Determinants and Inverses of Circulant Matrices with Jacobsthal and Jacobsthal-Lucas Numbers

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Abstract

Let $n \geq 3$ and $\mathbb{J}_n := \mathrm{circ}(J_1, J_2, \ldots, J_n)$ and $\mathbb{J}_n := \mathrm{circ}(j_0, j_1, \ldots, j_{n-1})$ be the $n \times n$ circulant matrices, associated with the nth Jacobsthal number J_n and the nth Jacobsthal-Lucas number j_n , respectively. The determinants of \mathbb{J}_n and \mathbb{J}_n are obtained in terms of the Jacobsthal and Jacobsthal-Lucas numbers. These imply that \mathbb{J}_n and \mathbb{J}_n are invertible. We also derive the inverses of \mathbb{J}_n and \mathbb{J}_n .

1 Introduction

The $n \times n$ circulant matrix $C_n := \text{circ}(c_0, c_1, \dots, c_{n-1})$, assoicated with the numbers c_0, \dots, c_{n-1} , is defined as

$$C_n := \begin{bmatrix} c_0 & c_1 & \dots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & \dots & c_{n-3} & c_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_2 & c_3 & \dots & c_0 & c_1 \\ c_1 & c_2 & \dots & c_{n-1} & c_0 \end{bmatrix}.$$
 (1)

Circulant matrices have a wide range of applications, for examples in signal processing, coding theory, image processing, digital image disposal, self-regress design and so on. Numerical solutions of the certain types of elliptic and parabolic partial differential equations with periodic boundary conditions often involve linear systems associated with circulant matrices [9-11].

*e-mail: dbozkurt@selcuk.edu.tr †e-mail: tamtiny@auburn.edu The eigenvalues and eigenvectors of C_n are well-known [14]:

$$\lambda_j = \sum_{k=0}^{n-1} c_k \omega^{jk}, \quad j = 0, \dots, n-1,$$

where $\omega := \exp(\frac{2\pi i}{n})$ and $i := \sqrt{-1}$ and the corresponding eigenvectors

$$v_j = (1, \omega^j, \omega^{2j}, \dots, \omega^{(n-1)j})^T, \quad j = 0, \dots, n-1.$$

Thus we have the determinants and inverses of nonsingular circulant matrices [1,3,4,14]:

$$\det(C_n) = \prod_{j=0}^{n-1} (\sum_{k=0}^{n-1} c_k \omega^{jk}),$$

and

$$C_n^{-1} = \operatorname{circ}(a_0, a_1, \dots, a_{n-1}),$$

where $a_j := \frac{1}{n} \sum_{k=0}^{n-1} \lambda_k \omega^{-kj}$, and $r = 0, 1, \ldots, n-1$ [4]. When n is getting large, the above formulas are not very handy to use. If there is some structure among c_0, \ldots, c_{n-1} , we may be able to get more explicit forms of the eigenvalues, determinants and inverses of C_n . Recently, studies on the circulant matrices involving interesting number sequences appeared. In [1] the determinants and inverses of the circulant matrices $\mathbb{A}_n = \text{circ}(F_1, F_2, \ldots, F_n)$ and $\mathbb{B}_n = \text{circ}(L_1, L_2, \ldots, L_n)$ are derived, where F_n and L_n are the nth Fibonacci and Lucas numbers, respectively. In [2] the r-circulant matrix is defined and its norm is computed. The norms of Toeplitz matrices [13] involving Fibonacci and Lucas numbers are obtained [5]. Miladinovic and Stanimirovic [6] gave an explicit formula of the Moore-Penrose inverse of singular generalized Fibonacci matrix. Lee and et al. found the factorizations and eigenvalues of Fibonacci and symmetric Fibonacci matrices [7].

When $n \geq 2$, the Jacobsthal and Jacobsthal-Lucas sequences $\{J_n\}$ and $\{j_n\}$ are defined by $J_n = J_{n-1} + 2J_{n-2}$ and $j_n = j_{n-1} + 2j_{n-2}$ with initial conditions $J_0 = 0$, $J_1 = 1$, $j_0 = 2$, and $j_1 = 1$, respectively. Let $\mathbb{J}_n := \mathrm{circ}(J_1, J_2, \ldots, J_n)$ and $\mathbb{J}_n := \mathrm{circ}(j_0, j_1, \ldots, j_{n-1})$. The aim of this paper is to establish some useful formulas for the determinants and inverses of \mathbb{J}_n and \mathbb{J}_n using the nice properties of the Jacobsthal and Jacobsthal-Lucas numbers. Question: How about eigenvalues? Matrix decompositions are derived for \mathbb{J}_n and \mathbb{J}_n in order to obtain the results.

2 Determinants of \mathbb{J}_n and \mathbb{J}_n

Recall that $\mathbb{J}_n := \operatorname{circ}(J_1, J_2, \dots, J_n)$ and $\mathbb{J}_n := \operatorname{circ}(j_0, j_1, \dots, j_{n-1})$, i.e., where J_k and j_k are the kth Jacobsthal and Jacobsthal-Lucas numbers, respectively, with the recurrence relations $J_k = J_{k-1} + 2J_{k-2}$, $j_k = j_{k-1} + 2j_{k-2}$, and the initial conditions $J_0 = 0$, $J_1 = 1$, $j_0 = 2$, and $j_1 = 1$ ($k \ge 2$). Let α and β be

the roots of $x^2 - x - 2 = 0$. Using the Binet formulas [8, p.40] for the sequences $\{J_n\}$ and $\{j_n\}$, one has

$$J_n = \frac{\alpha^n - \beta^n}{3} = \frac{1}{3} [2^n - (-1)^n]$$
 (2)

and

$$j_n = \alpha^n + \beta^n = 2^n + (-1)^n.$$
 (3)

Theorem 1 Let $n \geq 3$. Then

$$\det(\mathbb{J}_n) = (1 - J_{n+1})^{n-2} (1 - J_n) + 2 \sum_{k=1}^{n-2} \left[J_k (1 - J_{n+1})^{k-1} (2J_n)^{n-k-1} \right].$$
 (4)

Proof. Obviously, $\det(\mathbb{J}_3) = 20$. It satisfies (4). For n > 3, we select the matrices P_n and Q_n so that when we multiply \mathbb{J}_n with P_n on the left and Q_n on the right we obtain a special upper triangular matrix that have nonzero entries only on the first two rows, main diagonal and super diagonal:

$$P_n := \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ -1 & 0 & 0 & 0 & \dots & 0 & 1 \\ -2 & 0 & 0 & 0 & \dots & 1 & -1 \\ 0 & 0 & 0 & 0 & \dots & -1 & -2 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 1 & -1 & \dots & 0 & 0 \\ 0 & 1 & -1 & -2 & \dots & 0 & 0 \end{bmatrix}$$
 (5)

and

$$Q_n := \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & \left(\frac{2J_n}{1-J_{n+1}}\right)^{n-2} & 0 & \dots & 0 & 0 \\ 0 & \left(\frac{2J_n}{1-J_{n+1}}\right)^{n-3} & 0 & \dots & 0 & -1 \\ 0 & \left(\frac{2J_n}{1-J_{n+1}}\right)^{n-4} & 0 & \dots & -1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \left(\frac{2J_n}{1-J_{n+1}}\right) & -1 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \end{bmatrix}.$$

Notice that we have the following equivalence:

and S_n is upper triangular, where

$$f_n := \sum_{k=1}^{n-1} J_{k+1} \left(\frac{2J_n}{1 - J_{n+1}} \right)^{n-k-1},$$

$$g_n := 1 - J_n + 2 \sum_{k=1}^{n-2} J_{n-k-1} \left(\frac{2J_n}{1 - J_{n+1}} \right)^k.$$

Then we have

$$\det(S_n) = \det(P_n) \det(\mathbb{J}_n) \det(Q_n) = (2J_n)^{n-2} g_n.$$

Since

$$\det(P_n) = \begin{cases} 1 & n \equiv 1, \text{ or } 2 \text{ (mod 4)} \\ -1, & n \equiv 0 \text{ or } 3 \text{ (mod 4)}, \end{cases}$$

and

$$\det(Q_n) = \begin{cases} \left(\frac{2J_n}{1 - J_{n+1}}\right)^{n-2}, & n \equiv 1 \text{ or } 2 \pmod{4} \\ -\left(\frac{2J_n}{1 - J_{n+1}}\right)^{n-2}, & n \equiv 0 \text{ or } 3 \pmod{4}, \end{cases}$$

for all n > 3,

$$\det(P_n)\det(Q_n) = \left(\frac{2J_n}{1 - J_{n+1}}\right)^{n-2}$$

and (4) follows.

Theorem 2 Let $n \geq 3$. Then

$$\det(\mathbf{J}_n) = (2 - j_n)^{n-2} (4 - j_{n-1}) + \sum_{k=2}^{n-1} \left[(2j_k - j_{k-1})(2 - j_n)^{k-2} (1 + 2j_{n-1})^{n-k} \right].$$
(6)

Proof. Since $\det(\mathbb{J}_3) = 104$, \mathbb{J}_3 satisfies (6). For n > 3, we select the matrices K_n and M_n so that when we multiply \mathbb{J}_n with K_n on the left and M_n on the right we obtain a special upper triangular matrix that have nonzero entries only on the first two rows, main diagonal and super diagonal:

$$K_n := \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 & \dots & 0 & 1 \\ -2 & 0 & 0 & 0 & \dots & 1 & -1 \\ 0 & 0 & 0 & 0 & \dots & -1 & -2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 1 & -1 & \dots & 0 & 0 \\ 0 & 1 & -1 & -2 & \dots & 0 & 0 \end{bmatrix}$$

$$(7)$$

and

$$M_n := \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & \left(\frac{1+2j_{n-1}}{2-j_n}\right)^{n-2} & 0 & \dots & 0 & 0 \\ 0 & \left(\frac{1+2j_{n-1}}{2-j_n}\right)^{n-3} & 0 & \dots & 0 & -1 \\ 0 & \left(\frac{1+2j_{n-1}}{2-j_n}\right)^{n-4} & 0 & \dots & -1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \left(\frac{1+2j_{n-1}}{2-j_n}\right) & 0 & \dots & 0 & 0 \\ 0 & 1 & -1 & \dots & 0 & 0 \end{bmatrix}.$$

We have

$$U_{n} = K_{n} \mathbb{I}_{n} M_{n}$$

$$\begin{bmatrix} 2 & y'_{n} & -j_{n-1} & -j_{n-2} & \dots & -j_{3} & -j_{2} \\ y_{n} & \frac{1}{2} j_{n-1} - j_{0} & \frac{1}{2} j_{n-2} - j_{n-1} & \dots & \frac{1}{2} j_{3} - j_{4} & \frac{1}{2} j_{2} - j_{3} \\ & & 1 + 2 j_{n-1} & j_{n} - 2 \\ & & & 1 + 2 j_{n-1} \end{bmatrix}$$

$$= \begin{bmatrix} & & & & & & & & & & & & & & & \\ & & & & & & & & & & \\ & & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & \\ & & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & \\ & & & & \\ & & &$$

and U_n is upper triangular, where

$$y_n := \frac{1}{2} \left[(4 - j_{n-1}) + \sum_{k=2}^{n-1} (2j_k - j_{k-1}) \left(\frac{1 + 2j_{n-1}}{2 - j_n} \right)^{n-k} \right],$$

$$y'_n := \sum_{k=1}^{n-1} j_k \left(\frac{1 + 2j_{n-1}}{2 - j_n} \right)^{n-k-1}.$$

Then we obtain

$$\det(U_n) = \det(K_n) \det(\mathbb{I}_n) \det(M_n) = 2(1 + 2j_{n-1})^{n-2} y_n.$$

Since

$$\det(K_n) = \begin{cases} 1, & n \equiv 1 \text{ or } 2 \pmod{4} \\ -1, & n \equiv 0 \text{ or } 3 \pmod{4}, \end{cases}$$

and

$$\det(M_n) = \begin{cases} \left(\frac{1+2j_{n-1}}{2-j_n}\right)^{n-2}, & n \equiv 1 \text{ or } 2 \pmod{4} \\ -\left(\frac{1+2j_{n-1}}{2-j_n}\right)^{n-2}, & n \equiv 0 \text{ or } 3 \pmod{4}, \end{cases}$$

for all n > 3,

$$\det(K_n)\det(M_n) = \left(\frac{1+2j_{n-1}}{2-j_n}\right)^{n-2}$$

and we have (6).

3 Inverses of \mathbb{J}_n and \mathbb{J}_n

We will use the well-known fact that the inverse of a nonsingular circulant matrix is also circulant [14, p.84] [12, p.33], [4, p.90-91].

Theorem 3 The matrix $\mathbb{J}_n = circ(J_1, J_2, \dots, J_n)$ is invertible when $n \geq 3$.

Proof. From Theorem 1, $\det(\mathbb{J}_3) = 20 \neq 0$ and $\det(\mathbb{J}_4) = -400 \neq 0$. Then \mathbb{J}_3 and \mathbb{J}_4 are invertible. Let $n \geq 5$. The Binet formula for Jacobsthal numbers gives $J_n = \frac{\alpha^n - \beta^n}{3}$, where $\alpha + \beta = 1$, $\alpha\beta = -2$ and $\alpha - \beta = 3$. Then we have

$$g(\omega^{k}) = \sum_{r=1}^{n} J_{r} \omega^{kr-k} = \sum_{r=1}^{n} \left(\frac{\alpha^{r} - \beta^{r}}{3} \right) \omega^{kr-k} = \frac{1}{3} \sum_{r=1}^{n} (\alpha^{r} - \beta^{r}) \omega^{kr-k}$$

$$= \frac{1}{3} \left[\frac{\alpha(1 - \alpha^{n})}{1 - \alpha \omega^{k}} - \frac{\beta(1 - \beta^{n})}{1 - \beta \omega^{k}} \right], \quad (1 - \alpha \omega^{k}, 1 - \beta \omega^{k} \neq 0)$$

$$= \frac{1}{3} \left(\frac{(\alpha - \beta) - (\alpha^{n+1} - \beta^{n+1}) + \alpha \beta \omega^{k} (\alpha^{n} - \beta^{n})}{1 - \alpha \omega^{k} - \beta \omega^{k} + \alpha \beta \omega^{2k}} \right)$$

$$= \frac{1 - J_{n+1} - 2J_{n}\omega^{k}}{1 - \omega^{k} - 2\omega^{2k}}, \quad k = 1, 2, \dots, n - 1.$$

If there existed ω^k $(k=1,2,\ldots,n-1)$ such that $g(\omega^k)=0$, then we would have $1-J_{n+1}-2J_n\omega^k=0$ for $1-\omega^k-2\omega^{2k}\neq 0$. Hence $\omega^k=\frac{1-J_{n+1}}{2J_n}$. It is well known that

$$\omega^k = \exp\left(\frac{2k\pi i}{n}\right) = \cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right)$$
 (8)

where $i:=\sqrt{-1}$. Since $\omega^k=\frac{1-J_{n+1}}{2J_n}$ is a real number, $\sin\left(\frac{2k\pi}{n}\right)=0$ so that $\omega^k=-1$ for $0<\frac{2k\pi}{n}<2\pi$. However u=-1 is not a root of the equation $1-J_{n+1}-2J_nu=0$ $(n\geq 5)$, a contradiction, i.e., $g(\omega^k)\neq 0$ for any ω^k , where $k=1,2,\ldots,n-1,\,n\geq 5$. Thus the proof is completed by [1, Lemma 1.1].

Lemma 4 Let $A = (a_{ij})$ be the $(n-2) \times (n-2)$ matrix defined by

$$a_{ij} = \begin{cases} 2J_n, & i = j \\ J_{n+1} - 1, & j = i+1 \\ 0, & otherwise. \end{cases}$$

Then $A^{-1} = (a_{ij}^{'})$ is given by

$$a'_{ij} := \begin{cases} \frac{(1 - J_{n+1})^{j-i}}{(2J_n)^{j-i+1}}, & j \ge i\\ 0, & otherwise. \end{cases}$$

Proof. Let $B = (b_{ij}) = AA^{-1}$. Clearly $b_{ij} = \sum_{k=1}^{n-2} a_{ik} a'_{kj}$. When i = j, we have

$$b_{ii} = 2J_n \cdot \frac{1}{2J_n} = 1.$$

If j > i, then

$$b_{ij} = \sum_{k=1}^{n-2} a_{ik} a'_{kj} = a_{i,i+1} a'_{i+1,j} + a_{ii} a'_{ij}$$

$$= (J_{n+1} - 1) \frac{(1 - J_{n+1})^{j-i-1}}{(2J_n)^{j-i}} + 2J_n \frac{(1 - J_{n+1})^{j-i}}{(2J_n)^{j-i+1}} = 0;$$

similar for j < i. Thus $AA^{-1} = I_{n-2}$.

Theorem 5 Let the matrix \mathbb{J}_n be $\mathbb{J}_n := circ(J_1, J_2, \dots, J_n)$ $(n \ge 3)$. Then the inverse of the matrix \mathbb{J}_n is

$$\mathbb{J}_n^{-1} = circ(m_1, m_2, \dots, m_n)$$

where

$$m_1 = \frac{J_{n+1} + (1 - 2J_{n-1})g_n - 1}{2g_n J_n^2}$$

$$m_2 = \frac{g_n - 1}{J_n g_n}$$

$$m_3 = \frac{1}{g_n} \left[(1 - J_n - g_n) \frac{(1 - J_{n+1})^{n-3}}{(2J_n)^{n-2}} + 2 \sum_{k=2}^{n-2} J_{n-k} \frac{(1 - J_{n+1})^{n-k-2}}{(2J_n)^{n-k-1}} \right]$$

$$m_4 = \frac{1}{g_n} \left(\frac{[(1 - J_n - g_n)(J_{n+2} - 1) - 4J_n J_{n-2}](1 - J_{n+1})^{n-4}}{(2J_n)^{n-2}} + 4 \sum_{k=1}^{n-4} J_k \frac{(1 - J_{n+1})^{k-1}}{(2J_n)^k} \right)$$

$$m_i = \frac{1}{g_n} \frac{(1 - J_{n+1})^{n-i}}{(2J_n)^{n-i+1}} \begin{cases} \frac{(1 - J_n - g_n)(2^{n+2} - 4)}{(2J_n)^2} - 2\left(J_{n-1} + \frac{J_{n-2}(1 - J_{n+1})}{J_n}\right), & n \text{ is odd} \\ -2\left(J_{n-1} + \frac{J_{n-2}(1 - J_{n+1})}{J_n}\right), & n \text{ is even} \end{cases}$$

$$for g_n = 1 - J_n + 2 \sum_{k=1}^{n-2} J_{n-k-1} \left(\frac{2J_n}{1 - J_{n+1}}\right)^k \text{ and } i = 5, 6, \dots, n.$$

Proof. Let

$$R_n = \begin{bmatrix} 1 & -f_n & \frac{f_n}{g_n}(J_n - 1) + J_n & J_{n-1} - 2\frac{f_n}{g_n}J_{n-2} & \dots & J_3 - 2\frac{f_n}{g_n}J_2 \\ 0 & 1 & -\frac{J_n - 1}{g_n} & \frac{2J_{n-2}}{g_n} & \dots & \frac{2J_2}{g_n} \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

and $G = diag(1, g_n)$ where $f_n = \sum_{k=1}^{n-1} J_{k+1} \left(\frac{2J_n}{1 - J_{n+1}}\right)^{n-k-1}$ and $g_n = 1 - J_n + 2\sum_{k=1}^{n-2} J_{n-k-1} \left(\frac{2J_n}{1 - J_{n+1}}\right)^k$. Then we can write

$$P_n \mathbb{J}_n Q_n R_n = G \oplus A$$

where $G \oplus A$ is the direct sum of the matrices G and A. Let $T_n = Q_n R_n$. Then we have

$$\mathbb{J}_n^{-1} = T_n(G^{-1} \oplus A^{-1})P_n.$$

Since the matrix \mathbb{J}_n is circulant, its inverse is circulant from Lemma 1.1 [1, p. 9791]. Let

$$\mathbb{J}_n^{-1} = circ(m_1, m_2, \dots, m_n).$$

Since the last row of the matrix T_n is

$$\left(0,1,\frac{1-J_n}{g_n}-1,\frac{2J_{n-2}}{g_n},\frac{2J_{n-3}}{g_n},\ldots,\frac{2J_3}{g_n},\frac{2J_2}{g_n}\right),$$

the last row entries of the matrix \mathbb{J}_n^{-1} are

$$m_{2} = \frac{g_{n} - 1}{J_{n}g_{n}}$$

$$m_{3} = \frac{1}{g_{n}} \left((1 - J_{n} - g_{n}) \frac{(1 - J_{n+1})^{n-3}}{(2J_{n})^{n-2}} + 2 \sum_{k=2}^{n-2} J_{n-k} \frac{(1 - J_{n+1})^{n-k-2}}{(2J_{n})^{n-k-1}} \right)$$

$$m_{4} = \frac{1}{g_{n}} \left(\frac{[(1 - J_{n} - g_{n})(J_{n+2} - 1) - 4J_{n}J_{n-2}](1 - J_{n+1})^{n-4}}{(2J_{n})^{n-2}} + 4 \sum_{k=1}^{n-4} J_{k} \frac{(1 - J_{n+1})^{k-1}}{(2J_{n})^{k}} \right)$$

$$m_5 = \frac{1}{g_n} \left[(1 - J_n - g_n)(2^{n+1} - 2)(1 + (-1)^{n-1}) \frac{(1 - J_{n+1})^{n-5}}{(2J_n)^{n-2}} + \frac{2}{g_n} \left(\sum_{k=1}^{n-4} J_{k+3} \frac{(1 - J_{n+1})^{k-1}}{(2J_n)^k} - \sum_{k=1}^{n-3} J_{k+2} \frac{(1 - J_{n+1})^{k-1}}{(2J_n)^k} - 2 \sum_{k=1}^{n-2} J_{k+1} \frac{(1 - J_{n+1})^{k-1}}{(2J_n)^k} \right)$$

$$m_n = \frac{1}{g_n} \left[(1 - J_n - g_n)(2^{n+1} - 2)(1 + (-1)^{n-1}) + 2J_{n-2} \left(-\frac{1}{2J_n} - 2\frac{1 - J_{n+1}}{(2J_n)^2} \right) + 2J_{n-3} \left(-\frac{2}{2J_n} \right) \right]$$

$$m_1 = \frac{J_{n+1} + (1 - 2J_{n-1})g_n - 1}{2g_n J^2}$$

where $g_n = 1 - J_n + 2 \sum_{k=1}^{n-2} J_{n-k-1} \left(\frac{2J_n}{1 - J_{n+1}} \right)^k$. If we rearrange m_5 , then

$$\begin{split} m_5 &= \frac{1}{g_n} \left[(1 - J_n - g_n)(2^{n+1} - 2)(1 + (-1)^{n-1}) \frac{(1 - J_{n+1})^{n-5}}{(2J_n)^{n-2}} \right. \\ &\quad + \frac{2}{g_n} \frac{(1 - J_{n+1})^{n-4}}{(2J_n)^{n-3}} \left[\left(-J_{n-1} \frac{(1 - J_{n+1})^{n-4}}{(2J_n)^{n-3}} - 2J_{n-1} \frac{(1 - J_{n+1})^{n-3}}{(2J_n)^{n-2}} - 2J_{n-2} \frac{(1 - J_{n+1})^{n-4}}{(2J_n)^{n-3}} \right) \right. \\ &\quad + \sum_{k=1}^{n-4} \underbrace{\left(J_{k+3} - J_{k+2} - 2J_{k+1} \right)}_{0} \frac{(1 - J_{n+1})^{k-1}}{(2J_n)^k} \right] \\ &= \frac{1}{g_n} \frac{(1 - J_{n+1})^{n-4}}{(2J_n)^{n-3}} \left\{ \begin{array}{c} \frac{(1 - J_n - g_n)(2^{n+2} - 4)}{(2J_n)^2} - 2\left(J_{n-1} + \frac{J_{n-2}(1 - J_{n+1})}{J_n} \right), & n \ is \ odd \\ -2\left(J_{n-1} + \frac{J_{n-2}(1 - J_{n+1})}{J_n} \right), & n \ is \ even. \\ \end{array} \right. \end{split}$$

Then

$$m_{i} = \frac{1}{g_{n}} \frac{(1 - J_{n+1})^{n-i}}{(2J_{n})^{n-i+1}} \begin{cases} \frac{(1 - J_{n} - g_{n})(2^{n+2} - 4)}{(2J_{n})^{2}} - 2\left(J_{n-1} + \frac{J_{n-2}(1 - J_{n+1})}{J_{n}}\right), & n \text{ is odd} \\ -2\left(J_{n-1} + \frac{J_{n-2}(1 - J_{n+1})}{J_{n}}\right), & n \text{ is even.} \end{cases}$$

and $i=5,6,\ldots,n$. Since the matrix \mathbb{J}_n^{-1} is a circulant matrix and its last row is known, the proof is completed. \blacksquare

A Hankel matrix $A = (a_{ij})$ is an $n \times n$ matrix such that $a_{i,j} = a_{i-1,j+1}$. It is closely related to the Toeplitz matrix in the sense that a Hankel matrix is an upside-down Toeplitz matrix.

Corollary 6 Let the matrix P_n be as in (5). Then

$$P_n^{-1} = \left[\begin{array}{c} H_1 \\ H_2 \end{array} \right],$$

where $H_1:=[1,0,\ldots,0]$ and H_2 is an $(n-1)\times n$ Hankel type in which the first row is $[J_n,J_{n-1},\ldots,J_1]$ and the last column is $[1,0,\ldots,0]^T$.

Proof. The matrix P_n^{-1} is computed easily by applying elementary row operations to the augmented matrix $[P_n:I_n]$.

Theorem 7 The matrix $\mathbb{I}_n = circ(j_0, j_1, \dots, j_{n-1})$ is invertible when $n \geq 3$.

Proof. We show that $\det(\mathbb{I}_3) = 104 \neq 0$ and $\det(\mathbb{I}_4) = -675 \neq 0$ by Theorem 2. Then \mathbb{I}_3 and \mathbb{I}_4 are invertible. Let $n \geq 5$. The Binet formula for Jacobsthal-Lucas numbers yields $j_n = \alpha^n + \beta^n$, where $\alpha + \beta = 1$ and $\alpha\beta = -2$. Then we have

$$h(\omega^{k}) = \sum_{r=1}^{n} j_{r} \omega^{kr-k} = \sum_{r=1}^{n} (\alpha^{r} + \beta^{r}) \omega^{kr-k}$$

$$= \frac{\alpha(1 - \alpha^{n})}{1 - \alpha \omega^{k}} + \frac{\beta(1 - \beta^{n})}{1 - \beta \omega^{k}}, \quad (1 - \alpha \omega^{k}, 1 - \beta \omega^{k} \neq 0)$$

$$= \left(\frac{(\alpha + \beta) - (\alpha^{n+1} + \beta^{n+1}) + \alpha \beta \omega^{k} (\alpha^{n} + \beta^{n}) - 2\alpha \beta \omega^{k}}{1 - \alpha \omega^{k} - \beta \omega^{k} + \alpha \beta \omega^{2k}}\right)$$

$$= \frac{1 - j_{n+1} + 2\omega^{k}(2 - j_{n})}{1 - \omega^{k} - 2\omega^{2k}}, \quad k = 1, 2, \dots, n - 1.$$

We are going to show that there is no ω^k , $k=1,2,\ldots,n-1$ such that $h(\omega^k)=0$. If $1-j_{n+1}+2\omega^k(2-j_n)=0$ for $1-\omega^k-2\omega^{2k}\neq 0$, then $\omega^k=\frac{j_{n+1}-1}{2(2-j_n)}$ would be a real number. By (8) we would have $\sin\left(\frac{2k\pi}{n}\right)=0$ so that $\omega^k=-1$ for $0<\frac{2k\pi}{n}<2\pi$. However u=-1 is not a root of the equation $1-j_{n+1}+2(2-j_n)u=0$ $(n\geq 5)$, a contradiction. i.e., $h(\omega^k)\neq 0$ for any ω^k , where $k=1,2,\ldots,n-1$ and $n\geq 5$. Thus the proof is completed by [1, Lemma 1.1].

Lemma 8 If the matrix $\mathbb{S} = (s_{ij})_{i,j=1}^{n-2}$ is of the form

$$s_{ij} = \begin{cases} 1 + 2j_{n-1}, & i = j\\ j_n - 2, & j = i + 1\\ 0, & otherwise, \end{cases}$$

then $\mathbb{S}^{-1}=(s_{ij}^{'})_{i,j=1}^{n-2}$ is given by

$$s'_{ij} = \begin{cases} \frac{(2-j_n)^{j-i}}{(1+2j_{n-1})^{j-i+1}}, & j \ge i\\ 0, & otherwise. \end{cases}$$

Proof. Let $\mathbb{B} := \mathbb{SS}^{-1} = (b_{ij})$ so that $b_{ij} = \sum_{k=1}^{n-2} s_{ik} s'_{kj}$. Clearly

$$b_{ii} = (1 + 2j_{n-1}) \cdot \frac{1}{1 + 2j_{n-1}} = 1.$$

If j > i, then

$$b_{ij} = \sum_{k=1}^{n-2} s_{ik} s'_{kj} = s_{i,i+1} s'_{i+1,j} + s_{ii} s'_{ij}$$

$$= (j_n - 2) \frac{(2 - j_n)^{j-i-1}}{(1 + 2j_{n-1})^{j-i}} + (1 + 2j_{n-1}) \frac{(2 - j_n)^{j-i}}{(1 + 2j_{n-1})^{j-i+1}} = 0;$$

similar for j < i. Thus $SS^{-1} = I_{n-2}$.

Theorem 9 Let $n \geq 3$. The inverse of the matrix \mathbb{I}_n is

$$\mathbf{J}_{n}^{-1} = circ(h_{0}, h_{1}, \dots, h_{n-1})$$

where

$$h_0 = \frac{1}{2y_n} \left(\frac{9j_n - 18 + (10 - 8j_{n-2})y_n}{(1 + 2j_{n-1})^2} \right)$$

$$h_1 = \frac{1}{2y_n} \left(\frac{4y_n - 9}{1 + 2j_{n-1}} \right)$$

$$h_2 = \frac{1}{2y_n} \left[(4 - j_{n-1} - 2y_n) \frac{(2 - j_n)^{n-3}}{(1 + 2j_{n-1})^{n-2}} + \sum_{k=2}^{n-2} (2j_{k+1} - j_k) \left(\frac{(2 - j_n)^{k-2}}{(1 + 2j_{n-1})^{k-1}} \right) \right]$$

$$h_3 = \frac{1}{2y_n} \left[((4 - j_{n-1} - 2y_n)(j_{n+1} - 1) - (2j_{n-1} - j_{n-2})(1 + 2j_{n-1}) \frac{(2 - j_n)^{n-4}}{(1 + 2j_{n-1})^{n-2}} + 2\sum_{k=1}^{n-4} \left((2j_{k+1} - j_k) \frac{(2 - j_n)^{k-1}}{(1 + 2j_{n-1})^k} \right) \right]$$

$$h_i := \frac{1}{2y_n} \left(\frac{(4 - j_{n-1} - 2y_n)(j_{n+1} + 8j_n - 2j_{n-1} - 9((-2)^n + 1)}{(1 + 2j_{n-1})^2} - 2j_n + j_{n-1} - \frac{(4j_{n-1} - 2j_{n-2})(2 - j_n)}{1 + 2j_{n-1}} \right) \frac{(2 - j_n)^{n-i-1}}{(1 + 2j_{n-1})^{n-i+2}}$$

$$for y_n = \frac{1}{2}(4 - j_{n-1}) + \frac{1}{2} \sum_{k=2}^{n-1} (2j_k - j_{k-1}) \left(\frac{1 + 2j_{n-1}}{2 - j_n} \right)^{n-k} \text{ and } i = 4, 5, \dots, n-1.$$

Proof. Let

and
$$\mathbb{G} = diag(2, y_n)$$
 where $y_n = \frac{1}{2}(4 - j_{n-1}) + \frac{1}{2} \sum_{k=2}^{n-1} (2j_k - j_{k-1}) \left(\frac{1 + 2j_{n-1}}{2 - j_n}\right)^{n-k}$ and $y'_n = \sum_{k=1}^{n-1} j_k \left(\frac{1 + 2j_{n-1}}{2 - j_n}\right)^{n-k-1}$. Then we obtain

$$K_n \mathbf{J}_n M_n Z_n = \mathbb{G} \oplus \mathbb{S}$$

where $\mathbb{G} \oplus \mathbb{S}$ is the direct sum of the matrices \mathbb{G} and \mathbb{S} . If $\mathbb{T}_n = M_n Z_n$, then we have

$$\mathbf{J}_n^{-1} = \mathbb{T}_n(\mathbb{G}^{-1} \oplus \mathbb{S}^{-1})K_n.$$

Since the matrix \mathfrak{I}_n is circulant, the inverse matrix \mathfrak{I}_n^{-1} is circulant from Lemma 1.1 [1, p. 9791]. Let

$$\mathbf{J}_{n}^{-1} = circ(h_{0}, h_{1}, \dots, h_{n-1}).$$

Since the last row of the matrix \mathbb{T}_n is

$$\left(0,1,\frac{2j_0-j_{n-1}}{2y_n}-1,\frac{2j_{n-1}-j_{n-2}}{2y_n},\frac{2j_{n-2}-j_{n-3}}{2y_n},\ldots,\frac{2j_4-j_3}{2y_n},\frac{2j_3-j_2}{2y_n}\right),$$

the last row elements of the matrix \mathfrak{I}_n^{-1} are

$$h_1 = \frac{1}{2y_n} \left(\frac{4y_n - 9}{1 + 2j_{n-1}} \right)$$

$$h_2 = \frac{1}{2y_n} \left[(4 - j_{n-1} - 2y_n) \frac{(2 - j_n)^{n-3}}{(1 + 2j_{n-1})^{n-2}} + \sum_{k=2}^{n-2} (2j_{k+1} - j_k) \left(\frac{(2 - j_n)^{k-2}}{(1 + 2j_{n-1})^{k-1}} \right) \right]$$

$$h_{3} = \frac{1}{2y_{n}} \left[((4 - j_{n-1} - 2y_{n})(j_{n+1} - 1) - (2j_{n-1} - j_{n-2})(1 + 2j_{n-1})) \frac{(2 - j_{n})^{n-4}}{(1 + 2j_{n-1})^{n-2}} + 2 \sum_{k=1}^{n-4} \left((2j_{k+1} - j_{k}) \frac{(2 - j_{n})^{k-1}}{(1 + 2j_{n-1})^{k}} \right) \right]$$

$$h_{i} = \frac{1}{2y_{n}} \left(\frac{(4 - j_{n-1} - 2y_{n})(j_{n+1} + 8j_{n} - 2j_{n-1} - 9((-2)^{n} + 1)}{(1 + 2j_{n-1})^{2}} - 2j_{n} + j_{n-1} - \frac{(4j_{n-1} - 2j_{n-2})(2 - j_{n})}{1 + 2j_{n-1}} \right) \frac{(2 - j_{n})^{n-i-1}}{(1 + 2j_{n-1})^{n-i+2}}$$

$$h_{0} = \frac{1}{2y_{n}} \left(\frac{9j_{n} - 18 + (10 - 8j_{n-2})y_{n}}{(1 + 2j_{n-1})^{2}} \right)$$

where $y_n = \frac{1}{2}(4-j_{n-1}) + \frac{1}{2}\sum_{k=2}^{n-1}\left(2j_k - j_{k-1}\right)\left(\frac{1+2j_{n-1}}{2-j_n}\right)^{n-k}$ and $i=4,5,\ldots,n-1$. Since the matrix \mathbb{J}_n^{-1} is a circulant matrix and its last row is known, the proof is completed.

Corollary 10 Let the matrix K_n be as in (7). Then

$$K_n^{-1} := \begin{bmatrix} 1 & 0 \\ C & D \end{bmatrix},$$

where

$$C := \left(\frac{j_{n-1}}{2} \ \frac{j_{n-2}}{2} \ \frac{j_{n-3}}{2} \ \dots \frac{j_1}{2}\right)_{(n-1)\times 1}^{T}$$

and D is the $(n-1)\times(n-1)$ Hankel matrix in which the first row is $[J_{n-1},J_{n-2},\ldots,J_1]$ and the last column is $[J_1,0,\ldots,0]^T$.

Proof. The matrix K_n^{-1} is obtained easily by applying elementary row operations to the augmented matrix $[K_n|I_n]$.

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