

HERMITE-HADAMARD TYPE INEQUALITIES FOR HARMONICALLY CONVEX FUNCTIONS VIA FRACTIONAL INTEGRALS

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ABSTRACT. In this paper, the author established Hermite-Hadamard's inequalities for harmonically convex functions via fractional integrals and obtained some Hermite-Hadamard type inequalities of these classes of functions.

1. INTRODUCTION

Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following inequality

$$(1.1) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}$$

holds. This double inequality is known in the literature as Hermite-Hadamard integral inequality for convex functions. Note that some of the classical inequalities for means can be derived from (1.1) for appropriate particular selections of the mapping f . Both inequalities hold in the reversed direction if f is concave. For some results which generalize, improve and extend the inequalities (1.1) we refer the reader to the recent papers (see [1, 2, 3, 4, 5]) and references therein.

In [1], Iscan gave definition of harmonically convexity as follows:

Definition 1. Let $I \subseteq \mathbb{R} \setminus \{0\}$ be a real interval. A function $f : I \rightarrow \mathbb{R}$ is said to be harmonically convex, if

$$(1.2) \quad f\left(\frac{xy}{tx + (1-t)y}\right) \leq tf(y) + (1-t)f(x)$$

for all $x, y \in I$ and $t \in [0, 1]$. If the inequality in (1.2) is reversed, then f is said to be harmonically concave.

The following result of the Hermite-Hadamard type holds.

Theorem 1 ([1]). Let $f : I \subseteq \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ be a harmonically convex function and $a, b \in I$ with $a < b$. If $f \in L[a, b]$ then the following inequalities hold

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \leq \frac{f(a) + f(b)}{2}.$$

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Lemma 1 ([1]). Let $f : I \subseteq \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ be a differentiable function on I° and $a, b \in I$ with $a < b$. If $f' \in L[a, b]$ then

$$(1.3) \quad \begin{aligned} & \frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \\ &= \frac{ab(b-a)}{2} \int_0^1 \frac{1-2t}{(tb+(1-t)a)^2} f' \left(\frac{ab}{tb+(1-t)a} \right) dt. \end{aligned}$$

In [1], Iscan proved the following results connected with the right part of (1.2)

Theorem 2. Let $f : I \subseteq (0, \infty) \rightarrow \mathbb{R}$ be a differentiable function on I° , $a, b \in I$ with $a < b$, and $f' \in L[a, b]$. If $|f'|^q$ is harmonically convex on $[a, b]$ for $q \geq 1$, then

$$(1.4) \quad \begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \right| \\ & \leq \frac{ab(b-a)}{2} \lambda_1^{1-\frac{1}{q}} [\lambda_2 |f'(a)|^q + \lambda_3 |f'(b)|^q]^{\frac{1}{q}}, \end{aligned}$$

where

$$\begin{aligned} \lambda_1 &= \frac{1}{ab} - \frac{2}{(b-a)^2} \ln \left(\frac{(a+b)^2}{4ab} \right), \\ \lambda_2 &= \frac{-1}{b(b-a)} + \frac{3a+b}{(b-a)^3} \ln \left(\frac{(a+b)^2}{4ab} \right), \\ \lambda_3 &= \frac{1}{a(b-a)} - \frac{3b+a}{(b-a)^3} \ln \left(\frac{(a+b)^2}{4ab} \right) \\ &= \lambda_1 - \lambda_2. \end{aligned}$$

Theorem 3. Let $f : I \subseteq (0, \infty) \rightarrow \mathbb{R}$ be a differentiable function on I° , $a, b \in I$ with $a < b$, and $f' \in L[a, b]$. If $|f'|^q$ is harmonically convex on $[a, b]$ for $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then

$$(1.5) \quad \begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \right| \\ & \leq \frac{ab(b-a)}{2} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} (\mu_1 |f'(a)|^q + \mu_2 |f'(b)|^q)^{\frac{1}{q}}, \end{aligned}$$

where

$$\begin{aligned} \mu_1 &= \frac{[a^{2-2q} + b^{1-2q} [(b-a)(1-2q) - a]]}{2(b-a)^2 (1-q)(1-2q)}, \\ \mu_2 &= \frac{[b^{2-2q} - a^{1-2q} [(b-a)(1-2q) + b]]}{2(b-a)^2 (1-q)(1-2q)}. \end{aligned}$$

We recall the following special functions and inequality

(1) The Beta function:

$$\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1} (1-t)^{y-1} dt, \quad x, y > 0,$$

(2) The hypergeometric function:

$${}_2F_1(a, b; c; z) = \frac{1}{\beta(b, c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a} dt, \quad c > b > 0, \quad |z| < 1 \quad (\text{see [14]}).$$

Lemma 2 ([6, 7]). *For $0 < \alpha \leq 1$ and $0 \leq a < b$, we have*

$$|a^\alpha - b^\alpha| \leq (b-a)^\alpha.$$

In the following, we will give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper.

Definition 2. *Let $f \in L[a, b]$. The Riemann-Liouville integrals $J_{a+}^\alpha f$ and $J_{b-}^\alpha f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad x > a$$

and

$$J_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \quad x < b$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$

and $J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x)$.

Because of the wide application of Hermite-Hadamard type inequalities and fractional integrals, many researchers extend their studies to Hermite-Hadamard type inequalities involving fractional integrals not limited to integer integrals. Recently, more and more Hermite-Hadamard inequalities involving fractional integrals have been obtained for different classes of functions; see [7, 8, 9, 10, 11, 12, 13].

The aim of this paper is to establish Hermite-Hadamard's inequalities for Harmonically convex functions via Riemann-Liouville fractional integral and some other integral inequalities using the identity is obtained for fractional integrals. These results have some relationships with [1].

2. MAIN RESULTS

Let $f : I \subseteq (0, \infty) \rightarrow \mathbb{R}$ be a differentiable function on I° , the interior of I , throughout this section we will take

$$\begin{aligned} & I_f(g; \alpha, a, b) \\ &= \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha+1)}{2} \left(\frac{ab}{b-a} \right)^\alpha \left\{ J_{1/a-}^\alpha (f \circ g)(1/b) + J_{1/b+}^\alpha (f \circ g)(1/a) \right\} \end{aligned}$$

where $a, b \in I$ with $a < b$, $\alpha > 0$, $g(x) = 1/x$ and Γ is Euler Gamma function.

Hermite–Hadamard's inequalities for Harmonically convex functions can be represented in fractional integral forms as follows:

Theorem 4. Let $f : I \subseteq (0, \infty) \rightarrow \mathbb{R}$ be a function such that $f \in L[a, b]$, where $a, b \in I$ with $a < b$. If f is a harmonically convex function on $[a, b]$, then the following inequalities for fractional integrals hold:

$$(2.1) \quad f\left(\frac{2ab}{a+b}\right) \leq \frac{\Gamma(\alpha+1)}{2} \left(\frac{ab}{b-a}\right)^\alpha \left\{ J_{1/a-}^\alpha(f \circ g)(1/b) + J_{1/b+}^\alpha(f \circ g)(1/a) \right\} \leq \frac{f(a) + f(b)}{2}$$

with $\alpha > 0$.

Proof. Since f is a harmonically convex function on $[a, b]$, we have for all $x, y \in [a, b]$ (with $t = 1/2$ in the inequality (1.2))

$$f\left(\frac{2xy}{x+y}\right) \leq \frac{f(x) + f(y)}{2}.$$

Choosing $x = \frac{ab}{tb+(1-t)a}$, $y = \frac{ab}{ta+(1-t)b}$, we get

$$(2.2) \quad f\left(\frac{2ab}{a+b}\right) \leq \frac{f\left(\frac{ab}{tb+(1-t)a}\right) + f\left(\frac{ab}{ta+(1-t)b}\right)}{2}.$$

Multiplying both sides of (2.2) by $t^{\alpha-1}$, then integrating the resulting inequality with respect to t over $[0, 1]$, we obtain

$$\begin{aligned} f\left(\frac{2ab}{a+b}\right) &\leq \frac{\alpha}{2} \left\{ \int_0^1 t^{\alpha-1} f\left(\frac{ab}{tb+(1-t)a}\right) dt + \int_0^1 t^{\alpha-1} f\left(\frac{ab}{ta+(1-t)b}\right) dt \right\} \\ &= \frac{\alpha}{2} \left(\frac{ab}{b-a} \right)^\alpha \left\{ \int_{1/b}^{1/a} \left(x - \frac{1}{b} \right)^{\alpha-1} f\left(\frac{1}{x}\right) dx + \int_{1/b}^{1/a} \left(\frac{1}{a} - x \right)^{\alpha-1} f\left(\frac{1}{x}\right) dx \right\} \\ &= \frac{\alpha\Gamma(\alpha)}{2} \left(\frac{ab}{b-a} \right)^\alpha \left\{ J_{1/a-}^\alpha(f \circ g)(1/b) + J_{1/b+}^\alpha(f \circ g)(1/a) \right\} \\ &= \frac{\Gamma(\alpha+1)}{2} \left(\frac{ab}{b-a} \right)^\alpha \left\{ J_{1/a-}^\alpha(f \circ g)(1/b) + J_{1/b+}^\alpha(f \circ g)(1/a) \right\}, \text{ where } g(x) = 1/x. \end{aligned}$$

and the first inequality is proved.

For the proof of the second inequality in (2.1) we first note that if f is a harmonically convex function, then, for $t \in [0, 1]$, it yields

$$f\left(\frac{ab}{tb+(1-t)a}\right) \leq tf(a) + (1-t)f(b)$$

and

$$f\left(\frac{ab}{ta+(1-t)b}\right) \leq tf(b) + (1-t)f(a).$$

By adding these inequalities we have

$$(2.3) \quad f\left(\frac{ab}{tb+(1-t)a}\right) + f\left(\frac{ab}{ta+(1-t)b}\right) \leq f(a) + f(b).$$

Then multiplying both sides of (2.3) by $t^{\alpha-1}$, and integrating the resulting inequality with respect to t over $[0, 1]$, we obtain

$$\int_0^1 f\left(\frac{ab}{tb + (1-t)a}\right) t^{\alpha-1} dt + \int_0^1 f\left(\frac{ab}{ta + (1-t)b}\right) t^{\alpha-1} dt \leq [f(a) + f(b)] \int_0^1 t^{\alpha-1} dt$$

i.e.

$$\Gamma(\alpha + 1) \left(\frac{ab}{b-a}\right)^\alpha \left\{ J_{1/a-}^\alpha (f \circ g)(1/b) + J_{1/b+}^\alpha (f \circ g)(1/a) \right\} \leq f(a) + f(b).$$

The proof is completed. \square

Lemma 3. *Let $f : I \subseteq (0, \infty) \rightarrow \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. Then the following equality for fractional integrals holds:*

$$(2.4) \quad \begin{aligned} & I_f(g; \alpha, a, b) \\ &= \frac{ab(b-a)}{2} \int_0^1 \frac{[t^\alpha - (1-t)^\alpha]}{(ta + (1-t)b)^2} f'\left(\frac{ab}{ta + (1-t)b}\right) dt. \end{aligned}$$

Proof. Let $A_t = ta + (1-t)b$. It suffices to note that

$$(2.5) \quad \begin{aligned} I_f(g; \alpha, a, b) &= \frac{ab(b-a)}{2} \int_0^1 \frac{[t^\alpha - (1-t)^\alpha]}{A_t^2} f'\left(\frac{ab}{A_t}\right) dt \\ &= \frac{ab(b-a)}{2} \int_0^1 \frac{t^\alpha}{A_t^2} f'\left(\frac{ab}{A_t}\right) dt - \frac{ab(b-a)}{2} \int_0^1 \frac{(1-t)^\alpha}{A_t^2} f'\left(\frac{ab}{A_t}\right) dt \\ &\quad I_1 + I_2. \end{aligned}$$

By integrating by part, we have

$$(2.6) \quad \begin{aligned} I_1 &= \frac{1}{2} \left[t^\alpha f\left(\frac{ab}{A_t}\right) \Big|_0^1 - \alpha \int_0^1 t^{\alpha-1} f\left(\frac{ab}{A_t}\right) dt \right] \\ &= \frac{1}{2} \left[f(b) - \alpha \left(\frac{ab}{b-a}\right)^\alpha \int_{1/b}^{1/a} \left(\frac{1}{a} - x\right)^{\alpha-1} f\left(\frac{1}{x}\right) dx \right] \\ &= \frac{1}{2} \left[f(b) - \Gamma(\alpha + 1) \left(\frac{ab}{b-a}\right)^\alpha J_{1/b+}^\alpha (f \circ g)(1/a) \right] \end{aligned}$$

and similarly we get,

$$\begin{aligned}
I_2 &= -\frac{1}{2} \left[(1-t)^\alpha f\left(\frac{ab}{A_t}\right) \Big|_0^1 + \alpha \int_0^1 (1-t)^{\alpha-1} f\left(\frac{ab}{A_t}\right) dt \right] \\
&= -\frac{1}{2} \left[-f(a) + \alpha \left(\frac{ab}{b-a}\right)^\alpha \int_{1/b}^{1/a} (x - \frac{1}{b})^{\alpha-1} f\left(\frac{1}{x}\right) dx \right] \\
(2.7) \quad &= \frac{1}{2} \left[f(a) - \Gamma(\alpha+1) \left(\frac{ab}{b-a}\right)^\alpha J_{1/a-}^\alpha (f \circ g)(1/b) \right].
\end{aligned}$$

Using (2.6) and (2.7) in (2.5), we get equality (2.4). \square

Remark 1. If Lemma 3, we let $\alpha = 1$, then equality (2.4) becomes equality (1.3) of Lemma 1.

Using lemma 3, we can obtain the following fractional integral inequality.

Theorem 5. Let $f : I \subseteq (0, \infty) \rightarrow \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a, b]$, where $a, b \in I^\circ$ with $a < b$. If $|f'|^q$ is harmonically convex on $[a, b]$ for some fixed $q \geq 1$, then the following inequality for fractional integrals holds:

$$\begin{aligned}
(2.8) \quad &|I_f(g; \alpha, a, b)| \\
&\leq \frac{ab(b-a)}{2} C_1^{1-1/q}(\alpha; a, b) (C_2(\alpha; a, b) |f'(b)|^q + C_3(\alpha; a, b) |f'(a)|^q)^{1/q},
\end{aligned}$$

where

$$\begin{aligned}
C_1(\alpha; a, b) &= \frac{b^{-2}}{\alpha+1} \left[{}_2F_1\left(2, 1; \alpha+2; 1 - \frac{a}{b}\right) + {}_2F_1\left(2, \alpha+1; \alpha+2; 1 - \frac{a}{b}\right) \right], \\
C_2(\alpha; a, b) &= \frac{b^{-2}}{\alpha+2} \left[\frac{1}{\alpha+1} \cdot {}_2F_1\left(2, 2; \alpha+3; 1 - \frac{a}{b}\right) + {}_2F_1\left(2, \alpha+2; \alpha+3; 1 - \frac{a}{b}\right) \right], \\
C_3(\alpha; a, b) &= \frac{b^{-2}}{\alpha+1} \left[{}_2F_1\left(2, 1; \alpha+3; 1 - \frac{a}{b}\right) + \frac{1}{\alpha+1} \cdot {}_2F_1\left(2, \alpha+1; \alpha+3; 1 - \frac{a}{b}\right) \right].
\end{aligned}$$

Proof. Let $A_t = ta + (1-t)b$. From Lemma..., using the property of the modulus, the power mean inequality and the harmonically convexity of $|f'|^q$, we find

$$\begin{aligned}
&|I_f(g; \alpha, a, b)| \\
&\leq \frac{ab(b-a)}{2} \int_0^1 \frac{|(1-t)^\alpha - t^\alpha|}{A_t^2} \left| f'\left(\frac{ab}{A_t}\right) \right| dt \\
&\leq \frac{ab(b-a)}{2} \left(\int_0^1 \frac{|(1-t)^\alpha - t^\alpha|}{A_t^2} dt \right)^{1-1/q} \left(\int_0^1 \frac{|(1-t)^\alpha - t^\alpha|}{A_t^2} \left| f'\left(\frac{ab}{A_t}\right) \right|^q dt \right)^{1/q} \\
&\leq \frac{ab(b-a)}{2} \left(\int_0^1 \frac{[(1-t)^\alpha + t^\alpha]}{A_t^2} dt \right)^{1-1/q} \left(\int_0^1 \frac{[(1-t)^\alpha + t^\alpha]}{A_t^2} [t |f'(b)|^q + (1-t) |f'(a)|^q] dt \right)^{1/q} \\
(2.9) \quad &\leq \frac{ab(b-a)}{2} C_1^{1-1/q}(\alpha; a, b) (C_2(\alpha; a, b) |f'(b)|^q + C_3(\alpha; a, b) |f'(a)|^q)^{1/q}.
\end{aligned}$$

calculating $C_1(\alpha; a, b)$, $C_2(\alpha; a, b)$ and $C_3(\alpha; a, b)$, we have

$$\begin{aligned} C_1(\alpha; a, b) &= \int_0^1 \frac{[1-t]^\alpha + t^\alpha}{A_t^2} dt \\ (2.10) \quad &= \frac{b^{-2}}{\alpha+1} \left[{}_2F_1 \left(2, 1; \alpha+2; 1 - \frac{a}{b} \right) + {}_2F_1 \left(2, \alpha+1; \alpha+2; 1 - \frac{a}{b} \right) \right], \end{aligned}$$

$$\begin{aligned} C_2(\alpha; a, b) &= \int_0^1 \frac{[1-t]^\alpha + t^\alpha}{A_t^2} t dt \\ (2.11) \quad &= \frac{b^{-2}}{\alpha+2} \left[\frac{1}{\alpha+1} {}_2F_1 \left(2, 2; \alpha+3; 1 - \frac{a}{b} \right) + {}_2F_1 \left(2, \alpha+2; \alpha+3; 1 - \frac{a}{b} \right) \right], \end{aligned}$$

$$\begin{aligned} C_3(\alpha; a, b) &= \int_0^1 \frac{[1-t]^\alpha + t^\alpha}{A_t^2} (1-t) dt \\ (2.12) \quad &= \frac{b^{-2}}{\alpha+1} \left[{}_2F_1 \left(2, 1; \alpha+3; 1 - \frac{a}{b} \right) + \frac{1}{\alpha+1} {}_2F_1 \left(2, \alpha+1; \alpha+3; 1 - \frac{a}{b} \right) \right], \end{aligned}$$

Thus, if we use (2.10), (2.11) and (2.12) in (2.9), we obtain the inequality of (2.8). This completes the proof. \square

When $0 < \alpha \leq 1$, using Lemma 2 and Lemma 3 we shall give another result for harmonically convex functions as follows.

Theorem 6. *Let $f : I \subseteq (0, \infty) \rightarrow \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a, b]$, where $a, b \in I^\circ$ with $a < b$. If $|f'|^q$ is harmonically convex on $[a, b]$ for some fixed $q \geq 1$, then the following inequality for fractional integrals holds:*

$$\begin{aligned} (2.13) \quad &|I_f(g; \alpha, a, b)| \\ &\leq \frac{ab(b-a)}{2} C_1^{1-1/q}(\alpha; a, b) (C_2(\alpha; a, b) |f'(b)|^q + C_3(\alpha; a, b) |f'(a)|^q)^{1/q}, \end{aligned}$$

where

$$\begin{aligned} &C_1(\alpha; a, b) \\ &= \frac{b^{-2}}{\alpha+1} \left[{}_2F_1 \left(2, \alpha+1; \alpha+2; 1 - \frac{a}{b} \right) - {}_2F_1 \left(2, 1; \alpha+2; 1 - \frac{a}{b} \right) \right. \\ &\quad \left. + {}_2F_1 \left(2, 1; \alpha+2; \frac{1}{2} \left(1 - \frac{a}{b} \right) \right) \right], \end{aligned}$$

$$\begin{aligned} &C_2(\alpha; a, b) \\ &= \frac{b^{-2}}{\alpha+2} \left[{}_2F_1 \left(2, \alpha+2; \alpha+3; 1 - \frac{a}{b} \right) - \frac{1}{\alpha+1} {}_2F_1 \left(2, 2; \alpha+3; 1 - \frac{a}{b} \right) \right. \\ &\quad \left. + \frac{1}{2(\alpha+1)} {}_2F_1 \left(2, 2; \alpha+3; \frac{1}{2} \left(1 - \frac{a}{b} \right) \right) \right], \end{aligned}$$

$$\begin{aligned}
& C_3(\alpha; a, b) \\
&= \frac{b^{-2}}{\alpha+2} \left[\frac{1}{\alpha+1} {}_2F_1 \left(2, \alpha+1; \alpha+3; 1 - \frac{a}{b} \right) - {}_2F_1 \left(2, 1; \alpha+3; 1 - \frac{a}{b} \right) \right. \\
&\quad \left. + {}_2F_1 \left(2, 1; \alpha+3; \frac{1}{2} \left(1 - \frac{a}{b} \right) \right) \right]
\end{aligned}$$

and $0 < \alpha \leq 1$.

Proof. Let $A_t = ta + (1-t)b$. From Lemma 3, using the property of the modulus, the power mean inequality and the harmonically convexity of $|f'|^q$, we find

$$\begin{aligned}
& |I_f(g; \alpha, a, b)| \\
&\leq \frac{ab(b-a)}{2} \int_0^1 \frac{|(1-t)^\alpha - t^\alpha|}{A_t^2} \left| f' \left(\frac{ab}{A_t} \right) \right| dt \\
&\leq \frac{ab(b-a)}{2} \left(\int_0^1 \frac{|(1-t)^\alpha - t^\alpha|}{A_t^2} dt \right)^{1-1/q} \left(\int_0^1 \frac{|(1-t)^\alpha - t^\alpha|}{A_t^2} \left| f' \left(\frac{ab}{A_t} \right) \right|^q dt \right)^{1/q} \\
&\leq \frac{ab(b-a)}{2} K_1^{1-1/q} \left(\int_0^1 \frac{|(1-t)^\alpha - t^\alpha|}{A_t^2} [t|f'(b)|^q + (1-t)|f'(a)|^q] dt \right)^{1/q} \\
&\leq \frac{ab(b-a)}{2} K_1^{1-1/q} (K_2 |f'(b)|^q + K_3 |f'(a)|^q)^{1/q}, \tag{2.14}
\end{aligned}$$

where

$$\begin{aligned}
K_1 &= \int_0^1 \frac{|(1-t)^\alpha - t^\alpha|}{A_t^2} dt, \\
K_2 &= \int_0^1 \frac{|(1-t)^\alpha - t^\alpha|}{A_t^2} t dt, \\
K_3 &= \int_0^1 \frac{|(1-t)^\alpha - t^\alpha|}{A_t^2} (1-t) dt.
\end{aligned}$$

Calculating K_1 , K_2 and K_3 , by Lemma 2, we have

$$\begin{aligned}
K_1 &= \int_0^1 \frac{|(1-t)^\alpha - t^\alpha|}{A_t^2} dt \\
&= \int_0^{1/2} \frac{(1-t)^\alpha - t^\alpha}{A_t^2} dt + \int_{1/2}^1 \frac{t^\alpha - (1-t)^\alpha}{A_t^2} dt \\
&= \int_0^1 \frac{t^\alpha - (1-t)^\alpha}{A_t^2} dt + 2 \int_0^{1/2} \frac{(1-t)^\alpha - t^\alpha}{A_t^2} dt
\end{aligned}$$

$$\begin{aligned}
&\leq \int_0^1 t^\alpha A_t^{-2} dt - \int_0^1 (1-t)^\alpha A_t^{-2} dt + 2 \int_0^{1/2} (1-2t)^\alpha A_t^{-2} dt \\
&= \int_0^1 t^\alpha A_t^{-2} dt - \int_0^1 (1-t)^\alpha A_t^{-2} dt + \int_0^1 (1-u)^\alpha b^{-2} \left(1-u\frac{1}{2}(1-\frac{a}{b})\right)^{-2} du \\
&= \frac{b^{-2}}{\alpha+1} \left[{}_2F_1 \left(2, \alpha+1; \alpha+2; 1-\frac{a}{b}\right) - {}_2F_1 \left(2, 1; \alpha+2; 1-\frac{a}{b}\right) \right. \\
&\quad \left. + {}_2F_1 \left(2, 1; \alpha+2; \frac{1}{2} \left(1-\frac{a}{b}\right)\right) \right] \\
(2.15) \quad &= C_1(\alpha; a, b)
\end{aligned}$$

and similarly we get

$$\begin{aligned}
K_2 &= \int_0^1 \frac{|(1-t)^\alpha - t^\alpha|}{A_t^2} t dt \\
&\leq \int_0^1 t^{\alpha+1} A_t^{-2} dt - \int_0^1 (1-t)^\alpha t A_t^{-2} dt + 2 \int_0^{1/2} (1-2t)^\alpha t A_t^{-2} dt \\
&= \frac{b^{-2}}{\alpha+2} \left[{}_2F_1 \left(2, \alpha+2; \alpha+3; 1-\frac{a}{b}\right) - \frac{1}{\alpha+1} {}_2F_1 \left(2, 2; \alpha+3; 1-\frac{a}{b}\right) \right. \\
&\quad \left. + \frac{1}{2(\alpha+1)} {}_2F_1 \left(2, 2; \alpha+3; \frac{1}{2} \left(1-\frac{a}{b}\right)\right) \right] \\
(2.16) \quad &= C_2(\alpha; a, b)
\end{aligned}$$

$$\begin{aligned}
K_3 &= \int_0^1 \frac{|(1-t)^\alpha - t^\alpha|}{A_t^2} (1-t) dt \\
&\leq \int_0^1 t^\alpha (1-t) A_t^{-2} dt - \int_0^1 (1-t)^{\alpha+1} A_t^{-2} dt + 2 \int_0^{1/2} (1-2t)^\alpha (1-t) A_t^{-2} dt \\
&= \frac{b^{-2}}{\alpha+2} \left[\frac{1}{\alpha+1} {}_2F_1 \left(2, \alpha+1; \alpha+3; 1-\frac{a}{b}\right) \right. \\
&\quad \left. - {}_2F_1 \left(2, 1; \alpha+3; 1-\frac{a}{b}\right) + {}_2F_1 \left(2, 1; \alpha+3; \frac{1}{2} \left(1-\frac{a}{b}\right)\right) \right] \\
(2.17) \quad &= C_3(\alpha; a, b).
\end{aligned}$$

Thus, if we use (2.15), (2.16) and (2.17) in (2.14), we obtain the inequality of (2.13). This completes the proof. \square

Remark 2. If we take $\alpha = 1$ in Theorem 6, then inequality (2.13) becomes inequality (1.4) of Theorem 2.

Theorem 7. Let $f : I \subseteq (0, \infty) \rightarrow \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a, b]$, where $a, b \in I^\circ$ with $a < b$. If $|f'|^q$ is harmonically convex on $[a, b]$ for some fixed $q > 1$, then the following inequality for fractional integrals holds:

$$(2.18) \quad |I_f(g; \alpha, a, b)| \leq \frac{a(b-a)}{2b} \left(\frac{1}{\alpha p + 1} \right)^{1/p} \left(\frac{|f'(b)|^q + |f'(a)|^q}{2} \right)^{1/q} \times \left[{}_2F_1^{1/p} \left(2p, 1; \alpha p + 2; 1 - \frac{a}{b} \right) + {}_2F_1^{1/p} \left(2p, \alpha p + 1; \alpha p + 2; 1 - \frac{a}{b} \right) \right],$$

where $1/p + 1/q = 1$.

Proof. Let $A_t = ta + (1-t)b$. From Lemma 3, using the Hölder inequality and the harmonically convexity of $|f'|^q$, we find

$$\begin{aligned} & |I_f(g; \alpha, a, b)| \\ & \leq \frac{ab(b-a)}{2} \int_0^1 \frac{(1-t)^\alpha}{A_t^2} \left| f' \left(\frac{ab}{A_t} \right) \right| dt + \int_0^1 \frac{t^\alpha}{A_t^2} \left| f' \left(\frac{ab}{A_t} \right) \right| dt \\ & \leq \frac{ab(b-a)}{2} \left\{ \left(\int_0^1 \frac{(1-t)^{\alpha p}}{A_t^{2p}} dt \right)^{1/p} \left(\int_0^1 \left| f' \left(\frac{ab}{A_t} \right) \right|^q dt \right)^{1/q} \right. \\ & \quad \left. + \left(\int_0^1 \frac{t^{\alpha p}}{A_t^{2p}} dt \right)^{1/p} \left(\int_0^1 \left| f' \left(\frac{ab}{A_t} \right) \right|^q dt \right)^{1/q} \right\} \\ & \leq \frac{ab(b-a)}{2} \left(K_4^{1/p} + K_5^{1/p} \right) \left(\int_0^1 [t|f'(b)|^q + (1-t)|f'(a)|^q] dt \right)^{1/q} \\ (2.19) \quad & \leq \frac{ab(b-a)}{2} \left(K_4^{1/p} + K_5^{1/p} \right) \left(\frac{|f'(b)|^q + |f'(a)|^q}{2} \right)^{1/q}. \end{aligned}$$

Calculating K_4 and K_5 , we have

$$\begin{aligned} K_4 &= \int_0^1 \frac{(1-t)^{\alpha p}}{A_t^{2p}} dt \\ (2.20) \quad &= \frac{b^{-2p}}{\alpha p + 1} {}_2F_1 \left(2p, 1; \alpha p + 2; 1 - \frac{a}{b} \right), \end{aligned}$$

$$\begin{aligned} K_5 &= \int_0^1 \frac{t^{\alpha p}}{A_t^{2p}} dt \\ (2.21) \quad &= \frac{b^{-2p}}{\alpha p + 1} {}_2F_1 \left(2p, \alpha p + 1; \alpha p + 2; 1 - \frac{a}{b} \right) \end{aligned}$$

Thus, if we use (2.20) and (2.21) in (2.19), we obtain the inequality of (2.18). This completes the proof. \square

Theorem 8. Let $f : I \subseteq (0, \infty) \rightarrow \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a, b]$, where $a, b \in I^\circ$ with $a < b$. If $|f'|^q$ is harmonically convex on $[a, b]$ for some fixed $q > 1$, then the following inequality for fractional integrals holds:

$$(2.22) \quad \begin{aligned} |I_f(g; \alpha, a, b)| \\ \leq \frac{b-a}{2(ab)^{1-1/p}} L_{2p-2}^{2-2/p}(a, b) \left(\frac{1}{\alpha q + 1} \right)^{1/q} \left(\frac{|f'(b)|^q + |f'(a)|^q}{2} \right)^{1/q}, \end{aligned}$$

where $1/p + 1/q = 1$ and $L_{2p-2}(a, b) = \left(\frac{b^{2p-1} - a^{2p-1}}{(2p-1)(b-a)} \right)^{1/(2p-2)}$ is $2p-2$ -Logarithmic mean.

Proof. Let $A_t = ta + (1-t)b$. From Lemma 3 and Lemma 2, using the Hölder inequality and the harmonically convexity of $|f'|^q$, we find

$$\begin{aligned} & |I_f(g; \alpha, a, b)| \\ & \leq \frac{ab(b-a)}{2} \int_0^1 \frac{|(1-t)^\alpha - t^\alpha|}{A_t^2} \left| f' \left(\frac{ab}{A_t} \right) \right| dt \\ & \leq \frac{ab(b-a)}{2} \left(\int_0^1 \frac{1}{A_t^{2p}} dt \right)^{1/p} \left(\int_0^1 |(1-t)^\alpha - t^\alpha|^q \left| f' \left(\frac{ab}{A_t} \right) \right|^q dt \right)^{1/q} \\ & \leq \frac{ab(b-a)}{2} \left(\int_0^1 \frac{1}{A_t^{2p}} dt \right)^{1/p} \left(\int_0^1 |1 - 2t|^{\alpha q} [t|f'(b)|^q + (1-t)|f'(a)|^q] dt \right)^{1/q} \\ & \leq \frac{ab(b-a)}{2} K_6^{1/p} (K_7 |f'(b)|^q + K_8 |f'(a)|^q)^{1/q}, \end{aligned} \quad (2.23)$$

where

$$\begin{aligned} K_6 &= \int_0^1 \frac{1}{A_t^{2p}} dt = b^{-2p} \int_0^1 \left(1 - t \left(1 - \frac{a}{b} \right) \right)^{-2p} dt \\ (2.24) \quad &= b^{-2p} {}_2F_1 \left(2p, 1; 2; 1 - \frac{a}{b} \right) = \frac{L_{2p-2}^{2p-2}(a, b)}{(ab)^{2p-1}}, \end{aligned}$$

$$\begin{aligned} K_7 &= \int_0^1 |1 - 2t|^{\alpha q} t dt \\ &= \int_0^{1/2} (1 - 2t)^{\alpha q} t dt + \int_{1/2}^1 (2t - 1)^{\alpha q} t dt \\ (2.25) \quad &= \frac{1}{2(\alpha q + 1)}, \end{aligned}$$

and

$$\begin{aligned}
 K_8 &= \int_0^1 |1-2t|^{\alpha q} (1-t) dt \\
 (2.26) \quad &= \frac{1}{2(\alpha q + 1)}.
 \end{aligned}$$

Thus, if we use (2.24), (2.25) and (2.26) in (2.23), we obtain the inequality of (2.22). This completes the proof. \square

Theorem 9. Let $f : I \subseteq (0, \infty) \rightarrow \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a, b]$, where $a, b \in I^\circ$ with $a < b$. If $|f'|^q$ is harmonically convex on $[a, b]$ for some fixed $q > 1$, then the following inequality for fractional integrals holds:

$$\begin{aligned}
 (2.27) \quad |I_f(g; \alpha, a, b)| &\leq \frac{a(b-a)}{2b} \left(\frac{1}{\alpha p + 1} \right)^{1/p} \\
 &\times \left(\frac{{}_2F_1(2q, 2; 3; 1 - \frac{a}{b}) |f'(b)|^q + {}_2F_1(2q, 1; 3; 1 - \frac{a}{b}) |f'(a)|^q}{2} \right)^{1/q},
 \end{aligned}$$

where $1/p + 1/q = 1$.

Proof. Let $A_t = ta + (1-t)b$. From Lemma 3 and Lemma 2, using the Hölder inequality and the harmonically convexity of $|f'|^q$, we find

$$\begin{aligned}
 &|I_f(g; \alpha, a, b)| \\
 &\leq \frac{ab(b-a)}{2} \int_0^1 \frac{|(1-t)^\alpha - t^\alpha|}{A_t^2} \left| f' \left(\frac{ab}{A_t} \right) \right| dt \\
 &\leq \frac{ab(b-a)}{2} \left(\int_0^1 |(1-t)^\alpha - t^\alpha|^p dt \right)^{1/p} \left(\int_0^1 \frac{1}{A_t^{2q}} \left| f' \left(\frac{ab}{A_t} \right) \right|^q dt \right)^{1/q} \\
 &\leq \frac{ab(b-a)}{2} \left(\int_0^1 |1-2t|^{\alpha p} dt \right)^{1/p} \left(\int_0^1 \frac{1}{A_t^{2q}} [t |f'(b)|^q + (1-t) |f'(a)|^q] dt \right)^{1/q} \\
 (2.28) \quad &\leq \frac{ab(b-a)}{2} K_9^{1/p} (K_{10} |f'(b)|^q + K_{11} |f'(a)|^q)^{1/q},
 \end{aligned}$$

where

$$(2.29) \quad K_9 = \int_0^1 |1-2t|^{\alpha p} dt = \frac{1}{\alpha p + 1}$$

$$\begin{aligned}
 K_{10} &= \int_0^1 t A_t^{-2q} dt = b^{-2q} \int_0^1 t \left(1 - t \left(1 - \frac{a}{b} \right) \right)^{-2q} dt \\
 (2.30) \quad &= \frac{1}{2b^{2q}} {}_2F_1 \left(2q, 2; 3; 1 - \frac{a}{b} \right)
 \end{aligned}$$

and

$$(2.31) \quad K_{11} = \int_0^1 (1-t) A_t^{-2q} dt = \frac{1}{2b^{2q}} \cdot {}_2F_1 \left(2q, 1; 3; 1 - \frac{a}{b} \right)$$

Thus, if we use (2.29), (2.30) and (2.31) in (2.28), we obtain the inequality of (2.27). This completes the proof. \square

Remark 3. If we take $\alpha = 1$ in Theorem 9, then inequality (2.27) becomes inequality (1.5) of Theorem 3.

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