

SOME IDENTITIES OF SYMMETRY FOR THE GENERALIZED q -EULER POLYNOMIALS

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ABSTRACT. By the symmetric properties of Dirichlet's type multiple $q-l$ -function, we establish various identities concerning the generalized higher-order q -Euler polynomials. Furthermore, we give some interesting relationship between the power sums and the generalized higher-order q -Euler polynomials.

1. INTRODUCTION

Let χ be a Dirichlet character with conductor $d \in \mathbb{N}$ with $d \equiv 1 \pmod{2}$. As is well known, the generalized higher-order Euler polynomials are defined by the generating function to be

$$\left(2 \sum_{a=0}^{d-1} \frac{(-1)^a \chi(a) e^{at}}{e^{dt} + 1} \right)^r e^{xt} = \sum_{n=0}^{\infty} E_{n,\chi}^{(r)}(x) \frac{t^n}{n!}. \quad (1.1)$$

When $x = 0$, $E_{n,\chi}^{(r)} = E_{n,\chi}^{(r)}(0)$ are called the generalized Euler numbers attached to χ of order $r (\in \mathbb{N})$.

For $q \in \mathbb{C}$ with $|q| < 1$, the q -number is defined by $[x]_q = \frac{1-q^x}{1-q}$.

Note that $\lim_{q \rightarrow 1} [x]_q = x$. In [7], Kim considered q -extension of generalized higher-order Euler polynomials attached to χ as follows:

$$\begin{aligned} F_{q,\chi}^{(r)}(t, x) &= [2]_q^r \sum_{m_1, \dots, m_r=0}^{\infty} (-q)^{m_1+\dots+m_r} (\prod_{i=1}^r \chi(m_i)) e^{[m_1+\dots+m_r+x]_q t} \\ &= \sum_{n=0}^{\infty} E_{n,\chi,q}^{(r)}(x) \frac{t^n}{n!}. \end{aligned} \quad (1.2)$$

Note that

$$\lim_{q \rightarrow 1} F_{q,\chi}^{(r)}(t, x) = \left(2 \sum_{a=0}^{d-1} \frac{(-1)^a \chi(a) e^{at}}{e^{dt} + 1} \right)^r e^{xt}.$$

For $s \in \mathbb{C}$ and $x \in \mathbb{R}$ with $x \neq 0, -1, -2, \dots$, Kim defined Dirichlet-type multiple $q-l$ -funtion which is given by

$$\begin{aligned} l_{q,r}(s, x | \chi) &= [2]_q^r \sum_{m_1, \dots, m_r=0}^{\infty} \frac{(-q)^{m_1+\dots+m_r} (\prod_{i=1}^r \chi(m_i))}{[m_1 + \dots + m_r + x]_q^s} \\ &= \frac{1}{\Gamma(s)} \int_0^{\infty} F_{q,\chi}^{(r)}(-t, x) t^{s-1} dt, \quad (\text{see [7]}). \end{aligned} \quad (1.3)$$

Applying the Laurent series and Cauchy residue theorem in (1.2) and (1.3), we get

$$l_{q,r}(-n, x|\chi) = E_{n,\chi,q}^{(r)}(x), \text{ where } n \in \mathbb{Z}_{\geq 0}. \quad (1.4)$$

When $x = 0$, $E_{n,\chi,q}^{(r)} = E_{n,\chi,q}^{(r)}(0)$ are called the generalized q -Euler numbers attached to χ of order r . From (1.2), we note that

$$\begin{aligned} E_{n,\chi,q}^{(r)}(x) &= \sum_{l=0}^n \binom{n}{l} q^{lx} E_{l,\chi,q}^{(r)} [x]_q^{n-l} \\ &= \left(q^x E_{\chi,q}^{(r)} + [x]_q \right)^n, \end{aligned} \quad (1.5)$$

with the usual convention about replacing $(E_{\chi,q}^{(r)})^n$ by $E_{n,\chi,q}^{(r)}$, (see [1-13]).

In this paper, we investigate properties of symmetry in two variables related to multiple $q-l$ -function which interpolates generalized higher-order q -Euler polynomials attached to χ at negative integers. From our investigation, we derive identities of symmetry in two variables related to generalized higher-order q -Euler polynomials attached to χ . Recently, several authors have studied q -extensions of Euler polynomials due to T. Kim (see [1, 2, 3, 9, 10, 11, 12, 13]).

2. SYMMETRY OF q -POWER SUM AND THE GENERALIZED q -EULER POLYNOMIALS

For $a, b \in \mathbb{N}$ with $a \equiv 1 \pmod{2}$ and $b \equiv 1 \pmod{2}$, we observe that

$$\begin{aligned} &\frac{1}{[2]_{q^a}^r} l_{q^a,r} \left(s, bx + \frac{b}{a}(j_1 + \cdots + j_r) | \chi \right) \\ &= [a]_q^s \sum_{n_1, \dots, n_r=0}^{\infty} \sum_{i_1, \dots, i_r=0}^{db-1} \frac{(-1)^{\sum_{l=1}^r (i_l+n_l)} q^{a \sum_{l=1}^r (i_l+b d n_l)} (\prod_{l=1}^r \chi(i_l))}{[ab(x+d \sum_{l=1}^r n_l) + b \sum_{l=1}^r j_l + a \sum_{l=1}^r i_l]_q^s} \end{aligned} \quad (2.1)$$

From (2.1), we have

$$\begin{aligned} &\frac{[b]_q^s}{[2]_{q^a}^r} \sum_{j_1, \dots, j_r=0}^{da-1} (-1)^{\sum_{l=1}^r j_l} (\prod_{l=1}^r \chi(j_l)) q^{b \sum_{l=1}^r j_l} l_{q^a,r}(s, bx + \frac{b}{a} \sum_{l=1}^r j_l | \chi) \\ &= [a]_q^s [b]_q^s \sum_{i_1, \dots, i_r=0}^{db-1} \sum_{j_1, \dots, j_r=0}^{da-1} \sum_{n_1, \dots, n_r=0}^{\infty} \frac{(-1)^{\sum_{l=1}^r (i_l+n_l+j_l)} (\prod_{l=1}^r \chi(j_l)) (\prod_{l=1}^r \chi(i_l))}{[ab(x+d \sum_{l=1}^r n_l) + \sum_{l=1}^r (bj_l + ai_l)]_q^s} \\ &\quad \times q^{\sum_{l=1}^r (ai_l + bj_l + abdn_l)}. \end{aligned} \quad (2.2)$$

By the same method as (2.2), we get

$$\begin{aligned}
 & \frac{[a]_q^s}{[2]_{q^b}^r} \sum_{j_1, \dots, j_r=0}^{db-1} (-1)^{\sum_{l=1}^r j_l} (\prod_{l=1}^r \chi(j_l)) q^{a \sum_{l=1}^r j_l} l_{q^b, r}(s, ax + \frac{a}{b} \sum_{l=1}^r j_l | \chi) \\
 &= [a]_q^s [b]_q^s \sum_{j_1, \dots, j_r=0}^{db-1} \sum_{i_1, \dots, i_r=0}^{da-1} \sum_{n_1, \dots, n_r=0}^{\infty} \frac{(-1)^{\sum_{l=1}^r (i_l + n_l + j_l)} (\prod_{l=1}^r \chi(j_l) \chi(i_l))}{[ab(x + d \sum_{l=1}^r n_l) + \sum_{l=1}^r (aj_l + bi_l)]_q^s} \\
 &\quad \times q^{\sum_{l=1}^r (aj_l + bi_l + abdn_l)}. \tag{2.3}
 \end{aligned}$$

Therefore, by (2.2) and (2.3), we obtain the following theorem.

Theorem 2.1. *For $a, b \in \mathbb{N}$ with $a \equiv 1 \pmod{2}$ and $b \equiv 1 \pmod{2}$, we have*

$$\begin{aligned}
 & [2]_{q^b}^r [b]_q^s \sum_{j_1, \dots, j_r=0}^{da-1} (-1)^{\sum_{l=1}^r j_l} (\prod_{l=1}^r \chi(j_l)) q^{b \sum_{l=1}^r j_l} l_{q^a, r}(s, bx + \frac{b}{a} \sum_{l=1}^r j_l | \chi) \\
 &= [2]_{q^a}^r [a]_q^s \sum_{j_1, \dots, j_r=0}^{db-1} (-1)^{\sum_{l=1}^r j_l} (\prod_{l=1}^r \chi(j_l)) q^{a \sum_{l=1}^r j_l} l_{q^b, r}(s, ax + \frac{a}{b} \sum_{l=1}^r j_l | \chi).
 \end{aligned}$$

From (1.4) and Theorem 2.1, we obtain the following theorem.

Theorem 2.2. *For $n \geq 0$ and $a, b \in \mathbb{N}$ with $a \equiv 1 \pmod{2}$ and $b \equiv 1 \pmod{2}$, we have*

$$\begin{aligned}
 & [2]_{q^b}^r [a]_q^n \sum_{j_1, \dots, j_r=0}^{da-1} (-1)^{\sum_{l=1}^r j_l} (\prod_{l=1}^r \chi(j_l)) q^{b \sum_{l=1}^r j_l} E_{n, \chi, q^a}^{(r)}(bx + \frac{b}{a} \sum_{l=1}^r j_l) \\
 &= [2]_{q^a}^r [b]_q^n \sum_{j_1, \dots, j_r=0}^{db-1} (-1)^{\sum_{l=1}^r j_l} (\prod_{l=1}^r \chi(j_l)) q^{a \sum_{l=1}^r j_l} E_{n, \chi, q^b}^{(r)}(ax + \frac{a}{b} \sum_{l=1}^r j_l).
 \end{aligned}$$

By (1.5), we easily get

$$\begin{aligned}
 E_{n, \chi, q}^{(r)}(x + y) &= (q^{x+y} E_{\chi, q}^{(r)} + [x + y]_q)^n \\
 &= (q^{x+y} E_{\chi, q}^{(r)} + q^x [y]_q + [x]_q)^n \\
 &= \sum_{i=0}^n \binom{n}{i} q^{ix} (q^y E_{\chi, q}^{(r)} + [y]_q)^i [x]_q^{n-i} \\
 &= \sum_{i=0}^n \binom{n}{i} q^{xi} E_{i, \chi, q}^{(r)}(y) [x]_q^{n-i}. \tag{2.4}
 \end{aligned}$$

From (2.4), we note that

$$\begin{aligned}
& \sum_{j_1, \dots, j_r=0}^{da-1} (-1)^{\sum_{l=1}^r j_l} q^{b \sum_{l=1}^r j_l} (\prod_{l=1}^r \chi(j_l)) E_{n,\chi,q^a}^{(r)}(bx + \frac{b}{a} \sum_{l=1}^r j_l) \\
&= \sum_{j_1, \dots, j_r=0}^{da-1} (-1)^{\sum_{l=1}^r j_l} q^{b \sum_{l=1}^r j_l} (\prod_{l=1}^r \chi(j_l)) \\
&\quad \times \sum_{i=0}^n \binom{n}{i} q^{ib \sum_{l=1}^r j_l} E_{i,\chi,q^a}^{(r)}(bx) \left[\frac{b(j_1 + \dots + j_r)}{a} \right]_{q^a}^{n-i} \\
&= \sum_{j_1, \dots, j_r=0}^{da-1} (-1)^{\sum_{l=1}^r j_l} q^{b \sum_{l=1}^r j_l} (\prod_{l=1}^r \chi(j_l)) \\
&\quad \times \sum_{i=0}^n \binom{n}{i} q^{(n-i)b \sum_{l=1}^r j_l} E_{n-i,\chi,q^a}^{(r)}(bx) \left[\frac{b}{a} \sum_{l=1}^r j_l \right]_{q^a}^i \\
&= \sum_{i=0}^n \binom{n}{i} \left(\frac{[b]_q}{[a]_q} \right)^i E_{n-i,\chi,q^a}^{(r)}(bx) \\
&\quad \times \sum_{j_1, \dots, j_r=0}^{da-1} (-1)^{\sum_{l=1}^r j_l} q^{b \sum_{l=1}^r (n-i+1) j_l} [j_1 + \dots + j_r]_{q^b}^i \\
&= \sum_{i=0}^n \binom{n}{i} \left(\frac{[b]_q}{[a]_q} \right)^i E_{n-i,\chi,q^a}^{(r)}(bx) S_{n,i,q^b}^{(r)}(ad|\chi),
\end{aligned} \tag{2.5}$$

where

$$S_{n,i,q}^{(r)}(ad|\chi) = \sum_{j_1, \dots, j_r=0}^{a-1} (-1)^{j_1 + \dots + j_r} (\prod_{l=1}^r \chi(j_l)) q^{\sum_{l=1}^r j_l(n-i+1)} \left[\sum_{l=1}^r j_l \right]_q^i. \tag{2.6}$$

From (2.5) and (2.6), we can derive the following equation.

$$\begin{aligned}
& [2]_{q^b}^r [a]_q^n \sum_{j_1, \dots, j_r=0}^{da-1} (-1)^{\sum_{l=1}^r j_l} q^{b \sum_{l=1}^r j_l} (\prod_{l=1}^r \chi(j_l)) E_{n,\chi,q^a}^{(r)}(bx + \frac{b}{a} \sum_{l=1}^r j_l) \\
&= [2]_{q^b}^r \sum_{i=0}^n \binom{n}{i} [a]_q^{n-i} [b]_q^i E_{n-i,\chi,q^a}^{(r)}(bx) S_{n,i,q^b}^{(r)}(ad|\chi).
\end{aligned} \tag{2.7}$$

By the same method as (2.7), we get

$$\begin{aligned}
& [2]_{q^a}^r [b]_q^n \sum_{j_1, \dots, j_r=0}^{db-1} (-1)^{\sum_{l=1}^r j_l} q^{a \sum_{l=1}^r j_l} (\prod_{l=1}^r \chi(j_l)) E_{n,\chi,q^b}^{(r)}(ax + \frac{a}{b} \sum_{l=1}^r j_l) \\
&= [2]_{q^a}^r \sum_{i=0}^n \binom{n}{i} [b]_q^{n-i} [a]_q^i E_{n-i,\chi,q^b}^{(r)}(ax) S_{n,i,q^a}^{(r)}(bd|\chi).
\end{aligned} \tag{2.8}$$

Therefore, by (2.7) and (2.8), we obtain the following theorem.

Theorem 2.3. For $n \geq 0$ and $a, b \in \mathbb{N}$ with $a \equiv 1 \pmod{2}$ and $b \equiv 1 \pmod{2}$, we have

$$\begin{aligned} & [2]_{q^b}^r \sum_{i=0}^n \binom{n}{i} [a]_q^{n-i} [b]_q^i E_{n-i, \chi, q^a}^{(r)}(bx) S_{n, i, q^b}^{(r)}(ad|\chi) \\ & = [2]_{q^a}^r \sum_{i=0}^n \binom{n}{i} [b]_q^{n-i} [a]_q^i E_{n-i, \chi, q^b}^{(r)}(ax) S_{n, i, q^a}^{(r)}(bd|\chi). \end{aligned}$$

Remark. It is not difficult to show that

$$\begin{aligned} & e^{[x]_q u} \sum_{m_1, \dots, m_r=0}^{\infty} q^{\sum_{l=1}^r m_l} (-1)^{\sum_{l=1}^r m_l} (\prod_{l=1}^r \chi(m_l)) e^{[y+\sum_{l=1}^r m_l]_q q^x(u+v)} \\ & = e^{-[x]_q u} \sum_{m_1, \dots, m_r=0}^{\infty} q^{\sum_{l=1}^r m_l} (-1)^{\sum_{l=1}^r m_l} (\prod_{l=1}^r \chi(m_l)) e^{[x+y+\sum_{l=1}^r m_l]_q(u+v)}. \end{aligned} \quad (2.9)$$

Thus, by (2.9), we get

$$\begin{aligned} & \sum_{k=0}^m \binom{m}{k} q^{kx} E_{n+k, \chi, q}^{(r)}(y) [x]_q^{m-k} \\ & = \sum_{k=0}^n \binom{n}{k} q^{-kx} E_{m+k, \chi, q}^{(r)}(x+y) [-x]_q^{n-k}. \end{aligned} \quad (2.10)$$

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