & 9 \\
2 & 4 & 3 & 10 & 1 & 3 & 7 & 9 & 10 & 0
\end{bmatrix}.
$$

The optimal permutation given by [30] is $x^* = (9, 4, 5, 10, 7, 2, 6, 3, 1, 8)$ and the optimal objective value given by [30] is $f^* = 2025$. Note that the optimal objective value given by [30] is not right, the correct optimal objective value is $f^* = 2027$. Table 2 records the numerical results of solving Example $[EX2]$ by Algorithm 3.
From Table 2, we see that the obtained third local minimizer of problem $[EX2]$ is the global one.

Example 4. [30] Consider the following quadratic assignment problem:

$$
\begin{aligned}
[EX3] \qquad & \min \quad f(x) := \sum_{i=1}^{15}\sum_{j=1}^{15}\sum_{k=1}^{15}\sum_{l=1}^{15} f_{ij}d_{kl}x_{ik}x_{jl} \\
& s.t. \quad \sum_{i=1}^{15} x_{ij} = 1, j = 1, \ldots, 15, \\
& \qquad\quad \sum_{j=1}^{15} x_{ij} = 1, i = 1, \ldots, 15, \\
& \qquad\quad x_{ij} \in \{0, 1\}, i, j = 1, \ldots, 15,
\end{aligned}
$$

Table 2: Numerical results for Example $[EX2]$

k	x_k	$f(x_k)$	local minimizer x_k^*	$f(x_k^*)$
1	$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{pmatrix}$	2777	$\begin{pmatrix} 7 \\ 6 \\ 10 \\ 4 \\ 5 \\ 2 \\ 9 \\ 8 \\ 1 \\ 3 \end{pmatrix}$	2434
2	$\begin{pmatrix} 1 \\ 2 \\ 4 \\ 7 \\ 9 \\ 6 \\ 5 \\ 8 \\ 10 \\ 3 \end{pmatrix}$	2676	$\begin{pmatrix} 6 \\ 9 \\ 3 \\ 5 \\ 8 \\ 1 \\ 7 \\ 9 \\ 2 \\ 4 \end{pmatrix}$	2388
3	$\begin{pmatrix} 3 \\ 1 \\ 6 \\ 2 \\ 4 \\ 10 \\ 9 \\ 7 \\ 5 \\ 8 \end{pmatrix}$	2672	$\begin{pmatrix} 9 \\ 4 \\ 5 \\ 10 \\ 7 \\ 2 \\ 6 \\ 3 \\ 1 \\ 8 \end{pmatrix}$	2027

where f_{ij} and d_{kl} are given by the following matrices, respectively:

$$F := (f_{ij})_{15\times15} = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 & 5 & 2 & 3 & 4 & 5 & 6 \\ 1 & 0 & 1 & 2 & 3 & 2 & 1 & 2 & 3 & 4 & 3 & 2 & 3 & 4 & 5 \\ 2 & 1 & 0 & 1 & 2 & 3 & 2 & 1 & 2 & 3 & 4 & 3 & 2 & 3 & 4 \\ 3 & 2 & 1 & 0 & 1 & 4 & 3 & 2 & 1 & 2 & 5 & 4 & 3 & 2 & 3 \\ 4 & 3 & 2 & 1 & 0 & 5 & 4 & 3 & 2 & 1 & 6 & 5 & 4 & 3 & 2 \\ 1 & 2 & 3 & 4 & 5 & 0 & 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 2 & 3 & 4 & 1 & 0 & 1 & 2 & 3 & 2 & 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 2 & 3 & 2 & 1 & 0 & 1 & 2 & 3 & 2 & 1 & 2 & 3 \\ 4 & 3 & 2 & 1 & 2 & 3 & 2 & 1 & 0 & 1 & 4 & 3 & 2 & 1 & 2 \\ 5 & 4 & 3 & 2 & 1 & 4 & 3 & 2 & 1 & 0 & 5 & 4 & 3 & 2 & 1 \\ 2 & 3 & 4 & 5 & 6 & 1 & 2 & 3 & 4 & 5 & 0 & 1 & 2 & 3 & 4 \\ 3 & 2 & 3 & 4 & 5 & 2 & 1 & 2 & 3 & 4 & 1 & 0 & 1 & 2 & 3 \\ 4 & 3 & 2 & 3 & 4 & 3 & 2 & 1 & 2 & 3 & 2 & 1 & 0 & 1 & 2 \\ 5 & 4 & 3 & 2 & 3 & 4 & 3 & 2 & 1 & 2 & 3 & 2 & 2 & 0 & 1 \\ 6 & 5 & 4 & 3 & 2 & 5 & 4 & 3 & 2 & 1 & 4 & 3 & 2 & 1 & 0 \end{bmatrix}$$

and

$$D := (d_{kl})_{15\times 15} = \begin{bmatrix} 0 & 10 & 0 & 5 & 1 & 0 & 1 & 2 & 2 & 2 & 2 & 0 & 4 & 0 & 0 \\ 10 & 0 & 1 & 3 & 2 & 2 & 2 & 3 & 2 & 0 & 2 & 0 & 10 & 5 & 0 \\ 0 & 1 & 0 & 10 & 2 & 0 & 2 & 5 & 4 & 5 & 2 & 2 & 5 & 5 & 5 \\ 5 & 3 & 10 & 0 & 1 & 1 & 5 & 0 & 0 & 2 & 1 & 0 & 2 & 5 & 0 \\ 1 & 2 & 2 & 1 & 0 & 3 & 5 & 5 & 5 & 1 & 0 & 3 & 0 & 5 & 5 \\ 0 & 2 & 0 & 1 & 3 & 0 & 2 & 2 & 1 & 5 & 0 & 0 & 2 & 5 & 10 \\ 1 & 2 & 2 & 5 & 5 & 2 & 0 & 6 & 0 & 1 & 5 & 5 & 5 & 1 & 0 \\ 2 & 3 & 5 & 0 & 5 & 2 & 6 & 0 & 5 & 2 & 10 & 0 & 5 & 0 & 0 \\ 2 & 2 & 4 & 0 & 5 & 1 & 0 & 5 & 0 & 0 & 10 & 5 & 10 & 0 & 2 \\ 2 & 0 & 5 & 2 & 1 & 5 & 1 & 2 & 0 & 0 & 0 & 4 & 0 & 0 & 5 \\ 2 & 2 & 2 & 1 & 0 & 0 & 5 & 10 & 10 & 0 & 0 & 5 & 0 & 5 & 0 \\ 0 & 0 & 2 & 0 & 3 & 0 & 5 & 0 & 5 & 4 & 5 & 0 & 3 & 3 & 0 \\ 4 & 10 & 5 & 2 & 0 & 2 & 5 & 5 & 10 & 0 & 0 & 3 & 0 & 10 & 2 \\ 0 & 5 & 5 & 5 & 5 & 5 & 1 & 0 & 0 & 0 & 5 & 3 & 10 & 0 & 4 \\ 0 & 0 & 5 & 0 & 5 & 10 & 0 & 0 & 2 & 5 & 0 & 0 & 2 & 4 & 0 \end{bmatrix}.$$

The optimal permutation given by [30] is $x^* = (1, 2, 13, 8, 9, 4, 3, 14, 7, 11, 10, 15, 6, 5, 12)$ and the optimal objective value given by [30] is $f^* = 1150$. Table 3 records the numerical results of solving Example [$EX3$] by Algorithm 3.

Table 3: Numerical results for Example [$EX3$]

k	x_k	$f(x_k)$	local minimizer x_k^*	$f(x_k^*)$
1	$\begin{pmatrix} 1 \\ 2 \\ 12 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 15 \\ 14 \\ 13 \\ 3 \\ 11 \\ 10 \\ 9 \end{pmatrix}$	1616	$\begin{pmatrix} 1 \\ 2 \\ 14 \\ 6 \\ 15 \\ 4 \\ 13 \\ 3 \\ 5 \\ 10 \\ 7 \\ 8 \\ 11 \\ 9 \\ 12 \end{pmatrix}$	1200

2	$\begin{pmatrix} 12 \\ 1 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 14 \\ 15 \\ 8 \\ 13 \\ 10 \\ 9 \\ 11 \\ 2 \end{pmatrix}$	1630	$\begin{pmatrix} 10 \\ 3 \\ 4 \\ 2 \\ 1 \\ 6 \\ 14 \\ 13 \\ 8 \\ 7 \\ 15 \\ 5 \\ 0 \\ 11 \\ 12 \end{pmatrix}$	1186
3	$\begin{pmatrix} 12 \\ 2 \\ 5 \\ 3 \\ 10 \\ 7 \\ 8 \\ 9 \\ 14 \\ 6 \\ 11 \\ 4 \\ 13 \\ 15 \\ 1 \end{pmatrix}$	1466	$\begin{pmatrix} 12 \\ 1 \\ 4 \\ 3 \\ 10 \\ 9 \\ 2 \\ 13 \\ 14 \\ 15 \\ 11 \\ 8 \\ 7 \\ 5 \\ 6 \end{pmatrix}$	1174
4	$\begin{pmatrix} 15 \\ 2 \\ 3 \\ 4 \\ 6 \\ 7 \\ 1 \\ 8 \\ 11 \\ 5 \\ 14 \\ 13 \\ 9 \\ 12 \\ 10 \end{pmatrix}$	1502	$\begin{pmatrix} 4 \\ 14 \\ 3 \\ 5 \\ 15 \\ 2 \\ 13 \\ 8 \\ 7 \\ 6 \\ 1 \\ 9 \\ 11 \\ 12 \\ 10 \end{pmatrix}$	1160

5	$\begin{pmatrix} 12 \\ 2 \\ 5 \\ 3 \\ 8 \\ 14 \\ 6 \\ 15 \\ 9 \\ 13 \\ 4 \\ 7 \\ 10 \\ 1 \\ 11 \end{pmatrix}$	1560	$\begin{pmatrix} 1 \\ 2 \\ 13 \\ 8 \\ 9 \\ 4 \\ 3 \\ 14 \\ 7 \\ 11 \\ 10 \\ 15 \\ 6 \\ 5 \\ 12 \end{pmatrix}$	1150

From Table 3, we see that 5 local minimizers are obtained by Algorithm 3 and the 5th local minimizer is the global one of problem $[EX3]$.

5. Conclusion

The quadratic assignment problem (QAP) has a wide range of practical applications. For this problem, we provided a new improvement method. We presented global optimality conditions for problem (QAP). Then we designed a new local optimization method by using necessary global optimality condition. Furthermore we provided a new global optimization method by combining sufficient global optimality condition, new local optimization method and auxiliary functions.

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