Products of 2 × 2 matrices related to non autonomous Fibonacci difference equations

Rafael Luís, Henrique M. Oliveira *

Center for Mathematical Analysis, Geometry and Dynamical Systems, Department of Mathematics, Technical Institute of Lisbon, University of Lisbon, Av. Rovisco Pais, 1049-001 Lisboa, Portugal

ARTICLE INFO	ABSTRACT
	A technique to compute arbitrary products of a class of Fibonacci 2 × 2 square matrices is proved in this work. General explicit solutions for non autonomous Fibonacci difference equations are obtained from these products. In the periodic non autonomous Fibonacci difference equations the monodromy matrix, the Floquet multipliers and the Binet's formulas are obtained. In the periodic case explicit solutions are obtained and the solutions are analyzed.

1. Introduction

1.1. Historical background

Difference equations can model effectively almost any physical and artificial phenomena [5]. One of the earliest recurrences, or in other words, a difference equation, giving us the Fibonacci sequence, was introduced in 1202 in the old "Liber Abaci" [15], a book about the abacus, by the famous Italian mathematician Leonardo Pisano better known as Fibonacci. The first numbers of this sequence are

$$0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

This example is related to ecology but one may find many applications of this sequence of numbers in various branches of science like pure and applied mathematics, in biology or in phyllotaxies, among many others.

If x_n represents a number in this sequence, with $n = 0, 1, 2, \ldots$, then the Fibonacci numbers satisfy the recurrence

$$q_{n+2} = q_{n+1} + q_n, \quad q_0 = 0, \quad q_1 = 1. \tag{1}$$

Obviously, we can look the above recurrence as a difference equation. The general solution of Eq. (1) is given by the Binet formula

$$q_n = \frac{1}{\sqrt{5}} \left(\phi^n - (1 - \phi)^n \right),$$

where $\phi = \frac{1 + \sqrt{5}}{2}$ is the well known golden ratio.

The Eq. (1) can be written in matrix notation

$$X_{n+1} = f(X_n), \tag{2}$$

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* Corresponding author.
E-mail addresses: rafel.luis.madeira@gmail.com (R. Luís), holiv@math.ist.utl.pt (H.M. Oliveira).

where $f(X_n) = AX_n$, $\quad A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, $\quad X_n = \begin{bmatrix} q_{n+1} \\ q_n \end{bmatrix}$ and $X_{n+1} = \begin{bmatrix} q_{n+2} \\ q_{n+1} \end{bmatrix}$.

It is easy to find the solution of Eq. (2) via the powers of the matrix A, i.e., $X_n = A^n X_0$ and using the eigenvalues of A, ϕ and $1 - \phi$.

Recently, Edson and Yayenie [3] studied a non-autonomous generalization of the Fibonacci recurrence. They considered the equation

$$x_{n+2} = a_n x_{n+1} + x_n, \tag{3}$$

with initial conditions x_0 and x_1 in which a_n is a 2-periodic non-zero sequence of non-negative integer numbers, i.e., $a_n = a_0$ if n is even and $a_n = a_1$ if n is odd. They found a Binet Formula for this equation using generating functions.

Now, let $\{a_n\}, n = 0, 1, 2, \ldots$ be a k-periodic non-zero sequence of non-negative integer numbers with fixed $k > 2$, i.e., $a_n = a_{n+k}$, for all $n = 0, 1, 2, \ldots$. It was stated in [3] that to find a Binet formula for Eq. (3), for all $n = 0, 1, 2, \ldots$, was an open problem.

Later on, Lewis joined the previous authors in another nice paper [4] on the same subject, where the k-periodic equation was treated using again generating functions. In that paper is developed an elegant new technique to obtain the solutions, which are not presented explicitly for the general case $\{a_n\}, n = 0, 1, 2, \ldots$.

In a series of papers [11–13] Mallik studied the solutions of linear difference equations with variable coefficients. Using classical techniques of iteration, he was able to find the solutions of certain special linear difference equations. In [11] he studied a second order equation and he was able to write expressions for the solutions. In [12] the author presented the solution of a linear difference equation of unbounded order. As special cases, the solutions of nonhomogeneous and homogeneous linear difference equations of order n with variable coefficients were obtained. From these solutions, Mallik was able to get the expressions for the product of companion matrices, and the power of a companion matrix. The closed form solution of the n-th order difference equation ($n \geqslant 3$) is presented in [13] using some combinatorial properties in the indices of the coefficients in an indirect manner. The results of Mallik, being very interesting and original, were new approaches to the classic problem of solving linear difference equations with variable coefficients.

In [9] de Jesus and Petronilho established algebraic conditions for the existence of a polynomial mapping using a monic orthogonal polynomial sequence (OPS). In particular in Section 5.1 of this paper is introduced a method to study such sequences with periodic coefficients that are related with difference equations with periodic coefficients. These results were developed in a recent work [14] by Petronilho to compute the solutions of the periodic case via orthogonal polynomials and a determinant of a tridiagonal matrix associated with the dynamics generated by Eq. 3. Again, an explicit expression of the solution for the general case $\{a_n\}, n = 0, 1, 2, \ldots$, is not presented. We point out that the work in [14] can be related to the recent method of computing generating functions via kneading determinants [1] applied to finite and infinite order difference equations. We suggest that this last method is a good way to tackle the problem of computing explicitly the solutions of this and other non-autonomous problems in future work.

1.2. Purpose and overview

Changing completely the perspective from the above mentioned literature we approach the problem in the framework of classic linear periodic difference equations using extensively linear algebra methods.

There exists a simpler method to find the solutions developed by Achille Marie Gaston Floquet (1847–1920) [7]. Floquet theory, first published in 1883 for periodic linear differential equations, was extended to difference equations being a long time classic and familiar in many textbooks [2,6,8,10]. In Floquet theory it is necessary to find explicitly a monodromy matrix and its eigenvalues, the Floquet multipliers.

We go further since we study the Eq. (3) in the general non-autonomous case for any complex initial conditions and considering arbitrary complex sequences of parameters $\{a_n\}$, $n = 0, 1, 2, \ldots$, not necessarily periodic. We consider $a_n \in \mathbb{C}$ since our method works with complex equations and orbits being not restricted to the original formulation on the natural numbers.[1]

Let us write the non-autonomous difference Eq. (3) in matrix form, namely

$$\begin{bmatrix} x_{n+2} \\ x_{n+1} \end{bmatrix} = \begin{bmatrix} a_n & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix}$$

for all $n = 0, 1, 2, \ldots$, or equivalently

$$X_{n+1} = A_n X_n, \tag{4}$$

where

$$A_n = \begin{bmatrix} a_n & 1 \\ 1 & 0 \end{bmatrix}, \quad X_n = \begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix}, \quad X_{n+1} = \begin{bmatrix} x_{n+2} \\ x_{n+1} \end{bmatrix}.$$

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[1] We consider the natural numbers as the non-negative integers.

The solution X_n of the general non-autonomous case is given by the matrix product

$$X_n = A_{n-1}A_{n-2}\ldots A_1 A_0 X_0 = \left(\prod_{i=0}^{n-1} A_{n-1-i}\right) X_0.$$

If we call

$$C_n = A_{n-1}A_{n-2}\ldots A_1 A_0 = \prod_{i=0}^{n-1} A_{n-1-i} \qquad (5)$$

we can write the solution of (4) in the form

$$X_n = C_n X_0.$$

where $X_0 \in \mathbb{C}^2$ is a given initial condition. Therefore the core of our work rests on techniques to obtain the products of the 2×2 matrices $A_n \in \mathbb{C}^{2\times2}$ that are obtained in the main Theorem 2.10 of this paper in Section 2. It is noticeable that the linear algebra approach is still complicated regarding that we have only products of very simple 2×2 matrices.

When the sequence of numbers $a_n, n = 0, 1, 2, \ldots$ is k-periodic with $a_k = a_0$ with k minimal, a similar iteration gives $X_k = C_k X_0$. In the periodic case, the matrix C_k is known as the monodromy matrix [2,6,8,10] of the periodic Eq. (4). Using the monodromy matrix one can construct the solution of Eq. (4) since

$$X_{mk} = C_k^m X_0$$

for any $m \geqslant 0$. For a general $n = mk + r$ with $r < k$ we have

$$X_n = \left(\prod_{i=0}^{r-1} A_{r-1-i}\right) C_k^m X_0$$

and, again, the solution of the periodic case is obtained computing matrix powers and products of matrices. The key of the periodic problem is to obtain the eigenvalues of the monodromy matrix, usually called Floquet multipliers in the field of dynamical systems. Moreover, the asymptotic behavior of the solutions can be studied via these numbers.

The problem of finding the Floquet multipliers is not easy, since we have to solve the characteristic equation

$$\det(C_k - \lambda I) = 0$$

and the entries of C_k satisfy the same recurrences of the original problem. Consequently, in Section 3 we approach this problem computing directly the eigenvalues of the monodromy matrix, i.e., the Floquet multipliers of the periodic equation.

In Section 4 we study more deeply the solutions of Eq. (3). We determine explicitly conditions for the periodicity or non-periodicity of the solutions. We remark here that the limits of the quotients of consecutive iterates exhibit periodicity when the iteration time n tends to infinity. Something similar to the convergence to the golden ratio of the quotients of consecutive iterates in the original Fibonacci problem.

Finally, using the techniques previously developed, we present some examples in Section 5. More specifically, we give a complete study of a 3-periodic and a 4-periodic equation where $k = 3$ and $k = 4$ respectively.

In a nutshell we can say that in this paper the main result rests mainly on the general technique to compute arbitrary products of some special 2×2 matrices. As usual in the field of linear difference equations most part of the theory can be seen as a reinterpretation of linear algebra results as we can see in [2,6,8,10]. We present explicit solutions of difference Eq. (3) for general sequences of complex numbers $\{a_n\}, n = 0, 1, 2, \ldots$, periodic or not, not obtained before in the literature. We provide abundant examples to help facilitate clarity.

2. Main result

We start this section by introducing some definitions and notations that will help us greatly in the statement of the results. As usual $\mathbb{N}$ represents the set of non-negative integers and $\mathbb{C}[t]$ the set of formal polynomials with complex coefficients in the indeterminate t.

Definition 2.1. Let $\mathcal{A} = \{a_i\}_{i\in\mathbb{N}}$ be a non-zero complex sequence of numbers not necessarily periodic. We define a 2×2 Fibonacci matrix $A_i \in \mathbb{C}^{2\times2}$ as the following

$$A_i = \begin{bmatrix} a_i & 1 \\ 1 & 0 \end{bmatrix}.$$

Notation 2.2. Let us use the vector notation $\mathfrak{a}_{i,j} = (a_i, a_{i+1}, \ldots, a_{j-1}, a_j)^T$, for some $i < j, i, j \in \mathbb{N}$. In the sequel, it will be evident that we need to indicate both the first index and the last index in this notation. When this vector has only one component the notation will be naturally simplified to $\mathfrak{a}_i = (\mathfrak{a}_i)^T = a_i$. Moreover, all the usual conventions about summations and products will be used.

Notation 2.3. To simplify the expressions with summations that we have in the following discussion we define for n and p even the multi summation operation

$$\widehat{\sum_{i_0,\ldots,i_{p-1}}^{n,p}}(\cdot) \triangleq \overbrace{\sum_{i_0=0}^{\frac{n-p}{2}}\sum_{i_1=i_0}^{\frac{n-p}{2}}\sum_{i_2=i_1+1}^{\frac{n-p}{2}+1}\sum_{i_3=i_2}^{\frac{n-p}{2}+1}\cdots\sum_{i_{p-4}=i_{p-5}+1}^{\frac{n-2}{2}-1}\sum_{i_{p-3}=i_{p-4}}^{\frac{n-2}{2}-1}\sum_{i_{p-2}=i_{p-3}+1}^{\frac{n-2}{2}}\sum_{i_{p-1}=i_{p-2}}^{\frac{n-2}{2}}}^{p\,\text{terms}}(\cdot), \tag{6}$$

when n and p are odd we write

$$\widehat{\sum_{i_0,\ldots,i_{p-1}}^{n,p}}(\cdot) \triangleq \overbrace{\sum_{i_0=0}^{\frac{n-p}{2}}\sum_{i_1=i_0}^{\frac{n-p}{2}}\sum_{i_2=i_1+1}^{\frac{n-p}{2}+1}\sum_{i_3=i_2}^{\frac{n-p}{2}+1}\cdots\sum_{i_{p-3}=i_{p-4}+1}^{\frac{n-1}{2}-1}\sum_{i_{p-2}=i_{p-3}}^{\frac{n-1}{2}-1}\sum_{i_{p-1}=i_{p-2}+1}^{\frac{n-1}{2}}}^{p\,\text{terms}}(\cdot). \tag{7}$$

In both cases the number of sums in each operator is p. Please note that the upper bound on the innermost summation determines the other upper bounds and not the other way around.

The introduction of this notation facilitates a great deal the writing of the proofs since large consecutive sums appear in the entries of the product of matrices. With this non-standard notation the lower indices and higher indices appear in consecutive pairs, starting at 0 for the pair $i_0 = 0$ and $i_1 = i_0$, having successfully pairs of the type $i_{2m} = i_{2m-1} + 1$ and $i_{2m+1} = i_{2m}$; each pair of upper and lower indices increases by one at each transition from odd indexed i_{2m-1} indices to even indexed i_{2m} indices. Clearly, in the case of odd n and p the innermost sum has only one term instead of a pair of sums. Please check examples 2.5 and 2.7, or the last sections of this paper, to see practical evaluations.

Definition 2.4. Let $n \in \mathbb{N}$ be even. We write $\chi_{n,p}(\mathfrak{a}_{0,n-1})t^p \in \mathbb{C}[t]$, with even degree p such that $0 < p \leqslant n$, as the following formal monomial in the indeterminate t

$$\chi_{n,p}(\mathfrak{a}_{0,n-1})t^p \triangleq t^p \widehat{\sum_{i_0,\ldots,i_{p-1}}^{n,p}}\prod_{l=0}^{\frac{p}{2}-1}a_{2i_{2l}}\prod_{l=0}^{\frac{p}{2}-1}a_{1+2i_{2l+1}}. \tag{8}$$

Finally, when we do not restrict the indices in $\mathfrak{a}$ to start at 0 but at $L \in \mathbb{N}$, we have $\chi_{n,p}(\mathfrak{a}_{L,L+n-1})$ such that

$$\chi_{n,p}(\mathfrak{a}_{L,L+n-1})t^p \triangleq t^p \widehat{\sum_{i_0,\ldots,i_{p-1}}^{n,p}}\prod_{l=0}^{\frac{p}{2}-1}a_{L+2i_{2l}}\prod_{l=0}^{\frac{p}{2}-1}a_{L+1+2i_{2l+1}}.$$

When $p = 0$, we consider by definition that $\chi_{n,0}(v) \triangleq 1$, for every even n and any vector v. Moreover, $\chi_{0,0}(\varnothing) = \chi_{0,0} = 1$.

The apparently complicated expression (8) is nothing but the sum of all possible products $a_{j_1}\ldots a_{j_p}$ with p factors, such that the first factor a_{j_1} has even index j_1, the second factor has odd index j_2 and so on, ending each product with an odd index factor a_{j_p} where j_p is odd.

Example 2.5. One can see that

$$\chi_{6,4}(\mathfrak{a}_{0,5}) = \widehat{\sum_{i_0,i_1,i_2,i_3}^{6,4}}\prod_{l=0}^{1}a_{2i_{2l}}\prod_{l=0}^{1}a_{1+2i_{2l+1}} = \sum_{i_0=0}^{1}\sum_{i_1=i_0}^{1}\sum_{i_2=i_1+1}^{2}\sum_{i_3=i_2}^{2}\prod_{l=0}^{1}a_{2i_{2l}}\prod_{l=0}^{1}a_{1+2i_{2l+1}}$$

$$= a_0a_1a_2a_3 + a_0a_1a_2a_5 + a_0a_1a_4a_5 + a_0a_3a_4a_5 + a_2a_3a_4a_5.$$

Obviously, when we shift the indices in $\mathfrak{a}$ by one unit, we have

$$\chi_{6,4}(\mathfrak{a}_{1,6}) = a_1a_2a_3a_4 + a_1a_2a_3a_6 + a_1a_2a_5a_6 + a_1a_4a_5a_6 + a_3a_4a_5a_6.$$

From a practical point of view, one can see that the construction of the number $\chi_{6,4}(\mathfrak{a}_{0,5})$ is equivalent to the combinatorial problem of finding all the possible configurations obtained from $(0, 1, 2, 3, 4, 5)$ when we cut a pair of consecutive numbers (not considering a_5 and a_0 consecutive) in the sequence of elements of the set $\mathcal{A}$. In this case, we have the configurations

$$\{(0, 1, 2, 3), (0, 1, 2, 5), (0, 1, 4, 5), (0, 3, 4, 5), (2, 3, 4, 5)\}.$$

One can formulate this problem, for example, as the possible configurations of people remaining when a pair of persons seating together leave a counter or the possible configurations of a row of distinguishable balls when two adjacent balls are taken.

Definition 2.6. Consider $n \in \mathbb{N}$ an odd number and $L \in \mathbb{N}$. Similarly to the even case, we define the p-degree formal monomial $\chi_{n,p}(\mathfrak{a}_{L,L+n-1})t^p \in \mathbb{C}[t]$, with $1 \leqslant p \leqslant n$, as

$$\chi_{n,p}(\mathfrak{a}_{L,L+n-1})t^p \triangleq t^p \widehat{\sum_{i_0,\ldots,i_{p-1}}^{n,p}}\prod_{l=0}^{\frac{p-1}{2}}a_{L+2i_{2l}}\prod_{l=0}^{\frac{p-1}{2}-1}a_{L+1+2i_{2l+1}}.$$

We remark again that the apparently complicated expression of $\chi_{n,p}(\mathfrak{a}_{0,n-1})$ is the sum of all possible products $a_{j_1} \ldots a_{j_p}$ with p factors, such that the first factor has even index, the second factor has odd index and so on, ending on a_{j_p} where j_p is an even index.

Example 2.7. For instance, one can see that

$$\chi_{5,3}(\mathfrak{a}_{0,4}) = \overset{\overbrace{5,3}}{\sum_{i_0,i_1,i_2}} \prod_{l=0}^{1} a_{2i_{2l}} \prod_{l=0}^{0} a_{1+2i_{2l+1}} = \sum_{i_0=0}^{1}\sum_{i_1=i_0}^{1} \sum_{i_2=i_1+1}^{2} \prod_{l=0}^{1} a_{2i_{2l}} \prod_{l=0}^{0} a_{1+2i_{2l+1}} = a_0 a_1 a_2 + a_0 a_1 a_4 + a_0 a_3 a_4 + a_2 a_3 a_4.$$

Again, this is the same combinatorial problem of finding all the possible configurations obtained from $(0,1,2,3,4)$ when we cut a pair of consecutive numbers. In this case the possible configurations are

$$\{(0,1,2),(0,1,4),(0,3,4),(2,3,4)\}.$$
Other easy cases are $\chi_{5,1}(\mathfrak{a}_{0,4}) = a_0 + a_2 + a_4$ or $\chi_{5,1}(\mathfrak{a}_{1,5}) = a_1 + a_3 + a_5$.

Definition 2.8. For the natural numbers n and L, we define the formal polynomial $\Omega(t) \in \mathbb{C}[t]$ by

$$\Omega_n(t, \mathfrak{a}_{L,L+n-1}) \triangleq \begin{cases} \displaystyle\sum_{j=0}^{\frac{n}{2}} \chi_{n,2j}(\mathfrak{a}_{L,L+n-1}) t^{2j}, n \text{ even}, \\ \displaystyle\sum_{j=0}^{\frac{n-1}{2}} \chi_{n,2j+1}(\mathfrak{a}_{L,L+n-1}) t^{2j+1}, n \text{ odd}. \end{cases} \tag{9}$$

The polynomial $\Omega_n(t, \mathfrak{a}_{0,n-1})$ has degree n and when $n = 0$ we write $\Omega_0(t) = \chi_{0,0}t^0 = 1$. The construction of $\Omega_n(t, \mathfrak{a}_{0,n-1})$ corresponds to the combinatorial problem of finding all the possible configurations when we cut all the possible pairs, from zero pairs to $\lfloor \frac{n}{2} \rfloor$ pairs, of consecutive numbers from a row of n integers.

Example 2.9. As an example, let us illustrate how to compute $\Omega_2(t, \mathfrak{a}_{0,1})$. Since

$$\Omega_2(t, \mathfrak{a}_{0,1}) = \chi_{2,0}(\mathfrak{a}_{0,1}) + \chi_{2,2}(\mathfrak{a}_{0,1}) t^2,$$

it follows that

$$\chi_{2,2}(\mathfrak{a}_{0,1}) = \overset{\overbrace{2,2}}{\sum_{i_0,i_1}} \prod_{l=0}^{0} a_{2i_{2l}} \prod_{l=0}^{0} a_{1+2i_{2l+1}} = a_0 a_1$$

and

$$\chi_{2,0}(\mathfrak{a}_{0,1}) = 1.$$

Hence, $\Omega_2(t, \mathfrak{a}_{0,1}) = 1 + a_0 a_1 t^2$. Notice that $\Omega_1(t, \mathfrak{a}_0) = a_0 t$ and $\Omega_0(t) = 1$. Finally we note that in the case of $i = 1$ the indices in $\chi_{2,2}(\mathfrak{a}_{0,1})$ are increased by 1, so $\Omega_2(t, \mathfrak{a}_{1,2}) = 1 + a_1 a_2 t^2$

In the next result we show how to compute the product matrix C_n defined in (5).

Theorem 2.10. Consider the formal polynomials $\Omega_n(t, \mathfrak{a}_{0,n-1}), \Omega_{n-1}(t, \mathfrak{a}_{1,n-1}), \Omega_{n-1}(t, \mathfrak{a}_{0,n-2}), \Omega_{n-2}(t, \mathfrak{a}_{1,n-2}) \in \mathbb{C}[t]$ defined in (9). For any $n \geqslant 2$ and $t = 1$, the product matrix of Fibonacci matrices defined in (5) is given by

$$C_n = \prod_{i=0}^{n-1} \begin{bmatrix} a_{n-1-i} & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \Omega_n(1, \mathfrak{a}_{0,n-1}) & \Omega_{n-1}(1, \mathfrak{a}_{1,n-1}) \\ \Omega_{n-1}(1, \mathfrak{a}_{0,n-2}) & \Omega_{n-2}(1, \mathfrak{a}_{1,n-2}) \end{bmatrix}.$$

Proof. The proof is by induction in n. When $n = 2$ we have

$$C_2 = \begin{bmatrix} \Omega_2(1, \mathfrak{a}_{0,1}) & \Omega_1(1, \mathfrak{a}_1) \\ \Omega_1(1, \mathfrak{a}_0) & \Omega_0(t) \end{bmatrix} = \begin{bmatrix} 1 + a_0 a_1 & a_1 \\ a_0 & 1 \end{bmatrix},$$

which is $\prod_{j=0}^{1} A_{1-j}$. We notice also that

$$C_3 = \begin{bmatrix} \Omega_3(1, \mathfrak{a}_{0,2}) & \Omega_2(1, \mathfrak{a}_{1,2}) \\ \Omega_2(1, \mathfrak{a}_{0,1}) & \Omega_1(1, \mathfrak{a}_1) \end{bmatrix} = \begin{bmatrix} a_0 + a_2 + a_0 a_1 a_2 & 1 + a_1 a_2 \\ 1 + a_0 a_1 & a_1 \end{bmatrix}.$$

It is an easy computation to show that this matrix is $\prod_{j=0}^{2} A_{2-j}$.

Now we have the hypothesis

$$C_n = \begin{bmatrix} \Omega_n(1, \mathfrak{a}_{0,n-1}) & \Omega_{n-1}(1, \mathfrak{a}_{1,n-1}) \\ \Omega_{n-1}(1, \mathfrak{a}_{0,n-2}) & \Omega_{n-2}(1, \mathfrak{a}_{1,n-2}) \end{bmatrix}.$$

One needs to show the induction step for the matrix C_{n+1},

$$C_{n+1} = \begin{bmatrix} c_{n+1}(1,1)c_{n+1}(1,2) \\ c_{n+1}(2,1)c_{n+1}(2,2) \end{bmatrix} = \begin{bmatrix} a_n & 1 \\ 1 & 0 \end{bmatrix} C_n = \begin{bmatrix} a_n\Omega_n(1,\mathfrak{a}_{0,n-1}) + \Omega_{n-1}(1,\mathfrak{a}_{0,n-2}) & a_n\Omega_{n-1}(1,\mathfrak{a}_{1,n-1}) + \Omega_{n-2}(1,\mathfrak{a}_{1,n-2}) \\ \Omega_n(1,\mathfrak{a}_{0,n-1}) & \Omega_{n-1}(1,\mathfrak{a}_{1,n-1}) \end{bmatrix}.$$

Automatically, the entries $c_{n+1}(2,1)$ and $c_{n+1}(2,2)$ are done. Now, we have to prove that

$$\Omega_{n+1}(1, \mathfrak{a}_{0,n}) = c_{n+1}(1,1) = a_n\Omega_n(1, \mathfrak{a}_{0,n-1}) + \Omega_{n-1}(1, \mathfrak{a}_{0,n-2}), \tag{10}$$

for any n.

We note that

$$\Omega_n(1, \mathfrak{a}_{0,n}) = c_{n+1}(1,2) = a_n\Omega_{n-1}(1, \mathfrak{a}_{1,n-1}) + \Omega_{n-2}(1, \mathfrak{a}_{1,n-2}),$$

holds immediately if the condition (10) is true.

We make use again of the formal indeterminate t to tackle the polynomial

$$a_n t\Omega_n(t, \mathfrak{a}_{0,n-1}) + \Omega_{n-1}(t, \mathfrak{a}_{0,n-2}),$$

computed for each degree of t. At the end of the proof, we will make $t = 1$ to obtain the desired equality.

Supposing first that n is even we note that

$$a_n t\Omega_n(t, \mathfrak{a}_{0,n-1}) + \Omega_{n-1}(t, \mathfrak{a}_{0,n-2}) =$$

$$a_n t\sum_{j=0}^{\frac{n}{2}} \chi_{n,2j}(\mathfrak{a}_{0,n-1})t^{2j} + \sum_{j=0}^{\frac{n}{2}} \chi_{n-1,2j+1}(\mathfrak{a}_{0,n-2})t^{2j+1}. \tag{11}$$

The term with the highest degree is

$$a_n t\chi_{n,n}(\mathfrak{a}_{0,n-1})t^n = t^{n+1}a_n\prod_{l=0}^{n-1}a_l = t^{n+1}\prod_{l=0}^{n}a_l = \chi_{n+1,n+1}(\mathfrak{a}_{0,n})t^{n+1}.$$

On the other hand, the term with lowest degree is given by

$$a_n t\chi_{n,0}(\mathfrak{a}_{0,n-1}) + \chi_{n-1,1}(\mathfrak{a}_{0,n-2})t = a_n t + \sum_{i=0}^{n-1}a_i t = \sum_{i=0}^{n}a_i t = \chi_{n+1,1}(\mathfrak{a}_{0,n})t.$$

Please note that n is even and p must also be even. We consider the sums in the same degree p in (11)

$$a_n t\chi_{n,p}(\mathfrak{a}_{0,n-1})t^p + \chi_{n-1,p+1}(\mathfrak{a}_{0,n-2})t^{p+1}.$$

Consequently

$$a_n\chi_{n,p}(\mathfrak{a}_{0,n-1}) + \chi_{n-1,p+1}(\mathfrak{a}_{0,n-2}) = \sum_{i_0,\ldots,i_{p-1}}^{\widehat{n,p}} a_n\prod_{l=0}^{\frac{p}{2}-1}a_{2i_{2l}}\prod_{l=0}^{\frac{p}{2}-1}a_{1+2i_{2l+1}} + \sum_{i_0,\ldots,i_p}^{\widehat{n-1,p+1}}\prod_{l=0}^{\frac{p}{2}}a_{2i_{2l}}\prod_{l=0}^{\frac{p}{2}-1}a_{1+2i_{2l+1}}.$$

Consider now

$$\chi_{n+1,p+1}(\mathfrak{a}_{0,n}) = \sum_{i_0,\ldots,i_p}^{\widehat{n-1,p+1}}\prod_{l=0}^{\frac{p}{2}}a_{2i_{2l}}\prod_{l=0}^{\frac{p}{2}-1}a_{1+2i_{2l+1}}.$$

We split this last member in two terms, one with factor a_n and the other without a_n. Hence,

$$\chi_{n+1,p+1}(\mathfrak{a}_{0,n}) = S_1 + S_2,$$

where

$$S_1 = \sum_{i_0,\ldots,i_{p-1}}^{\widehat{n,p}}\sum_{i_p=i_{p-1}+1}^{\frac{n}{2}-1}\prod_{l=0}^{\frac{p}{2}}a_{2i_{2l}}\prod_{l=0}^{\frac{p}{2}-1}a_{1+2i_{2l+1}}$$

and

$$S_2 = \sum_{i_0,\ldots,i_{p-1}}^{\widehat{n,p}} a_n \prod_{l=0}^{\frac{p}{2}-1} a_{2i_{2l}} \prod_{l=0}^{\frac{p}{2}-1} a_{1+2i_{2l+1}} = a_n \chi_{n,p}(\mathfrak{a}_{0,n-1}).$$

On the other hand

$$S_1 = \sum_{i_0,\ldots,i_{p-1}}^{\widehat{n,p}} \sum_{i_p=i_{p-1}+1}^{\frac{n}{2}-1} \prod_{l=0}^{\frac{p}{2}} a_{2i_{2l}} \prod_{l=0}^{\frac{p}{2}-1} a_{1+2i_{2l+1}}. \tag{12}$$

Now $p+1$ is odd. The innermost summation upper bound forces all the other upper bounds in all the summations to be decreasing one unit by steps of two summations. The nominal upper bounds do not matter if bigger than the forced upper bounds since the summations cannot be done due to the lack of possible summands when the upper bounds exceed the forced ones. The nominal upper bounds of all summations $\sum_{i_0,\ldots,i_{p-1}}^{\widehat{n,p}} \sum_{i_p=i_{p-1}+1}^{\frac{n}{2}-1}$ in (12), from inner summation to outer summation, are

$$\overbrace{\frac{n}{2}-1, \frac{n}{2}-1, \frac{n}{2}-1, \ldots, \frac{n-p}{2}, \frac{n-p}{2}}^{p+1}$$

therefore the actual upper bounds that really matter for the computation, from innermost summation to the outermost summation, are

$$\overbrace{\frac{n}{2}-1, \frac{n}{2}-2, \frac{n}{2}-2, \ldots, \frac{n-p-2}{2}, \frac{n-p-2}{2}}^{p+1}$$

corresponding to odd values $n-1$ and $p+1$ in expression (7). This leads to the conclusion that

$$S_1 = \sum_{i_0,\ldots,i_p}^{\widehat{n-1,p+1}} \prod_{l=0}^{\frac{p}{2}} a_{2i_{2l}} \prod_{l=0}^{\frac{p}{2}-1} a_{1+2i_{2l+1}} = \chi_{n-1,p+1}(\mathfrak{a}_{0,n-2}).$$

Adding all the possible values of $\chi_{n+1,2j+1}(\mathfrak{a}_{0,n}) t^{2j+1}$ and putting $t = 1$ we get

$$\Omega_{n+1}(1, \mathfrak{a}_{0,n}) = a_n \Omega_n(1, \mathfrak{a}_{0,n-1}) + \Omega_{n-1}(1, \mathfrak{a}_{0,n-2}).$$

The reasonings for n odd and p odd are exactly the same. For sake of completeness we present here the corresponding induction step for odd n.

Let n be odd. Then, one can verify that

$$a_n t \Omega_n(t, \mathfrak{a}_{0,n-1}) + \Omega_{n-1}(t, \mathfrak{a}_{0,n-2}) = a_n t \sum_{j=0}^{\frac{n-1}{2}} \chi_{n,2j+1}(\mathfrak{a}_{0,n-1}) t^{2j+1} + \sum_{j=0}^{\frac{n-1}{2}} \chi_{n-1,2j}(\mathfrak{a}_{0,n-2}) t^{2j}. \tag{13}$$

The term with the highest degree is

$$a_n t \chi_{n,n}(\mathfrak{a}_{0,n-1}) t^n = t^{n+1} a_n \prod_{l=0}^{n-1} a_l = t^{n+1} \prod_{l=0}^{n} a_l = \chi_{n+1,n+1}(\mathfrak{a}_{0,n}) t^{n+1},$$

while the term with lowest degree is

$$\chi_{n-1,0}(\mathfrak{a}_{0,n-2}) t^0 = 1 = \chi_{n+1,0}(\mathfrak{a}_{0,n}).$$

Let us consider the sums with the same degree p in (13)

$$a_n t \chi_{n,p}(\mathfrak{a}_{0,n-1}) t^p + \chi_{n-1,p+1}(\mathfrak{a}_{0,n-2}) t^{p+1}.$$

Hence, we have to study the coefficients of t^{p+1}

$$a_n \chi_{n,p}(\mathfrak{a}_{0,n-1}) + \chi_{n-1,p+1}(\mathfrak{a}_{0,n-2}) = \sum_{i_0,\ldots,i_{p-1}}^{\widehat{n,p}} a_n \prod_{l=0}^{\frac{p-1}{2}} a_{2i_{2l}} \prod_{l=0}^{\frac{p-1}{2}-1} a_{1+2i_{2l+1}} + \sum_{i_0,\ldots,i_p}^{\widehat{n-1,p+1}} \prod_{l=0}^{\frac{p-1}{2}} a_{2i_{2l}} \prod_{l=0}^{\frac{p-1}{2}} a_{1+2i_{2l+1}}.$$

Consider now

$$\chi_{n+1,p+1}(\mathfrak{a}_{0,n}) = \sum_{i_0,\ldots,i_p}^{\widehat{n+1,p+1}} \prod_{l=0}^{\frac{p-1}{2}} a_{2\,i_{2l}} \prod_{l=0}^{\frac{p-1}{2}} a_{1+2\,i_{2l+1}}.$$

Splitting this last member in two terms, one with factor a_n and the other without this factor we have

$$\chi_{n+1,p+1}(\mathfrak{a}_{0,n}) = \widetilde{S}_1 + \widetilde{S}_2,$$

where

$$\widetilde{S}_1 = \sum_{i_0,\dots,i_{p-1}}^{\widehat{n,p}} \sum_{i_p=i_{p-1}}^{\frac{n-1}{2}-1} \prod_{l=0}^{\frac{p-1}{2}} a_{2\ i_{2l}} \prod_{l=0}^{\frac{p-1}{2}} a_{1+2\ i_{2l+1}}$$

and

$$\widetilde{S}_2 = \sum_{i_0,\dots,i_{p-1}}^{\widehat{n,p}} a_n \prod_{l=0}^{\frac{p-1}{2}} a_{2\ i_{2l}} \prod_{l=0}^{\frac{p-1}{2}-1} a_{1+2\ i_{2l+1}}.$$

One can verify that $\widetilde{S}_2 = a_n \chi_{n,p}(\mathfrak{a}_{0,n-1})$ and $\widetilde{S}_1$ can be written as

$$\widetilde{S}_1 = \sum_{i_0,\dots,i_{p-1}}^{\widehat{n,p}} \sum_{i_p=i_{p-1}}^{\frac{n-1-2}{2}} \prod_{l=0}^{\frac{p-1}{2}} a_{2i_{2l}} \prod_{l=0}^{\frac{p-1}{2}} a_{1+2i_{2l+1}},$$

a similar reasoning to the one used to simplify (12) gives

$$\widetilde{S}_1 = \sum_{i_0,\dots,i_p}^{\widehat{n-1,p+1}} \prod_{l=0}^{\frac{p-1}{2}} a_{2i_{2l}} \prod_{l=0}^{\frac{p-1}{2}} a_{1+2i_{2l+1}} = \chi_{n-1,p+1}(\mathfrak{a}_{0,n-2}).$$

Adding all the possible values of $\chi_{n+1,2j+1}(\mathfrak{a}_{0,n})t^{2j+1}$ and putting $t = 1$ we get

$$\Omega_{n+1}(1,\mathfrak{a}_{0,n}) = a_n \Omega_n(1,\mathfrak{a}_{0,n-1}) + \Omega_{n-1}(1,\mathfrak{a}_{0,n-2}),$$

as desired. □

Obviously, the result on general non-autonomous difference equations can be further developed for the general non-periodic case. In the present work its main use is on the study of periodic systems. Anyway, we present here a very simple example for the general non-autonomous case.

Example 2.11. Consider the sequence of Fibonacci matrices $\mathcal{A} = \{a_i\}_{i \in \mathbb{N}}$ with $a_i = 0$ when i is odd. For n odd we have

$$C_n = \begin{bmatrix} \Omega_n(1,\mathfrak{a}_{0,n-1}) & \Omega_{n-1}(1,\mathfrak{a}_{1,n-1}) \\ \Omega_{n-1}(1,\mathfrak{a}_{0,n-2}) & \Omega_{n-2}(1,\mathfrak{a}_{1,n-2}) \end{bmatrix} = \begin{bmatrix} \sum_{j=0}^{\frac{n-1}{2}} a_{2j} & 1 \\ 1 & 0 \end{bmatrix},$$

where $\Omega_{n-2}(1,\mathfrak{a}_{1,n-2}) = \sum_{j=0}^{\frac{n-3}{2}} a_{1+2j} = 0$. For n even we have

$$C_n = \begin{bmatrix} \Omega_n(1,\mathfrak{a}_{0,n-1}) & \Omega_{n-1}(1,\mathfrak{a}_{1,n-1}) \\ \Omega_{n-1}(1,\mathfrak{a}_{0,n-2}) & \Omega_{n-2}(1,\mathfrak{a}_{1,n-2}) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \sum_{j=0}^{\frac{n-2}{2}} a_{2j} & 1 \end{bmatrix}.$$

In this simple example the asymptotic behavior of the solutions depends only on the convergence of the series $\sum_{j=0}^{\infty} a_{2j}$.

3. Periodic Fibonacci difference equation

In this section we study the periodic generalized Fibonacci difference Eq. 4 using the monodromy matrix and its Floquet multipliers.

Definition 3.1. Consider the periodic non-zero sequence of complex numbers $\mathcal{A} = \{a_i\}_{i \in \mathbb{N}}$ such that $a_{i+k} = a_i$ for any $i \in \mathbb{N}$, some fixed $k \in \mathbb{Z}^+$ and a periodic sequence of 2×2 Fibonacci matrices

$$A_i = \begin{bmatrix} a_i & 1 \\ 1 & 0 \end{bmatrix}.$$

We define the monodromy matrix $C_k \in \mathbb{C}^{2 \times 2}$ by

$$C_k \triangleq \prod_{i=0}^{k-1} A_{k-1-i}.$$

[Theorem 2.10](#) naturally applies to the monodromy matrix. The next step is to determine the Floquet multipliers, which are the eigenvalues of C_k. For that purpose we need to compute the determinant and trace of C_k.

Since $C_k = \prod_{i=0}^{k-1} \begin{bmatrix} a_{k-1-i} & 1 \\ 1 & 0 \end{bmatrix}$, it follows that $\det C_k = (-1)^k$. By [Theorem 2.10](#) the trace[2] of the monodromy matrix C_k is

$$tr C_k = \Omega_k(1, a_{0,k-1}) + \Omega_{k-2}(1, a_{1,k-2}).$$

To understand better this trace we use again the formal indeterminate t.

Definition 3.2. The formal polynomial $T_k(t) \in \mathbb{C}[t]$ is given by

$$T_k(t) \triangleq \Omega_k(t, a_{0,k-1}) + \Omega_{k-2}(t, a_{1,k-2}).$$

$T_k(t)$ is defined such that the trace of C_k is $T_k(1)$.

Notation 3.3. We write the polynomial $T_k(t)$ as

$$T_k(t) = \sum_{j=0}^{\frac{k}{2}} \Psi_{2j}(a_{0,k-1}) t^{2j},$$

when k is even and

$$T_k(t) = \sum_{j=0}^{\frac{k-1}{2}} \Psi_{2j+1}(a_{0,k-1}) t^{2j+1},$$

when k is odd.

Consider the case of even k (similar reasonings hold for the odd case), we have

$$T_k(t) = \sum_{j=0}^{\frac{k}{2}} \chi_{k,2j}(a_{0,k-1}) t^{2j} + \sum_{j=0}^{\frac{k}{2}-1} \chi_{k-2,2j}(a_{1,k-2}) t^{2j} = \sum_{j=0}^{\frac{k}{2}} \left(\chi_{k,2j}(a_{0,k-1}) + \chi_{k-2,2j}(a_{1,k-2}) \right) t^{2j},$$

subject to the convention that

$$\chi_{k-2,2j}(v) = 0, \quad \text{if } 2j > k-2.$$

Now we write

$$T_k(t) = \sum_{j=0}^{\frac{k}{2}} \Psi_{2j}(a_{0,k-1}) t^{2j},$$

where $\Psi_{2j}(a_{0,k-1})$ is the coefficient of t^{2j} in $T_k(t)$ such that

$$\Psi_{2j}(a_{0,k-1}) = \chi_{k,2j}(a_{0,k-1}) + \chi_{k-2,2j}(a_{1,k-2}).$$

Please note that the coefficient of the highest degree (t^k) is

$$\Psi_k(a_{0,k-1}) = \chi_{k,k}(a_{0,k-1}) + \underbrace{\chi_{k-2,k}(a_{1,k-2})}_{0} = \prod_{i=0}^{k-1} a_i.$$

We focus our attention on the monomials in the formal indeterminate t. We remember that k is even and $2j < k$. Hence, the coefficient of t^{2j} is

$$\Psi_{2j}(a_{0,k-1}) = \overbrace{\sum_{i_0,\dots,i_{2j-1}}^{k,2j}}^{} \prod_{l=0}^{j-1} a_{2i_{2l}} \prod_{l=0}^{j-1} a_{1+2i_{2l+1}} + \overbrace{\sum_{i_0,\dots,i_{2j-1}}^{k-2,2j}}^{} \prod_{l=0}^{j-1} a_{1+2i_{2l}} \prod_{l=0}^{j-1} a_{2+2i_{2l+1}}.$$

We realize that this rather long expression is nothing but the sum of all possible products with $2j$ factors of coefficients a_{m_s} such that the first factor can have index m_1 even or odd, the second factor has index odd or even, respectively, alternating always the parity of the indices along the product of coefficients. For instance if $k \geqslant 6$ we have in the coefficient of t^4 summands which are products of the form $a_0 a_1 a_2 a_3, a_0 a_1 a_4 a_5, a_0 a_3 a_4 a_5, a_2 a_3 a_4 a_5$ or $a_1 a_2 a_3 a_4$, but we do not have the forbidden products $a_0 a_2 a_3 a_4$ or $a_1 a_2 a_3 a_5$, where we would have two even consecutive indices, in the first case, and odd, in the second case.

We can obtain explicitly the Floquet multipliers in the following result.

---

[2] The constant A in [4] is basically $(-1)^k tr C_k$ which is very dificult to compute without [Theorem 2.10](#).

Proposition 3.4. *The Floquet multipliers of C_k are given by*

$$\Phi_k^{\pm} = \frac{1}{2}\left(T_k(1) \pm \sqrt{(T_k(1))^2 - 4(-1)^k}\right).$$

Proof. Since the eigenvalues of any 2×2 matrix $A \in \mathbb{C}^{2\times2}$ are given by

$$\lambda_{1,2} = \frac{trA \pm \sqrt{(trA)^2 - 4\det A}}{2},$$

it follows that

$$\Phi_k^{\pm} = \frac{T_k(1) \pm \sqrt{(T_k(1))^2 - 4(-1)^k}}{2}. \qquad \square$$

Now let $C_k = J\Lambda J^{-1}$ be the Jordan canonical form of the monodromy matrix C_k. Assume that when k is even we have $|trC_k| \neq 2$. Hence, we can write, without loss of generality, the matrices J, J^{-1} and Λ as

$$J = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}, \quad J^{-1} = \begin{bmatrix} J_{11}^{-1} & J_{12}^{-1} \\ J_{21}^{-1} & J_{22}^{-1} \end{bmatrix}, \quad \Lambda = \begin{bmatrix} \Phi_k^- & 0 \\ 0 & \Phi_k^+ \end{bmatrix}.$$

So, we have

$$C_k^n = J\Lambda^n J^{-1} = \begin{bmatrix} J_{11}(\Phi_k^-)^n J_{11}^{-1} + J_{12}(\Phi_k^+)^n J_{21}^{-1} & J_{11}(\Phi_k^-)^n J_{12}^{-1} + J_{12}(\Phi_k^+)^n J_{22}^{-1} \\ J_{21}(\Phi_k^-)^n J_{11}^{-1} + J_{22}(\Phi_k^+)^n J_{21}^{-1} & J_{21}(\Phi_k^-)^n J_{12}^{-1} + J_{22}(\Phi_k^+)^n J_{22}^{-1} \end{bmatrix}.$$

If k is even and $|trC_k| = 2$, then the eigenvalues of C_k are either 1 or -1 with algebraic multiplicity 2. In this case the matrix Λ^n is either

$$\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \text{or} \begin{bmatrix} (-1)^n & n(-1)^{n-1} \\ 0 & (-1)^n \end{bmatrix}.$$

In the following corollary of Theorem 2.10 and Proposition 3.4 we write explicitly the solution of Eq. (3).

Corollary 3.5. *The solution of the k-periodic generalized Fibonacci difference Eq. (3) is given by*

1. *Case k odd or k even and $|trC_k| \neq 2$:*

$$x_{nk} = J_{21}(J_{11}^{-1}x_1 + J_{21}^{-1}x_0)(\Phi_k^-)^n + J_{22}(J_{21}^{-1}x_1 + J_{22}^{-1}x_0)(\Phi_k^+)^n, \quad x_{nk+1} = J_{11}(J_{11}^{-1}x_1 + J_{21}^{-1}x_0)(\Phi_k^-)^n + J_{12}(J_{21}^{-1}x_1 + J_{22}^{-1}x_0)(\Phi_k^+)^n$$

and

$$x_{nk+(i+2)} = a_i x_{nk+(i+1)} + x_{nk+i}, i \in \{0, 1, 2, \ldots, k-3\},$$

for all $n = 0, 1, 2, \ldots$.
2. *Case k even and $trC_k = 2$:*

$$x_{nk} = \left(J_{21}J_{11}^{-1} + (nJ_{21} + J_{22})J_{21}^{-1}\right)x_1 + \left(J_{21}J_{12}^{-1} + (nJ_{21} + J_{22})J_{22}^{-1}\right)x_0, x_{nk+1}$$
$$= \left(J_{11}J_{11}^{-1} + (nJ_{11} + J_{12})J_{21}^{-1}\right)x_1 + \left(J_{11}J_{12}^{-1} + (nJ_{11} + J_{12})J_{22}^{-1}\right)x_0,$$

and

$$x_{nk+(i+2)} = a_i x_{nk+(i+1)} + x_{nk+i}, i \in \{0, 1, 2, \ldots, k-3\},$$

for all $n = 0, 1, 2, \ldots$.
3. *Case k even and $trC_k = -2$:*

$$x_{nk} = (-1)^n\left(J_{21}J_{11}^{-1} + (-nJ_{21} + J_{22})J_{21}^{-1}\right)x_1 + (-1)^n\left(J_{21}J_{12}^{-1} + (-nJ_{21} + J_{22})J_{22}^{-1}\right)x_0,$$
$$x_{nk+1} = (-1)^n\left(J_{11}J_{11}^{-1} + (-nJ_{11} + J_{12})J_{21}^{-1}\right)x_1 + (-1)^n\left(J_{11}J_{12}^{-1} + (-nJ_{11} + J_{12})J_{22}^{-1}\right)x_0,$$

and

$$x_{nk+(i+2)} = a_i x_{nk+(i+1)} + x_{nk+i}, i \in \{0, 1, 2, \ldots, k-3\},$$

for all $n = 0, 1, 2, \ldots$.

Remark 3.6. Notice that when $x_0 = 0, x_1 = 1$ and $k = 1$ we have $\Phi_1^+ = \phi$, $\Phi_1^- = 1 - \phi, J_{21} = -\phi, J_{11}^{-1} = -\frac{1}{\sqrt{5}}(1 - \phi), J_{22} = 1 - \phi$ and $J_{21}^{-1} = \frac{1}{\sqrt{5}}\phi$ leading to

$$x_n = \frac{1}{\sqrt{5}}(\phi^n - (1 - \phi)^n), \quad n = 0, 1, 2, \ldots$$

4. The structure of the solution

In this section we study the solutions of Eq. (3), namely we determine explicitly the conditions for the periodicity of the solutions in the case of k-periodic Fibonacci equations. The sequence of quotients of consecutive iterates, such that $q_0 = \frac{x_1}{x_0}, \ldots, q_n = \frac{x_{n+1}}{x_n}$ approaches a periodic cycle with period P, where P is a multiple of k (it depends on the Floquet multipliers), which is related to the convergence to the golden ratio of the quotients of consecutive iterates in the classic autonomous Fibonacci equation.

4.1. Odd period

From Corollary 3.5 it follows that

$$\frac{x_{nk+1}}{x_{nk}} = \frac{\alpha(\Phi_k^-)^n + \beta(\Phi_k^+)^n}{\delta(\Phi_k^-)^n + \gamma(\Phi_k^+)^n},$$

where $\alpha = J_{11}(J_{11}^{-1}x_1 + J_{21}^{-1}x_0), \beta = J_{12}(J_{21}^{-1}x_1 + J_{22}^{-1}x_0), \delta = J_{21}(J_{11}^{-1}x_1 + J_{21}^{-1}x_0)$ and $\gamma = J_{22}(J_{21}^{-1}x_1 + J_{22}^{-1}x_0)$.

Let the period of Eq. (3) be odd, i.e., k is an odd number. Then

$$\Phi_k^\pm = \frac{trC_k \pm \sqrt{(trC_k)^2 + 4}}{2}.$$

Let us assume first that $trC_k \neq 0$. Hence, either $\left|\frac{\Phi_k^-}{\Phi_k^+}\right| < 1$ or $\left|\frac{\Phi_k^-}{\Phi_k^+}\right| > 1$.

If $\left|\frac{\Phi_k^-}{\Phi_k^+}\right| = |r| < 1$, we have

$$\frac{x_{nk+1}}{x_{nk}} = \frac{\alpha r^n + \beta}{\delta r^n + \gamma} \longrightarrow \frac{\beta}{\gamma}.$$

Let $\frac{\beta}{\gamma} = L_0$. Similarly, one can show that $\frac{x_{nk+2}}{x_{nk+1}} \longrightarrow \frac{a_0 L_0 + 1}{L_0} = L_1$. Analogously, $\frac{x_{nk+3}}{x_{nk+2}} \longrightarrow \frac{a_1 L_1 + 1}{L_1} = L_2$. More generally, $\frac{x_{nk+(i+2)}}{x_{nk+(i+1)}} \longrightarrow \frac{a_i L_i + 1}{L_i} = L_{i+1}, i \in \{0, 1, 2, \ldots, k - 2\}$. Notice that $L_{k+i} = L_i$, for all $i = 0, 1, 2, \ldots$. Hence, one can consider the following k-periodic cycle as the limit of the quotients of the solution of Eq. (3)

$$\{L_0, L_1, L_2, \ldots, L_{k-1}\}. \tag{14}$$

Remark 4.1. Notice that, if there exists an $i \in \{0, 1, 2, \ldots, k - 1\}$ such that $L_i = 0$, then the k-periodic cycle (14) is unbounded since $L_{i+1} \to \infty$.

On the other hand, if $\left|\frac{\Phi_k^-}{\Phi_k^+}\right| > 1$ one can show that $\frac{x_{nk+1}}{x_{nk}} \longrightarrow \frac{\alpha}{\delta} = \widetilde{L}_0, \frac{x_{nk+2}}{x_{nk+1}} \longrightarrow \frac{a_0 \widetilde{L}_0 + 1}{\widetilde{L}_0} = \widetilde{L}_1$ and more generally $\frac{x_{nk+(i+2)}}{x_{nk+(i+1)}} \longrightarrow \frac{a_i \widetilde{L}_i + 1}{\widetilde{L}_i} = \widetilde{L}_{i+1}, i \in \{0, 1, 2, \ldots, k - 2\}$, yielding the following k-periodic cycle as the limit of the quotients of the solutions

$$\{\widetilde{L}_0, \widetilde{L}_1, \widetilde{L}_2, \ldots, \widetilde{L}_{k-1}\}.$$

Secondly, if $trC_k = 0$, then the eigenvalues of C_k are 1 and -1. Hence, the solutions of Eq. (3) are given by

$$x_{kn} = J_{21}(J_{11}^{-1}x_1 + J_{21}^{-1}x_0)(-1)^n + J_{22}(J_{21}^{-1}x_1 + J_{22}^{-1}x_0), \quad x_{kn+1} = J_{11}(J_{11}^{-1}x_1 + J_{21}^{-1}x_0)(-1)^n + J_{12}(J_{21}^{-1}x_1 + J_{22}^{-1}x_0)$$

and

$$x_{nk+(i+2)} = a_i x_{nk+(i+1)} + x_{nk+i}, \quad i \in \{0, 1, 2, \ldots, k - 3\}$$

for all $n = 0, 1, 2, \ldots$.

A simple computation shows that $\{x_0, x_1, x_2, \ldots, x_{2k-1}\}$ is a $2k$-periodic solution of Eq. (3), where

$$x_0 = J_{21}(J_{11}^{-1}x_1 + J_{21}^{-1}x_0) + J_{22}(J_{21}^{-1}x_1 + J_{22}^{-1}x_0),$$
$$x_1 = -J_{11}(J_{11}^{-1}x_1 + J_{21}^{-1}x_0) + J_{12}(J_{21}^{-1}x_1 + J_{22}^{-1}x_0),$$
$$x_{i+2} = a_i x_{i+1} + x_i, i \in \{0, 1, 2, \ldots, k - 3\},$$
$$x_k = -J_{21}(J_{11}^{-1}x_1 + J_{21}^{-1}x_0) + J_{22}(J_{21}^{-1}x_1 + J_{22}^{-1}x_0),$$
$$x_{k+1} = J_{11}(J_{11}^{-1}x_1 + J_{21}^{-1}x_0) + J_{12}(J_{21}^{-1}x_1 + J_{22}^{-1}x_0),$$
$$x_{k+(i+2)} = a_i x_{k+(i+1)} + x_{k+i}, i \in \{0, 1, 2, \ldots, k - 3\}.$$

4.2. Even period

Let the period of Eq. (3) be even. First let us assume that $|trC_k| < 2$. Then

$$\Phi_k^{\pm} = \frac{trC_k \pm \Im\sqrt{4 - (trC_k)^2}}{2},$$

with $|\Phi_k^+| = |\Phi_k^-| = 1$ and $\Im$ being the imaginary unit. Hence, the eigenvalues of C_k lie on the unit circle. Moreover, the matrix Λ is periodic being

$$\frac{trC_k \pm \Im\sqrt{4 - (trC_k)^2}}{2} = e^{\Im\theta},$$

with $\theta = \arctan\frac{\sqrt{4-(trC_k)^2}}{trC_k}, \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Let $P = \frac{2\pi}{\theta}$ be the period of the monodromy matrix, i.e., $C_k^P = C_k$. Hence, the solutions of Eq. (3) are periodic. Let us now determine this period.

From Corollary 3.5 it follows that

$$x_0 = J_{21}(J_{11}^{-1}x_1 + J_{21}^{-1}x_0) + J_{22}(J_{21}^{-1}x_1 + J_{22}^{-1}x_0),$$
$$x_1 = J_{11}(J_{11}^{-1}x_1 + J_{21}^{-1}x_0) + J_{12}(J_{21}^{-1}x_1 + J_{22}^{-1}x_0),$$
$$x_{i+2} = a_i x_{i+1} + x_i, i \in \{0, 1, 2, \ldots, k-3\},$$
$$x_k = J_{21}(J_{11}^{-1}x_1 + J_{21}^{-1}x_0)\Phi_k^- + J_{22}(J_{21}^{-1}x_1 + J_{22}^{-1}x_0)\Phi_k^+,$$
$$x_{k+1} = J_{11}(J_{11}^{-1}x_1 + J_{21}^{-1}x_0)\Phi_k^- + J_{12}(J_{21}^{-1}x_1 + J_{22}^{-1}x_0)\Phi_k^+,$$
$$x_{k+(i+2)} = a_i x_{k+(i+1)} + x_{k+i}, i \in \{0, 1, 2, \ldots, k-3\},$$
$$\cdots$$
$$x_{(P-1)k} = J_{21}(J_{11}^{-1}x_1 + J_{21}^{-1}x_0)(\Phi_k^-)^{P-1} + J_{22}(J_{21}^{-1}x_1 + J_{22}^{-1}x_0)(\Phi_k^+)^{P-1},$$
$$x_{(P-1)k+1} = J_{11}(J_{11}^{-1}x_1 + J_{21}^{-1}x_0)(\Phi_k^-)^{P-1} + J_{12}(J_{21}^{-1}x_1 + J_{22}^{-1}x_0)(\Phi_k^+)^{P-1},$$
$$x_{(P-1)k+(i+2)} = a_i x_{(P-1)k+(i+1)} + x_{(P-1)k+i}, i \in \{0, 1, 2, \ldots, k-3\}.$$

Since the monodromy matrix is P-periodic, it follows that $(\Phi_k^{\pm})^{P+i} = (\Phi_k^{\pm})^i$, for all $i = 0, 1, 2, \ldots$. Hence, $x_{kP+i} = x_i$, for all $i = 0, 1, 2, \ldots$. This implies that the minimal period of the cycle is kP.

In conclusion, if $|trC_k| < 2$, the following cycle is a kP-periodic solution of Eq. (3)

$$\{x_0, x_1, x_2, \ldots, x_{kP-1}\}.$$

Second, if $|trC_k| = 2$, it follows that $\left|\frac{\Phi_k^-}{\Phi_k^+}\right| = 1$. In this case we will have a k-periodic cycle as the limit of the quotients of the solution.

Third, if $|trC_k| > 2$, it follows that we will have either $\left|\frac{\Phi_k^-}{\Phi_k^+}\right| < 1$ or $\left|\frac{\Phi_k^-}{\Phi_k^+}\right| > 1$. Following the same ideas as in the odd case we will have a k-periodic cycle as the limit of the quotients of the solution of Eq. (3).

5. Applications and examples

In this section we study some examples of the periodic generalized Fibonacci difference Eq. (3). We start by a 3-periodic equation.

Example 5.1. Let the period of Eq. (3) be 3, i.e., $k = 3$. The monodromy matrix is given by

$$C_3 = \begin{bmatrix} a_0 + a_2 + a_0 a_1 a_2 & 1 + a_1 a_2 \\ 1 + a_0 a_1 & a_1 \end{bmatrix},$$

yielding the following representation

$$C_3^n = J\Lambda^n J^{-1}$$

with

$$J = \begin{bmatrix} \frac{\Delta - 2a_1 - \sqrt{4+\Delta^2}}{2(1+a_0 a_1)} & \frac{\Delta - 2a_1 + \sqrt{4+\Delta^2}}{2(1+a_0 a_1)} \\ 1 & 1 \end{bmatrix}, J^{-1} = \begin{bmatrix} -\frac{1+a_0 a_1}{\sqrt{4+\Delta^2}} & \frac{\Delta - 2a_1 + \sqrt{4+\Delta^2}}{2\sqrt{4+\Delta^2}} \\ \frac{1+a_0 a_1}{\sqrt{4+\Delta^2}} & -\frac{\Delta - 2a_1 - \sqrt{4+\Delta^2}}{2\sqrt{4+\Delta^2}} \end{bmatrix}, \Lambda^n = \begin{bmatrix} \left(\frac{\Delta - \sqrt{4+\Delta^2}}{2}\right)^n & 0 \\ 0 & \left(\frac{\Delta + \sqrt{4+\Delta^2}}{2}\right)^n \end{bmatrix},$$

where $\Delta = a_0 + a_1 + a_2 + a_0 a_1 a_2$.

1. Case $\Delta = trC_3 = 0$. This occurs when $a_0 = -\frac{a_1+a_2}{1+a_1a_2}$. Hence, the general solution is given by

$$x_{3n} = \frac{1 + (-1)^n + (1-(-1)^n)a_1}{2}x_0 + \frac{(1-(-1)^n)(1-a_1^2)}{2(1+a_1a_2)}x_1,$$

$$x_{3n+1} = \frac{1-(-1)^n}{2}(1+a_1a_2)x_0 + \frac{1+(-1)^n - (1-(-1)^n)a_1}{2}x_1$$

and

$$x_{3n+2} = a_0 x_{3n+1} + x_{3n},$$

for all $n = 0, 1, 2, \ldots$. It is straightforward to see that this general solution is, in fact, a 6-periodic solution of the form

$$\left\{x_0, x_1, a_0x_1 + x_0, a_1x_0 + \frac{1-a_1^2}{2(1+a_1a_2)}x_1, (1+a_1a_2)x_0 - a_1x_1, a_0((1+a_1a_2)x_0 - a_1x_1) + a_1x_0 + \frac{1-a_1^2}{2(1+a_1a_2)}x_1\right\}.$$

Notice that this cycle can be not bounded if $a_1a_2 = -1$. In Fig. 1 we can see an example of this situation. Using a 3-periodic sequence of parameters we present in the plane (n, x_n) a 6-periodic solution of Eq. (3).

2. Case $\Delta = trC_3 \neq 0$. The general solution of the 3-periodic equation is given by

$$x_{3n} = \frac{1+a_0a_1}{\sqrt{4+\Delta^2}}\left((\Phi_3^+)^n - (\Phi_3^-)^n\right)x_1 + \left(\frac{\Delta - 2a_1 + \sqrt{4+\Delta^2}}{2\sqrt{4+\Delta^2}}(\Phi_3^-)^n + \frac{-\Delta + 2a_1 + \sqrt{4+\Delta^2}}{2\sqrt{4+\Delta^2}}(\Phi_3^+)^n\right)x_0,$$

$$x_{3n+1} = \frac{1+a_1a_2}{\sqrt{4+\Delta^2}}\left((\Phi_3^+)^n - (\Phi_3^-)^n\right)x_0 + \left(\frac{-\Delta + 2a_1 + \sqrt{4+\Delta^2}}{2\sqrt{4+\Delta^2}}(\Phi_3^-)^n + \frac{\Delta - 2a_1 + \sqrt{4+\Delta^2}}{2\sqrt{4+\Delta^2}}(\Phi_3^+)^n\right)x_1,$$

and

$$x_{3n+2} = a_0x_{3n+1} + x_{3n},$$

for all $n = 0, 1, 2, \ldots$, where $\Phi_3^\pm = \frac{\Delta \pm \sqrt{4+\Delta^2}}{2}$.

- Case $\Delta > 0$. If the trace of the monodromy matrix is positive, then the limit of the quotients of the solution converge to the following 3-periodic cycle

$$\left\{L_0, \frac{a_0L_0 + 1}{L_0}, \frac{a_1(a_0L_0 + 1) + L_0}{a_0L_0 + 1}\right\}, \tag{15}$$

where

$$L_0 = \frac{2(1+a_1a_2)x_0 + (\Delta - 2a_1 + \sqrt{4+\Delta^2})x_1}{2(1+a_0a_1)x_1 + (-\Delta + 2a_1 + \sqrt{4+\Delta^2})x_0}.$$

- Finally, $\Delta < 0$. In this case we obtain a cycle as in (15) with

$$L_0 = \frac{-2(1+a_1a_2)x_0 + (-\Delta + 2a_1 + \sqrt{4+\Delta^2})x_1}{-2(1+a_0a_1)x_1 + (\Delta - 2a_1 + \sqrt{4+\Delta^2})x_0}.$$

In Fig. 2 we have an example of this case. Using a 3-periodic sequence of parameters we plot in the plane $(n, \frac{x_{n+1}}{x_n})$ a 3-periodic cycle for the quotients of the solutions.
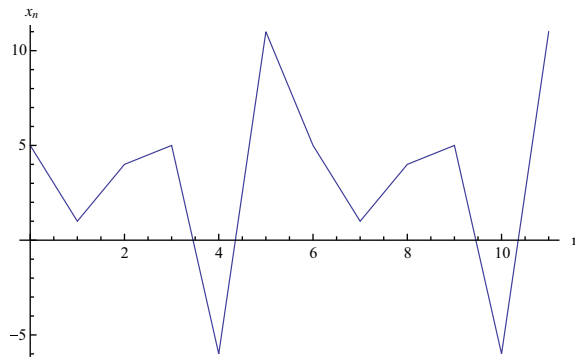

Fig. 1. A 6-periodic solution $(\{5, 1, 4, 5, -6, 11\})$ of a 3-periodic equation. The values of the parameters are $a_0 = -1, a_1 = 1$ and $a_2 = -2$ with initial conditions $x_0 = 5$ and $x_1 = 1$. In this case the eigenvalues of C_3 are 1 and -1 since $trC_k = 0$.

Example 5.2. Let us now consider a 4-periodic equation. The monodromy matrix is given by

$$C_4 = \begin{bmatrix} 1 + a_0 a_1 + a_0 a_3 + a_2 a_3 + a_0 a_1 a_2 a_3 & a_1 + a_3 + a_1 a_2 a_3 \\ a_0 + a_2 + a_0 a_1 a_2 & 1 + a_1 a_2 \end{bmatrix}.$$

1. Consider the case $trC_4 = 2$. This occurs when $a_0 = -\frac{a_2(a_1+a_3)}{a_1+a_3+a_1 a_2 a_3}$. Hence, the monodromy matrix can be simplified as

$$C_4 = \begin{bmatrix} 1 - a_1 a_2 & a_1 + a_3 + a_1 a_2 a_3 \\ -\frac{a_1^2 a_2^2}{a_3 + a_1(1+a_2 a_3)} & 1 + a_1 a_2 \end{bmatrix},$$

yielding the following representation

$$C_4^n = \begin{bmatrix} \frac{a_1+a_3+a_1 a_2 a_3}{a_1 a_2} & \frac{-a_1-a_3-a_1 a_2 a_3}{a_1^2 a_2^2} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -\frac{a_1^2 a_2^2}{a_3 + a_1(1+a_2 a_3)} & a_1 a_2 \end{bmatrix}.$$

It follows that

$$x_{4n} = (1 + n a_1 a_2)x_0 - \frac{a_1^2 a_2^2 n}{a_3 + a_1(1 + a_2 a_3)} x_1,$$

$$x_{4n+1} = (a_1 + a_3 + a_1 a_2 a_3)n x_0 + (1 - a_1 a_2 n)x_1,$$

$$x_{4n+2} = -\frac{a_2(a_1 + a_3)}{a_1 + a_3 + a_1 a_2 a_3} x_{4n+1} + x_{4n},$$

$$x_{4n+3} = a_1 x_{4n+2} + x_{4n+1}.$$

With some computations we see that $\frac{x_{4n+1}}{x_{4n}} \to L_0$, $\frac{x_{4n+2}}{x_{4n+1}} \to L_1 = \frac{a_0 L_0 + 1}{L_0}$, $\frac{x_{4n+3}}{x_{4n+2}} \to L_2 = \frac{a_1 L_1 + 1}{L_1}$ and $\frac{x_{4n+4}}{x_{4n+3}} \to L_3 = \frac{a_2 L_2 + 1}{L_2}$, where

$$L_0 = \frac{(a_1 + a_3 + a_1 a_3)^2 x_0 - a_1 a_2(a_1 + a_3 + a_1 a_3)x_1}{a_1 a_2(a_1 + a_3 + a_1 a_3)x_0 - (a_1 a_2)^2 x_1}.$$

Hence, $\{L_0, L_1, L_2, L_3\}$ is a 4-periodic cycle of the limiting process of the quotients of the solution of the equation. See Fig. 3 for a concrete example.

2. Consider $trC_4 = -2$. This occurs when $a_0 = -\frac{4 + a_2(a_1 + a_3)}{a_1 + a_3 + a_1 a_2 a_3}$. The general solution is given by

$$x_{4n} = (-1)^n \left((1 - (2 + a_1 a_2)n)x_0 + \frac{n(2 + a_1 a_2)^2}{a_3 + a_1(1 + a_2 a_3)} x_1 \right),$$

$$x_{4n+1} = (-1)^n ((1 + (2 + a_1 a_2)n)x_1 - (a_1 + a_3 + a_1 a_2 a_3)n x_0),$$

$$x_{4n+2} = a_0 x_{4n+1} + x_{4n}$$

and

$$x_{4n+3} = a_1 x_{4n+2} + x_{4n+1},$$

for all $n = 0, 1, 2, \ldots$. Hence, $\frac{x_{4n+1}}{x_{4n}} \to L_0$, $\frac{x_{4n+2}}{x_{4n+1}} \to L_1 = \frac{a_0 L_0 + 1}{L_0}$, $\frac{x_{4n+3}}{x_{4n+2}} \to L_2 = \frac{a_1 L_1 + 1}{L_1}$ and $\frac{x_{4n+4}}{x_{4n+3}} \to L_3 = \frac{a_2 L_2 + 1}{L_2}$, where
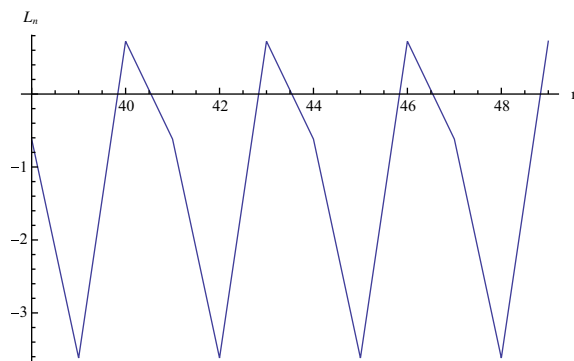

Fig. 2. A 3-periodic cycle $\{-3.61803, 0.723607, -0.618034\}$ for the ratios of the solution of a 3-periodic equation. The values of the parameters are $a_0 = 1, a_1 = -2$ and $a_2 = -2$ with initial conditions $x_0 = 5$ and $x_1 = 1$.

$$L_0 = \frac{(a_1 + a_3 + a_1 a_2 a_3)((2 + a_1 a_2) x_1 - (a_1 + a_3 + a_1 a_2 a_3) x_0)}{(2 + a_1 a_2)^2 x_1 - (2 + a_1 a_2)(a_1 + a_3 + a_1 a_2 a_3) x_0}.$$

With initial conditions $x_0 = 5$ and $x_1 = 1$, for instance, we can find a 4-periodic solution $\{10.625, -0.211765, 0.277775, 1.60003\}$ when $a_0 = -\frac{6}{17}, a_1 = 5, a_2 = -2$ and $a_3 = 10$ and an unbounded cycle $\{1, 0, \infty, -1\}$ when $a_0 = -1, a_1 = 1, a_2 = -1$ and $a_3 = 2$.

3. Consider $|trC_4| \neq 2$. The general solution of the 4-periodic equation is given by

$$x_{4n} = \frac{2(a_0 + a_2 + a_0 a_1 a_2)\left(\left(\Phi_4^+\right)^n - \left(\Phi_4^-\right)^n\right)}{2\sqrt{\Delta^2 - 4}} x_1$$
$$+ \frac{(\Delta - 2 - 2a_1 a_2 + \sqrt{\Delta^2 - 4})(\Phi_4^-)^n - (\Delta - 2 - 2a_1 a_2 - \sqrt{\Delta^2 - 4})(\Phi_4^+)^n}{2\sqrt{\Delta^2 - 4}} x_0,$$

$$x_{4n+1} = \frac{2(a_1 + a_3 + a_1 a_2 a_3)\left(\left(\Phi_4^+\right)^n - \left(\Phi_4^-\right)^n\right)}{2\sqrt{\Delta^2 - 4}} x_0$$
$$+ \frac{(\Delta - 2 - 2a_1 a_2 + \sqrt{\Delta^2 - 4})(\Phi_4^+)^n - (\Delta - 2 - 2a_1 a_2 - \sqrt{\Delta^2 - 4})(\Phi_4^-)^n}{2\sqrt{\Delta^2 - 4}} x_1,$$

$$x_{4n+2} = a_0 x_{4n+1} + x_{4n}$$

and

$$x_{4n+3} = a_1 x_{4n+2} + x_{4n+1},$$

for all $n = 0, 1, 2, \ldots$, where Δ is the trace of C_4, i.e., $\Delta = 2 + a_0 a_1 + a_1 a_2 + a_0 a_3 + a_2 a_3 + a_0 a_1 a_2 a_3$ and $\Phi_4^\pm = \frac{\Delta \pm \sqrt{\Delta^2 - 4}}{2}$.

- Case $|\Delta| > 2$. The quotients of the solution converge to a 4-periodic cycle of the form

$$\left\{ L_0, \frac{a_0 L_0 + 1}{L_0}, a_1 + \frac{L_0}{a_0 L_0 + 1}, a_2 + \frac{a_0 L_0 + 1}{a_1(a_0 L_0 + 1) + L_0} \right\},$$

where $L_0 = \lim \frac{x_{4n+1}}{x_{4n}}$. Notice that this cycle may be unbounded.

- Case $|\Delta| < 2$. Since we have a pair of complex conjugated Floquet multipliers, the solution of the 4-periodic equation is periodic with period $4P, P = \frac{2\pi}{\theta}$ where $\theta = \arctan \frac{\sqrt{4 - \Delta^2}}{\Delta}, \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$. See SubSection 4.2 for the computations of this cycle.

  Let us present a concrete example to illustrate this case. Let us consider a 4-periodic sequence of parameters $a_n = a_{n+4}$, for all $n = 0, 1, 2, \ldots$ such that $a_0 = -1, a_1 = 1, a_2 = -1$ and $a_3 = -1$. With some computations we see that we have $\theta = \arctan \sqrt{3} = \frac{\pi}{3}$ and $P = 6$. Hence the monodromy matrix C_4 is 6-periodic. Consequently, we have a 24-periodic cycle as the solution of this equation. A straightforward computation shows that this cycle is given by

$$\{x_0, x_1, x_0 - x_1, x_0, -x_1, x_0 + x_1, -x_0 - 2x_1, -x_1, -x_0 - x_1, x_0, -2x_0 - x_1, -x_0 - x_1, -x_0, -x_1, -x_0 + x_1, -x_0, x_1, -x_0 - x_1, x_0$$
$$+ 2x_1, x_1, x_0 + x_1, -x_0, 2x_0 + x_1, x_0 + x_1\}.$$

Notice that any cyclic permutation of this last sequence of parameters leads necessarily to a 24-periodic solution.
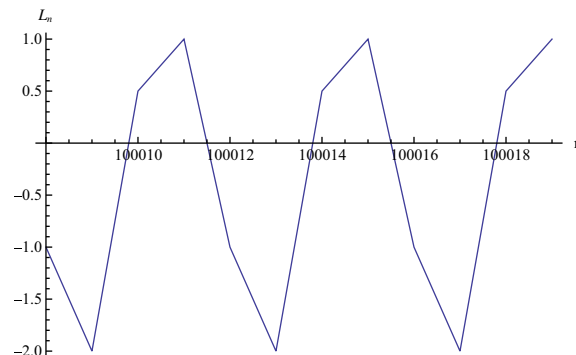

Fig. 3. The 4-periodic cycle corresponding to the limit process of the ratios of the solution of a 4-periodic generalized Fibonacci equation. In this example we consider $a_0 = -1, a_1 = 1, a_2 = -1$ and $a_3 = -2$ with initial conditions $x_0 = 5$ and $x_1 = 1$ originating the cycle $\{-1, -2, 0.5, 1\}$.

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Acknowledgement

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