# CONVERGENCE OF RATIONAL BERNSTEIN OPERATORS 

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#### Abstract

In this paper we discuss convergence properties and error estimates of rational Bernstein operators introduced by P. Piţul and P. Sablonnière. It is shown that the rational Bernstein operators $R_{n}$ converge to the identity operator if and only if $\Delta_{n}$, the maximal difference between two consecutive nodes of $R_{n}$, is converging to zero for $n \rightarrow \infty$. Error estimates in terms of $\Delta_{n}$ are provided. Moreover a Voronovskaja theorem is presented which is based on the explicit computation of higher order moments for the rational Bernstein operator.


## 1. Introduction

Let $C[0,1]$ be the set of all continuous real-valued functions on the interval $[0,1]$. The classical Bernstein operator $B_{n}: C[0,1] \rightarrow C[0,1]$ is defined by

$$
B_{n} f(x)=\sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^{k}(1-x)^{n-k}
$$

see e.g. [11], [14. In [21], P. Piţul and P. Sablonnière introduced rational Bernstein operators which are positive operators of the form

$$
\begin{equation*}
R_{n} f(x):=\sum_{k=0}^{n} f\left(x_{n, k}\right) \bar{w}_{n, k}\binom{n}{k} \frac{x^{k}(1-x)^{n-k}}{Q_{n-1}(x)} \tag{1}
\end{equation*}
$$

where $Q_{n-1}(x)$ is a given strictly positive polynomial over $[0,1]$ of degree $\leq n-1$. Further it is assumed that $Q_{n-1}$ has two additional properties: (i) the Bernstein coefficients $w_{n-1, k}$ in the representation

$$
\begin{equation*}
Q_{n-1}(x)=\sum_{k=0}^{n-1} w_{n-1, k-1}\binom{n-1}{k} x^{k}(1-x)^{n-1-k} \tag{2}
\end{equation*}
$$

are strictly positive and (ii) the sequence $w_{n-1, k}, k=0, \ldots, n-1$ satisfies the inequality

$$
\begin{equation*}
\frac{w_{n-1, k-1} w_{n-1, k+1}}{w_{n-1, k}^{2}}<\left(\frac{k+1}{k}\right)\left(\frac{n-k}{n-k-1}\right) \tag{W}
\end{equation*}
$$

[^0]Key words and phrases: Rational approximants, Bernstein operator, positive operator.
for $k=1, \ldots, n-1$. Then, according to the results in [21], there exist positive weights $\bar{w}_{n, k}, k=0, \ldots, n$, and increasing nodes $0=x_{n, 0}<x_{n, 1}<\ldots<x_{n, n}=1$ such that $R_{n}$ reproduces the constant function $e_{0}(x)=1$ and the linear function $e_{1}(x)=x$, i.e. that

$$
\begin{equation*}
R_{n} e_{j}=e_{j} \text { for } j=0,1 \tag{3}
\end{equation*}
$$

The weights $\bar{w}_{n, k}$ and the nodes $x_{n, k}$ are uniquely defined through the condition (3) and they are given by the formula

$$
\begin{aligned}
x_{n, k} & =\frac{k w_{n, k-1}}{k w_{n, k-1}+(n-k) w_{n, k}} \text { for } k=1, . ., n-1 \\
\bar{w}_{n, k} & =\frac{k}{n} w_{n, k-1}+\left(1-\frac{k}{n}\right) w_{n, k} \text { for } k=1, \ldots, n-1
\end{aligned}
$$

and the conditions $x_{n, 0}=0$ and $x_{n, n}=1$ and $w_{n, 0}=Q_{n-1}(0)$ and $w_{n, n}=Q_{n-1}(1)$. It was shown in [21] that the rational Bernstein operators $R_{n}$ have the same shape preserving properties as the classical Bernstein operator $B_{n}$. Moreover it was proved that $R_{n}$ converges to the identity operator and that a Voronovskaja-type theorem holds under the additional assumption that there exists a positive continuous function $\varphi$ such that

$$
w_{n-1, k}=\varphi\left(\frac{k}{n-1}\right)\binom{n-1}{k} \text { for } k=0, \ldots, n-1
$$

for all natural numbers $n$. The main purpose of this article is to study the convergence of the rational Bernstein operators in the general case. Our main result states that the operators $R_{n}$ converge to the identity operator if and only if

$$
\begin{equation*}
\Delta_{n}=\sup _{k=0, ., n-1}\left|x_{n, k+1}-x_{n, k}\right| \tag{4}
\end{equation*}
$$

converges to 0 . The main innovation in the present article is the computation and estimation of the moments

$$
R_{n}\left(e_{1}-x\right)^{r}(x) \text { and } R_{n}\left(e_{r}\right)(x)-x^{r}
$$

for the rational Bernstein operator $R_{n}$ where $e_{r}(x)=x^{r}$. For example, we shall prove the inequality

$$
\left|R_{n}\left(e_{2}\right)(x)-x^{2}\right| \leq \sup _{0 \leq k \leq n-1}\left|x_{n, k+1}-x_{n, k}\right| \cdot x(1-x)
$$

which implies the convergence of $R_{n}$ to the identity operator provided that $\Delta_{n} \rightarrow 0$. Convergence results and error estimates of O. Shisha and B. Mond for positive operators are used for explicit error estimates. Results of R.G. Mamedov lead to a Voronovskaja theorem for rational Bernstein operators in the general setting. We shall illustrate the results by examples which are of different type as those in [21].

The paper is organized as follows: in the second section we shall recall briefly the basic construction of the rational Bernstein operators as given in [21]. We shall show that
there are many natural examples of rational Bernstein operators: starting with nodes $0=x_{n, 0}<x_{n, 1}<\ldots<x_{n, n-1}<x_{n, n}=1$ and a positive constant $\gamma_{n-1,0}>0$ we define

$$
\gamma_{n-1, k}:=\gamma_{n-1,0} \prod_{l=1}^{k} \frac{1-x_{n, l}}{x_{n, l}}
$$

for $k=1, \ldots, n$ and $Q_{n-1}(x):=\sum_{k=0}^{n-1} \gamma_{n-1, k} x^{k}(1-x)^{n-1-k}$. Then $Q_{n-1}$ satisfies property (W) and

$$
R_{n} f(x)=\sum_{k=0}^{n} f\left(x_{n, k}\right)\left(\gamma_{n-1, k}+\gamma_{n-1, k-1}\right) \frac{x^{k}(1-x)^{n-k}}{Q_{n-1}(x)}
$$

is a rational Bernstein operator $R_{n}$ fixing $e_{0}$ and $e_{1}$. In Section 3 we compute the expressions $R_{n}\left(e_{r}\right)(x)-x^{r}$ explicitly and obtain the above-mentioned criterion for the convergence of $R_{n}$. Section 4 is devoted to error estimates. In Section 5 we prove a Voronovskaja result. In Section 6 we discuss the special case of rational Bernstein operators of [21] and improve some results. Further we present a sequence of rational Bernstein operators $R_{n}$ converging to the identity operator where the polynomials $Q_{n}(x)$ converges pointwise to a discontinuous function. In the final Section 7, we shall comment on links between rational Bernstein operators and general results about Bernstein operators fixing two functions in the framework of extended Chebyshev systems.

By $C^{r}[0,1]$ we shall denote the set of all $r$ times continuously differentiable functions on the unit interval $[0,1]$ and $\mathbb{N}$ will denote the set of all natural numbers.

## 2. Rational Bernstein operators

For convenience of the reader we recall the basic construction of the rational Bernstein operator $R_{n}$ as outlined in [21]. Let $Q_{n-1}$ be a polynomial of degree $\leq n-1$. Instead of the representation (22) it is more convenient to work with

$$
\begin{equation*}
Q_{n-1}(x)=\sum_{k=0}^{n-1} \gamma_{n-1, k} x^{k}(1-x)^{n-1-k} \tag{5}
\end{equation*}
$$

so $\gamma_{n-1, k}=w_{n-1, k}\binom{n-1}{k}$. Since $x^{k}(1-x)^{n-1-k}=x^{k}(1-x)^{n-k}+x^{k+1}(1-x)^{n-1-k}$ we infer that

$$
Q_{n-1}(x)=\sum_{k=0}^{n}\left(\gamma_{n-1, k}+\gamma_{n-1, k-1}\right) x^{k}(1-x)^{n-k}
$$

with the convention that $\gamma_{n-1,-1}=0$ and $\gamma_{n-1, n}=0$. In view of (1) the requirement $R_{n} 1=1$ is then equivalent to

$$
Q_{n-1}(x)=\sum_{k=0}^{n} \bar{w}_{n, k}\binom{n}{k} x^{k}(1-x)^{n-k}
$$

and we conclude that

$$
\bar{w}_{n, k}\binom{n}{k}=\gamma_{n-1, k}+\gamma_{n-1, k-1}
$$

Further we want that $R_{n} e_{1}=e_{1}$ for the linear function $e_{1}(x)=x$ which is equivalent to the identity

$$
\begin{equation*}
x Q_{n-1}(x)=\sum_{k=0}^{n} x_{n, k} \cdot \bar{w}_{n, k}\binom{n}{k} x^{k}(1-x)^{n-k} \tag{6}
\end{equation*}
$$

Inserting $x=0$ implies that that $x_{n, 0}=0$. From the identity

$$
x Q_{n-1}(x)=\sum_{k=0}^{n-1} \gamma_{n-1, k} \cdot x^{k+1}(1-x)^{n-1-k}=\sum_{k=1}^{n} \gamma_{n-1, k-1} x^{k}(1-x)^{n-k}
$$

and (6) we infer that for $k=1, \ldots n$

$$
\begin{equation*}
x_{n, k}=\frac{\gamma_{n-1, k-1}}{\bar{w}_{n, k}}=\frac{\gamma_{n-1, k-1}}{\gamma_{n-1, k}+\gamma_{n-1, k-1}}=\frac{\frac{\gamma_{n-1, k-1}}{\gamma_{n-1, k}}}{1+\frac{\gamma_{n-1, k-1}}{\gamma_{n-1, k}}} . \tag{7}
\end{equation*}
$$

Hence, given the polynomial $Q_{n-1}(x)$, there is at most one choice for the nodes $x_{n, k}$ and the weights $\bar{w}_{n, k}$ such that $R_{n}$ fixes $e_{0}$ and $e_{1}$. However, in general the numbers $x_{n, k}$ defined by (7) are not in the interval $[0,1]$, and they are in general not increasing numbers, for see Example 19 in Section 6. From formula (17) and the fact that $f(x)=x /(1+x)$ is strictly increasing we derive that $x_{n, k}$ is strictly increasing if and only if

$$
\frac{\gamma_{n-1, k-1}}{\gamma_{n-1, k}}=\frac{w_{n-1, k-1}}{w_{n, k}} \frac{k}{n-k} \text { is strictly increasing. }
$$

This is exactly condition (W). The construction of the rational Bernstein operator $R_{n}$ has the disadvantage that one has to check the condition (W) for the Bernstein coefficients of the polynomial $Q_{n-1}$ which in general might be cumbersome.
Example 1. Take $Q_{n-1}(x)=1+x^{2}$, then straightforward calculations show that

$$
\begin{aligned}
\gamma_{n-1, k} & =\binom{n-1}{k}\left(1+\frac{k(k-1)}{(n-1)(n-2)}\right) \\
\gamma_{n-1, k-1}+\gamma_{n-1, k} & =\binom{n}{k} \frac{n(n-1)+k(k-1)}{n(n-1)} \\
x_{n, k} & =\frac{k}{n-2} \frac{(n-1)(n-2)+(k-1)(k-2)}{n(n-1)+k(k-1)}
\end{aligned}
$$

and the rational Bernstein operator $R_{n}$ is given by

$$
R_{n} f(x)=\sum_{k=1}^{n-1}\binom{n}{k} \frac{n(n-1)+k(k-1)}{n(n-1)} f\left(x_{n, k}\right) \frac{x^{k}(1-x)^{n-k}}{1+x^{2}} .
$$

We now change our point of view: instead of starting with the polynomial $Q_{n-1}$ we just start with an increasing sequence

$$
0=x_{n, 0}<x_{n, 1}<\ldots<x_{n, n-1}<x_{n, n}=1
$$

We use equation (7) to define $\frac{\gamma_{n-1, k-1}}{\gamma_{n-1, k}}$. Clearly (7) is equivalent to

$$
x_{n, k}\left(1+\frac{\gamma_{n-1, k-1}}{\gamma_{n-1, k}}\right)=\frac{\gamma_{n-1, k-1}}{\gamma_{n-1, k}} \text { and } \frac{\gamma_{n-1, k-1}}{\gamma_{n-1, k}}=\frac{x_{n, k}}{1-x_{n, k}}
$$

which is a recursion formula for $\gamma_{n-1, k}$ provided we have defined $\gamma_{n-1,0}$. Hence define

$$
\begin{equation*}
\gamma_{n-1, k}:=\gamma_{n-1,0} \prod_{l=1}^{k} \frac{1-x_{n, l}}{x_{n, l}} . \tag{8}
\end{equation*}
$$

These remarks lead to the following statement:
Proposition 2. Let $0=x_{n, 0}<x_{n, 1}<\ldots<x_{n, n-1}<x_{n, n}=1$. Let $\gamma_{n-1,0}>0$ and define $\gamma_{n-1, k}$ by (8) for $k=1, \ldots, n$ and define

$$
Q_{n-1}(x)=\sum_{k=0}^{n-1} \gamma_{n-1, k} x^{k}(1-x)^{n-1-k}
$$

Then $Q_{n-1}$ satisfies property $(W)$ and the operator

$$
R_{n} f(x)=\sum_{k=0}^{n} f\left(x_{n, k}\right)\left(\gamma_{n-1, k}+\gamma_{n-1, k-1}\right) \frac{x^{k}(1-x)^{n-k}}{Q_{n-1}(x)}
$$

is the rational Bernstein operator $R_{n}$ fixing $e_{0}$ and $e_{1}$.
Proof. There is not much to show: by definition of $\gamma_{n-1, k}$ we see that $\frac{\gamma_{n-1, k-1}}{\gamma_{n-1, k}}=\frac{x_{n, k}}{1-x_{n, k}}$. This clearly implies that (7) holds, so the nodes of the operator $R_{n}$ are just the given numbers $x_{n, k}$. Since $x_{n, k}$ 's are increasing we see that $\frac{\gamma_{n-1, k-1}}{\gamma_{n-1, k}}$ are increasing and therefore property (W) holds.

## 3. Convergence of rational Bernstein operators

The following result is of central importance:
Theorem 3. Let $R_{n}$ be the rational Bernstein operator for the polynomial $Q_{n-1}(x)$ of degree $\leq n-1$ satisfying (i) and (ii) in the introduction and let $x_{n, k}$ be defined by (7). Then the following identity

$$
\begin{equation*}
R_{n}\left(e_{s}\right)(x)-x^{s}=\frac{x(1-x)}{Q_{n-1}(x)} \sum_{l=0}^{s-2} x^{l} \sum_{k=0}^{n-1} \gamma_{n-1, k}\left(x_{n, k+1}^{s-1-l}-x_{n, k}^{s-1-l}\right) x^{k}(1-x)^{n-1-k} \tag{9}
\end{equation*}
$$

holds for the polynomial $e_{s}(x)=x^{s}$ and $s \geq 1$.

Proof. At first we note that (7) implies that

$$
\gamma_{n-1, k}+\gamma_{n-1, k-1}=\gamma_{n-1, k-1} \frac{1-x_{n, k}}{x_{n, k}}+\gamma_{n-1, k-1}=\gamma_{n-1, k-1} \frac{1}{x_{n, k}}
$$

It follows that

$$
R_{n} f=f(0) \gamma_{n-1,0} \frac{(1-x)^{n}}{Q_{n-1}(x)}+\sum_{k=1}^{n} f\left(x_{n, k}\right) \frac{\gamma_{n-1, k-1}}{x_{n, k}} \frac{x^{k}(1-x)^{n-k}}{Q_{n-1}(x)}
$$

Let $s \geq 1$ and $e_{s}(x)=x^{s}$. Since $x_{n, 0}=0$ and $e_{s}\left(x_{n, 0}\right)=x_{n, 0}^{s}=0$ we have

$$
\begin{equation*}
Q_{n-1}(x) R_{n}\left(e_{s}\right)(x)=\sum_{k=1}^{n} \gamma_{n-1, k-1} x_{n, k}^{s-1} \cdot x^{k}(1-x)^{n-k} \tag{10}
\end{equation*}
$$

Using an index transformation we arrive at

$$
\begin{equation*}
Q_{n-1}(x) R_{n}\left(e_{s}\right)(x)=x \sum_{k=0}^{n-1} \gamma_{n-1, k} x_{n, k+1}^{s-1} \cdot x^{k}(1-x)^{n-1-k} . \tag{11}
\end{equation*}
$$

Writing $x^{k}(1-x)^{n-1-k}=x^{k}(1-x)^{n-k}+x^{k+1}(1-x)^{n-1-k}$ we obtain

$$
\begin{aligned}
Q_{n-1}(x) R_{n}\left(e_{s}\right)(x)= & x \sum_{k=0}^{n-1} \gamma_{n-1, k} x_{n, k+1}^{s-1} \cdot x^{k}(1-x)^{n-k} \\
& +x \sum_{k=0}^{n-1} \gamma_{n-1, k} x_{n, k+1}^{s-1} \cdot x^{k+1}(1-x)^{n-1-k} .
\end{aligned}
$$

The second sum is equal to $x \sum_{k=1}^{n} \gamma_{n-1, k-1} x_{n, k}^{s-1} \cdot x^{k}(1-x)^{n-k}$. Using the convention $\gamma_{n-1, n}=\gamma_{n-1,-1}=0$ and the fact that $x_{n, 0}=0$ we obtain

$$
Q_{n-1}(x) R_{n}\left(e_{s}\right)(x)=x \sum_{k=0}^{n}\left(\gamma_{n-1, k} x_{n, k+1}^{s-1}+\gamma_{n-1, k-1} x_{n, k}^{s-1}\right) \cdot x^{k}(1-x)^{n-k}
$$

On the other hand, we have the trivial identity

$$
\gamma_{n-1, k}\left(x_{n, k+1}^{s-1}-x_{n, k}^{s-1}\right)=\gamma_{n-1, k}\left(x_{n, k+1}^{s-1}+\frac{x_{n, k}}{1-x_{n, k}} x_{n, k}^{s-2}\left(x_{n, k}-1\right)\right)
$$

and using $\gamma_{n-1, k}=\gamma_{n-1, k-1} \frac{1-x_{n, k}}{x_{n, k}}$

$$
\begin{aligned}
\gamma_{n-1, k}\left(x_{n, k+1}^{s-1}-x_{n, k}^{s-1}\right) & =\gamma_{n-1, k}\left(x_{n, k+1}^{s-1}+\frac{\gamma_{n-1, k-1}}{\gamma_{n-1, k}} x_{n, k}^{s-2}\left(x_{n, k}-1\right)\right) \\
& =\gamma_{n-1, k} x_{n, k+1}^{s-1}+\gamma_{n-1, k-1} x_{n, k}^{s-1}-\gamma_{n-1, k-1} x_{n, k}^{s-2}
\end{aligned}
$$

It follows that

$$
\begin{aligned}
Q_{n-1}(x) R_{n}\left(e_{s}\right)(x)= & x \sum_{k=0}^{n} \gamma_{n-1, k}\left(x_{n, k+1}^{s-1}-x_{n, k}^{s-1}\right) \cdot x^{k}(1-x)^{n-k} \\
& +x \sum_{k=0}^{n} \gamma_{n-1, k-1} x_{n, k}^{s-2} \cdot x^{k}(1-x)^{n-k}
\end{aligned}
$$

As $\gamma_{n-1, n}=0$, the indices in the first sum range only up to $n-1$. The first summand of the second sum is zero. Using (10) for $s-1$ instead of $s$ for the second sum we arrive

$$
\begin{aligned}
Q_{n-1}(x) R_{n}\left(e_{s}\right)(x)= & x(1-x) \sum_{k=0}^{n-1} \gamma_{n-1, k}\left(x_{n, k+1}^{s-1}-x_{n, k}^{s-1}\right) \cdot x^{k}(1-x)^{n-1-k} \\
& +x \cdot Q_{n-1}(x) R_{n}\left(e_{s-1}\right)(x)
\end{aligned}
$$

Now use this formula inductively and recall that $R_{n}\left(e_{1}\right)=e_{1}$ leading to the statement in theorem.

Corollary 4. The rational Bernstein operators $R_{n}$ satisfy the inequality

$$
\left|R_{n}\left(e_{2}\right)(x)-x^{2}\right| \leq \sup _{0 \leq k \leq n-1}\left|x_{n, k+1}-x_{n, k}\right| \cdot x(1-x)
$$

Proof. From Theorem 3 for $s=2$ we see that

$$
\begin{equation*}
R_{n}\left(e_{2}\right)(x)-x^{2}=\frac{x(1-x)}{Q_{n-1}(x)} \sum_{k=0}^{n-1} \gamma_{n-1, k}\left(x_{n, k+1}-x_{n, k}\right) x^{k}(1-x)^{n-1-k} \tag{12}
\end{equation*}
$$

and then the statement is obvious since $\gamma_{n-1, k}$ is positive.
Corollary 5. The rational Bernstein operators $R_{n}$ converges to the identity operator if and only if

$$
\begin{equation*}
\Delta_{n}:=\sup _{0 \leq k \leq n-1}\left|x_{n, k+1}-x_{n, k}\right| \tag{13}
\end{equation*}
$$

converges to 0 .
Proof. If $\Delta_{n}$ converges to zero it follows that $R_{n} e_{2}$ converges uniformly to $e_{2}$ and Korovkin's theorem shows that $R_{n}$ converges to the identity operator. Conversely, suppose that $R_{n}$ converges to the identity operator and suppose that $\Delta_{n}$ does not converge to 0 . Then there exists $\delta>0$ and a subsequence $\left(n_{l}\right)_{l}$ such that $\Delta_{n_{l}} \geq 2 \delta$. Hence for each $l$ there $k_{n, l} \in\left\{0, \ldots, n_{l}-1\right\}$ such that

$$
\begin{equation*}
\left|x_{n_{l}, k_{l}+1}-x_{n_{l}, k_{l}}\right| \geq \delta . \tag{14}
\end{equation*}
$$

Since $x_{n, k} \in[0,1]$ we can pass to a subsequence of $x_{n_{l}, k_{l}}$ which converges to some point $x_{0}$ and we can pass again to a subsequence of the subsequence such that $x_{n_{l_{r}}, k_{l_{r}}}$ converges to $x_{0}$ and $x_{n_{l r}, k_{l_{r}}+1}$ converges to $x_{1}$. From (14) it follows that $\left|x_{1}-x_{0}\right| \geq \delta$, and since
$x_{n_{l}, k_{l}} \leq x_{n_{l}, k_{l}+1}$ we infer that $x_{0} \leq x_{1}$. Now we take a natural number $r_{0}$ such that $\left|x_{0}-x_{n_{l}, k_{l r}}\right|<\delta / 3$ and $\left|x_{1}-x_{n_{l}, k_{l r}+1}\right|<\delta / 3$ for all $r \geq r_{0}$. From the monotonicity of $x_{n, k}$ for $k=0, \ldots, n_{l}-1$ it follows that $x_{n_{l}, k} \notin\left[x_{0}+\delta / 3, x_{1}-\delta / 3\right]$ for all $k=0, \ldots, n_{l_{r}}$ and $l \geq l_{0}$. Now construct a continuous non-zero function $f$ with support in $\left[x_{0}+\delta / 3, x_{1}-\delta / 3\right]$ such that $f(\xi) \neq 0$ for some $\xi \in\left[x_{0}+\delta / 3, x_{1}-\delta / 3\right]$. Then $B_{n_{l}} f(x)=0$ for all $x \in$ $[0,1]$. By assumption, $B_{n l_{r}} f(\xi)$ converges to $f(\xi) \neq 0$. Since $B_{n l_{r}} f(\xi)=0$ we obtain a contradiction completing the proof.

Corollary 6. The following inequality holds for all $x \in[0,1]$ and for all natural numbers $s \geq 2$ :

$$
0 \leq x^{s}<R_{n}\left(e_{s}\right)(x)
$$

Proof. The right hand side in (9) is strictly positive for $x \in[0,1]$ and $s \geq 2$. Alternatively, one may argue that the function $e_{s}$ is convex, and the result follows from the remarks in [21, p. 46].

In the rest of this section we shall prove some inequalities which will be needed in Section 5:

Proposition 7. The following inequality holds

$$
0 \leq R_{n}\left(e_{3}\right)(x)-x^{3} \leq 3 \cdot\left(R_{n}\left(e_{2}\right)(x)-x^{2}\right)
$$

Proof. From Theorem 3 for $s=3$ we see that

$$
R_{n}\left(e_{3}\right)(x)-x^{3}=\frac{x(1-x)}{Q_{n-1}(x)} \sum_{k=0}^{n-1} \gamma_{n-1, k} A_{k} x^{k}(1-x)^{n-1-k}
$$

where

$$
A_{k}=x_{n, k+1}^{2}-x_{n, k}^{2}+x\left(x_{n, k+1}-x_{n, k}\right)=\left(x_{n, k+1}-x_{n, k}\right)\left(x_{n, k+1}+x_{n, k}+x\right) \geq 0 .
$$

Since $0 \leq x_{n, k+1}+x_{n, k}+x \leq 3$ we obtain

$$
0 \leq R_{n}\left(e_{3}\right)(x)-x^{3} \leq 3 \frac{x(1-x)}{Q_{n-1}(x)} \sum_{k=0}^{n-1} \gamma_{n-1, k}\left(x_{n, k+1}-x_{n, k}\right) x^{k}(1-x)^{n-1-k}
$$

and the last expression is equal to $3\left(R_{n}\left(e_{2}\right)(x)-x^{2}\right)$. The proof is complete.
Proposition 8. Let $r$ be a natural number. Then the expression

$$
A:=\frac{x}{Q_{n-1}(x)} \sum_{k=0}^{n-1}\left(x-x_{n, k+1}\right)^{r} \gamma_{n-1, k} x^{k}(1-x)^{n-1-k}
$$

is equal to

$$
B:=\sum_{l=0}^{r}\binom{r}{l} x^{r-l}(-1)^{l}\left[R_{n}\left(e_{l+1}\right)(x)-x^{l+1}\right]
$$

Proof. Since $\left(x-x_{n, k+1}\right)^{r}=\sum_{l=0}^{r}\binom{r}{l} x^{r-l}(-1)^{l} x_{n, k+1}^{l}$ it is easy to see that

$$
A=\sum_{l=0}^{r}\binom{r}{l} x^{r-l}(-1)^{l} \frac{x}{Q_{n-1}(x)} \sum_{k=0}^{n-1} \gamma_{n-1, k} x_{n, k+1}^{l} x^{k}(1-x)^{n-1-k}
$$

Using (11) we see that

$$
A=\sum_{l=0}^{r}\binom{r}{l} x^{r-l}(-1)^{l} R_{n}\left(e_{l+1}\right)
$$

and the result follows from the fact that

$$
\sum_{l=0}^{r}\binom{r}{l} x^{r-l}(-1)^{l} x^{l+1}=x(x+(-x))^{r}=0
$$

For the Bernstein operator $B_{n}$ it is well known that

$$
B_{n} e_{2}(x)-x^{2}=\frac{x(1-x)}{n}
$$

is a polynomial of degree $\leq 2$. For rational Bernstein operators the expression $R_{n} e_{2}(x)-x^{2}$ is a never polynomial except that $Q_{n-1}(x)=1$, the case of the classical Bernstein operator. Indeed, suppose that $R_{n} e_{2}(x)-x^{2}=p_{s}(x)$ for some polynomial $p_{s}(x)$ of degree $s$. Then by (12)

$$
x(1-x) \sum_{k=0}^{n-1} \gamma_{n-1, k}\left(x_{n, k+1}-x_{n, k}\right) x^{k}(1-x)^{n-1-k}=p_{s}(x) Q_{n-1}(x)
$$

which shows that $p_{s}(x) Q_{n-1}(x)$ has degree $\leq n+1$. Hence $s \leq 2$ and clearly $x(1-x)$ must be a factor of $p_{s}(x)$. Hence $p_{s}(x)=A x(1-x)$. By uniqueness of the representation (5) we infer that $\gamma_{n-1, k}\left(x_{n, k+1}-x_{n, k}\right)=A \gamma_{n-1, k}$, so $x_{n, k+1}-x_{n, k}=A$, and we arrive at the classical Bernstein operator.

## 4. Error estimates for rational Bernstein operators

Next we derive quantitative convergence results for $R_{n}$. By estimates of O. Shisha and B. Mond (we refer to Theorem 8.1 in [21]) we conclude that

$$
\begin{equation*}
\left|R_{n} f(x)-f(x)\right| \leq\left(1+\frac{1}{h} \sqrt{R_{n}\left(e_{1}-x\right)^{2}(x)}\right) \omega_{1}(f, h) \tag{15}
\end{equation*}
$$

for all $f \in C[0,1]$ and $h>0$ where $\omega_{1}(f, h)$ is the first modulus of continuity defined by

$$
\omega_{1}(f, h)=\sup _{|x-y| \leq h}|f(x)-f(y)| .
$$

Since

$$
R_{n}\left(e_{1}-x\right)^{2}(x)=R_{n}\left(e_{2}\right)(x)-2 x R_{n} e_{1}(x)+x^{2}=R_{n} e_{2}(x)-x^{2}
$$

we obtain from (15) for $h:=\sqrt{\Delta_{n}}$, defined in (13), and from Corollary 4 the following result:

Theorem 9. The rational Bernstein operators $R_{n}$ satisfies the following inequality:

$$
\begin{equation*}
\left|R_{n} f(x)-f(x)\right| \leq(1+\sqrt{x(1-x)}) \omega_{1}\left(f, \sqrt{\Delta_{n}}\right) \tag{16}
\end{equation*}
$$

for all $f \in C[0,1]$.
Similarly, Theorem 8.2 in [21] provides us with the estimate

$$
\left|R_{n} f(x)-f(x)\right| \leq\left(1+\frac{1}{2 h^{2}} R_{n}\left(e_{1}-x\right)^{2}(x)\right) \omega_{2}(f, h)
$$

for all $f \in C[0,1]$ and $h>0$ where $\omega_{2}(f, h)$ is the second modulus of continuity defined by

$$
\omega_{2}(f, h)=\sup _{|\delta| \leq h}\{|f(x+\delta)-2 f(x)+f(x-\delta)|: x \pm h \in[a, b]\}
$$

Taking $h=\sqrt{\Delta_{n}}$ we obtain
Theorem 10. The rational Bernstein operators $R_{n}$ satisfy the following inequality

$$
\begin{equation*}
\left|R_{n} f(x)-f(x)\right| \leq\left(1+\frac{1}{2} x(1-x)\right) \omega_{2}\left(f, \sqrt{\Delta_{n}}\right) \tag{17}
\end{equation*}
$$

for all $f \in C[0,1]$.

## 5. Voronovskaja's Theorem

The classical Voronovskaja theorem states the following:
Theorem 11. Let $f:[0,1] \rightarrow \mathbb{R}$ be bounded and differentiable in a neighborhood of $x$ and has second derivative $f^{\prime \prime}(x)$. Then

$$
\lim _{n \rightarrow \infty} n \cdot\left(R_{n} f(x)-f(x)\right)=\frac{x(1-x)}{2} f^{\prime \prime}(x)
$$

We shall need the following generalization due to R.G. Mamedov [16], see also [13] and [22] for quantitative estimates and higher order of differentiability.
Theorem 12. Let $f \in C^{2}[0,1]$ and $L_{n}: C[0,1] \rightarrow C[0,1]$ be a sequence of positive operators such that $L_{n} e_{j}=e_{j}$ for $j=0,1$ and

$$
\lim _{n \rightarrow \infty} \frac{L_{n}\left(e_{1}-x\right)^{4}(x)}{L_{n}\left(e_{1}-x\right)^{2}(x)}=0
$$

for each $x \in[0,1]$. Then

$$
\frac{L_{n} f(x)-f(x)}{L_{n}\left(e_{1}-x\right)^{2}(x)} \rightarrow \frac{1}{2} f^{\prime \prime}(x)
$$

when $n \rightarrow \infty$.

The classical proof of the Voronovskaja theorem requires the computation of the moments of order $r$ of the Bernstein operator $B_{n}$ :

$$
B_{n}\left[\left(e_{1}-x\right)^{r}\right](x)=\sum_{k=0}\left(\frac{k}{n}-x\right)^{r}\binom{n}{k} x^{k}(1-x)^{n-k}=: \frac{1}{n^{r}} T_{n, r}(x) .
$$

It is well known that $T_{n, r}(x)$ is a polynomial of degree $r$ in the variable $x$ and one can determine $T_{n, r}(x)$ recursively by the formula

$$
T_{n, r+1}(x)=x(1-x)\left[T_{n, r}^{\prime}(x)+n s T_{n, r-1}(x)\right]
$$

see [14]. From this it is not difficult to show that for each $r \in \mathbb{N}$ there exists a constant $A_{r}>0$ such that

$$
\begin{equation*}
B_{n}\left[\left(e_{1}-x\right)^{r}\right](x) \leq \sqrt{A_{r}} \frac{1}{\sqrt{n}^{r}}, \tag{18}
\end{equation*}
$$

see e.g. [22]. In passing we mention that in the recent article [12] the following inequality was established: for $r \in \mathbb{N}$ there exists a constant $K_{r}>0$ such that

$$
B_{n}\left[\left(e_{1}-x\right)^{r+1}\right](x) \leq \frac{K_{r}}{\sqrt{n}} B_{n}\left[\left(e_{1}-x\right)^{r}\right](x)
$$

which clearly implies (18).
In the case of the rational Bernstein operator the moments $R_{n}\left[\left(e_{1}-x\right)^{r}\right](x)$ are not polynomials in the variable $x$ as we have seen already at the end of Section 4 for $r=2$. Nonetheless, we can compute them explicitly but the formulae are much more complicated. Indeed, if we use the binomial theorem for $\left(e_{1}-x\right)^{r}$ we obtain

$$
R_{n}\left[\left(e_{1}-x\right)^{r}\right](x)=\sum_{s=0}^{r}\binom{r}{s}(-x)^{r-s} R_{n}\left(e_{s}\right)(x)
$$

and since $0=(x-x)^{r}=\sum_{s=0}^{r}\binom{r}{s}(-x)^{r-s} x^{s}$ we have

$$
\begin{equation*}
R_{n}\left[\left(e_{1}-x\right)^{r}\right](x)=\sum_{s=2}^{r}\binom{r}{s}(-x)^{r-s}\left[R_{n}\left(e_{s}\right)(x)-x^{s}\right] \tag{19}
\end{equation*}
$$

where we used the fact that $R_{n}\left(e_{s}\right)=e_{s}$ for $s=0,1$. Theorem 3 provides then an explicit formula for the moments. But in view of Theorem 12 we have to estimate

$$
\frac{R_{n}\left(e_{1}-x\right)^{4}(x)}{R_{n}\left(e_{1}-x\right)^{2}(x)}
$$

and it is therefore not sufficient just to estimate the moments.
Theorem 13. The fourth moment satisfies the following inequality;

$$
R_{n}\left(e_{1}-x\right)^{4}(x) \leq \Delta_{n} \cdot\left[R_{n}\left(e_{1}-x\right)^{2}(x)\right]\left(6 x^{2}-15 x+12+\Delta_{n}\right) .
$$

Proof. Formula (19) shows that $R_{n}\left(e_{1}-x\right)^{4}(x)$ is equal to

$$
\left(R_{n}\left(e_{4}\right)(x)-x^{4}\right)-4 x\left(R_{n}\left(e_{3}\right)(x)-x^{3}\right)+6 x^{2}\left(R_{n}\left(e_{2}\right)(x)-x^{2}\right) .
$$

By Theorem 3 we can calculate each summand explicitly and we obtain

$$
\begin{equation*}
R_{n}\left(e_{1}-x\right)^{4}(x)=\frac{x(1-x)}{Q_{n-1}(x)} \sum_{k=0}^{n-1} \gamma_{n-1, k} x^{k}(1-x)^{n-1-k} \cdot H_{k} \tag{20}
\end{equation*}
$$

with

$$
\begin{aligned}
H_{k}= & x_{n, k+1}^{3}-x_{n, k}^{3}+x\left(x_{n, k+1}^{2}-x_{n, k}^{2}\right)+x^{2}\left(x_{n, k+1}-x_{n, k}\right) \\
& -4 x\left(x_{n, k+1}^{2}-x_{n, k}^{2}+x\left(x_{n, k+1}-x_{n, k}\right)\right)+6 x^{2}\left(x_{n, k+1}-x_{n, k}\right)
\end{aligned}
$$

which simplifies to

$$
H_{k}=\left(x_{n, k+1}^{3}-x_{n, k}^{3}\right)-3 x\left(x_{n, k+1}^{2}-x_{n, k}^{2}\right)+3 x^{2}\left(x_{n, k+1}-x_{n, k}\right) .
$$

We write $H_{k}=\left(x_{n, k+1}-x_{n, k}\right) A_{k}$ with

$$
A_{k}=x_{n, k+1}^{2}+x_{n, k+1} x_{n, k}+x_{n, k}^{2}-3 x\left(x_{n, k+1}+x_{n, k}\right)+3 x^{2} .
$$

A straightforward calculation shows that

$$
A_{k}=3\left(x-\frac{1}{2}\left(x_{n, k+1}+x_{n, k}\right)\right)^{2}+\frac{1}{4}\left(x_{n, k+1}-x_{n, k}\right)^{2} \geq 0 .
$$

Hence $A_{k}$ is positive and and it is easy to see that

$$
\begin{equation*}
R_{n}\left(e_{1}-x\right)^{4}(x) \leq \Delta_{n} \frac{x(1-x)}{Q_{n-1}(x)} \sum_{k=0}^{n-1} \gamma_{n-1, k} x^{k}(1-x)^{n-1-k} \cdot A_{k} \tag{21}
\end{equation*}
$$

We write now

$$
\begin{equation*}
A_{k}=3\left(x-x_{n, k+1}\right)^{2}+3\left(x-x_{n, k+1}\right)\left(x_{n, k+1}-x_{n, k}\right)+\left(x_{n, k+1}-x_{n, k}\right)^{2} . \tag{22}
\end{equation*}
$$

Proposition 8 applied to the case $r=2$ and 7 show that

$$
\begin{aligned}
& \frac{x}{Q_{n-1}(x)} \sum_{k=0}^{n-1} \gamma_{n-1, k}\left(x-x_{n, k+1}\right)^{2} x^{k}(1-x)^{n-1-k} \\
= & R_{n}\left(e_{3}\right)(x)-x^{3}-2 x\left[R_{n}\left(e_{2}\right)(x)-x^{2}\right] \leq(3-2 x)\left[R_{n}\left(e_{2}\right)(x)-x^{2}\right] .
\end{aligned}
$$

Formula (21) and (22) in connection with the last inequality and the simple estimates $\left|x-x_{n, k+1}\right| \leq 1$ and $x_{n, k+1}-x_{n, k} \leq \Delta_{n}$ lead to

$$
\begin{aligned}
R_{n}\left(e_{1}-x\right)^{4}(x) \leq & \Delta_{n}(1-x) \cdot 3(3-2 x) \cdot\left[R_{n}\left(e_{1}\right)(x)-x^{2}\right] \\
& +\Delta_{n}\left(3+\Delta_{n}\right) \frac{x(1-x)}{Q_{n-1}(x)} \sum_{k=0}^{n-1} \gamma_{n-1, k}\left(x_{n, k+1}-x_{n, k}\right) x^{k}(1-x)^{n-1-k}
\end{aligned}
$$

It follows that

$$
R_{n}\left(e_{1}-x\right)^{4}(x) \leq \Delta_{n}\left[R_{n}\left(e_{2}\right)(x)-x^{2}\right]\left((1-x)(9-6 x)+3+\Delta_{n}\right)
$$

and the statement is now obvious since $R_{n}\left(e_{2}\right)(x)-x^{2}=R\left(e_{1}-x\right)^{2}(x)$.
Using Theorem 12 and Theorem 13 we obtain
Theorem 14. Let $f \in C^{2}[0,1]$ and assume that $\Delta_{n} \rightarrow 0$ for the rational Bernstein operators $R_{n}: C[0,1] \rightarrow C[0,1]$. Then

$$
\frac{R_{n} f(x)-f(x)}{R_{n}\left(e_{1}-x\right)^{2}(x)} \rightarrow \frac{1}{2} f^{\prime \prime}(x) .
$$

## 6. Special classes of rational Bernstein operators

In [21] error estimates and convergence results have been given for rational Bernstein operators $R_{n}$ under the assumption that there exists a positive function $\varphi \in C[0,1]$ such that

$$
Q_{n-1}(x):=B_{n-1} \varphi(x)=\sum_{k=0}^{n-1} \varphi\left(\frac{k}{n-1}\right)\binom{n-1}{k} x^{k}(1-x)^{n-1-k}
$$

where $B_{n-1}$ is the classical Bernstein operator of degree $n-1$. Then $Q_{n-1}$ has clearly positive Bernstein coefficients but in general one has to assume in addition that property (W) is satisfied.

It is shown in [21, p. 42] that property (W) is satisfied provided that $n$ is sufficiently large and $\varphi \in C^{2}[0,1]$. Later in this section we shall show that it suffices to assume only that $\varphi \in C^{1}[0,1]$, and we shall show by example that the result is not true for a Lipschitz function. Now we cite from [21] the following result:

Theorem 15. Suppose that $\varphi \in C[0,1]$ such that $Q_{n-1}(x)=B_{n-1} \varphi(x)$ satisfies property (W). Then

$$
\left|R_{n} f(x)-f(x)\right| \leq\left(1+\frac{1}{2} \sqrt{\frac{\max _{x \in[0,1]} \varphi(x)}{\min _{x \in[0,1]} \varphi(x)}}\right) \omega_{1}\left(f, \frac{1}{\sqrt{n}}+\frac{1}{2 m} \omega_{1}\left(\varphi, \frac{1}{n-1}\right)\right)
$$

We want to show that Theorem 15 can be derived and improved from our previous results. Indeed we want to show:

Theorem 16. Suppose that $\varphi \in C[0,1]$ such that $Q_{n-1}(x)=B_{n-1} \varphi(x)$ satisfies property (W). Then

$$
\left|R_{n} f(x)-f(x)\right| \leq(1+\sqrt{x(1-x)}) \omega_{1}\left(f, \frac{1}{\sqrt{n}}+\frac{1}{2 m} \omega_{1}\left(\varphi, \frac{1}{n-1}\right)\right)
$$

Obviously the result is better since $\sqrt{x(1-x)} \leq 1 / 2$ and $\min _{x \in[0,1]} \varphi(x) \leq \max _{x \in[0,1]} \varphi(x)$. We need the following result which is implicitly contained in [21]:

Proposition 17. Let $\varphi \in C[0,1]$ positive and $Q_{n-1}(x)=B_{n-1} \varphi(x)=\sum_{k=0}^{n-1} \gamma_{n-1, k} x^{k}$ with $\gamma_{n-1, k}=\varphi(k /(n-1))\binom{n-1}{k}$. If one defines

$$
x_{n, k}:=\frac{\gamma_{n-1, k-1}}{\gamma_{n-1, k-1}+\gamma_{n-1, k}}=\frac{k \varphi\left(\frac{k-1}{n-1}\right)}{k \varphi\left(\frac{k-1}{n-1}\right)+(n-k) \varphi\left(\frac{k}{n-1}\right)}
$$

then

$$
\begin{equation*}
\Delta_{n}=\sup _{k=0, \ldots \ldots n-1}\left|x_{n, k+1}-x_{n, k}\right| \leq \frac{1}{2 m} \omega_{1}\left(\varphi, \frac{1}{n-1}\right)+\frac{1}{n}, \tag{23}
\end{equation*}
$$

where $m=\min _{x \in[0,1]} \varphi(x)$.
Proof. Define

$$
\psi_{h}(x)=\frac{x \varphi(x-h)}{x \varphi(x-h)+(1-x+h) \varphi(x)} .
$$

Put $h=1 /(n-1)$ and $x=k /(n-1)$ then

$$
\begin{equation*}
x_{n, k}=\psi_{\frac{1}{n-1}}\left(\frac{k}{n-1}\right) . \tag{24}
\end{equation*}
$$

Similarly,

$$
\begin{equation*}
\frac{k}{n}=\tau_{\frac{1}{n-1}}\left(\frac{k}{n-1}\right) \text { for } \tau_{h}(x)=\frac{x}{1+h} . \tag{25}
\end{equation*}
$$

We want to estimate $x_{n, k}-\frac{k}{n}$ and therefore we look at

$$
\begin{aligned}
\psi_{h}(x)-\frac{x}{1+h} & =x \frac{(1+h) \varphi(x-h)-x \varphi(x-h)-(1-x+h) \varphi(x)}{(1+h)(x \varphi(x-h)+(1-x+h) \varphi(x))} \\
& =\frac{x \cdot(1-x+h) \cdot(\varphi(x-h)-\varphi(x))}{(1+h) \cdot(x \varphi(x-h)+(1-x+h) \varphi(x))}
\end{aligned}
$$

Further we can estimate with $m:=\min _{y \in[0,1]} \varphi(y)$

$$
x \varphi(x-h)+(1-x+h) \varphi(x) \geq(1+h) m
$$

and we obtain that

$$
\left|\psi_{h}(x)-\frac{x}{1+h}\right| \leq \frac{x(1-x+h)}{(1+h)^{2} m} \omega_{1}(\varphi, h) \leq \frac{1}{4 m} \omega_{1}(\varphi, h) .
$$

where we used that $4 x(1-x+h) \leq(1+h)^{2}$ for all $x \in[0,1]$ and $h>0$. Using (24) and (25) it follows that for all $k=0, \ldots . n$ and all $n$ the following inequality

$$
\left|x_{n, k}-\frac{k}{n}\right| \leq \frac{1}{4 m} \omega_{1}\left(\varphi, \frac{1}{n-1}\right)
$$

holds. Since $x_{n, k+1}-x_{n, k}=x_{n, k+1}-\frac{k+1}{n}+\frac{1}{n}+\frac{k}{n}-x_{n, k}$ we can estimate

$$
\begin{equation*}
\left|x_{n, k+1}-x_{n, k}\right| \leq \frac{1}{2 m} \omega_{1}\left(\varphi, \frac{1}{n-1}\right)+\frac{1}{n} . \tag{26}
\end{equation*}
$$

Proof of Theorem 16: Formula (23), the inequality $\sqrt{a+b} \leq \sqrt{a}+\sqrt{b}$ for positive numbers $a, b$, and (26) imply that

$$
\sqrt{\Delta_{n}} \leq \frac{1}{\sqrt{n}}+\sqrt{\frac{\omega_{1}\left(\varphi, \frac{1}{n-1}\right)}{2 m}}=\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{2 m}} \sup _{|x-y| \leq \frac{1}{n-1}} \sqrt{|\varphi(x)-\varphi(y)|}
$$

Further

$$
\sup _{|x-y| \leq \frac{1}{n-1}} \sqrt{|\varphi(x)-\varphi(y)|}=\sup _{|x-y| \leq \frac{1}{n-1}} \frac{|\varphi(x)-\varphi(y)|}{\sqrt{|\varphi(x)+\varphi(y)|}} \leq \frac{\omega_{1}\left(\varphi, \frac{1}{n-1}\right)}{\sqrt{2 m}} .
$$

and $\sqrt{\Delta_{n}} \leq \frac{1}{\sqrt{n}}+\frac{1}{2 m} \omega_{1}\left(\varphi, \frac{1}{n-1}\right)$. Further (16) and the trivial estimate $\omega_{1}(f, \delta) \leq \omega_{1}\left(f, \delta^{\prime}\right)$ for $\delta \leq \delta^{\prime}$ leads to

$$
\left|R_{n} f(x)-f(x)\right| \leq(1+\sqrt{x(1-x)}) \omega_{1}\left(f, \frac{1}{\sqrt{n}}+\frac{1}{2 m} \omega_{1}\left(\varphi, \frac{1}{n-1}\right)\right)
$$

which is the above estimate.
Finally we shall prove:
Theorem 18. Let $\varphi \in C[0,1]$ be strictly positive. If $\varphi \in C^{1}[0,1]$ then $Q_{n-1}(x):=$ $B_{n-1} \varphi(x)$ satisfies property (W) for sufficiently large $n \in \mathbb{N}$. If $\varphi$ is Lipschitz continuous then $a+\varphi$ satisfies property ( $W$ ) for sufficiently large $n \in \mathbb{N}$ and sufficiently large $a>0$.

Proof. We use the notations from the proof of Proposition 17, In view of (24) it suffices to show that the function $x \longmapsto \psi_{h}(x)$ is increasing if $h>0$ is sufficiently small, or equivalently, that for $\delta>0$ and $h>0$ sufficiently small and for all $x \in[0,1]$ the inequality

$$
\begin{equation*}
\psi_{h}(x)=\frac{x \varphi(x-h)}{c_{h}(\varphi)(x)}<\psi_{h}(x+\delta)=\frac{(x+\delta) \varphi(x+\delta-h)}{c_{h}(\varphi)(x+\delta)} \tag{27}
\end{equation*}
$$

holds where

$$
c_{h}(\varphi)(x):=x \varphi(x-h)+(1-x+h) \varphi(x) .
$$

Note that $c_{h}(\varphi)(x)$ converges to $\varphi(x)$ uniformly in $x$ when $h$ tends to zero. Inequality (27) means that

$$
D(x, h, \delta):=x \varphi(x-h) c_{h}(\varphi)(x+\delta)-x \varphi(x+\delta-h) c_{h}(\varphi)(x)
$$

satisfies the inequality

$$
\begin{equation*}
D(x, h, \delta)<\delta \varphi(x+\delta-h) c_{h}(\varphi)(x) \tag{28}
\end{equation*}
$$

By inserting and subtracting $x \varphi(x-h) c_{h}(\varphi)(x)$ we conclude that

$$
\begin{aligned}
\frac{D(x, h, \delta)}{\delta}= & x \varphi(x-h) \frac{c_{h}(\varphi)(x+\delta)-c_{h}(\varphi)(x)}{\delta} \\
& +x c_{h}(\varphi)(x) \frac{\varphi(x-h)-\varphi(x+\delta-h)}{\delta}
\end{aligned}
$$

If $\varphi \in C^{1}[0,1]$ we can find $\xi_{x, h, \delta} \in[x-h, x-h+\delta]$ and $\eta_{x, h, \delta} \in[x, x+\delta]$ with

$$
\begin{aligned}
\varphi(x-h)-\varphi(x+\delta-h) & =\varphi^{\prime}\left(\xi_{x, h, \delta}\right) \cdot \delta \\
c_{h}(\varphi)(x+\delta)-c_{h}(\varphi)(x) & =c_{h}(\varphi)^{\prime}\left(\eta_{x, h, \delta}\right) \cdot \delta
\end{aligned}
$$

It follows that

$$
\frac{D(x, h, \delta)}{\delta}=x \varphi(x-h) c_{h}(\varphi)^{\prime}\left(\eta_{x, h, \delta}\right)-x \cdot c_{h}(\varphi)(x) \varphi^{\prime}\left(\xi_{x, h, \delta}\right)
$$

In order to show (28) we note that $c_{h}(\varphi)(x)$ converges to $\varphi(x)$, and $c_{h}(\varphi)^{\prime}(x)$ converges to $\varphi^{\prime}(x)$ for $h \rightarrow 0$. Hence $D(x, h, \delta) / \delta$ converges to 0 for $h \rightarrow 0$ and $\delta \rightarrow 0$, and (28) holds since

$$
\frac{D(x, h, \delta)}{\delta}<\frac{1}{2} m^{2} \leq \frac{m}{2} \varphi(x)^{2} \leq \varphi(x+\delta-h) c_{h}(\varphi)(x)
$$

for $m:=\min _{x \in[0,1]} \varphi(x)$ and $h$ sufficiently small.
Now assume that $\varphi$ is only Lipschitz continuous. Clearly $c_{h}(\varphi)$ is Lipschitz continuous and there exist $M>0$ and $N>0$ such that

$$
\begin{aligned}
|\varphi(x-h)-\varphi(x+\delta-h)| & \leq M \delta \\
\left|c_{h}(\varphi)(x+\delta)-c_{h}(\varphi)(x)\right| & \leq N \delta
\end{aligned}
$$

where $N$ does not depend on $h$. It follows that $|D(x, h, \delta)| / \delta$ is bounded for all $x \in[0,1]$ and $h>0$ and $\delta>0$. If we replace now $\varphi$ by $a+\varphi$ we see that

$$
\begin{aligned}
c_{h}(a+\varphi)(x) & =x[a+\varphi(x-h)]+(1-x+h)[a+\varphi(x)] \\
& =a(1+h)+c_{h}(\varphi)(x) .
\end{aligned}
$$

Then

$$
\begin{aligned}
D(x, h, \delta, a+\varphi)= & x(a+\varphi(x-h)) \cdot\left(a(1+h)+c_{h}(\varphi)\right)(x+\delta) \\
& -x(a+\varphi(x+\delta-h))\left(a(1+h)+c_{h}(\varphi)\right)(x)
\end{aligned}
$$

can be simplified to

$$
\begin{aligned}
D(x, h, \delta, a+\varphi)= & D(x, h, \delta, \varphi)+a x\left[c_{h}(\varphi)(x+\delta)-c_{h}(\varphi)(x)\right] \\
& +x a(1+h)[\varphi(x-h)-\varphi(x+\delta-h)]
\end{aligned}
$$

On the other hand

$$
(a+\varphi(x+\delta-h)) c_{h}(a+\varphi)(x) \geq a^{2}(1+h)
$$

and by taking $a>0$ sufficiently large we obtain the desired inequality.

We shall give an example of a positive function $\varphi \in C[0,1]$ such that $Q_{2 n}(x)=B_{2 n} \varphi$ does not satisfy property (W):

Example 19. Let $\varphi_{a}(x)=a+\left|x-\frac{1}{2}\right|$ for $a>0$. Then

$$
Q_{2 n, a}(x):=B_{2 n} \varphi_{a}(x)=\sum_{k=0}^{2 n}\left(a+\left|\frac{k}{2 n}-\frac{1}{2}\right|\right)\binom{2 n}{k} x^{k}(1-x)^{2 n-k}
$$

has strictly positive Bernstein coefficients, and it satisfies property (W) if and only if $a>\frac{1}{2}$.
Proof. It follows that

It follows that

$$
\frac{\gamma_{2 n-1, n-1}}{\gamma_{2 n-1, n}}=\frac{n a+\frac{1}{2}}{(n+1) a} \text { and } \frac{\gamma_{2 n-1, n}}{\gamma_{2 n-1, n+1}}=\frac{(n+1) a}{n a+\frac{1}{2}} .
$$

If $\frac{\gamma_{2 n-1, k-1}}{\gamma_{2 n-1, k}}$ is increasing then necessarily $\frac{\gamma_{2 n-1, n-1}}{\gamma_{2 n-1, n}}<1$ and this implies that $n a+\frac{1}{2}<$ $(n+1) a$, which means that $\frac{1}{2}<a$. Conversely, this condition implies that $\frac{\gamma_{2 n-1, n-1}}{\gamma_{2 n-1, n}}<$ $\frac{\gamma_{2 n-1, n}}{\gamma_{2 n-1, n+1}}$. It is not difficult to see that the coefficients are increasing.

Next we want to show by example that the positive polynomials $Q_{n-1}(x)$ may not converge in general to a continuous function even if the Bernstein operators $R_{n}$ converge to the identity. In particular there does not exists in this case a continuous function $\varphi$ with $Q_{n-1}=B_{n-1} \varphi$ for all $n \in \mathbb{N}$.

Example 20. The rational Bernstein operator $R_{n}$ associated to the nodes $x_{n, k}=\sqrt{\frac{k}{n}}$ for $k=0, \ldots, n$ converges to the identity operator but the associated polynomials $Q_{n-1}(x)$ defined by

$$
Q_{n-1}(x)=(1-x)^{n}+\sum_{k=1}^{n-1}\binom{n-1}{k} \prod_{l=1}^{k} \frac{\sqrt{\frac{l}{n}}}{1+\sqrt{\frac{l}{n}}} x^{k}(1-x)^{n-1-k}
$$

do not converge to a continuous function, in particular $Q_{n-1}$ is not equal to $B_{n-1} \varphi$ for some continuous function $\varphi \in C[0,1]$.
Proof. Clearly $1 / \sqrt{n} \leq\left|x_{n, 1}-x_{n, 0}\right| \leq \Delta_{n}$ and

$$
\left|x_{n, k+1}-x_{n, k}\right|=\sqrt{\frac{k+1}{n}}-\sqrt{\frac{k}{n}}=\frac{\frac{k+1}{n}-\frac{k}{n}}{\sqrt{\frac{k+1}{n}}+\sqrt{\frac{k}{n}}} \leq \frac{1}{\sqrt{n}} \frac{1}{\sqrt{k+1}}
$$

Next we consider for $l=1, . ., n-1$

$$
\frac{1-x_{n, l}}{x_{n, l}}=\frac{1-x_{n, l}^{2}}{x_{n, l}\left(1+x_{n, l}\right)}=\frac{1-\frac{l}{n}}{\sqrt{\frac{l}{n}}} \frac{1}{1+\sqrt{\frac{l}{n}}}=\frac{n-l}{l} \frac{\sqrt{l}}{\sqrt{n}+\sqrt{l}} .
$$

Since $2 \sqrt{l} \leq \sqrt{n}+\sqrt{l}$ we can estimate the last factor by $1 / 2$. It follows that

$$
\gamma_{n-1, k}=\prod_{l=1}^{k} \frac{1-x_{n, l}}{x_{n, l}} \leq\binom{ n-1}{k} \frac{1}{2^{k}}
$$

and

$$
Q_{n-1}(x) \leq \sum_{k=0}^{n-1}\binom{n-1}{k} \frac{1}{2^{k}} x^{k}(1-x)^{n-1-k}=\left(1-\frac{x}{2}\right)^{n} .
$$

Then $Q_{n-1}(x)$ converges to 0 for $0<x \leq 1$ but $Q_{n-1}(0)=1$ for all $n$.

## 7. Final Comments

We want to comment on rational Bernstein operators $R_{n}$ from a different point of view: Given a strictly positive polynomial $Q_{n-1}(x)$ we consider the space

$$
E_{n}=\left\{\frac{p(x)}{Q_{n-1}(x)}: p(x) \text { is a polynomial of degree } \leq n\right\} .
$$

Then $E_{n}$ is an extended Chebyshev space over any interval $[a, b]$, meaning that each nonzero function $f \in E_{n}$ has at most $n$ zeros (including multiplicities) in $[a, b]$. We call a system of functions $P_{n, k}, k=0, \ldots, n$ in an $n+1$ dimensional linear space $E_{n}$ of $C^{n}[a, b]$ a Bernstein basis, if each $P_{n, k}$ has exactly $k$ zeros in $a$ and $n-k$ zeros in $b$. Thus the system of functions

$$
\frac{x^{k}(1-x)^{n-k}}{Q_{n}(x)}, k=0, \ldots, n-1
$$

is a Bernstein basis in $E_{n}$ for $[0,1]$. Bernstein bases in extended Chebyshev spaces have been studied by many authors, see [6], [7], [8], [9], ,10], [17], [18].

Recently, Bernstein operators for an extended Chebyshev space $E_{n}$ of dimension $n+1$ have been introduced by J. M. Aldaz, O. Kounchev and the author which by definition are operators of the form

$$
S_{n} f(x)=\sum_{k=0}^{n} f\left(x_{n, k}\right) \alpha_{n, k} p_{n, k}(x)
$$

where $p_{n, k}(x), k=0, \ldots, n$, is a Bernstein basis for $E_{n}$. The nodes $x_{n, k}$ and the weights $\alpha_{n, k}$ are chosen such that $S_{n} f_{0}=f_{0}$ and $S_{n} f_{1}=f_{1}$ where $f_{0}$ is a strictly positive function in $E_{n}$ and $f_{1} \in E_{n}$ has the property that $f_{1} / f_{0}$ is strictly increasing. We refer to [1], [2], [3], 4] and [19] for a systematic study (existence of Bernstein operators fixing two functions and shape preserving properties) and to [20] for a discussion of Schoenberg-type
operators in the setting of extended Chebyshev space. It seems to be a difficult task to establish convergence results of Bernstein operators in the setting of extended Chebyshev spaces, and the rational Bernstein operators considered here seems to be the simplest non-trivial example beyond the classical case of Bernstein operators.

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