

**Evaluation of Matrix-variate Gamma and Beta Integrals  
as Multiple Integrals and Kober Fractional Integral Operators  
in the Complex Matrix Variate Case**

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**Abstract**

Explicit evaluations of matrix-variate gamma and beta integrals in the complex domain by using conventional procedures is extremely difficult. Such an evaluation will reveal the structure of these matrix-variate integrals. In this article, explicit evaluations of matrix-variate gamma and beta integrals in the complex domain for the order of the matrix  $p = 1, 2$  are given. Then fractional integral operators of the Kober type are given for some specific cases of the arbitrary function. A formal definition of fractional integrals in the complex matrix-variate case was given by the author earlier as the M-convolution of products and ratios, where Kober operators become a special class of fractional integral operators.

**Keywords** Fractional integrals, complex matrix-variate case, matrix-variate gamma and beta integrals, M-convolutions, products and ratios, Kober fractional operators.

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**1. Introduction**

There is intensified activity in the area of fractional calculus in recent years due to its many applications in engineering, physical, biological and social sciences. Solutions coming out of fractional differential equations are seen to describe real-life and experimental situations much better compared to the solutions of conventional differential equations. The reason seems to be that fractional derivatives

are certain types of integrals and thereby describe global activity whereas conventional derivatives describe local activity. Fractional calculus in the scalar and matrix-variate case in the complex domain was not available in the literature but recently given in [8],[9]. Fractional calculus for the real scalar variable case is available in many books and articles, see for example [1],[2],[4],[11],[14]. Solutions of fractional differential equations in the real scalar case is available in many books and journals, see for example, [3]. Fractional integral operators in the scalar and real matrix-variate cases may be seen from [7],[12]. A pathway extension may be seen from [6],[10].

This author has given a formal definition of fractional integrals in the real and complex matrix-variate case recently as M-convolutions of products and ratios, see [8], [9]. Let  $\tilde{X}_1$  and  $\tilde{X}_2$  be  $p \times p$  hermitian positive definite matrices and let  $\tilde{U}_1 = \tilde{X}_2^{\frac{1}{2}} \tilde{X}_1 \tilde{X}_2^{\frac{1}{2}}$  and  $\tilde{U}_2 = \tilde{X}_2^{\frac{1}{2}} \tilde{X}_1^{-1} \tilde{X}_2^{\frac{1}{2}}$ . Then  $\tilde{U}_1$  is called the product and  $\tilde{U}_2$  is called the ratio. M-convolutions correspond to Mellin convolutions of products and ratios in the scalar variable cases. Product will lead to type-2 or right-sided fractional integrals and the ratio will lead to type-1 or left-sided fractional integrals. Fractional integrals of the Kober type or Kober operators are denoted by  $K_{2,\tilde{U}}^{-\alpha} f$  and  $K_{1,\tilde{U}}^{-\alpha} f$  respectively, where  $\alpha$  indicates the order of the integrals. As per the new definition introduced, these Kober operators have the following representations:

$$K_{2,\tilde{U}}^{-\alpha} f = \frac{|\det(\tilde{U})|^\beta}{\tilde{\Gamma}_p(\alpha)} \int_{\tilde{V} > \tilde{U}} |\det(\tilde{V})|^{-\beta-\alpha} |\det(\tilde{V} - \tilde{U})|^{\alpha-p} f(\tilde{V}) d\tilde{V}. \quad (1.1)$$

$$K_{1,\tilde{U}}^{-\alpha} f = \frac{|\det(\tilde{U})|^{-\alpha-\beta}}{\tilde{\Gamma}_p(\alpha)} \int_{\tilde{V} < \tilde{U}} |\det(\tilde{V})|^\beta |\det(\tilde{U} - \tilde{V})|^{\alpha-p} f(\tilde{V}) d\tilde{V}. \quad (1.2)$$

In this paper we will examine (1.1) and (1.2) for many cases of the arbitrary function  $f$  after evaluating matrix-variate gamma and beta integrals explicitly for the cases  $p = 1, 2$  in the complex domain. Explicit evaluations of matrix-variate integrals are difficult and the evaluations given in this article reveal the structure of these matrix-variate integrals in the real and complex cases.

We will use the following standard notations. All matrices appearing are  $p \times p$  with elements in the complex domain unless otherwise stated. Determinant of  $(\cdot)$  will be denoted by  $\det(\cdot)$  and  $|\det(\cdot)|$  will denote the absolute value of the determinant of  $(\cdot)$ . Matrices in the complex domain will be written with a tilde,  $\tilde{X}$ . Constant matrices will not be written with a tilde whether in the real or complex domain.  $\text{tr}(X)$  is the trace of  $X$ ,  $(d\tilde{X}) = (d\tilde{x}_{ij})$  is the matrix of differentials  $d\tilde{x}_{ij}$ 's.

Let  $\tilde{X} = X_1 + iX_2$  where  $X_1$  and  $X_2$  are real  $m \times n$  matrices and  $i = \sqrt{-1}$ . Then  $d\tilde{X} = dX_1 \wedge dX_2$  where

$$dX_1 = \prod_{i=1}^m \prod_{j=1}^n \wedge dx_{ij1} \text{ and } dX_2 = \prod_{i=1}^m \prod_{j=1}^n \wedge dx_{ij2}$$

where  $x_{ij1}$  and  $x_{ij2}$  are the  $(i, j)$ -th elements in  $X_1$  and  $X_2$  respectively, and  $\wedge$  denotes the wedge product. For any  $p \times p$  matrix  $B = B_1 + iB_2$  in the complex domain, the determinant will be a complex number of the form  $\det(B) = b + ic$  where  $b$  and  $c$  are real scalar quantities. Then the absolute value of the determinant will be of the form  $|\det(B)| = [(b + ic)(b - ic)]^{\frac{1}{2}} = [b^2 + c^2]^{\frac{1}{2}}$ . Note that the conjugate of  $B_1 + iB_2$  is  $B_1 - iB_2$ .

We need a few basic results on Jacobians of matrix transformations in the complex domain. These results, other results and properties may be seen from [5]. The results that we need will be listed here as lemmas.

**Lemma 1.1.** *Let  $\tilde{X}$  and  $\tilde{Y}$  be  $m \times n$  matrices in the complex domain. Let  $A$  be  $m \times m$  and  $B$  be  $n \times n$  nonsingular constant matrices in the sense of free of the elements in  $\tilde{X}$  and  $\tilde{Y}$ . Let  $C$  be a constant  $m \times n$  matrix. Then*

$$\tilde{Y} = A\tilde{X}B + C, \det(A) \neq 0, \det(B) \neq 0 \Rightarrow d\tilde{Y} = |\det(AA^*)|^n |\det(BB^*)|^m d\tilde{X}, \quad (1.3)$$

where  $A^*$  and  $B^*$  denote the conjugate transposes of  $A$  and  $B$  respectively.

When  $A = A^*$ , where  $A^*$  denotes the conjugate transpose of  $A$ , then it is called a hermitian matrix. The next result is about the transformation of a hermitian matrix to a hermitian matrix.

**Lemma 1.2.** *Let  $\tilde{X}$  and  $\tilde{Y}$  be  $p \times p$  hermitian matrices and let  $A$  be a nonsingular constant matrix. Then*

$$\tilde{Y} = A\tilde{X}A^* \Rightarrow d\tilde{Y} = \begin{cases} |\det(A)|^{2p} d\tilde{X} \\ |\det(AA^*)|^p d\tilde{X} \end{cases} \quad (1.4)$$

The next result is on a decomposition of the hermitian positive definite matrix  $\tilde{X} = \tilde{X}^* > O$ .

**Lemma 1.3.** *Let  $\tilde{X}$  be a  $p \times p$  hermitian positive definite matrix. Let  $\tilde{T}$  be a  $p \times p$  lower triangular matrix with diagonal elements  $t_{jj}$ 's being real and positive.*

Consider the unique representation  $\tilde{X} = \tilde{T}\tilde{T}^*$ . Then

$$\tilde{X} = \tilde{T}\tilde{T}^* \Rightarrow d\tilde{X} = 2^p \left\{ \prod_{j=1}^p t_{jj}^{2(p-j)+1} \right\} d\tilde{T}. \quad (1.5)$$

Next we define a complex matrix variate gamma function, denoted by  $\tilde{\Gamma}_p(\alpha)$  and defined as

$$\tilde{\Gamma}_p(\alpha) = \pi^{\frac{p(p-1)}{2}} \Gamma(\alpha) \Gamma(\alpha - 1) \dots \Gamma(\alpha - p + 1), \quad \Re(\alpha) > p - 1, \quad (1.6)$$

with the following integral representation:

$$\tilde{\Gamma}_p(\alpha) = \int_{\tilde{Y} > O} |\det(\tilde{Y})|^{\alpha-p} e^{-\text{tr}(\tilde{Y})} d\tilde{Y}. \quad (1.7)$$

By using Lemma 1.3 and (1.6) we can evaluate the integral in (1.7). Then a matrix-variate gamma density, in the complex domain, can be defined as follows:

$$f(\tilde{X}) = \frac{|\det(B)|^\alpha}{\tilde{\Gamma}_p(\alpha)} |\det(\tilde{X})|^{\alpha-p} e^{-\text{tr}(B\tilde{X})}, \quad \tilde{X} = \tilde{X}^* > O, \quad \Re(\alpha) > p - 1 \quad (1.8)$$

and  $f(\tilde{X}) = 0$  elsewhere, where  $B = B^* > O$  is a constant hermitian positive definite matrix.

**Lemma 1.4.** *Let  $\tilde{X}$  be a nonsingular matrix and let  $\tilde{Y} = \tilde{X}^{-1}$ . Then*

$$\tilde{Y} = \tilde{X}^{-1} \Rightarrow d\tilde{Y} = \begin{cases} |\det(\tilde{X}\tilde{X}^*)|^{-2p} d\tilde{X} & \text{for a general } \tilde{X} \\ |\det(\tilde{X}\tilde{X}^*)|^{-p} & \text{for } \tilde{X} = \tilde{X}^* \text{ or } \tilde{X} = -\tilde{X}^*. \end{cases} \quad (1.9)$$

We need complex matrix variate beta function and its integral representations. The complex matrix variate beta function will be denoted and defined as follows:

$$\tilde{B}_p(\alpha, \beta) = \frac{\tilde{\Gamma}_p(\alpha)\tilde{\Gamma}_p(\beta)}{\tilde{\Gamma}_p(\alpha + \beta)}, \quad \Re(\alpha) > p - 1, \Re(\beta) > p - 1 \quad (1.10)$$

$$= \int_{O < \tilde{X} < I} |\det(\tilde{X})|^{\alpha-p} |\det(I - \tilde{X})|^{\beta-p} d\tilde{X} \quad (1.11)$$

$$= \int_{\tilde{U} > 0} |\det(\tilde{U})|^{\alpha-p} |\det(I + \tilde{U})|^{-(\alpha+\beta)} d\tilde{U} \quad (1.12)$$

for  $\Re(\alpha) > p - 1, \Re(\beta) > p - 1$  where, in general,  $\int_{A < \tilde{X} < B} f(\tilde{X}) d\tilde{X}$  will mean the integral of a real-valued scalar function  $f(\tilde{X})$  of complex matrix argument  $\tilde{X}$  and

the integral is taken over all  $\tilde{X}$  such that  $A = A^* > O, B = B^* > O, \tilde{X} = \tilde{X}^* > O, \tilde{X} - A > O, B - \tilde{X} > O$ , where  $A$  and  $B$  are constant matrices.

## 2. Explicit Evaluations of Gamma and Beta Integrals in the Matrix Variate Case in the Complex Domain

We will try to evaluate explicitly matrix-variate gamma and beta integrals in the complex domain by using conventional procedures. This will reveal the structure of the integrals. When the real  $p \times p$  matrix  $X$  is symmetric and positive definite then for  $p = 1$  it reduces to a real scalar positive variable  $x > 0$ . If the matrix  $\tilde{X}$  is in the complex domain and if it is hermitian positive definite then also for  $p = 1$  it reduces to a real scalar positive variable  $x > 0$ . Hence for  $p = 1$  the matrix-variate gamma and beta integrals in the real positive definite or hermitian positive definite cases reduce to the ordinary gamma and beta integrals in the real scalar case.

### 2.1. Matrix-Variate Gamma in the Real Case, $p = 2$

For  $p = 2$

$$\Gamma_2(\alpha) = \pi^{\frac{1}{2}} \Gamma(\alpha) \Gamma(\alpha - \frac{1}{2}) = \int_{X > O} |X|^{\alpha - \frac{3}{2}} e^{-\text{tr}(X)} dX.$$

Let

$$X = \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix} > O \Rightarrow x_1 > 0, x_3 > 0, x_1 x_3 - x_2^2 > 0.$$

Hence we have to evaluate a triple integral over  $x_1, x_2, x_3$  subject to the conditions  $x_1 > 0, x_3 > 0, x_1 x_3 - x_2^2 > 0$ . The integral, denoted by  $\Gamma_2$  is the following:

$$\Gamma_2 = \int \int \int [x_1 x_3 - x_2^2]^{\alpha - \frac{3}{2}} e^{-(x_1 + x_3)} dx_1 \wedge dx_2 \wedge dx_3.$$

Let  $x_2 = \sqrt{x_1 x_3} r$  for fixed  $x_1, x_3$ , then the Jacobian is  $\sqrt{x_1 x_3}$ . The integral becomes

$$\Gamma_2 = \int \int \int (x_1 x_3)^{\alpha - \frac{3}{2}} (1 - r^2)^{\alpha - \frac{3}{2}} e^{-(x_1 + x_3)} \sqrt{x_1 x_3} dx_1 \wedge dx_3 \wedge dr$$

for  $-1 < r < 1, x_1 > 0, x_3 > 0$ . Integral over  $r$  gives

$$\begin{aligned} \int_{-1}^1 (1 - r^2)^{\alpha - \frac{3}{2}} dr &= 2 \int_0^1 (1 - r^2)^{\alpha - \frac{3}{2}} dr, \quad u = r^2 \\ &= \frac{\Gamma(\frac{1}{2}) \Gamma(\alpha - \frac{1}{2})}{\Gamma(\alpha)} = \sqrt{\pi} \frac{\Gamma(\alpha - \frac{1}{2})}{\Gamma(\alpha)}. \end{aligned}$$

But

$$\int_{x_1>0} x_1^{\alpha-1} e^{-x_1} dx_1 = \int_{x_3>0} x_3^{\alpha-1} e^{-x_3} dx_3 = \Gamma(\alpha).$$

Hence the integral is

$$\begin{aligned} \Gamma_2 &= [\Gamma(\alpha)]^2 \pi^{\frac{1}{2}} \frac{\Gamma(\alpha - \frac{1}{2})}{\Gamma(\alpha)} \\ &= \pi^{\frac{1}{2}} \Gamma(\alpha) \Gamma(\alpha - \frac{1}{2}), \quad \Re(\alpha) > \frac{1}{2} \\ &= \Gamma_2(\alpha) \end{aligned}$$

and hence the result is verified.

## 2.2. Matrix-Variate Gamma Integral in the Complex Case, $p = 2$

From our notation

$$\tilde{\Gamma}_2(\alpha) = \pi \Gamma(\alpha) \Gamma(\alpha - 1), \quad \Re(\alpha) > 1.$$

Consider the matrix

$$\tilde{X} = \begin{bmatrix} x_1 & x_2 + iy_2 \\ x_2 - iy_2 & x_3 \end{bmatrix} = \tilde{X}^*$$

where  $*$  denotes the conjugate transpose. When  $\tilde{X} > O$  or hermitian positive definite then we have  $x_1 > 0, x_3 > 0, x_1 x_3 - (x_2^2 + y_2^2) > 0$ . Hence integration is to be done under these conditions. Let the integral be denoted by  $\tilde{\Gamma}_2$ . Then

$$\begin{aligned} \tilde{\Gamma}_2 &= \int_{\tilde{X} > O} |\det(\tilde{X})|^{\alpha-2} e^{-\text{tr}(\tilde{X})} d\tilde{X} \\ &= \int \int \int \int [x_1 x_3 - (x_2^2 + y_2^2)]^{\alpha-2} e^{-(x_1+x_3)} dx_1 \wedge dx_3 \wedge dx_2 \wedge dy_2. \end{aligned}$$

Let  $x_2 = \sqrt{x_1 x_3} r \cos \theta, y_2 = \sqrt{x_1 x_3} r \sin \theta$  then the Jacobian is  $x_1 x_3 r$  and the quadruple integral becomes

$$\tilde{\Gamma}_2 = \int_{\theta=0}^{2\pi} \int_{r=0}^1 \int_{x_1>0} \int_{x_3>0} (x_1 x_3)^{\alpha-1} r (1-r^2)^{\alpha-2} e^{-(x_1+x_3)} dx_1 \wedge dx_3 \wedge dr \wedge d\theta.$$

Integral over  $\theta$  gives  $2\pi$ . Put  $u = r^2$ . Integral over  $r$  gives

$$2 \int_0^1 r (1-r^2)^{\alpha-2} dr = \int_0^1 (1-u)^{\alpha-2} du = \frac{\Gamma(1) \Gamma(\alpha-1)}{\Gamma(\alpha)}.$$

Integrals over  $x_1$  and  $x_3$  give one  $\Gamma(\alpha)$  each. Then the quadruple integral gives

$$\pi[\Gamma(\alpha)]^2 \frac{\Gamma(1)\Gamma(\alpha-1)}{\Gamma(\alpha)} = \pi \Gamma(\alpha)\Gamma(\alpha-1) = \tilde{\Gamma}_2(\alpha)$$

for  $\Re(\alpha) > 1$  which verifies the result.

### 2.3. Matrix-Variate Beta Integral in the Real Case, $p = 2$

The general definition of real matrix-variate beta function and an integral representation are the following:

$$B_p(\alpha, \beta) = \frac{\Gamma_p(\alpha)\Gamma_p(\beta)}{\Gamma_p(\alpha+\beta)} = \int_O^I |X|^{\alpha-\frac{p+1}{2}} |I-X|^{\beta-\frac{p+1}{2}} dX$$

for  $X > O$ ,  $\Re(\alpha) > \frac{p-1}{2}$ ,  $\Re(\beta) > \frac{p-1}{2}$ . As mentioned earlier, for  $p = 1$  the real and complex matrix-variate cases coincide with the real scalar variable case. Hence we look into the case  $p = 2$ .

$$\begin{aligned} B_2(\alpha, \beta) &= \frac{\Gamma_2(\alpha)\Gamma_2(\beta)}{\Gamma_2(\alpha+\beta)} \\ &= \pi^{\frac{1}{2}} \frac{\Gamma(\alpha)\Gamma(\alpha-\frac{1}{2})\Gamma(\beta)\Gamma(\beta-\frac{1}{2})}{\Gamma(\alpha+\beta)\Gamma(\alpha+\beta-\frac{1}{2})} \end{aligned}$$

for  $\Re(\alpha) > \frac{1}{2}$ ,  $\Re(\beta) > \frac{1}{2}$ . Let

$$X = \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix} \text{ then } I - X = \begin{bmatrix} 1-x_1 & -x_2 \\ -x_2 & 1-x_3 \end{bmatrix}$$

and the integral representation becomes

$$\int_O^I |X|^{\alpha-\frac{3}{2}} |I-X|^{\beta-\frac{3}{2}} dX = \int_O^I [x_1x_3-x_2^2]^{\alpha-\frac{3}{2}} [(1-x_1)(1-x_3)-x_2^2]^{\beta-\frac{3}{2}} dx_1 \wedge dx_3 \wedge dx_2.$$

But

$$(x_1x_3 - x_2^2)^{\alpha-\frac{3}{2}} = x_3^{\alpha-\frac{3}{2}} [x_1 - \frac{x_2^2}{x_3}]^{\alpha-\frac{3}{2}}$$

which means that  $x_1 > \frac{x_2^2}{x_3}$ . Now,

$$[(1-x_1)(1-x_3) - x_2^2]^{\beta-\frac{3}{2}} = (1-x_3)^{\beta-\frac{3}{2}} [1-x_1 - \frac{x_2^2}{1-x_3}]^{\beta-\frac{3}{2}}$$

which means that  $x_1 < 1 - \frac{x_2^2}{1-x_3}$ . That is,

$$\frac{x_2^2}{x_3} < x_1 < 1 - \frac{x_2^2}{1-x_3}$$

which means that  $0 < u < b$  where  $u = x_1 - \frac{x_2^2}{x_3}$ ,  $b = 1 - \frac{x_2^2}{x_3(1-x_3)}$ . Then the integrand will reduce to the following factors:

$$\begin{aligned} & x_3^{\alpha-\frac{3}{2}}(1-x_3)^{\beta-\frac{3}{2}}u^{\alpha-\frac{3}{2}}b^{\beta-\frac{3}{2}}\left[1-\frac{u}{b}\right]^{\beta-\frac{3}{2}} \\ &= x_3^{\alpha-\frac{3}{2}}(1-x_3)^{\beta-\frac{3}{2}}v^{\alpha-\frac{3}{2}}(1-v)^{\beta-\frac{3}{2}}b^{\alpha+\beta-2} \end{aligned}$$

where  $v = \frac{u}{b}$ . Put  $z = \frac{x_2}{\sqrt{x_3(1-x_3)}}$  for fixed  $x_3$ . The integral over  $z$  gives

$$\begin{aligned} \int_z (1-z^2)^{\alpha+\beta-2} dz &= 2 \int_{z>0} (1-z^2)^{\alpha+\beta-2} dz \\ &= \int_{w>0} w^{\frac{1}{2}-1} (1-w)^{\alpha+\beta-2} dw, w = z^2 \\ &= \frac{\Gamma(\frac{1}{2})\Gamma(\alpha+\beta-1)}{\Gamma(\alpha+\beta-\frac{1}{2})}, \Re(\alpha+\beta) > 1. \end{aligned}$$

Now the integrals over  $x_3, v$  and  $w$  give the following:

$$\begin{aligned} \int_0^1 x_3^{\alpha-1} (1-x_3)^{\beta-1} dx_3 &= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \Re(\alpha) > 0, \Re(\beta) > 0 \\ \int_0^1 v^{\alpha-\frac{3}{2}} (1-v)^{\beta-\frac{3}{2}} dv &= \frac{\Gamma(\alpha-\frac{1}{2})\Gamma(\beta-\frac{1}{2})}{\Gamma(\alpha+\beta-1)}, \Re(\alpha) > \frac{1}{2}, \Re(\beta) > \frac{1}{2} \\ 2 \int_{z>0} (1-z^2)^{\alpha+\beta-2} dz &= \frac{\Gamma(\frac{1}{2})\Gamma(\alpha+\beta-1)}{\Gamma(\alpha+\beta-\frac{1}{2})}. \end{aligned}$$

Hence the total integral, by taking the product, is

$$\pi^{\frac{1}{2}} \frac{\Gamma(\alpha)\Gamma(\alpha-\frac{1}{2})\Gamma(\beta)\Gamma(\beta-\frac{1}{2})}{\Gamma(\alpha+\beta)\Gamma(\alpha+\beta-\frac{1}{2})} = \frac{\Gamma_2(\alpha)\Gamma_2(\beta)}{\Gamma_2(\alpha+\beta)} = B_2(\alpha, \beta).$$

Hence the result is verified for the real case. Explicit evaluations for  $p \geq 3$  will be difficult and it is simpler to use matrix methods directly starting from  $p \geq 2$ .

#### 2.4. Matrix-variate Beta Integral in the Complex Case: $p = 2$



Let

$$\tilde{X} = \begin{bmatrix} x_1 & x_2 + iy_2 \\ x_2 - iy_2 & x_3 \end{bmatrix}, \quad x_1 > 0, x_3 > 0$$

and

$$\begin{aligned} |\det(\tilde{X})|^{\alpha-2} &= [x_1 x_3 - (x_2^2 + y_2^2)]^{\alpha-2} \\ |\det(I - \tilde{X})|^{\beta-2} &= [(1 - x_1)(1 - x_3) - (x_2^2 + y_2^2)]^{\beta-2}. \end{aligned}$$

Steps parallel to the ones in the real case will go through and we have the factors

$$x_3^{\alpha-2}(1 - x_3)^{\beta-2}v^{\alpha-2}(1 - v)^{\beta-2}b^{\alpha+\beta-3}, \quad b = 1 - \frac{(x_2^2 + y_2^2)}{x_3(1 - x_3)}.$$

Put  $z_1 = \frac{x_2}{\sqrt{x_3(1-x_3)}}$ ,  $z_2 = \frac{y_2}{\sqrt{x_3(1-x_3)}}$  for fixed  $x_3$ . Then  $b = 1 - (z_1^2 + z_2^2)$ . Put  $z_1 = r \cos \theta$ ,  $z_2 = r \sin \theta$ ,  $0 \leq r \leq 1$ ,  $0 \leq \theta \leq 2\pi$ . The Jacobian is  $r$ .

$$\begin{aligned} \int_{z_1} \int_{z_2} b^{\alpha+\beta-3} dz_1 \wedge dz_2 &= \int_0^{2\pi} \int_{r=0}^1 r(1 - r^2)^{\alpha+\beta-3} dr \wedge d\theta \\ &= \pi \int_0^1 2r(1 - r^2)^{\alpha+\beta-3} dr \\ &= \pi \frac{\Gamma(1)\Gamma(\alpha + \beta - 2)}{\Gamma(\alpha + \beta - 1)}, \quad \Re(\alpha + \beta) > 2. \end{aligned}$$

Now, the integral over  $x_3, v, z_1, z_2$  give

$$\begin{aligned} \int_0^1 x_3^{\alpha-1}(1 - x_3)^{\beta-1} dx_3 &= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}, \quad \Re(\alpha) > 0, \Re(\beta) > 0 \\ \int_0^1 v^{\alpha-2}(1 - v)^{\beta-2} dv &= \frac{\Gamma(\alpha - 1)\Gamma(\beta - 1)}{\Gamma(\alpha + \beta - 2)}, \quad \Re(\alpha) > 1, \Re(\beta) > 1 \\ \int_{z_1} \int_{z_2} b^{\alpha+\beta-3} dz_1 \wedge dz_2 &= \pi \frac{\Gamma(\alpha + \beta - 2)}{\Gamma(\alpha + \beta - 1)}. \end{aligned}$$

Then the product gives

$$\pi \frac{\Gamma(\alpha)\Gamma(\alpha - 1)\Gamma(\beta)\Gamma(\beta - 1)}{\Gamma(\alpha + \beta)\Gamma(\alpha + \beta - 1)} = \frac{\tilde{\Gamma}_2(\alpha)\tilde{\Gamma}_2(\beta)}{\tilde{\Gamma}_2(\alpha + \beta)} = \tilde{B}_2(\alpha, \beta)$$

for  $\Re(\alpha) > 1, \Re(\beta) > 1$ , and hence the result is verified. Explicit evaluations for  $p \geq 3$  will be difficult and it is simpler to use matrix methods directly starting from  $p \geq 2$ .

### 3. Some Special Cases of Fractional Integrals in the Complex Matrix-variate Case

Let  $f(\tilde{V}) = |\det(\tilde{V})|^{-\gamma}$ . Consider Kober operator of the second kind when  $f(\tilde{V})$  is as given above.

$$\begin{aligned}
K_{2,\tilde{U}}^{-\alpha} f &= \frac{|\det(\tilde{U})|^\beta}{\tilde{\Gamma}_p(\alpha)} \int_{\tilde{V} > \tilde{U}} |\det(\tilde{V})|^{-\beta-\alpha} |\det(\tilde{V} - \tilde{U})|^{\alpha-p} f(\tilde{V}) d\tilde{V} \\
&= \frac{|\det(\tilde{U})|^\beta}{\tilde{\Gamma}_p(\alpha)} \int_{\tilde{V} > \tilde{U}} |\det(\tilde{V})|^{-\beta-\alpha-\gamma} |\det(\tilde{V} - \tilde{U})|^{\alpha-p} d\tilde{V} \\
&= \frac{|\det(\tilde{U})|^\beta}{\tilde{\Gamma}_p(\alpha)} \int_{\tilde{W} > O} |\det(\tilde{W})|^{\alpha-p} |\det(\tilde{W} + \tilde{U})|^{-\beta-\alpha-\gamma} d\tilde{W}, \quad \tilde{W} = \tilde{V} - \tilde{U} \\
&= \frac{|\det(\tilde{U})|^{-\alpha-\gamma}}{\tilde{\Gamma}_p(\alpha)} \int_{\tilde{W} > O} |\det(\tilde{W})|^{\alpha-p} |\det(I + \tilde{U}^{-\frac{1}{2}} \tilde{W} \tilde{U}^{-\frac{1}{2}})|^{-\beta-\alpha-\gamma} d\tilde{W} \\
&= \frac{|\det(\tilde{U})|^{-\gamma}}{\tilde{\Gamma}_p(\alpha)} \int_{\tilde{T} > O} |\det(\tilde{T})|^{\alpha-p} |\det(I + \tilde{T})|^{-\beta-\alpha-\gamma} d\tilde{T}, \quad \tilde{T} = \tilde{U}^{-\frac{1}{2}} \tilde{W} \tilde{U}^{-\frac{1}{2}} \\
&= |\det(\tilde{U})|^{-\gamma} \frac{\tilde{\Gamma}_p(\beta + \gamma)}{\tilde{\Gamma}_p(\alpha + \beta + \gamma)}, \quad \Re(\alpha) > p - 1, \quad \Re(\beta + \gamma) > p - 1.
\end{aligned}$$

Consider Kober operator of the first kind.

$$K_{1,\tilde{U}}^{-\alpha} f = \frac{|\det(\tilde{U})|^{-\alpha-\beta}}{\tilde{\Gamma}_p(\alpha)} \int_{\tilde{V} < \tilde{U}} |\det(\tilde{V})|^\beta |\det(\tilde{U} - \tilde{V})|^{\alpha-p} f(\tilde{V}) d\tilde{V}.$$

**Special case 1:**  $f(\tilde{V}) = |\det(\tilde{V})|^\gamma$ . Then

$$\begin{aligned}
K_{1,\tilde{U}}^{-\alpha} f &= \frac{|\det(\tilde{U})|^{-\alpha-\beta}}{\tilde{\Gamma}_p(\alpha)} \int_{\tilde{V} < \tilde{U}} |\det(\tilde{V})|^{\beta+\gamma} |\det(\tilde{U})|^{\alpha-p} |\det(I - \tilde{U}^{-\frac{1}{2}} \tilde{V} \tilde{U}^{-\frac{1}{2}})|^{\alpha-p} d\tilde{V} \\
&= \frac{|\det(\tilde{U})|^\gamma}{\tilde{\Gamma}_p(\alpha)} \int_{O < \tilde{W} < I} |\det(\tilde{W})|^{\beta+\gamma} |\det(I - \tilde{W})|^{\alpha-p} d\tilde{W}, \quad \tilde{W} = \tilde{U}^{-\frac{1}{2}} \tilde{V} \tilde{U}^{-\frac{1}{2}} \\
&= |\det(\tilde{U})|^\gamma \frac{\tilde{\Gamma}_p(\beta + \gamma + p)}{\tilde{\Gamma}_p(\alpha + \beta + \gamma + p)}
\end{aligned}$$

for  $\Re(\alpha) > p - 1, \Re(\beta + \gamma) > -1$ , by evaluating by using a type-1 beta integral.

**Special case 2:**  $f(\tilde{V}) = |\det(I - \tilde{V})|^\gamma$ .

$$\begin{aligned} K_{1,\tilde{U}}^{-\alpha} f &= \frac{|\det(\tilde{U})|^{-\alpha-\beta}}{\tilde{\Gamma}_p(\alpha)} \int_{\tilde{V} < \tilde{U}} |\det(\tilde{V})|^\beta |\det(I - \tilde{V})|^{-\gamma} |\det(\tilde{U} - \tilde{V})|^{\alpha-p} d\tilde{V} \\ &= \frac{1}{\tilde{\Gamma}_p(\alpha)} \int_{O < \tilde{W} < I} |\det(\tilde{W})|^\beta |\det(I - \tilde{W})|^{\alpha-p} |\det(I - \tilde{U}^{-\frac{1}{2}} \tilde{W} \tilde{U}^{-\frac{1}{2}})|^{-\gamma} d\tilde{W}, \end{aligned}$$

for  $\tilde{W} = \tilde{U}^{-\frac{1}{2}} \tilde{V} \tilde{U}^{-\frac{1}{2}}$ . Now, evaluating by using Example 6.4 of [5] we have

$$K_{1,\tilde{U}}^{-\alpha} f = \frac{\tilde{\Gamma}_p(\beta + p)}{\tilde{\Gamma}_p(\alpha + \beta + p)} {}_2F_1(\beta + p, \gamma; \alpha + \beta + p; \tilde{U}), O < \tilde{U} < I$$

for  $\Re(\alpha) > p - 1, \Re(\beta) > -1$ .

**Special case 3:**  $f(\tilde{V}) = |\det(\tilde{V})|^\gamma |\det(I - \tilde{V})|^{-\delta}$ . Then, following through the steps in special cases 1 and 2 we have

$$K_{1,\tilde{U}}^{-\alpha} f = |\det(\tilde{U})|^\gamma \frac{\tilde{\Gamma}_p(\beta + \gamma + p)}{\tilde{\Gamma}_p(\alpha + \beta + \gamma + p)} {}_2F_1(\beta + \gamma + p, \delta; \alpha + \beta + \gamma + p; \tilde{U})$$

for  $O < \tilde{U} < I, \Re(\alpha) > p - 1, \Re(\beta + \gamma) > -1$ .

**Special case 4:** We can have a hypergeometric series for  $f(\tilde{V})$ . For the meaning of the symbol  $[a]_K$ , zonal polynomial  $\tilde{C}_K(\tilde{V})$  and the partition  $K$ , see for example [5],[13]. Let

$$\begin{aligned} f(\tilde{V}) &= {}_rF_s(a_1, \dots, a_r; b_1, \dots, b_s; \tilde{V}) \\ &= \sum_{k=0}^{\infty} \sum_K \frac{[a_1]_K \dots [a_r]_K}{[b_1]_K \dots [b_s]_K} \frac{\tilde{C}_K(\tilde{V})}{k!} \end{aligned}$$

for  $s \geq r$  or  $r = s + 1$  and  $\|\tilde{V}\| < 1$  where  $\|(\cdot)\|$  denotes a norm of  $(\cdot)$ . Then

$$\begin{aligned} K_{1,\tilde{U}}^{-\alpha} f &= \sum_{k=0}^{\infty} \sum_K \frac{[a_1]_K \dots [a_r]_K}{[b_1]_K \dots [b_s]_K k!} \frac{|\det(\tilde{U})|^{-\alpha-\beta}}{\tilde{\Gamma}_p(\alpha)} \\ &\quad \times \int_{\tilde{V} < \tilde{U}} |\det(\tilde{V})|^\beta |\det(\tilde{U} - \tilde{V})|^{\alpha-p} C_K(\tilde{V}) d\tilde{V}. \end{aligned}$$

Take out  $\tilde{U}$  from  $|\det(\tilde{U} - \tilde{V})|^{\alpha-p}$  and make the transformation  $\tilde{W} = \tilde{U}^{-\frac{1}{2}}\tilde{V}\tilde{U}^{-\frac{1}{2}}$ . Then the integral part becomes

$$\begin{aligned} & \frac{1}{\tilde{\Gamma}_p(\alpha)} \int_{O < \tilde{W} < I} |\det(\tilde{W})|^\beta |\det(I - \tilde{W})|^{\alpha-p} \tilde{C}_K(\tilde{U}^{\frac{1}{2}}\tilde{W}\tilde{U}^{\frac{1}{2}}) \\ &= \frac{\tilde{\Gamma}_p(\beta + p)}{\tilde{\Gamma}_p(\alpha + \beta + p)} \frac{[\beta + p]_K}{[\alpha + \beta + p]_K} \tilde{C}_K(\tilde{U}) \end{aligned}$$

by using (6.1.21) of [5]. Now, writing the result as a hypergeometric function we have

$$K_{1,\tilde{U}}^{-\alpha} f = \frac{\tilde{\Gamma}_p(\beta + p)}{\tilde{\Gamma}_p(\alpha + \beta + p)} {}_{r+1}F_{s+1}(a_1, \dots, a_r, \beta + p; b_1, \dots, b_s, \alpha + \beta + p; \tilde{U})$$

for  $s \geq r$  or  $r = s + 1$  and  $\|\tilde{U}\| < 1$ .

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