# Parallel computation of real solving bivariate polynomial systems by zero-matching method ${ }^{\sqrt{2}}$ 

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#### Abstract

We present a new algorithm for solving the real roots of a bivariate polynomial system $\Sigma=\{f(x, y), g(x, y)\}$ with a finite number of solutions by using a zero-matching method. The method is based on a lower bound for bivariate polynomial system when the system is non-zero. Moreover, the multiplicities of the roots of $\Sigma=0$ can be obtained by a given neighborhood. From this approach, the parallelization of the method arises naturally. By using a multidimensional matching method this principle can be generalized to the multivariate equation systems.


Keywords: Bivariate polynomial system; Zero-matching method; Real roots; Symbolic-numerical computation; Parallel computation

## 1. Introduction

Considering the following system:

$$
\begin{equation*}
\Sigma=\{f(x, y), g(x, y)\} \tag{1}
\end{equation*}
$$

we assume that $f(x, y), g(x, y) \in \mathbb{Q}[x, y]$, where $\mathbb{Q}$ is the field of rational numbers. We call the zero-dimension if the bivariate polynomial system (1) has a finite number of solutions.

Real solving bivariate polynomial system in a real field is an active area of research. It is equivalent to finding the intersections of $f(x, y)$ and $g(x, y)$ in the real plane. The problem is closely related to computing the topology of a plane real algebraic curve and other important operations in non-linear computational geometry and ComputerAided Geometric Design[1, 15, 13, 10, 18]. Another field of application is the quantifier elimination [7, 17]. There are several algorithms that tackle this problem such as the Gröber basis method[19, 23], the resultant method [26], the characteristic set method [5], and the subdivision method[3, 21]. However, the procedure of these techniques is very complicated. In this paper, we propose an efficient approach to remedy these drawbacks.

In this paper, we propose a zero-matching method to solve the real roots of an equation system like (1). The basic idea of zero-matching method is as follows: First projecting the roots of $\Sigma$ to the $x$-axis, gives the roots $\left\{x_{1}, \cdots, x_{u}\right\}$, and the $y$-axis, gives the roots $\left\{y_{1}, \cdots, y_{v}\right\}$, respectively. Subsequently, for every root $x_{i}$, and for every $y_{j}$ is backsubstituted in $f(x, y)$ and $g(x, y)$. To that end, for some root $x_{i}$ there is the corresponding one or more roots $y_{j}$ to be determined satisfying $\Sigma$. The main contribution of our method is that how to determine the real roots of $\Sigma=0$ and the multiplicities of the roots. Moreover, our approach that has given solutions to this situation can be the design of parallelized algorithms.

[^0]In [9], Diochnos et al presented three algorithms to real solving bivariate systems and analyzed their asymptotic bit complexities. Among the three algorithms, the difference is the way they match solutions. The method of specialized Rational Univariate Representation (RUR) based on fast GCD computations of polynomials with coefficients in an extension field to achieve efficiency (hence the name G_RUR) has the lowest complexity and performs best in numerous experiments. The GRUR method projects the roots to the $x$-axis and $y$-axis, for each $x$-coordinate $\alpha$ computes the $\operatorname{GCD} h(\alpha, y)$ of the square-free parts of $f(\alpha, y)$ and $g(\alpha, y)$, and isolates the roots of $h(\alpha, y)=0$ based on computations of algebraic numbers and the rur techniques. Our algorithm only uses resultant computation and real solving for univariate polynomial equations with rational coefficients.

The hybrid method proposed by Hong et al $[16]$ that projects the roots of $\Sigma$ to the $x$-axis and $y$-axis respectively and uses the improved slope-based Hansen-Sengupta to determine whether the boxes formed by the projection intervals contain a root of $\Sigma$. The numerical method only works for simple roots of $\Sigma$. When the system has multiple roots, the rur technique is used to isolate the roots. Compared with this method, our approach also computes two resultants of the same total degrees. However, our method is a complete one, their numerical iteration method needs to use the rur technique to find multiple roots.

In[2], Bekker et al presented a Combinatorial Optimization Root Selection method (hence the name cors) to match the roots of a system of polynomial equations. However, the method is only suitable for solving a small system of polynomial equations, and does not work for the multiple roots. Recently, Cheng et al $[4]$ proposed a local generic position method to solve the bivariate polynomial equation system. The method can be used to represent the roots of a bivariate equation system as the linear combination of the roots of two univariate equations. Moreover, the multiplicities of the roots of the bivariate polynomial system are also derived. However, the method is very complicated to extend to solve the multivariate equation systems. Our method can solve the larger systems and easily generalize to the multivariate equation systems.

The rest of this paper is organized as follows. In Section 2, we give some notations, a lower bound for bivariate polynomial equation if it is non-zero, and how to determine the root multiplicity. In Section 3, we propose the algorithm to real solving the bivariate polynomial system and give a detailed example. In section 4, we present some comparisons of our algorithm. The final section concludes this paper.

## 2. Notations and main results

### 2.1. Notations

In what follows $\mathbf{D}$ is a ring, $\mathbf{F}$ is a commutative field of characteristic zero and $\overline{\mathbf{F}}$ its algebraic closure. Typically $\mathbf{D}=\mathbb{Z}, \mathbf{F}=\mathbb{Q}$ and $\overline{\mathbf{F}}=\overline{\mathbb{Q}}$.

In this paper, we consider the zero-dimensional bivariate polynomial system as follows:

$$
\left\{\begin{array}{l}
f(x, y)=\sum_{0 \leq i \leq n} \sum_{0 \leq j \leq m} a_{i, j} x^{i} y^{j}=0  \tag{2}\\
g(x, y)=\sum_{0 \leq i \leq p} \sum_{0 \leq j \leq q} b_{i, j} x^{i} y^{j}=0
\end{array} .\right.
$$

Throughout this paper, note that $d e g_{x}=\max (n, p), d e g_{y}=\max (m, q), N=\max \left(\|f\|_{1},\|g\|_{1}\right)$, where the $\|f\|_{1}$ and $\|g\|_{1}$ are the one norm of the vector ( $a_{00}, a_{01}, \cdots, a_{0 m}, \cdots, a_{n 0}, \cdots, a_{n m}$ ) and ( $b_{00}, b_{01}, \cdots, b_{0 q}, \cdots, b_{p 0}, \cdots, b_{p q}$ ), so $\|f\|_{1}=\Sigma_{i} \Sigma_{j}\left|a_{i j}\right|$, and $\|g\|_{1}=\Sigma_{i} \Sigma_{j}\left|b_{i j}\right|$, respectively. $M=\max \left(\|t\|_{1},\|T\|_{1}\right)$, where the $t(x)$ and $T(y)$ are the no extraneous factors in resultant polynomial of $\Sigma$. $|\Sigma|$ denotes that the bivariate polynomial system $\Sigma$ has been assigned values to two variables.

Let $\pi$ be the projection map from the $\Sigma$ to the $x$-axis:

$$
\begin{equation*}
\pi: \mathbb{R}^{2} \rightarrow \mathbb{R}, \quad \text { such that } \pi(x, y)=x . \tag{3}
\end{equation*}
$$

For a zero-dimensional system $\Sigma$ defined in (2), let $t(x) \in \mathbb{Q}[x]$ be the resultant of $f(x, y)$ and $g(x, y)$ with respect to $y$ :

$$
\begin{equation*}
t(x)=\operatorname{Res}_{y}(f(x, y), g(x, y)) . \tag{4}
\end{equation*}
$$

Since $\Sigma$ is zero-dimensional, we have $t(x) \not \equiv 0$. Then $\pi(\mathbf{V}(\Sigma)) \subseteq \mathbf{V}(t(x))$, where $\mathbf{V}\left(f_{1}, \cdots, f_{m}\right)$ is the set of common real zeros of $f_{i}=0$. If $t(x)$ is irreducible, then denote the highest degree by $\operatorname{deg}_{t}$. Let the real roots of $t(x)=0$ be

$$
\begin{equation*}
\alpha_{1}<\alpha_{2}<\cdots<\alpha_{u} . \tag{5}
\end{equation*}
$$

By using the same method, let $T(y) \in \mathbb{Q}[y]$ be the resultant of $f(x, y)$ and $g(x, y)$ with respect to $x$ :

$$
\begin{equation*}
T(y)=\operatorname{Res}_{x}(f(x, y), g(x, y)) \tag{6}
\end{equation*}
$$

If $\mathrm{T}(\mathrm{y})$ is irreducible, then denote the highest degree by $d e g_{T}$. Let the real roots of $T(y)$ be as follows:

$$
\begin{equation*}
\beta_{1}<\beta_{2}<\cdots<\beta_{v} . \tag{7}
\end{equation*}
$$

We observe that the above projection map may generate extraneous roots. Fortunately, we can easily discard these extraneous factors by computing the determinant of the sub-matrix of the coefficient matrix. Moreover, if the resultant is irreducible, then it is no extraneous factors. However, when the resultant is reducible, it may suffer from the extraneous factors. The method of removing extraneous factors mentioned can be adapted to the resultant for the bivariate polynomial system [27]. It is the following theorem to remove the extraneous roots.
Theorem 2.1. $\Sigma$ is defined as in (2). If the resultants of $\Sigma$ for one variable is reducible, denoted by tem, then the resultant of bivariate polynomial system is the only some irreducible factors in which the other variable appear.
Proof. The proof can be given similarly to that in Proposition 4.6 of Chapter 3 of [8].

### 2.2. A lower bound for $|\Sigma|$, if $\Sigma \neq 0$

The purpose of this subsection is to prove the following theorem.
Theorem 2.2. $\Sigma$ is defined as in (2). Let $\alpha, \beta$ be two approximate real algebraic numbers. Denote by the integer $s=\operatorname{deg}_{t} \cdot \operatorname{deg}_{T}$, and $N$ as above. If $|\Sigma| \neq 0$, then

$$
\begin{equation*}
|\Sigma| \geq N^{1-s} M^{-c \cdot s} \tag{8}
\end{equation*}
$$

where $c$ is the constant satisfying certain conditions, $|\Sigma|$ is the following two cases:
(a) If $f(\alpha, \beta)=0$ or $g(\alpha, \beta)=0$, then $|\Sigma|=\max \{|f(\alpha, \beta)|,|g(\alpha, \beta)|\} ;$
(b) If $f(\alpha, \beta) \neq 0$ and $g(\alpha, \beta) \neq 0$, then $|\Sigma|=\min \{|f(\alpha, \beta)|,|g(\alpha, \beta)|\}$.

Before giving the proof of theorem[2.2, we recall two lemmas:
Lemma 2.1. ([20]], lemma 3) Let $\alpha_{1}, \ldots, \alpha_{q}$ be algebraic numbers of exact degree of $d_{1}, \ldots, d_{q}$ respectively. Define $D=\left[\mathbb{Q}\left(\alpha_{1}, \ldots, \alpha_{q}\right): \mathbb{Q}\right]$. Let $P \in \mathbb{Z}\left[x_{1}, \ldots, x_{q}\right]$ have degree at most $N_{h}$ in $x_{h}(1 \leq h \leq q)$. If $P\left(\alpha_{1}, \ldots, \alpha_{q}\right) \neq 0$, then

$$
\left|P\left(\alpha_{1}, \ldots, \alpha_{q}\right)\right| \geq\|P\|_{1}^{1-D} \prod_{h=1}^{q} M\left(\alpha_{h}\right)^{-D N_{h} / d_{h}}
$$

where the $M\left(\alpha_{h}\right)$ is the Mahler measure of $\alpha_{h}$.
Proof. See the Lemma 4 of [20].
Lemma 2.2. Let $\alpha$ be an algebraic number. Denote by the $M(\alpha)$ of the Mahler measure of $\alpha$. If $P$ is a polynomial over $\mathbb{Z}$, then

$$
M(\alpha) \leq\|P\|_{1} .
$$

Proof. For any polynomial $P=\sum_{i=0}^{d} p_{i} \in \mathbb{Z}[x]$ of degree $d$ with the all roots $\sigma^{(1)}, \cdots, \sigma^{(d)}$, we define the measure $M(P)$ by

$$
M(P)=\left|p_{d}\right| \Pi_{i=1}^{d} \max \left\{1,\left|\sigma^{(i)}\right|\right\}
$$

The Mahler measure of an algebraic number is defined to be the Mahler measure of its minimal polynomial over $\mathbb{Q}$. We know from Landau ( $[14]$, p. 154, Thm. 6. 31) that for each algebraic number $\alpha$

$$
M(\alpha) \leq\|P\|_{2},
$$

where $\|P\|_{2}=\left(\sum_{i=0}^{d}\left|p_{i}\right|^{2}\right)^{1 / 2}$. It is very easy to get that $\|P\|_{2} \leq\|P\|_{1}$. This completes the proof of the lemma.

## Now we turn to give the proof of Theorem 2.2

Proof. From the assumption of the theorem, since $\Sigma$ is defined as in (2). Let the pair $(\alpha, \beta)$ be corresponding value to the variable $x$ and $y$ for $\Sigma$ respectively. We have the following equations:

$$
\begin{align*}
& f(\alpha, \beta)=\sum_{0 \leq i \leq n} \sum_{0 \leq j \leq m} a_{i, j} \alpha^{i} \beta^{j}  \tag{9a}\\
& g(\alpha, \beta)=\sum_{0 \leq i \leq p} \sum_{0 \leq j \leq q} b_{i, j} \alpha^{i} \beta^{j} . \tag{9b}
\end{align*}
$$

At first, we consider the lower bound for the equation (9a). Define $k=[\mathbb{Q}(\alpha, \beta): \mathbb{Q}]$. Denote by $|f|=|f(\alpha, \beta)|$, and $r, t$ by the exact degree of algebraic numbers $\alpha, \beta$ respectively. From Lemma (2.1), if $|f| \neq 0$, then

$$
|f| \geq\|f\|_{1}^{1-k} M(\alpha)^{-k n / r} M(\beta)^{-k m / t}
$$

We observe that $M(\alpha)$ and $M(\beta)$ derive from $t(x)$ and $T(y)$ respectively. From Lemma (2.2), we can get the following inequality:

$$
M(\alpha) \leq\|t\|_{1}, M(\beta) \leq\|T\|_{1} .
$$

So we can obtain that

$$
\begin{equation*}
|f| \geq\|f\|^{1-k}\|t\|_{1}^{-k n / r}\|T\|_{1}^{-k m / t} . \tag{10}
\end{equation*}
$$

By using the same technique as above, we can obtain the lower bound for the equation (9b). Denote by $|g|=|g(\alpha, \beta)|$. If $|g| \neq 0$, then

$$
\begin{equation*}
|g| \geq\|g\|^{1-k}\|t\|_{1}^{-k n / r}\|T\|_{1}^{-k m / t} . \tag{11}
\end{equation*}
$$

Since we have the following two cases:
(a) If $f(\alpha, \beta)=0$ or $g(\alpha, \beta)=0$, then $|\Sigma|=\max \{|f(\alpha, \beta)|,|g(\alpha, \beta)|\}$;
(b) If $f(\alpha, \beta) \neq 0$ and $g(\alpha, \beta) \neq 0$, then $|\Sigma|=\min \{|f(\alpha, \beta)|,|g(\alpha, \beta)|\}$.

Hence we are able to obtain the lower bound for the bivariate polynomial system. From the above assumption, we can get the following parameters:

$$
\begin{align*}
& k=[\mathbb{Q}(\alpha, \beta): \mathbb{Q}] \leq \operatorname{deg}\{t(x)\} \cdot \operatorname{deg}\{T(y)\}=\operatorname{deg}_{t} \cdot \operatorname{deg}_{T},  \tag{12a}\\
& N=\max \left\{\|f\|_{1},\|g\|_{1}\right\}, M=\max \left\{\|t\|_{1},\|T\|_{1}\right\}, r=\operatorname{deg}_{t}, t=\operatorname{deg}_{T} . \tag{12b}
\end{align*}
$$

Combined with the equation (12a) and (12b), it is obvious that $s=k$ and the constant $c=\frac{\operatorname{deg}_{t}}{r}+\frac{\operatorname{deg}_{T}}{t}+1$. Finally, note that the constant $c$ satisfies both cases. This proves the theorem.

As the corollary of Theorem 2.2, we have
Corollay 2.1. Under the same condition of Theorem 2.2 if $|\Sigma|<N^{1-s} M^{-c \cdot s}$, then $|\Sigma|=0$. We say that $\alpha$ is associated with $\beta$ for the real root of $\Sigma$. Denote by the $\varepsilon=N^{1-s} M^{-c \cdot s}$ for the rest of this paper.

Proof. The proof is very easy by contradiction.

### 2.3. Root multiplicity

The results of this subsection can be provided for the root multiplicity of $\Sigma$. We follow the approach and terminology of [8] and [11].

Let $C_{f}, C_{g}$ be $f, g$ corresponding affine algebraic plane curves, defined by the equations $\Sigma$. Let $I=<f, g>$ be the ideal that they generate in $\mathbf{F}[x, y]$, and so the associated quotient ring is $\mathcal{A}=\overline{\mathbf{F}}[x, y] / I$. Let the distinct intersection points, which are the distinct roots of $(\Sigma)$, be $C_{f} \cap C_{g} \subset\left\{S_{i j}=\left(\alpha_{i}, \beta_{j}\right)\right\}_{1 \leq i \leq u, 1 \leq j \leq v}$.

The multiplicity of a point $S_{i j}$ is

$$
\operatorname{mult}\left(S_{i j}: \mathcal{C}_{f} \cap C_{g}\right)=\operatorname{dim}_{\overline{\mathbf{F}}} \mathcal{A}_{S_{i j}}<\infty,
$$

where $\mathcal{A}_{S_{i j}}$ is the local ring obtained by localizing $\mathcal{A}$ at the maximal ideal $I=\left\langle x-\alpha_{i}, y-\beta_{j}\right\rangle$.
If $\mathcal{A}_{S_{i j}}$ is a finite dimensional vector space, then $S_{i j}=\left(\alpha_{i}, \beta_{j}\right)$ is an isolated zero of $I$ and its multiplicity is called the intersection number of the two curves. The finite $\mathcal{A}$ can be decomposed as a direct sum $\mathcal{A}=\mathcal{A}_{S_{11}} \bigoplus \mathcal{A}_{S_{12}} \bigoplus \ldots$ $\bigoplus \mathcal{A}_{S_{u v}}$ and thus $\operatorname{dim}_{\overline{\mathbf{F}}} \mathcal{A}=\sum_{i=1}^{u v} \operatorname{mult}\left(S_{i j}: C_{f} \cap C_{g}\right)$.
Proposition 2.1. ([1]], Proposition 1) Let $f, g \in \mathbf{F}[x, y]$ be two coprime curves, and let $p \in \overline{\mathbf{F}}^{2}$ be a point. Then

$$
\operatorname{mult}(p: f g) \geq \operatorname{mult}(p: f) \operatorname{mult}(p: g),
$$

where equality holds if and only if $C_{f}$ and $C_{g}$ have no common tangents at $p$.
Proposition 2.2. Let us obtain the real roots of $\Sigma=0$ in (5) and (7). If the two matching pairs $\left(\alpha_{i}, \beta_{j}\right)$ and $\left(\alpha_{i+1}, \beta_{j+1}\right)($ for $1 \leq i \leq u, 1 \leq j \leq v)$ are satisfying $\Sigma=0,\left|\alpha_{i}-\alpha_{i+1}\right|<\varepsilon$ and $\left|\beta_{j}-\beta_{j+1}\right|<\varepsilon$, then the $\left(\alpha_{i}, \beta_{j}\right)$ is multiple root of $\Sigma=0$.
Proof. From Theorem 2.2 and Corollary 2.1, it is obvious that $\Sigma=0$ if and only if

$$
|\Sigma|<\varepsilon .
$$

Therefore, the error controlling is less than $\varepsilon$ in numerical computation. Under the assumption of the proposition, we get $\left|\alpha_{i}-\alpha_{i+1}\right|<\varepsilon$ and $\left|\beta_{j}-\beta_{j+1}\right|<\varepsilon$. So we are able to obtain that $\left|\alpha_{i}-\alpha_{i+1}\right|=0$ and $\left|\beta_{j}-\beta_{j+1}\right|=0$ int the truncated error. This proves the proposition.

From Corollary 2.1, the two-tuple $(\alpha, \beta)$ is the real root of $\Sigma=0$. This method is called a zero-matching method. The technique is a posteriori method to match the solutions for the bivariate system. It can be generalized easily to real solving the multivariate polynomial systems.

## 3. Derivation of the Algorithm

The aim of this section is to describe an algorithm for real solving bivariate polynomial equations by using zeromatching method. We first find the parameters $N, c$ and $s$, then obtain the no extraneous factors $t(x)$ and $T(y)$ with the resultant elimination methods, and real solving two univariate polynomials, and finally match the real roots for the systems.

### 3.1. Description of algorithm

Algorithm 1 is to discard the extraneous factors from the resultant method, algorithm 2 is to obtain the solutions of bivariate polynomial systems.

```
Algorithm 1 NoExtrRes( \(\Sigma\), var)
Input: \(\{f(x, y), g(x, y)\}, v a r\) is one variable.
Output: No extraneous factors resultant of \(\Sigma\).
    tem \(\leftarrow \operatorname{Res}_{\text {var }}\{f(x, y), g(x, y)\}\);
    if tem is irreducible then
        return tem;
    else
        tem \(\leftarrow\) Res \(\cdot\) extraneousfacotrs;
        return Res.
    end if
```

Now we can give the algorithm 2 to compute the real roots for $\Sigma=0$.
The parallelization of the algorithm that we have just described can be easily done because it performs the same computations on different steps of data without the necessity of communication between the processors. Observe that the Step 1 and Step 2, Step 6 and Step 7 of the algorithm can be easily paralleled, respectively.

Now we get a theorem about the computational complexity of the whole algorithm.

```
Algorithm 2 ZMm \((\Sigma)\)
Input: \(\Sigma=\{f(x, y), g(x, y)\}\) is a zero-dimensional bivariate polynomial system.
Output: A set for the real roots of \(\Sigma=0\).
    Project on the \(x\)-axis such that \(t(x)=\operatorname{Res}_{y}(f(x, y), g(x, y))\);
    Project on the \(y\)-axis such that \(T(y)=\operatorname{Res}_{x}(f(x, y), g(x, y))\);
    Discard the extraneous factors from \(t(x)\) and \(T(y)\) by using Algorithm
    Find the parameters \(N\) and \(s\), and Compute \(c\) according to the Theorem 2.2,
    Obtain the lower bound \(\varepsilon\) by Corollary 2.1,
    Solve the real roots of the resultant \(t(x)\) for the set \(\mathbf{S}_{\mathbf{x}}=\left\{\alpha_{1}, \alpha_{2}, \cdots, \alpha_{u}\right\}\);
    Solve the real roots of the resultant \(T(y)\) for the set \(\mathbf{S}_{\mathbf{y}}=\left\{\beta_{1}, \beta_{2}, \cdots, \beta_{v}\right\}\);
    Match the real root pair to get the solving set \(\mathbf{S}=\left\{\left(\alpha_{i}, \beta_{j}\right), 1 \leq i \leq u, 1 \leq j \leq v\right\}\) by Corollary 2.1,
    Check the root multiplicity of the set \(\mathbf{S}\) by Proposition 2.2
```

Theorem 3.1. Algorithm 2 works correctly as specified and its complexity includes as follows:
(a) $O(d \tau+d l g d)$ for computation of real solving univariate polynomial, where $d$ is the degree of corresponding polynomial, $\tau=1+\max _{i \leq d} l g\left|a_{i}\right|$ and $a_{i}$ is the coefficients.
(b) $O(u v)$ for matching the solutions of bivariate polynomial system.

Proof. Correctness of the algorithm follows from theorem 2.2
(a) The number of arithmetic operations required to isolate all real roots is the number of real root isolation of univariate polynomial by using subdivision-based Descartes' rule of sign. Using exactly the same arguments we know that they perform the same number of steps, that is $O(d \tau+d l g d)$.
(b) As indicated before, the problem of matching the real roots of polynomial system mainly relies on the scale of solutions of every variable, respectively.

### 3.2. A small example in detail

Example 3.1. We propose a simple example $f(x, y)=x^{2}-y^{2}-3$ and $g(x, y)=3 x^{2}-2 y^{3}-1$ to illustrate our algorithms.
Step 1: $t(x)=4 * x^{6}-45 * x^{4}+114 * x^{2}-109$;
Step 2: $T(y)=\left(-2 * y^{3}+8+3 * y^{2}\right)^{2}$;
Step 3: Discard the extraneous factors $T(y)=-3 * y^{2}-8+2 * y^{3}$;
Step 4: Obtain the parameters $N=5, c=2, s=4$;
Step 5: Obtain the lower bound $\varepsilon=.1280 \times 10^{-4}$;
Step 6: Solve the real roots of the resultant $t(x)$ for the set $\mathbf{S}_{\mathbf{x}}=\{-2.858288520,2.858288520\}$;
Step 7: Solve the real roots of the resultant $T(y)$ for the set $\mathbf{S}_{\mathbf{y}}=\{2.273722337\}$;
Step 8: Combine the the pairs from $\mathbf{S}_{\mathbf{x}}$ and $\mathbf{S}_{\mathbf{y}}$ respectively, Substitute the pairs into $\Sigma$ for variables $x$ and $y$, determine whether less than the lower bound $\varepsilon$, finally we find that the pairs $\mathbf{S}=\{\{x=-2.858288520, y=$ 2.273722337\}, $\{x=2.858288520, y=2.273722337\}\}$ are the solutions for $\Sigma$;

Step 9: The multiplicity of the root of the system is one.

### 3.3. Generalization and applications

As for the generalization of the algorithm to real solving the multivariate equation systems case, we have to say that the situation is completely analogous to the bivariate case. However, its key technique is to transform the multivariate polynomial equations to the corresponding univariate polynomial equations. We can consider the Dixon Resultant Method to break this problem [6]. However, we observe that how to improve the projection algorithm in resultant methods is the significant challenge.

Moreover, our algorithm is applicable for rapidly computing the minimum distance between two objects collision detection [25]. This also enables us to improve the complexity of computing the topology of a real plane algebraic curve [9].

## 4. Some comparisons

We have implemented the above algorithms as a software package zmm in Maple 12. For problems of small size like the example of Section 3, any method can obtain the solutions in little time. But when the size of the problems is not small the differences appear clearly. Extensive experiments with this package show that this approach is efficient and stable, especially for larger and more complex bivariate polynomial systems.

We compare our method with lGP [4], Isolate [23], discoverer [24], and Grur [9]. LGP is a software package for root isolation of bivariate polynomial systems with local generic position method. Isolate is a tool to solve general equation systems based on the Realsolving c library by Rouillier. DISCOVERER is a tool for solving semi-algebraic systems. GRUR is a tool to solve bivariate equation systems. The following examples run in the same platform of Maple 12 under Windows and amd Athlon $(\mathrm{tm}) 2.70 \mathrm{ghz}, 2.00 \mathrm{~GB}$ of main memory. We did three sets of experiments. The precision in these experiments is set to be high. In three tables, where '?' represents that the computation is not finished.

In Table 1 the results are given both $f$ and $g$ are randomly generated dense polynomials with the same degree and with integer coefficients between -20 and 20. The command of Maple is as follows:
randpoly $([x, y]$, coeffs $=\operatorname{rand}(-20 . .20)$, dense, degree $=10)$.

| $\begin{aligned} & \frac{\pi}{U} \\ & \frac{\tilde{v}}{\omega} \end{aligned}$ | deg |  | $\begin{aligned} & 0 \\ & 0 \\ & \text { On } \\ & 0 \\ & 0 \end{aligned}$ | Average Time(sec) |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | $f$ | $g$ |  | ZMM | LGP | Isolate | DISCOVERER | GRUR |
| S1 | 4 | 7 | 2 | 0.031 | 0.031 | 0.047 | 0.313 | 2.734 |
| S2 | 6 | 8 | 6 | 0.415 | 1.328 | 0.500 | 1.828 | 247.203 |
| S3 | 7 | 8 | 6 | 1.204 | 2.734 | 1.500 | 7.047 | 382.640 |
| S4 | 8 | 9 | 6 | 4.211 | 8.906 | 4.672 | 20.437 | 2714.438 |
| S5 | 9 | 10 | 2 | 4.070 | 8.485 | 4.687 | 89.235 | 1645.312 |
| S6 | 10 | 7 | 6 | 1.805 | 3.860 | 2.109 | 22.250 | 978.421 |
| S7 | 10 | 11 | 4 | 21.078 | 43.734 | 22.828 | ? | ? |
| S8 | 12 | 11 | 2 | 26.945 | 54.969 | 29.094 | ? | ? |
| S9 | 12 | 13 | 4 | 118.266 | 241.734 | 123.469 | ? | ? |
| S10 | 13 | 11 | 1 | 15.446 | 31.485 | 17.796 | ? | ? |
| S11 | 14 | 10 | 8 | 63.914 | 200.828 | 68.594 | ? | ? |

In Table 2 the results are given both $f$ and $g$ are randomly generated sparse polynomials in the same degree, with sparsity default, and with integer coefficients between -20 and 20. The command of Maple is as follows: randpoly $([x, y]$, coeffs $=\operatorname{rand}(-20 . .20)$, sparse, degree $=10)$.

|  | deg |  | $\begin{aligned} & \text { O. } \\ & \text { On } \\ & \text { On } \end{aligned}$ | Average Time(sec) |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | $f$ | $g$ |  | ZMM | LGP | Isolate | DISCOVERER | GRUR |
| S1 | 5 | 6 | 1 | 0.015 | 0.032 | 0.015 | 0.141 | 1.032 |
| S2 | 6 | 7 | 3 | 0.040 | 0.062 | 0.047 | 0.188 | 5.375 |
| S3 | 7 | 5 | 3 | 0.024 | 0.047 | 0.047 | 0.265 | 2.688 |
| S4 | 8 | 6 | 5 | 0.031 | 0.031 | 0.047 | 0.094 | 1.031 |
| S5 | 9 | 8 | 2 | 0.047 | 0.172 | 0.078 | 1.828 | 51.000 |
| S6 | 10 | 11 | 3 | 0.063 | 0.297 | 0.125 | 0.656 | 11.110 |
| S7 | 11 | 9 | 2 | 0.164 | 0.609 | 0.375 | 3.938 | 877.875 |
| S8 | 12 | 13 | 2 | 1.141 | 2.593 | 1.453 | 6.703 | 1607.719 |
| S9 | 13 | 11 | 4 | 2.508 | 5.344 | 2.969 | ? | ? |
| S10 | 15 | 17 | 1 | 0.532 | 1.234 | 1.266 | ? | ? |
| S11 | 20 | 17 | 4 | 18.180 | 39.688 | 20.235 | ? | ? |

In Table 3 the results are given is done with polynomial systems with multiple roots. We randomly generate a polynomial $h(x, y, z)$ and take $f(x, y)=\operatorname{Res}_{z}\left(h, h_{z}\right), g(x, y)=f_{y}(x, y)$. Since $f(x, y)$ is the projection of a space curve to the $x y$-plane, it most probably has singular points and $f=g=0$ is an equation system with multiple roots. The command of Maple is as follows:
$h:=\operatorname{randpoly}([x, y, z]$, coeffs $=\operatorname{rand}(-5 . .5), \operatorname{degree}=5) ; f:=\operatorname{resultant}(h, \operatorname{diff}(h, z), z) ; g:=\operatorname{diff}(f, y)$.

| $\begin{aligned} & \frac{\pi}{0} \\ & \stackrel{\rightharpoonup}{\omega} \\ & \stackrel{N}{6} \end{aligned}$ | deg |  | $\begin{aligned} & 0.0 \\ & 0.0 \\ & 0 \\ & 0 \\ & 0 \end{aligned}$ | Average Time(sec) |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | $f$ | $g$ |  | ZMM | LGP | Isolate | DISCOVERER | GRUR |
| S1 | 3 | 2 | 2 | 0.016 | 0.016 | 0.016 | 0.016 | 0.062 |
| S2 | 4 | 3 | 2 | 0. | 0.032 | 0.031 | 0.016 | 0.094 |
| S3 | 4 | 6 | 7 | 0.024 | 0.016 | 0.047 | 0.109 | 1.109 |
| S4 | 5 | 4 | 3 | 0. | 0.016 | 0. | 0.016 | 0.109 |
| S5 | 6 | 5 | 2 | 0.015 | 0. | 0. | 0.016 | 0.063 |
| S6 | 9 | 8 | 2 | 0.016 | 0.046 | 0.032 | 0.015 | 0.063 |
| S7 | 12 | 11 | 3 | 0.109 | 0.234 | 0.187 | 0.063 | 0.094 |
| S8 | 13 | 12 | 7 | 2.875 | 137.641 | 3.141 | 1.328 | 207.094 |
| S9 | 14 | 13 | 4 | 0.860 | 2.891 | 0.953 | 0.141 | 0.3110 |
| S10 | 19 | 18 | 1 | 0.672 | 1.547 | 0.797 | 22.156 | 1520.812 |
| S11 | 16 | 15 | 5 | 7.945 | 27.047 | 9.000 | ? | ? |

From the Table 1, 2 and 3, we have the following observations.
In the first two cases, the equations are randomly generated and hence may have no multiple roots. For systems without multiple roots, zmm is the fastest method, which is significantly faster than LGP and Isolate. Both zmm and lGP compute two resultants and isolate their real roots. LGP is slow, because the polynomials obtained by the shear map are usually dense and with large coefficients [4]. discoverer and GRUR generally work for equation systems with degrees not higher than ten within reasonable time.

For systems with multiple roots, in the sparse and low degree cases, all methods are fast. Note that our method
is quite stable for equation systems with and without multiple roots. LGP and Isolate are also quite stable, but slower than zmm for bivariate equation systems.

We also observe that all methods spend more time with sparse and dense polynomials than polynomials with multiple roots in the same high degree. This phenomenon needs further exploration.
Remark 4.1. Of course, we should mention that DISCOVERER and Isolate can be used to solve general polynomial equations and even inequalities. Here our comparison is limited to the bivariate case. In further work, we would like to consider solving multivariate polynomial equations.

Remark 4.2. As is well known, the parallel algorithm is well suited for the implementation on parallel computers that allows the increase of the calculation speed. If our algorithm have been fully parallelized by using a large enough number of processors for each case, the real solutions of all the examples will have been computed in a couple of seconds.

## 5. Conclusion

In this paper, we propose a zero-matching method to real solving bivariate polynomial equation systems. The basic idea of this method is to find the lower bound for bivariate polynomial system when the system is non-zero. Moreover, we provide an algorithm for discarding extraneous factors with resultant and show how to construct a parallelized algorithm for real solving the bivariate polynomial system. An efficient method for multiplicities of the roots is also derived. The complexity of our method has increased steadily with the growth of bivariate polynomial system. Extensive experiments show that our approach is efficient and stable. The result of this paper can be extended to real solving of bivariate polynomial equations with more than two polynomials by using the resultant method. Furthermore, our method can be generalized easily to multivariate polynomial systems.

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