Integral relations for solutions of confluent Heun equations

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Abstract: Firstly, we construct kernels of integral relations among solutions of the confluent Heun equation (CHE) and its limit, the reduced CHE (RCHE). In both cases we generate additional kernels by systematically applying substitutions of variables. Secondly, we establish integral relations between known solutions of the CHE that are power series and solutions that are series of special functions; and similarly for solutions of the RCHE. Thirdly, by using one of the integral relations as an integral transformation we obtain a new series solution of the spheroidal wave equation. From this solution we construct new solutions of the general CHE, and show that these are suitable for solving the radial part of the two-center problem in quantum mechanics. Finally, by applying a limiting process to kernels for the CHEs we obtain kernels for two double-confluent Heun equations. As a result, we deal with kernels of four equations of the Heun family, each equation presenting a distinct structure of singularities. In addition, we find that the known kernels for the Mathieu equation are special instances of kernels of the RCHE.

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1. Introductory remarks

Recently we have found that the transformations of variables which preserve the form of the general Heun equation correspond to transformations which preserve the form of the equation for the kernels of integral relations among solutions of the Heun equation [1]. In fact, by using the known transformations of the Heun equation [2, 3] we have found prescriptions for transforming kernels and, in this manner, we have generated several new kernels for the equation.

The above correspondence can be extended to the confluent equations of the Heun family, that is, to the (single) confluent, double-confluent, biconfluent and triconfluent Heun equations [4, 5], as well as to the reduced forms of such equations [6, 7]. In the present study we consider only the confluent Heun equation (CHE) and equations connected to the CHE by limiting processes. Specifically:

- we deal with the construction and transformations of integral kernels for CHE and its limit called reduced confluent Heun equation (RCHE);
- from some of these kernels we establish integral relations between known solutions for the CHE;
- using one of the relations as an integral transformation we obtain new solutions in series of confluent hypergeometric functions for the CHE;
- we show that the previous solutions are suitable to solve the radial part of the Schrödinger equation for an electron in the field of two Coulomb centres [8] (two-centre problem);
- finally, from kernels of the CHE and RCHE we find kernels for the double-confluent Heun equation (DHE) and for the reduced DHE (RDHE).

We write the CHE as [9]

$$z(z - z_0)\frac{d^2U}{dz^2} + (B_1 + B_2 z)\frac{dU}{dz} + \left[B_3 - 2\omega\eta(z - z_0) + \omega^2 z(z - z_0)\right]U = 0, \tag{1}$$

where z_0, B_i, η and ω are constants. This equation is called *generalized spheroidal wave equation* by Leaver [9] but sometimes such expression refers to a particular case of the CHE [5, 10]. Excepting the special case represented by the Mathieu equation, the CHE is the most studied of the confluent Heun equations and embraces the (ordinary) spheroidal equation as a particular case [5]. However, further studies are necessary due to the recent emergence of several classes of quantum two-state systems ruled by the CHE [11]. On the other side, the reduced confluent Heun equation (RCHE) is written as

$$z(z - z_0)\frac{d^2U}{dz^2} + (B_1 + B_2 z)\frac{dU}{dz} + \left[B_3 + q(z - z_0)\right]U = 0, \tag{2}$$

where z_0, B_i and q ($q \neq 0$) are constants. The RCHE describes the angular part of the Schrödinger equation for an electron in the field of a point electric dipole [12, 13]. It appears as well in the study of two-level systems [14], polymer dynamics [15] and theory of gravitation [16]. The form (2) for the RCHE results from the CHE (1) by means of the limits

$$\omega \to 0, \quad \eta \to \infty \quad \text{such that} \quad 2\eta\omega = -q, \qquad \text{[Whittaker-Ince limit]}. \tag{3}$$

In both equations, $z = 0$ and $z = z_0$ are regular singular points with exponents $(0, 1 + B_1/z_0)$ and $(0, 1 - B_2 - B_1/z_0)$, respectively, that is, from ascending power series solutions we find

$$z \to 0: \quad U(z) \sim 1 \quad \text{or} \quad U(z) \sim z^{1 + \frac{B_1}{z_0}}; \qquad z \to z_0: \quad U(z) \sim 1 \quad \text{or} \quad U(z) \sim (z - z_0)^{1 - B_2 - \frac{B_1}{z_0}}. \tag{4}$$

In contrast, at the irregular singular point $z = \infty$, the behaviour of the solutions is different for each equation since

$$z \to \infty: \quad U(z) \sim e^{\pm i\omega z}\, z^{\mp i\eta - (B_2/2)} \text{ for the CHE (1) and } U(z) \sim e^{\pm 2i\sqrt{qz}}\, z^{(1/4) - (B_2/2)} \text{ for the RCHE (2)}, \tag{5}$$

as follow from the normal and the subnormal Thomé solutions [17] for the CHE and RCHE, respectively.

According to the concepts of Ref. [7], the s-rank of the singularity at $z = \infty$ is 2 for the CHE, and 3/2 for the RCHE. However, more important is the fact that the solutions exhibit the above behavior predicted by the normal or subnormal Thomé solutions, and the fact that the Whittaker-Ince limit (3) may generate solutions to the RCHE. In effect, most of the known solutions for the RCHE [18–20] has been obtained from solutions of the CHE by means of the limit (3). Despite this, the main part of the present study is restricted to integral relations concerning the CHE. Relations for RCHE are relegated to an appendix. In appendices we also present kernels for double-confluent Heun equations which are obtained by taking $z_0 = 0$ in Eqs. (1) and (2).

Integral relations are important because, in principle, they make possible the transformation of known solutions into solutions with different properties. However, apart from the Mathieu equation, only in rare cases this task has been accomplished successfully. One case is constituted by the expansions of the Lamé functions in series of associated Legendre functions [21], obtained by Erdélyi from Fourier-Jacobi series for the Lamé equation; however, as far we are aware, his solutions have not been extended for the general Heun equation (of which Lamé equation is a particular case). Another example is a Leaver expansion in series of irregular confluent hypergeometric functions for the CHE [9], obtained from a power series; the integral transformation was originally constructed for a particular case of CHE but the expansion has been generalized for any CHE.

To establish integral relations for solutions it is necessary to get appropriate integral kernels. To this end, in section 2 we proceed as in case of the general Heun equation [1]. In other words, firstly we insert into the integral connecting two solutions a weight function $w(z, t)$ which allows to write the CHE and the equation for its kernels in terms of differential operators functionally identical (respecting z and t). In this manner, by examining each variable substitution which leaves invariant the form of the CHE (one variable, z) we find prescriptions for the variables transformations which preserve the form of the equation for the kernels (two variables, z and t). By using these substitutions, we may systematically convert a given (initial) kernel into new kernels. As initial kernels we use the ones obtained as limits of kernels of the general Heun equation [1], adapting them to the form (1) for the CHE.

In section 3 we find integral relations which transform the Jaffé power-series solutions [22] into expansions in series of irregular confluent hypergeometric functions, including the aforementioned solution given by Leaver. In the second place, we find that the power-series solutions of Baber and Hassé [23] are transformed into expansions in series of regular confluent hypergeometric functions. These are integral transformations among known solutions of the CHE. In both examples, power-series solutions are converted into series of confluent hypergeometric functions. However, there are the non-integral transformations (involving only substitutions of variables) which do not modify the type of series: these transform, for example, a power-series solution into another power-series solution, and an expansion in series of hypergeometric functions into another expansion in series hypergeometric functions. Integral relations among these two types of modified series demand the use of kernels transformed in accordance with the prescriptions mentioned in the previous paragraph.

Analogously, in section 4 we apply an integral transformation to an asymptotic (Thomé) solution of the spheroidal equation and obtain a new solution in series of irregular confluent hypergeometric functions. That solution is extended to any CHE (not just the spheroidal equation); then, by substitutions of variables, we obtain a group of solutions for the CHE with domains of convergence different of the ones of the asymptotic solutions. Therefore, by combining integral and non-integral transformations we get new solutions for the CHE; as a test, we show that some of these solutions afford bounded and convergent solutions to the radial part of the two-center problem.

In section 5, we present concluding remarks and mention open issues. In appendix A we write some formulas concerning special functions, while in appendix B we discuss the convergence of asymptotic solutions for the CHE. In appendices C, D and E we obtain, respectively, kernels for the reduced CHE (RCHE), for the double-confluent Heun equation (DHE) and for the reduced DHE (RDHE).

2. Kernels for the confluent Heun equation

In this section we regard kernels for the CHE (1). In particular,

- in section 2.1 we get the correspondences among substitutions of variables which preserve the form of the CHE and the substitutions which preserve the equation of the kernels of the CHE;
- in section 2.2 we construct a group of kernels with an arbitrary constant of separation λ, given by products of two confluent hypergeometric functions and elementary functions;
- in section 2.3 we find another group of kernels with an arbitrary constant of separation λ, given by products of confluent hypergeometric functions and Gauss hypergeometric functions (and elementary functions);
- in sections 2.4, 2.5 and 2.6, by taking suitable values for λ we get kernels given by products of elementary and special functions; thus, in sections 2.4 and 2.5 we find products of elementary and confluent hypergeometric functions and, in section 2.6, products of elementary and Gauss hypergeometric functions.

Later on, in section 4, we will need kernels for the ordinary spheroidal wave equation [5]

$$\frac{d}{dx}\left[\left(1 - x^2\right) \frac{dX(x)}{dx} \right] + \left[\gamma^2 (1 - x^2) + \bar{\lambda} - \frac{\mu^2}{1 - x^2} \right] X(x) = 0. \tag{6}$$

Such kernels are obtained from the ones of the CHE through the substitutions

$$x = 1 - 2z, \qquad X(x) = z^{\mu/2} \, (z-1)^{\mu/2} U(z), \tag{7}$$

which give

$$z(z-1)\frac{d^2 U}{dz^2} + \left[-(\mu+1) + 2(\mu+1)z\right]\frac{dU}{dz} + \left[\mu(\mu+1) - \bar{\lambda} + 4\gamma^2 z(z-1)\right]U = 0, \tag{8a}$$

that is, the CHE (1) with

$$z_0 = 1, \qquad B_2 = -2B_1 = 2(\mu+1), \qquad B_3 = \mu(\mu+1) - \bar{\lambda}, \qquad \eta = 0, \qquad \omega^2 = 4\gamma^2. \tag{8b}$$

Thus, the spheroidal equation (6) will be treated as a CHE with $z_0 = 1$, $\eta = 0$ and $B_2 = -2B_1$, namely,

$$z(z-1)\frac{d^2 U}{dz^2} + (B_1 - 2B_1 z)\frac{dU}{dz} + \left[B_3 + \omega^2 z(z-1)\right]U = 0. \tag{9}$$

2.1. Transformations of the CHE and its kernels

Defining the operator L_z by

$$L_z = z[z - z_0]\frac{\partial^2}{\partial z^2} + [B_1 + B_2 z]\frac{\partial}{\partial z} + \left[\omega^2 z(z - z_0) - 2\omega\eta z\right] \tag{10}$$

and interpreting this as an ordinary differential operator, the CHE (1) reads

$$[L_z + B_3 + 2\eta\omega z_0]U(z) = 0. \tag{11}$$

The adjoint operator $\bar{L}_z$ corresponding to L_z is [24]

$$\bar{L}_z = z(z - z_0)\frac{\partial^2}{\partial z^2} + [-2z_0 - B_1 + (4 - B_2)z]\frac{\partial}{\partial z} + \left[\omega^2 z(z - z_0) - 2\omega\eta z + 2 - B_2\right]. \tag{12}$$

On the other side, if $U(z)$ is a known solution of the CHE, we seek new solutions $\mathcal{U}(z)$ having the form

$$\mathcal{U}(z) = \int_{t_1}^{t_2} K(z,t)U(t)dt = \int_{t_1}^{t_2} w(z,t)G(z,t)U(t)dt = \int_{t_1}^{t_2} t^{-1 - \frac{B_1}{z_0}}(t - z_0)^{B_2 + \frac{B_1}{z_0} - 1}G(z,t)U(t)dt, \tag{13}$$

$$w(z,t) = t^{-1 - \frac{B_1}{z_0}}(t - z_0)^{B_2 + \frac{B_1}{z_0} - 1},$$

where the kernel $K(z,t)$ or $G(z,t)$ is determined from a partial differential equation. The general theory is usually established for the function $K(z,t)$ [24], but to study the transformations of kernels we will deal with $G(z,t)$. If the integration endpoints t_1 and t_2 are independent of z, by applying L_z to integral (13) we find

$$L_z\mathcal{U}(z) = \int_{t_1}^{t_2} [L_z K(z,t)]U(t)dt = \int_{t_1}^{t_2} U(t)\left[L_z - \bar{L}_t\right]K(z,t)dt + \int_{t_1}^{t_2} U(t)\bar{L}_t K(z,t)dt, \tag{14}$$

$\bar{L}_t$ being obtained from $\bar{L}_z$ by replacing z with t. Now we demand that

$$\left[L_z - \bar{L}_t\right]K(z,t) = 0 \quad \Leftrightarrow \quad [L_z - L_t]G(z,t) = 0. \tag{15}$$

Thence, by using the Lagrange identity

$$U(t)\bar{L}_t K(z,t) - K(z,t)L_t U(t) = \frac{\partial}{\partial t}P(z,t),$$

where the bilinear concomitant $P(z,t)$ is given by

$$\begin{aligned} P(z,t) &= t(t - z_0)\left[U(t)\frac{\partial K(z,t)}{\partial t} - K(z,t)\frac{dU(t)}{dt}\right] - [(B_2 - 2)t + B_1 + z_0]U(t)K(z,t) \\ &= t^{-\frac{B_1}{z_0}}(t - z_0)^{B_2 + \frac{B_1}{z_0}}\left[U(t)\frac{\partial G(z,t)}{\partial t} - G(z,t)\frac{dU(t)}{dt}\right], \end{aligned} \tag{16}$$

Eq. (14) reduces to

$$L_z\mathcal{U}(z) = \int_{t_1}^{t_2}\left[K(z,t)L_t U(t) + \frac{\partial P(z,t)}{\partial t}\right]dt = -(B_3 + 2\eta\omega z_0)\int_{t_1}^{t_2} K(z,t)U(t)dt + \int_{t_1}^{t_2}\frac{\partial P(z,t)}{\partial t}dt,$$

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where in the last step we have used equation (11). Using equation (13) as well, this yields

$$[L_z + B_3 + 2\eta\omega z_0]\mathcal{U}(z) = P(z, t_2) - P(z, t_1). \tag{17}$$

Therefore, $\mathcal{U}(z)$ is also a solution of the CHE if: (i) the kernel satisfies Eq. (15), (ii) the integral (13) exists and (iii) the right-hand side of Eq. (17) vanishes.

Now let us examine the transformations of the solutions $U(z)$ and kernels $G(z,t)$. If $U(z) = U(B_1, B_2, B_3; z_0, \omega, \eta; z)$ denotes one solution of the CHE, the following transformations [18, 25, 26] – T_1, T_2, T_3 and T_4 – leave invariant the form of the CHE:

$$T_1 U(z) = z^{1 + \frac{B_1}{z_0}} U(C_1, C_2, C_3; z_0, \omega, \eta; z), \qquad T_2 U(z) = (z - z_0)^{1 - B_2 - \frac{B_1}{z_0}} U(B_1, D_2, D_3; z_0, \omega, \eta; z),$$

$$T_3 U(z) = U(B_1, B_2, B_3; z_0, -\omega, -\eta; z), \qquad T_4 U(z) = U(-B_1 - B_2 z_0, B_2, B_3 + 2\eta\omega z_0; z_0, -\omega, \eta; z_0 - z), \tag{18}$$

where

$$C_1 = -B_1 - 2z_0, \qquad C_2 = 2 + B_2 + \frac{2B_1}{z_0}, \qquad C_3 = B_3 + \left[1 + \frac{B_1}{z_0}\right]\left[B_2 + \frac{B_1}{z_0}\right],$$

$$D_2 = 2 - B_2 - \frac{2B_1}{z_0}, \qquad D_3 = B_3 + \frac{B_1}{z_0}\left(\frac{B_1}{z_0} + B_2 - 1\right). \tag{19}$$

By composition of these transformations, from an initial solution we may generate a group containing up to 16 solutions. To get the corresponding transformations for the kernels, we notice that the operators L_z and L_t (which appear in the CHE (11) and in $[L_z - L_t]G(z,t) = 0$) have the same functional form. Hence, if $G(z,t) = G(B_1, B_2; z_0, \omega, \eta; z, t)$ is a solution of the Eq. (15), we find that the transformations R_1, R_2, R_3 and R_4, given by

$$R_1 G(z,t) = (zt)^{1 + \frac{B_1}{z_0}} G(C_1, C_2; z_0, \omega, \eta; z, t), \qquad R_2 G(z,t) = [(z - z_0)(t - z_0)]^{1 - B_2 - \frac{B_1}{z_0}} G(B_1, D_2; z_0, \omega, \eta; z, t),$$

$$R_3 G(z,t) = G(B_1, B_2; z_0, -\omega, -\eta; z, t), \qquad R_4 G(z,t) = G(-B_1 - B_2 z_0, B_2; z_0, -\omega, \eta; z_0 - z, z_0 - t) \tag{20}$$

do not change the form of the kernel equation (15). These transformations may generate a group containing up to 16 kernels when applied to an initial kernel.

For another version of the CHE we have obtained initial kernels as limits of kernels for the general Heun equation [1]. For the version (1), in the following we reobtain these kernels by solving Eq. (15) and use the transformations (20) to generate groups of kernels closed under such transformations.

2.2. First group of kernels: products of two confluent hypergeometric functions

Kernels with products of two confluent hypergeometric functions have already appeared in a paper [27] which considers a particular problem obeying a CHE. Here we are concerned with the general case.

In the first place we show that the kernel equation $[L_z - L_t]G(z,t) = 0$, given in (15), is satisfied by 16 of such products, denoted by $G_1^{(i,j)}$ and defined as $(i, j = 1, 2, 3, 4)$

$$G_1^{(i,j)}(z,t) = e^{-i\omega(z+t)}\varphi^i(\xi) \times \bar{\varphi}^j(\zeta), \tag{21}$$

where $\varphi^i(\xi)$ and $\bar{\varphi}^j(\zeta)$ are the confluent hypergeometric functions (A.2), having the following arguments and parameters:

$$\varphi^i(\xi): \quad \xi = -\frac{2i\omega}{z_0}(z - z_0)(t - z_0), \qquad a = \frac{B_2}{2} - i\eta - \lambda, \qquad c = B_2 + \frac{B_1}{z_0}; \tag{22a}$$

$$\bar{\varphi}^j(\zeta): \quad \zeta = \frac{2i\omega}{z_0}zt, \quad a = \lambda, \quad c = -\frac{B_1}{z_0}, \tag{22b}$$

where λ is an arbitrary constant of separation. In the second place, by the transformation R_3 we may get another set of kernels, $G_2^{(i,j)}$, given by

$$G_2^{(i,j)}(z,t) = R_3 G_1^{(i,j)}(z,t) = G_1^{(i,j)}(z,t)\big|_{(\eta,\omega)\mapsto(-\eta,-\omega)}, \tag{23}$$

The transformations R_1, R_2 and R_4 are superfluous in this case.

To obtain the kernels (21), first we write

$$G(z,t) = e^{-i\omega(z+t)}f(z,t), \tag{24}$$

in Eq. (15). This leads to

$$z(z - z_0)\frac{\partial^2 f}{\partial z^2} + \left[B_1 + (B_2 + 2i\omega z_0)z - 2i\omega z^2\right]\frac{\partial f}{\partial z}$$

$$-t(t - z_0)\frac{\partial^2 f}{\partial t^2} - \left[B_1 + (B_2 + 2i\omega z_0)t - 2i\omega t^2\right]\frac{\partial f}{\partial t} - 2i\omega\left(\frac{B_2}{2} - i\eta\right)(z - t)f = 0. \tag{25}$$

Then, by the substitutions

$$\xi = -\frac{2i\omega(z-z_0)(t-z_0)}{z_0}, \qquad \zeta = \frac{2i\omega zt}{z_0}, \quad f = X(\xi)Y(\zeta) \tag{26}$$

we find the confluent hypergeometric equations

$$\xi \frac{d^2 X}{d\xi^2} + \left[B_2 + \frac{B_1}{z_0} - \xi\right]\frac{dX}{d\xi} - \left[\frac{B_2}{2} - i\eta - \lambda\right]X = 0, \qquad \zeta\frac{d^2 Y}{d\zeta^2} + \left[-\frac{B_1}{z_0} - \zeta\right]\frac{dY}{d\zeta} - \lambda Y = 0, \tag{27}$$

where λ is the constant of separation. The solutions for the above equations are: $X(\xi) = \varphi^i(\xi)$ with $a = (B_2/2) - i\eta - \lambda$ and $c = B_2 + (B_1/z_0)$; and $Y(\zeta) = \bar{\varphi}^j(\zeta)$ with $a = \lambda$ and $c = -B_1/z_0$. Inserting these solutions into (24) and (26) we find the kernels (21). Thence, the kernels given by regular confluent hypergeometric functions are

$$G_1^{(1,1)}(z,t) = e^{-i\omega(z+t)}\Phi\left[\frac{B_2}{2} - i\eta - \lambda, B_2 + \frac{B_1}{z_0}; -\frac{2i\omega}{z_0}(z-z_0)(t-z_0)\right]\Phi\left[\lambda, -\frac{B_1}{z_0}; \frac{2i\omega}{z_0}zt\right], \tag{28}$$

$$G_1^{(1,2)}(z,t) = e^{-i\omega(z+t)+\frac{2i\omega}{z_0}zt}[zt]^{1+\frac{B_1}{z_0}}\Phi\left[\frac{B_2}{2} - i\eta - \lambda, B_2 + \frac{B_1}{z_0}; -\frac{2i\omega}{z_0}(z-z_0)(t-z_0)\right]\Phi\left[1-\lambda, 2+\frac{B_1}{z_0}; -\frac{2i\omega}{z_0}zt\right], \tag{29}$$

$$\begin{aligned}
G_1^{(2,1)}(z,t) &= e^{-i\omega(z+t)-\frac{2i\omega}{z_0}(z-z_0)(t-z_0)}[(z-z_0)(t-z_0)]^{1-B_2-\frac{B_1}{z_0}} \\
&\times \Phi\left[1+i\eta+\lambda-\frac{B_2}{2}, 2-B_2-\frac{B_1}{z_0}; \frac{2i\omega}{z_0}(z-z_0)(t-z_0)\right]\Phi\left[\lambda, -\frac{B_1}{z_0}; \frac{2i\omega}{z_0}zt\right],
\end{aligned} \tag{30}$$

$$\begin{aligned}
G_1^{(2,2)}(z,t) &= e^{i\omega(z+t)}[zt]^{1+\frac{B_1}{z_0}}[(z-z_0)(t-z_0)]^{1-B_2-\frac{B_1}{z_0}} \\
&\times \Phi\left[1+i\eta+\lambda-\frac{B_2}{2}, 2-B_2-\frac{B_1}{z_0}; \frac{2i\omega}{z_0}(z-z_0)(t-z_0)\right]\Phi\left[1-\lambda, 2+\frac{B_1}{z_0}; -\frac{2i\omega}{z_0}zt\right].
\end{aligned} \tag{31}$$

The remaining kernels are obtained by replacing one or both functions Φ by Ψ. In this manner we obtain the 16 kernels. The transformations R_1, R_2 and R_4 are superfluous because they simply rearrange these kernels. For instance,

$$\begin{aligned}
R_1 G_1^{(1,1)} &= e^{-i\omega(z+t)+\frac{2i\omega}{z_0}zt}[zt]^{1+\frac{B_1}{z_0}}\Phi\left[2+\frac{B_1}{z_0} - \lambda_1, 2+\frac{B_1}{z_0}; -\frac{2i\omega}{z_0}zt\right] \\
&\times \Phi\left[\frac{B_1}{z_0} + \frac{B_2}{2} + 1 - i\eta - \lambda_1, B_2 + \frac{B_1}{z_0}; -\frac{2i\omega}{z_0}(z-z_0)(t-z_0)\right],
\end{aligned}$$

where we have transformed λ into λ_1. By putting $\lambda_1 = \lambda + 1 + (B_1/z_0)$, we see that $R_1 G_1^{(1,1)} = G_1^{(1,2)}$.

2.3. Second group: product of hypergeometric and confluent hypergeometric functions

Now we find a group of kernels $G^{(i,j)}$ given by products of the confluent hypergeometric functions φ^i written in (A.2) with the Gauss hypergeometric functions F^j written in (A.5), (A.6) and (A.7). These kernels take the form

$$G^{(i,j)}(z,t) = e^{-i\omega(z+t)}(z+t-z_0)^{-\lambda}\varphi^i(\xi) \times F^j(\zeta), \qquad [i=1,\cdots,4; \qquad j=1,\cdots,6] \tag{32}$$

where λ is a constant of separation, whereas the arguments and parameters for the hypergeometric functions are

$$\varphi^i(\xi): \ \xi = 2i\omega(z+t-z_0), \qquad a = \frac{B_2}{2} - i\eta - \lambda, \qquad c = B_2 - 2\lambda; \tag{33a}$$

$$F^j(\zeta): \ \zeta = \frac{zt}{z_0(z+t-z_0)}, \qquad a = \lambda, \ b = B_2 - 1 - \lambda, \qquad c = -\frac{B_1}{z_0}. \tag{33b}$$

We can show that the transformations R_i do not generate new kernels.

The above kernels are constructed by inserting

$$G(z,t) = e^{-i\omega(z+t)}f(z,t) = e^{-i\omega(z+t)}g(\xi,\zeta) \tag{34}$$

into $[L_z - L_t]G(z,t) = 0$, where ξ and ζ are defined in Eqs. (33a) and (33b) . Thus we find

$$\xi\left[\xi\frac{\partial^2 g}{\partial\xi^2} + (B_2 - \xi)\frac{\partial g}{\partial\xi} - \left(\frac{B_2}{2} - i\eta\right)g\right] + \zeta(1-\zeta)\frac{\partial^2 g}{\partial\zeta^2} + \left(-\frac{B_1}{z_0} - B_2\zeta\right)\frac{\partial g}{\partial\zeta} = 0.$$

The separation of variables $g(\xi, \zeta) = X(\xi)Y(\zeta)$ leads to

$$\xi \frac{d^2 X}{d\xi^2} + [B_2 - \xi] \frac{dX}{d\xi} - \left[\frac{B_2}{2} - i\eta - \frac{\bar{\lambda}}{\xi} \right] X = 0, \qquad \zeta(1-\zeta) \frac{d^2 Y}{d\zeta^2} - \left[\frac{B_1}{z_0} + B_2 \zeta \right] \frac{dY}{d\zeta} - \bar{\lambda} Y = 0, \tag{35}$$

where $\bar{\lambda}$ is a constant of separation. Putting $\bar{\lambda} = \lambda(B_2 - 1 - \lambda)$, we find that $Y(\zeta)$ is given by hypergeometric functions $Y(\zeta) = F^j(\zeta)$ as in Eqs. (32) and (33b), while $X(\xi)$ obeys the equation

$$\xi \frac{d^2 X}{d\xi^2} + [B_2 - \xi] \frac{dX}{d\xi} - \left[\frac{B_2}{2} - i\eta - \frac{\lambda(B_2 - 1 - \lambda)}{\xi} \right] X = 0.$$

The substitution $X(\xi) = \xi^{-\lambda} \bar{X}(x)$ gives the confluent hypergeometric equation

$$\xi \frac{d^2 \bar{X}}{d\xi^2} + \left[B_2 - 2\lambda - \xi \right] \frac{d\bar{X}}{d\xi} - \left[\frac{B_2}{2} - i\eta - \lambda \right] \bar{X} = 0, \tag{36}$$

whose solutions are $\bar{X}(\xi) = \varphi^i(\xi)$. In this manner, by inserting the previous solutions for $\bar{X}(\xi)$ and $Y(\zeta)$ into

$$G(z, t) = e^{-i\omega(z+t)} (z + t - z_0)^{-\lambda} \bar{X}(\xi) Y(\zeta) \tag{37}$$

we obtain kernels having the form (32).

The kernels $G^{(1,j)}$ and $G^{(2,j)}$ in terms of regular confluent hypergeometric functions are

$$G^{(1,j)}(z, t) = e^{-i\omega(z+t)} [z + t - z_0]^{-\lambda} F^j(\zeta) \Phi \left[\frac{B_2}{2} - i\eta - \lambda, B_2 - 2\lambda; 2i\omega(z + t - z_0) \right], \tag{38}$$

$$G^{(2,j)}(z, t) = e^{i\omega(z+t)} [z + t - z_0]^{1-B_2+\lambda} F^j(\zeta) \Phi \left[1 + i\eta + \lambda - \frac{B_2}{2}, 2 + 2\lambda - B_2; -2i\omega(z + t - z_0) \right], \tag{39}$$

whereas $G^{(3,j)}$ and $G^{(4,j)}$ in terms of irregular functions are obtained by substituting $\Psi(a, c; u)$ for $\Phi(a, c; u)$, that is,

$$G^{(3,j)}(z, t) = G^{(1,j)}(z, t) \big|_{\Phi \mapsto \Psi}, \qquad G^{(4,j)}(z, t) = G^{(2,j)}(z, t) \big|_{\Phi \mapsto \Psi}. \tag{40}$$

The functions $F^j(\zeta)$ are given by

$$F^1(\zeta) = F \left[\lambda, B_2 - 1 - \lambda; -\frac{B_1}{z_0}; \frac{zt}{z_0(z+t-z_0)} \right], \tag{41}$$

$$F^2(\zeta) = \left[\frac{zt}{z_0(z+t-z_0)} \right]^{1+\frac{B_1}{z_0}} F \left[\lambda + 1 + \frac{B_1}{z_0}, B_2 + \frac{B_1}{z_0} - \lambda; 2 + \frac{B_1}{z_0}; \frac{zt}{z_0(z+t-z_0)} \right], \tag{42}$$

$$F^3(\zeta) = F \left[\lambda, B_2 - 1 - \lambda; B_2 + \frac{B_1}{z_0}; \frac{(z-z_0)(t-z_0)}{z_0(z_0-z-t)} \right], \tag{43}$$

$$F^4(\zeta) = \left[\frac{(z-z_0)(t-z_0)}{z_0(z+t-z_0)} \right]^{1-B_2-\frac{B_1}{z_0}} F \left[-\lambda - \frac{B_1}{z_0}, \lambda + 1 - B_2 - \frac{B_1}{z_0}; 2 - B_2 - \frac{B_1}{z_0}; \frac{(z-z_0)(t-z_0)}{z_0(z_0-z-t)} \right], \tag{44}$$

$$F^5(\zeta) = \left[\frac{z_0(z+t-z_0)}{zt} \right]^{\lambda} F \left[\lambda, \lambda + 1 + \frac{B_1}{z_0}; 2 + 2\lambda - B_2; \frac{z_0(z+t-z_0)}{zt} \right], \tag{45}$$

$$F^6(\zeta) = \left[\frac{z_0(z+t-z_0)}{zt} \right]^{B_2-1-\lambda} F \left[B_2 + \frac{B_1}{z_0} - \lambda, B_2 - 1 - \lambda; B_2 - 2\lambda; \frac{z_0(z+t-z_0)}{zt} \right]. \tag{46}$$

By using the explicit form for the kernels and the fact that the separation constant is arbitrary, it is possible to show that the transformations R_i simply rearrange the previous kernels. For instance, we get

$$R_3 G^{(1,j)}(z, t) = e^{i\omega(z+t)} [z + t - z_0]^{-\lambda_3} \Phi \left[\frac{B_2}{2} + i\eta - \lambda_3, B_2 - 2\lambda_3; -2i\omega(z + t - z_0) \right] H^j(\zeta),$$

where $H^j(\zeta)$ is obtained by substituting λ_3 for λ in $F^j(\zeta)$. Thence, putting $\lambda_3 = B_2 - \lambda - 1$ and taking into account that $F(a, b; c; u) = F(b, a; c; u)$, we find that $H^5(\zeta) = F^6(\zeta)$, $H^6(\zeta) = F^5(\zeta)$ and $H^j(\zeta) = F^j(\zeta)$ if $j = 1, 2, 3, 4$. For this reason, $R_3 G^{(1,j)}$ is equivalent to $G^{(2,j)}$.

2.4. Third group: confluent hypergeometric functions

An initial set has the form

$$G_1^{(i)}(z,t) = e^{-i\omega(z+t)}\varphi^i(\xi), \qquad [i=1,2,3,4] \tag{47}$$

where the $\varphi^i(\xi)$ denote the four solutions (A.2) for the confluent hypergeometric equation with the following argument and parameters:

$$\xi = -\frac{2i\omega}{z_0}(z-z_0)(t-z_0), \qquad a = \frac{B_2}{2} - i\eta, \qquad c = B_2 + \frac{B_1}{z_0}. \tag{48}$$

The set (47) is obtained by putting $\lambda = 0$ and Y constant in (27). Besides this, from (47) we form four sets by using the rules R_2 and R_4, namely,

$$G_1^{(i)}(z,t), \qquad G_2^{(i)}(z,t) = R_1 G_1^{(i)}(z,t), \qquad G_3^{(i)}(z,t) = R_4 G_2^{(i)}(z,t), \qquad G_4^{(i)}(z,t) = R_2 G_3^{(i)}(z,t). \tag{49}$$

The four pairs in terms of regular confluent hypergeometric functions $\Phi(a,c;u)$ read

$$G_1^{(1)}(z,t) = e^{-i\omega(z+t)}\Phi\left[\frac{B_2}{2} - i\eta, B_2 + \frac{B_1}{z_0}; -\frac{2i\omega}{z_0}(z-z_0)(t-z_0)\right],$$

$$G_1^{(2)}(z,t) = e^{i\omega(z+t)-\frac{2i\omega zt}{z_0}}[(z-z_0)(t-z_0)]^{1-B_2-\frac{B_1}{z_0}}\Phi\left[1+i\eta - \frac{B_2}{2}, 2 - B_2 - \frac{B_1}{z_0}; \frac{2i\omega}{z_0}(z-z_0)(t-z_0)\right]; \tag{50}$$

$$G_2^{(1)}(z,t) = e^{-i\omega(z+t)}[zt]^{1+\frac{B_1}{z_0}}\Phi\left[1-i\eta + \frac{B_1}{z_0} + \frac{B_2}{2}, B_2 + \frac{B_1}{z_0}; -\frac{2i\omega}{z_0}(z-z_0)(t-z_0)\right],$$

$$G_2^{(2)}(z,t) = e^{i\omega(z+t)-\frac{2i\omega zt}{z_0}}[zt]^{1+\frac{B_1}{z_0}}[(z-z_0)(t-z_0)]^{1-B_2-\frac{B_1}{z_0}}\Phi\left[i\eta - \frac{B_1}{z_0} - \frac{B_2}{2}, 2 - B_2 - \frac{B_1}{z_0}; \frac{2i\omega}{z_0}(z-z_0)(t-z_0)\right]; \tag{51}$$

$$G_3^{(1)}(z,t) = e^{-i\omega(z+t)}[(z-z_0)(t-z_0)]^{1-B_2-\frac{B_1}{z_0}}\Phi\left(1-i\eta - \frac{B_1}{z_0} - \frac{B_2}{2}, -\frac{B_1}{z_0}; \frac{2i\omega zt}{z_0}\right),$$

$$G_3^{(2)}(z,t) = e^{-i\omega(z+t)+\frac{2i\omega zt}{z_0}}(zt)^{1+\frac{B_1}{z_0}}[(z-z_0)(t-z_0)]^{1-B_2-\frac{B_1}{z_0}}\Phi\left(i\eta + \frac{B_1}{z_0} + \frac{B_2}{2}, 2 + \frac{B_1}{z_0}; -\frac{2i\omega zt}{z_0}\right); \tag{52}$$

$$G_4^{(1)}(z,t) = e^{-i\omega(z+t)}\Phi\left(\frac{B_2}{2} - i\eta, -\frac{B_1}{z_0}; \frac{2i\omega zt}{z_0}\right),$$

$$G_4^{(2)}(z,t) = e^{-i\omega(z+t)+\frac{2i\omega zt}{z_0}}(zt)^{1+\frac{B_1}{z_0}}\Phi\left(1+i\eta - \frac{B_2}{2}, 2 + \frac{B_1}{z_0}; -\frac{2i\omega zt}{z_0}\right). \tag{53}$$

We get the pairs in terms of irregular confluent hypergeometric functions by replacing $\Phi(a,c;u)$ by $\Psi(a,c;u)$:

$$G_i^{(3)}(z,t) = G_i^{(1)}(z,t)\Big|_{\Phi\to\Psi}, \qquad G_i^{(4)}(z,t) = G_i^{(2)}(z,t)\Big|_{\Phi\to\Psi}, \qquad [i=1,2,3,4]. \tag{54}$$

Thus, by using also R_3, we find that this group is constituted by 32 kernels. Notice that R_1, R_2 and R_4 generate only four sets of kernels instead of 16 sets because in some cases these transformations rearrange the kernels of a given set in a different order: we can test this by computing, for example, $R_2 G_1^{(i)}$ or $R_2 G_2^{(i)}$. Notice that kernels whose arguments of the hypergeometric functions are $\pm(2i\omega zt/z_0)$ have been known since long [28].

For the spheroidal equation ($\eta = 0$, $z_0 = 1$, $B_2 = -2B_1$), sixteen of the previous kernels reduce to four kernels in terms of elementary functions, namely,

$$G_1^{(\pm)}(z,t) = e^{\pm i\omega(z+t)\mp 2i\omega zt}, \qquad G_2^{(\pm)}(z,t) = e^{\pm i\omega(z+t)\mp 2i\omega zt}[zt(z-1)(t-1)]^{1+B_1}. \tag{55}$$

For instance,

$$G_1^{(1)}(z,t) \propto G_1^{(4)}(z,t) \propto G_1^{(+)}(z,t), \qquad G_4^{(1)}(z,t) \propto G_4^{(4)}(z,t) \propto G_1^{(-)}(z,t),$$

$$G_2^{(2)}(z,t) \propto G_2^{(4)}(z,t) \propto G_2^{(+)}(z,t), \qquad G_3^{(2)}(z,t) \propto G_3^{(4)}(z,t) \propto G_2^{(-)}(z,t).$$

The kernel $G_2^{(-)}(z,t)$ will be used in section 4.1.

2.5. Fourth group: confluent hypergeometric functions again

To obtain new kernels given by confluent hypergeometric functions we take

$$G^{(i)}_1(z,t) = G^{(i,1)}(z,t)\big|_{\lambda=0},$$

where the $G^{(i,1)}$ denote the kernels (32) with $j = 1$. Since the above choice for λ eliminates the Gauss hypergeometric function $[F(0, b; c; \zeta) = 1]$ we find

$$G^{(i)}_1(z,t) = e^{-i\omega(z+t)}\varphi^i(\xi), \qquad [i = 1, 2, 3, 4] \tag{56a}$$

where $\varphi^i(\xi)$ denote the solutions (A.2) for the confluent hypergeometric equation with

$$\xi = 2i\omega(z + t - z_0), \qquad a = \frac{B_2}{2} - i\eta, \qquad c = B_2 \qquad [\text{see Eq. (33a)}]. \tag{56b}$$

Other choices for λ also lead to kernels in terms of confluent hypergeometric functions. However, such kernels are obtained from the initial set (56a) by using the transformations R_i. In this manner we find four sets, namely,

$$G^{(i)}_1(z,t), \quad G^{(i)}_2(z,t) = R_1 G^{(i)}_1(z,t), \quad G^{(i)}_3(z,t) = R_2 G^{(i)}_2(z,t), \quad G^{(i)}_4(z,t) = R_1 G^{(i)}_3(z,t), \tag{57}$$

since R_4 does not generate new kernels. The kernels given by regular confluent hypergeometric functions are

$$G^{(1)}_1(z,t) = e^{-i\omega(z+t)}\Phi\left[\frac{B_2}{2} - i\eta, B_2; 2i\omega(z + t - z_0)\right],$$
$$G^{(2)}_1(z,t) = e^{i\omega(z+t)}[z + t - z_0]^{1-B_2}\Phi\left[1 + i\eta - \frac{B_2}{2}, 2 - B_2; -2i\omega(z + t - z_0)\right]; \tag{58}$$

$$G^{(1)}_2(z,t) = e^{-i\omega(z+t)}[zt]^{1+\frac{B_1}{z_0}}\Phi\left[1 - i\eta + \frac{B_1}{z_0} + \frac{B_2}{2}, 2 + B_2 + \frac{2B_1}{z_0}; 2i\omega(z + t - z_0)\right],$$
$$G^{(2)}_2(z,t) = e^{i\omega(z+t)}[zt]^{1+\frac{B_1}{z_0}}[z + t - z_0]^{-1-B_2-\frac{2B_1}{z_0}}\Phi\left[i\eta - \frac{B_1}{z_0} - \frac{B_2}{2}, -B_2 - \frac{2B_1}{z_0}; -2i\omega(z + t - z_0)\right]; \tag{59}$$

$$G^{(1)}_3(z,t) = e^{-i\omega(z+t)}[zt]^{1+\frac{B_1}{z_0}}[(z - z_0)(t - z_0)]^{1-B_2-\frac{B_1}{z_0}}\ \Phi\left[2 - i\eta - \frac{B_2}{2}, 4 - B_2; 2i\omega(z + t - z_0)\right],$$
$$G^{(2)}_3(z,t) = e^{i\omega(z+t)}[zt]^{1+\frac{B_1}{z_0}}[(z - z_0)(t - z_0)]^{1-B_2-\frac{B_1}{z_0}}[z + t - z_0]^{B_2-3}\Phi\left[i\eta - 1 + \frac{B_2}{2}, B_2 - 2; -2i\omega(z + t - z_0)\right]; \tag{60}$$

$$G^{(1)}_4(z,t) = e^{-i\omega(z+t)}[(z - z_0)(t - z_0)]^{1-B_2-\frac{B_1}{z_0}}\Phi\left[1 - i\eta - \frac{B_1}{z_0} - \frac{B_2}{2}, 2 - B_2 - \frac{2B_1}{z_0}; 2i\omega(z + t - z_0)\right],$$
$$G^{(2)}_4(z,t) = e^{i\omega(z+t)}[(z - z_0)(t - z_0)]^{1-B_2-\frac{B_1}{z_0}}[z + t - z_0]^{B_2+\frac{2B_1}{z_0}-1}\Phi\left[i\eta + \frac{B_1}{z_0} + \frac{B_2}{2}, B_2 + \frac{2B_1}{z_0}; -2i\omega(z + t - z_0)\right]. \tag{61}$$

Replacing $\Phi(a, c; u)$ by $\Psi(a, c; u)$ and using the transformation R_3, once more we get a group with 32 kernels. Some particular cases of these kernels are already known [4]. Furthermore, if $\eta = 0$ this group can be expressed in terms of Bessel functions by means of (A.12).

2.6. Fifth group: hypergeometric functions

To get kernels given by hypergeometric functions we take $G^{(i)}_1(z,t) = G^{(1,i)}(z,t)|_{\lambda=(B_2/2)-i\eta}$, where $G^{(1,i)}$ are the kernels given in (38). In fact, for this choice for λ we obtain $\Phi(0, c; \xi) = 1$ and, thence,

$$G^{(i)}_1(z,t) = e^{-i\omega(z+t)}[z + t - z_0]^{i\eta-\frac{B_2}{2}}F^i(\zeta), \qquad \zeta = zt/[z_0(z + t - z_0)], \tag{62}$$

where the hypergeometric functions $F^i(\zeta)$ are obtained by putting $\lambda = (B_2/2) - i\eta$ in Eqs. (41-46). Explicitly

$$G^{(1)}_1(z,t) = e^{-i\omega(z+t)}[z + t - z_0]^{i\eta-\frac{B_2}{2}}\ F\left[\frac{B_2}{2} - i\eta, \frac{B_2}{2} + i\eta - 1; -\frac{B_1}{z_0}; \frac{zt}{z_0(z+t-z_0)}\right], \tag{63}$$

$$G^{(2)}_1(z,t) = e^{-i\omega(z+t)}[z + t - z_0]^{i\eta-1-\frac{B_1}{z_0}-\frac{B_2}{2}}[zt]^{1+\frac{B_1}{z_0}}\ F\left[1 - i\eta + \frac{B_1}{z_0} + \frac{B_2}{2}, i\eta + \frac{B_1}{z_0} + \frac{B_2}{2}; 2 + \frac{B_1}{z_0}; \frac{zt}{z_0(z+t-z_0)}\right], \tag{64}$$

$$G^{(3)}_1(x,t) = e^{-i\omega(z+t)} \left[z+t-z_0\right]^{i\eta-\frac{B_2}{2}} F\left[\tfrac{B_2}{2}-i\eta, \tfrac{B_2}{2}+i\eta-1; B_2 + \tfrac{B_1}{z_0}; \tfrac{(z-z_0)(t-z_0)}{z_0(z_0-z-t)}\right],$$ (65)

$$\begin{aligned}
G^{(4)}_1(x,t) &= e^{-i\omega(z+t)} \left[z+t-z_0\right]^{i\eta-1+\frac{B_1}{z_0}+\frac{B_2}{2}} \left[(z-z_0)(t-z_0)\right]^{1-B_2-\frac{B_1}{z_0}} \\
&\quad \times\ F\left[i\eta-\tfrac{B_1}{z_0}-\tfrac{B_2}{2}, 1-i\eta-\tfrac{B_1}{z_0}-\tfrac{B_2}{2}; 2-B_2-\tfrac{B_1}{z_0}; \tfrac{(z-z_0)(t-z_0)}{z_0(z_0-z-t)}\right],
\end{aligned}$$ (66)

$$G^{(5)}_1(x,t) = e^{-i\omega(z+t)} \left[zt\right]^{i\eta-\frac{B_2}{2}} F\left[\tfrac{B_2}{2}-i\eta, 1-i\eta+\tfrac{B_1}{z_0}+\tfrac{B_2}{2}; 2-2i\eta; \tfrac{z_0(z+t-z_0)}{zt}\right],$$ (67)

$$G^{(6)}_1(x,t) = e^{-i\omega(z+t)} \left[z+t-z_0\right]^{2i\eta-1} \left[zt\right]^{1-i\eta-\frac{B_2}{2}} F\left[i\eta+\tfrac{B_1}{z_0}+\tfrac{B_2}{2}, i\eta-1+\tfrac{B_2}{2}; 2i\eta; \tfrac{z_0(z+t-z_0)}{zt}\right].$$ (68)

The transformations R_1, R_2 and R_4 at most rearrange the preceding kernels. For example,

$$R_1 G^{(1)}_1(z,t) = G^{(2)}_1(z,t), \qquad R_2 G^{(1)}_1(z,t) = G^{(1)}_1(z,t), \qquad R_4 G^{(1)}_1(z,t) = G^{(3)}_1(z,t).$$

However, we find six additional kernels $G^{(j)}_2(z,t)$ by using the transformations R_3 as

$$G^{(i)}_2(z,t) = R_3 G^{(i)}_1(z,t)$$ (69)

So, $G^{(i)}_2$ is obtained by replacing (η,ω) by $(-\eta,-\omega)$ in $G^{(i)}_1$.

3. Integral relations between known solutions

In this section we use some kernels to obtain integral relations among solutions of the CHE. We find that:

- the Jaffé solutions in power series are tranformed into Leaver's expansions in series of irregular confluent hypergeometric series;
- the Baber-Hassé solutions in power series are transformed into solutions given by series of regular confluent hypergeometric functions.

Relations for solutions generated by transformations of the CHEs may be obtained by transforming also the kernel, since each transformation of a solution corresponds to a transformation of a kernel.

3.1. Jaffé's solutions in power series and Leaver's solutions

By $U^J_1(z)$ and $U^L_1(z)$ we denote respectively the Jaffé [22] and the Leaver [9] solutions for the CHE, namely,

$$U^J_1(z) = e^{i\omega z} z^{-i\eta-\frac{B_2}{2}} \sum_{n=0}^{\infty} a^1_n \left(\tfrac{z-z_0}{z}\right)^n,$$ (70a)

$$U^L_1(z) = e^{i\omega z} \sum_{n=0}^{\infty} a^1_n\, \Gamma\left(n+B_2+\tfrac{B_1}{z_0}\right) \Psi\left(n+i\eta+\tfrac{B_2}{2}, -\tfrac{B_1}{z_0}; -2i\omega z\right),$$ (70b)

where the recurrence relations for the a^1_n are ($a^1_{-1}=0$)

$$\begin{aligned}
(n+1)\left[n+B_2+\tfrac{B_1}{z_0}\right] a^1_{n+1} &+ \left[-2n\left(n+B_2+\tfrac{B_1}{z_0}+i\eta-i\omega z_0\right) + B_3 + \left(B_2+\tfrac{B_1}{z_0}\right)\left(i\omega z_0 - i\eta - \tfrac{B_2}{2}\right)\right] a^1_n \\
&+ \left[n-1+i\eta+\tfrac{B_2}{2}\right]\left[n+\tfrac{B_2}{2}+\tfrac{B_1}{z_0}+i\eta\right] a^1_{n-1} = 0.
\end{aligned}$$

The convergence of solutions (70a) and (70b) is discussed in the Leaver paper [9], where it is used the minimal solution for the coefficients a^1_n. In fact, three-term recurrence relations as the above ones admit two independent solutions, say, f_n and g_n. If $\lim_{n\to\infty}(f_n/g_n) = 0$, f_n is called minimal solution [29, 30]. In addition, it is necessary to suppose that there is an arbitrary parameter in the CHE. The series converges only for special values of that parameter, determined from a transcendental (characteristic) equation which results from the recurrence relations [9].

By supposing that $U^J_1(z)$ converges for $|z| \geq |z_0|$ and by using Eq. (13), we find the relation

$$U^L_1(z) = C_1 \int_{z_0}^{\infty} t^{-1-\frac{B_1}{z_0}} \left[t-z_0\right]^{B_2+\frac{B_1}{z_0}-1} G^{(4)}_4(z,t) U^J_1(t)\, dt, \qquad \mathrm{Re}\left[n+B_2+\tfrac{B_1}{z_0}\right] > 0, \qquad \mathrm{Re}[i\omega z] < 0,$$ (71)

10

where C_1 is a constant and $G(z,t) = G_4^{(4)}(z,t)$ is the kernel indicated in (54). In fact, by setting $y = t/z_0$, we find that the right-hand side of (71) is equivalent to

$$e^{-i\omega z} z^{1+\frac{B_1}{z_0}} \sum_{n=0}^{\infty} a_n^1 \int_{z_0}^{\infty} dy \left[e^{2i\omega zy} \, (y-1)^{n+B_2+\frac{B_1}{z_0}-1} \, y^{-n-i\eta-\frac{B_2}{2}} \, \Psi \left(1 + i\eta - \frac{B_2}{2}, 2 + \frac{B_1}{z_0}; -2i\omega zy \right) \right].$$

Then, by using the integral [31]

$$\int_1^{\infty} e^{-ay} (y-1)^{\mu-1} y^{\alpha+k-\mu-\frac{1}{2}} \Psi \left(\frac{1}{2} + \alpha - k, 2\alpha + 1; ay \right) dy = \Gamma(\mu) e^{-a} \Psi \left(\frac{1}{2} + \alpha + \mu - k, 2\alpha + 1; a \right), \tag{72}$$

$$[\text{Re } \mu > 0, \ \text{Re } a > 0],$$

we obtain the relation (71).

For the bilinear concomitant (16) we find

$$P_1(z,t) = z^{1+\frac{B_1}{z_0}} e^{i\omega z (\frac{2t}{z_0}-1)} \, t^{-i\eta-\frac{B_2}{2}} \, (t-z_0)^{B_2+\frac{B_1}{z_0}} \sum_{n=0}^{\infty} a_n \left(\frac{t-z_0}{t} \right)^n$$

$$\times \left\{ \left[2i\omega \left(\frac{z}{z_0} - 1 \right) t - \frac{nz_0}{t-z_0} + \frac{1}{t} \left(i\eta + 1 + \frac{B_1}{z_0} + \frac{B_2}{2} \right) \right] \Psi + t \frac{\partial \Psi}{\partial t} \right\},$$

where

$$\Psi = \Psi \left(1 + i\eta - \frac{B_2}{2}, 2 + \frac{B_1}{z_0}; -\frac{2i\omega zt}{z_0} \right), \qquad \frac{\partial \Psi}{\partial t} = \frac{2i\omega z}{z_0} \left(1 + i\eta - \frac{B_2}{2} \right) \Psi \left(2 + i\eta - \frac{B_2}{2}, 3 + \frac{B_1}{z_0}; -\frac{2i\omega zt}{z_0} \right).$$

Since $\Psi(a,b;y) = y^{-a}$ when $|y| \to \infty$ and $\text{Re}(i\omega z) < 0$, the exponential factor assures that $P_1(z,t)$ vanishes when $t/z_0 \to \infty$. On the other hand, the condition $\text{Re}[B_2 + \frac{B_1}{z_0}] > 0$ assures that $P_1(z,t)$ vanishes also for $t = z_0$ since $(t-z_0)^{B_2+B_1/z_0} \to 0$.

In this manner, we have extended the results of Leaver [9] who has considered only relations between solutions with $i\eta = \pm(B_2/2 - 1)$. Notice also that the conditions given in (71) are necessary only to assure the integral relation between the solutions. In fact the Leaver solutions can be derived directly from the differential equation without imposing those conditions [9].

For the present case the transformation T_1 is ineffective and, so, from (U_1^J, U_1^L) we can obtain only 8 pairs of solutions by composition of the transformations (18); to each pair corresponds a kernel generated by the transformations (20). For example, taking $U_2^J(z) = T_2 U_1^J(z)$ and $U_2^L(z) = T_2 U_1^L(z)$, we find

$$U_2^J(z) = e^{i\omega z} (z-z_0)^{1-B_2-\frac{B_2}{2}} z^{-i\eta-1+\frac{B_1}{z_0}+\frac{B_2}{2}} \sum_{n=0}^{\infty} a_n^2 \left(\frac{z-z_0}{z} \right)^n, \tag{73a}$$

$$U_2^L(z) = e^{i\omega z} (z-z_0)^{1-B_2-\frac{B_2}{2}} \sum_{n=0}^{\infty} a_n^2 \Gamma \left(n+2-B_2-\frac{B_1}{z_0} \right) \Psi \left(n + i\eta + 1 - \frac{B_2}{2} - \frac{B_1}{z_0}, -\frac{B_1}{z_0}; -2i\omega z \right), \tag{73b}$$

where the recurrence relations for a_n^2 are ($a_{-1}^2 = 0$)

$$(n+1) \left[n+2-B_2-\frac{B_1}{z_0} \right] a_{n+1}^2 + \left[-2n \left(n+2+i\eta-i\omega z_0 - B_2 - \frac{B_1}{z_0} \right) + B_3 + \left(2 - B_2 + \frac{B_1}{z_0} \right) (i\omega z_0 - i\eta) + \frac{B_1}{z_0} \right.$$

$$\left. + \left(1 - \frac{B_2}{2} \right) \left(1 + \frac{B_1}{z_0} - \frac{B_2}{2} \right) \right] a_n^2 + \left[n+1+i\eta - \frac{B_2}{2} \right] \left[n+i\eta - \frac{B_2}{2} - \frac{B_1}{z_0} \right] a_{n-1}^2 = 0.$$

Using Eq. (13), we find that

$$U_2^L(z) = C_2 \int_{z_0}^{\infty} dt \, t^{-1-\frac{B_1}{z_0}} [t-z_0]^{B_2+\frac{B_1}{z_0}-1} U_2^J(t) R_2 G_4^{(4)}(z,t), \qquad \text{Re} \left[n+2-B_2-\frac{B_1}{z_0} \right] > 0, \quad \text{Re}[i\omega z] < 0,$$

where C_2 is a constant, $G_4^{(4)}(z,t)$ is the kernel indicated in (54), and the transformation R_2 is given in (20); then,

$$R_2 G_4^{(4)} = e^{-i\omega(z+t)+\frac{2i\omega zt}{z_0}} [(z-z_0)(t-z_0)]^{1-B_2-\frac{B_1}{z_0}} (zt)^{1+\frac{B_1}{z_0}} \Psi \left(i\eta + \frac{B_1}{z_0} + \frac{B_2}{2}, 2 + \frac{B_1}{z_0}; -\frac{2i\omega zt}{z_0} \right).$$

We have supposed that the Jaffé solutions converge for $|z| \geq |z_0|$, but we must be careful about the point $z = \infty$, since [9]

$$\lim_{z \to \infty} U_1^J(z) = e^{i\omega z} z^{-i\eta-\frac{B_2}{2}} \sum_{n=0}^{\infty} a_n^1 \quad \text{with} \quad \lim_{n \to \infty} \frac{a_{n+1}^1}{a_n^1} = 1 - \frac{\sqrt{-2i\omega z_0}}{\sqrt{n}} + \frac{i(\eta - \omega z_0) - (3/4)}{n}, \tag{74}$$

where the ratio a_{n+1}^1/a_n^1 holds for the minimal solution of the recurrence relations. Thus, the D'Alambert test is inconclusive as to the convergence of $\sum a_n^1$. For the radial part of the two-center problem we could use the Raabe test for convergence, as in Eq. (106).

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3.2. Solutions in power series and solutions in series of confluent hypergeometric functions

We find another pair of solutions for the CHE which are again connected by the integral (13). By one side we have the Baber-Hassé expansion [18, 23]

$$U_1^{\text{baber}}(z) = e^{i\omega z} \sum_{n=0}^{\infty} a_n^1 (z - z_0)^n, \tag{75a}$$

where the coefficients satisfy the relations $(a_{-1}^1 = 0)$

$$z_0 \left(n + B_2 + \tfrac{B_1}{z_0} \right) (n+1) a_{n+1}^1 + \beta_n^1 a_n^1 + 2i\omega \left(n + i\eta + \tfrac{B_2}{2} - 1 \right) a_{n-1}^1 = 0, \tag{75b}$$

with $\beta_n^1 = n(n + B_2 - 1 + 2i\omega z_0) + B_3 + i\omega z_0 [B_2 + B_1/z_0]$. The minimal solutions for a_n^1 yield solutions convergent for any finite value of z. On the other side, if $(B_2/2) - i\eta$ is not zero or negative integer we have the solution [18]

$$U_1(z) = e^{-i\omega z} \sum_{n=0}^{\infty} b_n^1 \; \Phi \left(\frac{B_2}{2} - i\eta, n + B_2; 2i\omega z \right), \tag{76a}$$

where the recurrence relations for b_n^1 are obtained from the previous ones by taking

$$b_n^1 = \frac{C(-z_0)^n \Gamma(n + B_2 + B_1/z_0)}{\Gamma(n + B_2)} a_n^1, \qquad C = \text{constant}.$$

This yields

$$-(n + B_2)(n+1) b_{n+1}^1 + \beta_n^1 b_n^1 - 2i\omega z_0 \frac{\left(n + B_2 + \tfrac{B_1}{z_0} - 1 \right)\left(n + i\eta + \tfrac{B_2}{2} - 1 \right)}{n + B_2 - 1} b_{n-1}^1 = 0. \tag{76b}$$

Now, if we insert $U_1^{\text{baber}}(t)$ and the kernel $G_4^{(1)}(z, t)$ given in (53) into Eq. (13), we find the solution $U_1(z)$, that is,

$$U_1 = K \int_{t_1}^{t_2} t^{-1 - \frac{B_1}{z_0}} [t - z_0]^{B_2 + \frac{B_1}{z_0} - 1} G_4^{(1)}(z, t) U_1^{\text{baber}}(t) dt, \qquad \text{Re} \left[n + B_2 + \frac{B_1}{z_0} \right] > 0, \quad \text{Re}[-\frac{B_1}{z_0}] > 0, \tag{77}$$

where K is a constant. In effect, by taking $t_1 = 0$ and $t_2 = z_0$, the above integral is proportional to

$$e^{-i\omega z} \sum_{n=0}^{\infty} (-z_0)^n a_n^{(1)} \int_0^1 d\left(\frac{t}{z_0} \right) \left[\left(\frac{t}{z_0} \right)^{-1 - \frac{B_1}{z_0}} \left(1 - \frac{t}{z_0} \right)^{n + B_2 - 1 + \frac{B_1}{z_0}} \Phi \left(\frac{B_2}{2} - i\eta, -\frac{B_1}{z_0}; 2i\omega z \frac{t}{z_0} \right) \right]$$

Then, by using the relation [31]

$$\int_0^1 \left[x^{\lambda - 1}(1 - x)^{2\mu - \lambda} \Phi \left(\tfrac{1}{2} + \mu - \nu, \lambda; yx \right) \right] dx = \frac{\Gamma(\lambda)\Gamma(1 + 2\mu - \lambda)}{\Gamma(1 + 2\mu)} \Phi \left(\tfrac{1}{2} + \mu - \nu, 1 + 2\mu; y \right), \tag{78}$$
$$[\text{Re}(\lambda) > 0, \quad \text{Re}(1 + 2\mu - \lambda) > 0],$$

we find the solution $U_1(z)$ given in (76a) provided that $\text{Re}[n + B_2 + (B_1/z_0)] > 0$ and $\text{Re}[-B_1/z_0] > 0$. On the other side, from $d\Phi(a, b; \xi)/d\xi = (a/b)\Phi(a + 1, b + 1; \xi)$ for $\xi = 2i\omega z t/z_0$, $a = B_2/2 - i\eta$ and $b = -B_1/z_0$, we find that the bilinear concomitant (16) is given by

$$
\begin{aligned}
P_1(z, t) &= -e^{-i\omega z} t^{-\frac{B_1}{z_0}} (t - z_0)^{B_2 + \frac{B_1}{z_0}} \\
&\times \left\{ \Phi(a, b; \xi) \sum_{n=1}^{\infty} n a_n^1 (t - z_0)^{n-1} + 2i\omega \left[\Phi(a, b; \xi) - \frac{az}{bz_0} \Phi(a + 1, b + 1; \xi) \right] \sum_{n=0}^{\infty} a_n^1 (t - z_0)^n \right\}.
\end{aligned}
$$

Therefore, $P_1(z, t = 0) = P_1(z, t = z_0) = 0$ due to the conditions $\text{Re}(-B_1/z_0) > 0$ and $\text{Re}(B_2 + B_1/z_0) > 0$.

Observe that from the pair $\left(U_1^{\text{baber}}, U_1 \right)$ we can obtain 16 pairs of solutions by using the four transformations (18) and composition of them: to each pair corresponds a kernel which is obtained by using the transformations (20).

4. New solutions for the confluent equation

In section 4.1, by an integral transformation we find a new solution in series of irregular confluent hypergeometric functions for the ordinary spheroidal equation. Then, in section 4.2 we extend that solution to the general case (no restriction on the parameters of the CHE). In this manner, we obtain an initial solution, $\mathcal{U}_1(z)$, which allows to generate

a group of solutions $\mathcal{U}_i(z)$ for the CHE by by means of transformations (18). Finally, in section 4.3, we show that the new solutions are suitable for the radial part of the two-center problem of the quantum mechanics.

Initially we make some comments on the recurrence relations and the ratio test for convergence. As in the preceding section, the three-term recurrence relations for the series coefficients b_n^i of $\mathcal{U}_i(z)$ have the form

$$\alpha_0^i\, b_1^i + \beta_0^i\, b_0^i = 0, \qquad \alpha_n^i\, b_{n+1}^i + \beta_n^i\, b_n^i + \gamma_n^i\, b_{n-1}^i = 0 \quad (n \geq 1) \tag{79}$$

where α_n^i, β_n^i and γ_n^i depend on the parameters of the differential equation and on the summation index n. By omitting the superscripts, these relations take the form

$$
\begin{bmatrix}
\beta_0 & \alpha_0 & 0 & & & & \\
\gamma_1 & \beta_1 & \alpha_1 & & & & \\
0 & \gamma_2 & \beta_2 & \alpha_2 & & & \\
& & \ddots & \ddots & \ddots & & \\
& & & \gamma_N & \beta_N & \alpha_N & \\
& & & & \gamma_{N+1} & \beta_{N+1} & \alpha_{N+1} \\
& & & & & \ddots & \ddots & \ddots
\end{bmatrix}
\begin{bmatrix}
b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_N \\ b_{N+1} \\ \vdots
\end{bmatrix}
=
\begin{bmatrix}
0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots
\end{bmatrix},
\tag{80}
$$

where we have split the matrix into blocks. This system of homogeneous linear equations has nontrivial solutions for b_n only if the determinant of the above tridiagonal matrix vanishes: this demands some arbitrary parameter in the matrix elements and, as a consequence, in the differential equation. The condition on the determinant can also be expressed by an (characteristic) equation given by the continued fraction [9]

$$\beta_0 = \frac{\alpha_0\gamma_1}{\beta_1-}\ \frac{\alpha_1\gamma_2}{\beta_2-}\ \frac{\alpha_2\gamma_3}{\beta_3-}\cdots. \tag{81}$$

The solution of the characteristic equation and the computation of the series coefficients are important aspects concerning applications of the CHE [32, 33]. The problem is simplified if $\gamma_{n=N+1}^i = 0$ for some $N \geq 0$; then, the series terminates at $n = N$ leading to a finite-series solution with $0 \leq n \leq N$ (see page 146 of [34]) which is called polynomial or quasi-polynomial solution. In this case, only the left upper block of the matrix is relevant.

On the other side, the convergence of a series like $\sum_{n=0}^{\infty} f_n(z)$ is obtained by computing the limit of

$$L(z) = \left| \frac{f_{n+1}(z)}{f_n(z)} \right| \quad \text{when} \quad n \to \infty. \tag{82a}$$

By the D'Alembert ratio test the series converges in the region where $L(z) < 1$ and diverges where $L_1(z) > 1$. If $L(z) = 1$, the D'Alembert test is inconclusive; however, by the Raabe test [35, 36], if

$$L(z) = 1 + \frac{A}{n} + O\left(\frac{1}{n^2}\right) \tag{82b}$$

(where A is a constant) the series converges if $A < -1$ and diverges if $A > -1$; the test is inconclusive if $A = -1$.

4.1. An integral transformation for the spheroidal equation

For the spheroidal equation in the form (9) we will find a solution $\mathcal{U}_1(z)$ given by

$$\mathcal{U}_1(z) = e^{i\omega z} z^{1+B_1} (z-1)^{1+B_1} \sum_{n=0}^{\infty} b_n^1 \Psi(2+B_1, 2+B_1-n; -2i\omega z) \qquad [B_1 \neq -2, -3, \cdots] \tag{83a}$$

where the coefficients b_n^1 satisfy the relations ($b_{-1}^1 = 0$)

$$-2i\omega(n+1)b_{n+1}^1 + \left[n\left(n+1+2i\omega\right) + i\omega(2+B_1) - B_1\left(1+B_1\right) + B_3\right]b_n^1 - n\left(n+B_1+1\right)b_{n-1}^1 = 0. \tag{2}$$

$\mathcal{U}_1(z)$ is not valid if $B_1 = -2, -3, \cdots$, because in these cases the function $\Psi(a, c; y)$ becomes a polynomial of fixed degree and, accordingly, (83a) is not a series expansion. This follows from the relation [37]

$$\Psi(-l, \alpha+1; y) = (-1)^l\, l!\, L_l^\alpha(y), \qquad [l = 0, 1, \cdots] \tag{84}$$

where the $L_l^{(\alpha)}(y)$ denote Laguerre polynomials of degree l. Besides this, the above expansion in general does not hold at $z = 0$ because in most cases $\Psi(a, c; y)$ goes to infinity at $z = 0$ [5]. The convergence of Ψ for $z \neq 0$ will be discussed later on.

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We get the expansion (83a) by applying an integral transformation to the asymptotic expansion $\mathcal{W}_2(z)$ given in Eq. (B.4). First, for the spheroidal equation, by writing $W(z) = \mathcal{W}_2(z)$ and $a_n^2 = b_n^1$, we find

$$W(z) = e^{i\omega z}\,(z-1)^{1+B_1}\sum_{n=0}^{\infty} b_n^1\, z^{-n-1}, \qquad \text{[Eq. (B.4) for the spheroidal equation]} \tag{85}$$

where the coefficients b_n^1 satisfy (2). In the second place, the solution $\mathcal{U}_1$ is obtained by inserting $U(t) = W(t)$ and $G(z,t) = G_2^-(z,t)$ – see Eq. (55) – into the right-hand side of Eq. (13), and by integrating from $t = 1$ to $t = \infty$, that is,

$$\mathcal{U}_1(z) \overset{(13)}{=} \int_1^{\infty} t^{-1-B_1}(t-1)^{-1-B_1} G_2^-(z,t)\,W(t)dt \overset{(55)}{=} e^{-i\omega z}[z(z-1)]^{1+B_1}\int_1^{\infty} e^{-i\omega t + 2i\omega z t}\,W(t)dt$$

which gives

$$\mathcal{U}_1(z) \overset{(85)}{=} e^{-i\omega z}[z(z-1)]^{1+B_1}\sum_{n=0}^{\infty} b_n^1 \int_1^{\infty} e^{2i\omega z t}(t-1)^{1+B_1}t^{-n-1}dt. \tag{86}$$

Thence, we obtain (83a) by using [5]

$$\int_1^{\infty} e^{-yt}(t-1)^{a-1}t^{c-a-1}dt = \Gamma(a)e^{-y}\Psi(a,c;y), \qquad [\operatorname{Re} a > 0, \ \ \operatorname{Re} y > 0].$$

The integrability conditions on the right-hand side require that

$$\operatorname{Re}[2+B_1] > 0 \text{ and } \operatorname{Re}[i\omega z] < 0. \tag{87}$$

On the other side, the bilinear concomitant (16) reads

$$\begin{aligned}
P(z,t) &= t^{-B_1}(t-1)^{-B_1}\left[W(t)\frac{\partial G_2^-(z,t)}{\partial t} - G_2^-(z,t)\frac{dW(t)}{dt}\right], \\
&= e^{i\omega z(2t-1)}[z(z-1)]^{1+B_1}(t-1)^{2+B_1}\left\{[2i\omega(z-1)t + B_1]\sum_{n=0}^{\infty} b_n^1 t^{-n-1} + \sum_{n=0}^{\infty} n b_n^1 t^{-n-1}\right\} \tag{88}
\end{aligned}$$

Since the series converge at $t = \infty$, the conditions (87) assure that $P(z,t = \infty) = 0$. However, the concomitant is undetermined at $t = 1$ because [for $\operatorname{Re}(2+B_1) > 0$] $P(z,t)$ is given by the product of the vanishing factor $(t-1)^{2+B_1}$ by a divergent series. Despite this, we can check directly [38] that $\mathcal{U}_1(z)$ is indeed a solution of the spheroidal equation (9) regardless of the conditions (87).

Now we use the ratio test to get the convergence of $\mathcal{U}_1$. Thus, when $n \to \infty$, we find that the minimal solution of (2) satisfies [38]

$$\frac{b_{n+1}^1}{b_n^1} \sim 1 + \frac{B_1}{n} \quad \Rightarrow \quad \frac{b_{n-1}^1}{b_n^1} \sim 1 - \frac{B_1}{n}. \tag{89}$$

To get the ratio between successive Ψ, we use the relation [5]

$$(a+1-c)\Psi(a,c-1;y) + (c-1+y)\Psi(a,c;y) - y\Psi(a,c+1;y) = 0.$$

Hence, by taking

$$a = 2 + B_1, \quad c = 2 + B_1 - n, \quad y = -2i\omega z, \quad \Psi_n(y) = \Psi(2+B_1, 2+B_1-n; -2i\omega z)$$

we obtain

$$(n+1)\frac{\Psi_{n+1}}{\Psi_n} - (n-1-B_1+2i\omega z) + 2i\omega z\frac{\Psi_{n-1}}{\Psi_n} = 0.$$

If z is bounded (that is, if $2i\omega z/n \to 0$), then when $n \to \infty$ this equation is satisfied by

$$\frac{\Psi_{n+1}}{\Psi_n} \sim 1 - \frac{1}{n}(B_1+2) \qquad \Leftrightarrow \qquad \frac{\Psi_{n-1}}{\Psi_n} \sim 1 + \frac{1}{n}(B_1+2) \text{ or} \tag{90}$$

$$\frac{\Psi_{n+1}}{\Psi_n} \sim \frac{2i\omega z}{n}\left(1 + \frac{B_1}{n}\right) \qquad \Leftrightarrow \qquad \frac{\Psi_{n-1}}{\Psi_n} \sim \frac{n}{2i\omega z}\left[1 - \frac{1}{n}(1+B_1)\right].$$

Only the first ratio is consistent with the fact that, if $|c| \to \infty$ while a and y remain fixed and bounded, then [37]

$$\Psi(a,c;y) = c^{-a}\left[(-1)^{-a} + \frac{\sqrt{2\pi}}{\Gamma(a)}\left(\frac{c}{ey}\right)^{c+a-\frac{3}{2}} y^{a-\frac{1}{2}} e^{y+a-\frac{3}{2}}\right]\left[1 + O\left(\frac{1}{|c|}\right)\right],$$

$$[c \to \infty; \ \ a \neq 0, -1, -2, \cdots; \ \ |\arg(\pm c)| < \pi].$$

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Thus, using (89) and (90), we find that

$$\text{when } n \to \infty, \qquad \frac{b_{n+1}^1 \Psi_{n+1}}{b_n^1 \Psi_n} = 1 - \frac{2}{n} + O\left(\frac{1}{n^2}\right) \quad \text{in} \quad \mathcal{U}_1. \tag{91a}$$

Therefore, by the Raabe test the series may converge for any finite value of z (the ratios (90) are valid if z is finite); however, we must exclude the point $z = 0$ because in general the function $\Psi(a, c; y)$ goes to infinity at $y = 0$. On the other side, since $\Psi(a, c; y) \sim y^{-a}$ when $y \to \infty$, we find that for $z \to \infty$

$$\mathcal{U}_1(z) \sim e^{i\omega z} z^{B_1} \sum_{n=0}^{\infty} b_n^1, \qquad \left|\frac{b_{n+1}^1}{b_n^1}\right| \overset{(89)}{=} 1 + \frac{\operatorname{Re} B_1}{n} + O\left(\frac{1}{n^2}\right) \quad \text{when} \quad n \to \infty. \tag{91b}$$

Thus, according to the Raabe test, the series $\sum b_n^1$ converges only if $\operatorname{Re}(B_1) < -1$, and this condition assures that $\mathcal{U}_1(z)$ converges at $z = \infty$.

4.2. Solutions for the confluent Heun equation

Now the solution $\mathcal{U}_1(z)$ for the spheroidal equation, given in (83a), is extended for any CHE. In fact, we can construct a group of solutions $\mathcal{U}_i(z)$ whose series coefficient b_n^i satisfy the relations (79). To this end, in the right-hand side of (83a) we perform the substitutions

$$z^{1+B_1}(z-1)^{1+B_1} \mapsto z^{1+\frac{B_1}{z_0}}(z-z_0)^{1-B_2-\frac{B_1}{z_0}}, \qquad \Psi(2+B_1, 2+B_1-n; -2i\omega z) \mapsto \Psi(\alpha, \beta-n; -2i\omega z),$$

where we have used the exponents $1 + B_1/z_0$ and $1 - B_2 - B_1/z_0$ because these are indicial exponents at $z = 0$ and $z = z_0$, respectively. By using the properties of $\Psi(a, c; y)$ we find that $\alpha = 2 + i\eta - B_2/2$ and $\beta = 2 + B_1/z_0$ [38]. Thus, $\mathcal{U}_1$ is given

$$\mathcal{U}_1(z) = e^{i\omega z} z^{1+\frac{B_1}{z_0}} [z - z_0]^{1-B_2-\frac{B_1}{z_0}} \sum_{n=0}^{\infty} b_n^1 \, \Psi\left(2 + i\eta - \frac{B_2}{2}, 2 + \frac{B_1}{z_0} - n; -2i\omega z\right), \qquad [i\eta - B_2/2 \neq -2, -3, \cdots] \tag{92a}$$

where the coefficients b_n^1 satisfy the recurrence relations (79) with [38]

$$\alpha_n^1 = -2i\omega z_0(n+1), \qquad \beta_n^1 = n\left[n + 1 - B_2 - \frac{2B_1}{z_0} + 2i\omega z_0\right] + \left[i\omega z_0 - 1 - \frac{B_1}{z_0}\right]\left[2 - B_2 - \frac{B_1}{z_0}\right] + 2 - B_2 + B_3,$$

$$\gamma_n^1 = -\left[n + i\eta - \frac{B_1}{z_0} - \frac{B_2}{2}\right]\left[n + 1 - B_2 - \frac{B_1}{z_0}\right]. \tag{92b}$$

By the transformations (18), $\mathcal{U}_1$ produces a group constituted by 16 solutions, $\mathcal{U}_i$. Eight of these can be constructed as

$$\begin{aligned} &\mathcal{U}_1(z), & \mathcal{U}_2(z) &= T_1 \mathcal{U}_1(z), & \mathcal{U}_3(z) &= T_2 \mathcal{U}_2(z), & \mathbb{U}_4(z) &= T_1 \mathcal{U}_3(z); \\ &\mathcal{U}_5(z) = T_4 \mathcal{U}_1(z), & \mathcal{U}_6(z) &= T_4 \mathcal{U}_2(z), & \mathcal{U}_7(z) &= T_4 \mathcal{U}_3(z), & \mathcal{U}_8(z) &= T_4 \mathcal{U}_4(z), \end{aligned} \tag{93}$$

while the others result by the transformation T_3 which changes (η, ω) by $(-\eta, -\omega)$ in the above solutions. Thus,

$$\mathcal{U}_2(z) = e^{i\omega z}[z - z_0]^{1-B_2-\frac{B_1}{z_0}} \sum_{n=0}^{\infty} b_n^2 \Psi\left(1 + i\eta - \frac{B_2}{2} - \frac{B_1}{z_0}, -\frac{B_1}{z_0} - n; -2i\omega z\right), \qquad [i\eta - \frac{B_2}{2} - \frac{B_1}{z_0} \neq -1, -2, \cdots] \tag{94a}$$

where, in the recurrence relations (79) for b_n^2,

$$\alpha_n^2 = -2i\omega z_0(n+1), \qquad \beta_n^2 = n\left[n + 3 - B_2 + 2i\omega z_0\right] + i\omega z_0\left[2 - B_2 - \frac{B_1}{z_0}\right] + 2 - B_2 + B_3],$$

$$\gamma_n^2 = -\left[n + i\eta + 1 - \frac{B_2}{2}\right]\left[n + 1 - B_2 - \frac{B_1}{z_0}\right] \tag{94b}$$

The third solution reads

$$\mathcal{U}_3(z) = e^{i\omega z} \sum_{n=0}^{\infty} b_n^3 \, \Psi\left(i\eta + \frac{B_2}{2}, -\frac{B_1}{z_0} - n; -2i\omega z\right), \qquad [i\eta + \frac{B_2}{2} \neq 0, -1, \cdots,] \tag{95a}$$

with

$$\alpha_n^3 = -2i\omega z_0(n+1), \qquad \beta_n^3 = n\left[n + 1 + B_2 + \frac{2B_1}{z_0} + 2i\omega z_0\right] + \left[B_2 + \frac{B_1}{z_0}\right]\left[1 + \frac{B_1}{z_0} + i\omega z_0\right] + B_3,$$

$$\gamma_n^3 = -\left[n + i\eta + \frac{B_2}{2} + \frac{B_1}{z_0}\right]\left[n - 1 + B_2 + \frac{B_1}{z_0}\right]. \tag{95b}$$

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At last, we write

$$\mathcal{U}_4(z) = e^{i\omega z} z^{1+\frac{B_1}{z_0}} \sum_{n=0}^{\infty} b_n^4 \Psi\left(1 + i\eta + \frac{B_2}{2} + \frac{B_1}{z_0}, 2 + \frac{B_1}{z_0} - n; -2i\omega z\right), \qquad [i\eta + B_2/2 + B_1/z_0 \neq -1, -2, \cdots] \quad (96a)$$

with

$$\alpha_n^4 = -2i\omega z_0(n+1), \qquad \beta_n^4 = n\left[n - 1 + B_2 + 2i\omega z_0\right] + i\omega z_0\left[B_2 + \frac{B_1}{z_0}\right] + B_3,$$

$$\gamma_n^4 = -\left[n + i\eta - 1 + \frac{B_2}{2}\right]\left[n - 1 + B_2 + \frac{B_1}{z_0}\right]. \quad (96b)$$

The relation (91a) is valid also for the present case, whereas for large values of z Eq. (91b) is replaced by [38]

$$\mathcal{U}_1(z) \sim e^{i\omega z}\, z^{-i\eta - \frac{B_2}{2}} \sum_{n=0}^{\infty} b_n^1, \qquad \left|\frac{b_{n+1}^1}{b_n^1}\right| = 1 + \frac{1}{n}\mathrm{Re}\left(i\eta - \frac{B_2}{2}\right) + O\left(\frac{1}{n^2}\right) \text{ if } n \to \infty. \quad (97)$$

Then, $\mathcal{U}_1$ converges at $z = \infty$ if $\mathrm{Re}(i\eta - B_2/2) < -1$. By using the transformations as in (93), we find that all the $\mathcal{U}_i$ converge for finite values of z, excepting possibly the points $z = 0$ (if $i = 1, 2, 3, 4$) and $z = z_0$ (if $i = 5, 6, 7, 8$). According to the Raabe test, these $\mathcal{U}_i$ converge also at $z = \infty$ if

$$\begin{aligned}
&\mathrm{Re}\left[i\eta - \frac{B_2}{2} + 1\right] < 0 : \mathcal{U}_1, \mathcal{U}_5; \qquad &\mathrm{Re}\left[i\eta - \frac{B_1}{z_0} - \frac{B_2}{2}\right] < 0 : \mathcal{U}_2, \mathcal{U}_8; \\
&\mathrm{Re}\left[i\eta + \frac{B_2}{2} - 1\right] < 0 : \mathcal{U}_3, \mathcal{U}_7; \qquad &\mathrm{Re}\left[i\eta + \frac{B_1}{z_0} + \frac{B_2}{2}\right] < 0 : \mathcal{U}_4, \mathcal{U}_6.
\end{aligned} \quad (98)$$

4.3. The radial part of the two-center problem

Now we consider the equations of the two-center problem of quantum mechanics, as the one describing the electron of the ionized hydrogen molecule. Using Leaver's conventions [9], the wave function ψ of the time-independent Schrödinger equation for an electron in the field of two Coulombian centers has the form

$$\psi = e^{im\varphi}\, \bar{R}(\lambda)\, \bar{S}(\mu), \qquad \lambda = \frac{r_1 + r_2}{2a}, \qquad \mu = \frac{r_1 - r_2}{2a}, \qquad m = 0, \pm 1, \pm 2, \cdots, \quad (99)$$

where r_1 and r_2 are the distances from the electron to the two nuclei, and $2a$ the intercenter distance. By the definitions

$$\begin{aligned}
\mathrm{S}(z) &= \bar{S}(\lambda) = z^{\frac{m}{2}}(2 - z)^{\frac{m}{2}} U^-(z), \qquad &z = \mu + 1, \qquad &[0 \leq z \leq 2], \\
\mathrm{R}(z) &= \bar{R}(\mu) = z^{\frac{m}{2}}(z - 2)^{\frac{m}{2}} U^+(z), \qquad &z = \lambda + 1, \qquad &[z \geq 2],
\end{aligned} \quad (100)$$

Leaver obtained CHEs in the form (1) for $U^\pm$, with the parameters ($\eta^\pm$ for η , $B_3^\pm$ for B_3)

$$z_0 = 2, \qquad \omega^2 = 2a^2 E, \qquad \omega\eta^\pm = -a(N_1 \pm N_2), \qquad B_1 = -2(m+1), \qquad B_2 = 2(m+1),$$

$$B_3^\pm = \omega^2 + 2a(N_1 \pm N_2) + m(m+1) - A_{lm}. \quad (101)$$

where A_{lm} is a separation constant, and N_1 and N_2 are the charges on the two nuclei. Thus, there are two CHEs, one for the "angular" coordinate μ and one for the "radial" coordinate λ. Each CHE is associated with a characteristic equation (81) which determines the possible values of the constants A_{lm} and E.

Now we consider $\mathrm{R}(z)$, the radial solution given in (100). For bound states ($E < 0$) we take

$$i\omega = -a\sqrt{2|E|} \quad \Rightarrow \quad i\eta = i\eta^+ = -(N_1 + N_2)/\sqrt{2|E|} \quad (102)$$

in order to assure that the factor $\exp(i\omega z)$ remains finite when $z \to \infty$. Then, if $|E|$ is finite,

$$\mathrm{Re}\left[i\eta - \frac{B_1}{z_0} - \frac{B_2}{2}\right] = \mathrm{Re}\left[i\eta + \frac{B_1}{z_0} + \frac{B_2}{2}\right] = -\frac{N_1 + N_2}{\sqrt{2|E|}} < 0,$$

and, consequently, four of the solutions listed in (98) converge at $z = \infty$. To get wavefunctions bounded also at $z = 2$, we select $\mathcal{U}_2$ if $m \leq 0$ and $\mathcal{U}_4$ if $m \geq 0$. Thus, we find

$$\mathrm{R}(z) = e^{-a\sqrt{2|E|}\, z}\, z^{-\frac{|m|}{2}} (z - 2)^{\frac{|m|}{2}} \sum_{n=0}^{\infty} b_n^2 \Psi\left(1 - \frac{N_1 + N_2}{\sqrt{2|E|}}, 1 - |m| - n; a\sqrt{8|E|}\, z\right) \quad (103a)$$

where the coefficients b_n^2 satisfy the relations (79) with

$$\alpha_n^2 = \sqrt{8|E|}\, a(n+1), \qquad \beta_n^2 = n\left[n+1+2|m|-2a\sqrt{8|E|}\right] + \left[|m|+1\right]\left[|m|-a\sqrt{8|E|}\right] +$$

$$2a\left[N_1+N_2-a|E|\right] - A_{lm}, \qquad \gamma_n^2 = -\left[n+|m|\right]\left[n+|m|-\frac{N_1+N_2}{\sqrt{2|E|}}\right]. \tag{103b}$$

The expansion (103a) holds only if

$$(N_1+N_2)/\sqrt{2|E|} \neq l+1, \qquad [l=0,2,\cdots] \tag{104}$$

a condition which assures that $\Psi(a,b;y)$ is not a polynomial of degree l in y.

The condition (104) is also required by the Jaffé expansions. In effect, by using the solutions U_1^J (if $m \geq 0$) and U_2^J (if $m \leq 0$) given in Eqs (70a) and (73a), respectively, we find

$$R^J(z) = e^{-a\sqrt{2|E|}\,z}\, z^{-1-\frac{|m|}{2}+\frac{N_1+N_2}{\sqrt{2|E|}}}\, (z-2)^{\frac{|m|}{2}} \sum_{n=0}^{\infty} a_n^1 \left(\frac{z-2}{z}\right)^n, \tag{105a}$$

where the recurrence relations for a_n^1 have the form (79) with

$$\alpha_n^1 = (n+1)(n+|m|+1), \qquad \beta_n^1 = -2n\left[n+1+|m|+a\sqrt{8|E|}-\frac{N_1+N_2}{\sqrt{2|E|}}\right] + \left[|m|+1\right]\left[\frac{N_1+N_2}{\sqrt{2|E|}}-a\sqrt{8|E|}-1\right] +$$

$$2a\left[N_1+N_2-a|E|\right]-A_{lm}, \qquad \gamma_n^1 = -\left[n+|m|-\frac{N_1+N_2}{\sqrt{2|E|}}\right]\left[n-\frac{N_1+N_2}{\sqrt{2|E|}}\right]. \tag{105b}$$

Thus, $\gamma_{l+1} = 0$ if $(N_1+N_2)/\sqrt{2|E|} = l+1$ and, then, $R^J(z)$ becomes a finite-series solution with $0 \leq n \leq l$, as stated after Eq. (81). In this case, the constant A_{lm} would be determined from the characteristic equation associated with the recurrence relations for a_n^1. However, if E and A_{lm} are both determined from the radial solution, we cannot satisfy the characteristic equation corresponding to the angular solutions (these are usually given by series where the summation begins at $n=0$ and, so, present recurrence relations having the form (80)). Therefore, also for the Jaffé solutions it is necessary that $(N_1+N_2)/\sqrt{2|E|} \neq l+1$. The same is true respecting Hylleraas' expansions in series of Laguerre polynomials [9, 39].

The convergence of solution (103a) follows immediately from the Raabe test. As to the Jaffé solution (105a), we have to examine its behavior at $z=\infty$. By using (101) together with (102), the expressions (74) imply that

$$\lim_{z\to\infty} R^J(z) = e^{-a\sqrt{2|E|}\,z}\, z^{\frac{N_1+N_2}{\sqrt{2|E|}}-1} \sum_{n=0}^{\infty} a_n^1, \quad \text{with} \quad \lim_{n\to\infty}\frac{a_{n+1}^1}{a_n^1} = 1 - \frac{1}{n}\left[\frac{3}{4} + 2\sqrt{an\sqrt{2|E|}} - 2a\sqrt{2|E|} + \frac{N_1+N_2}{\sqrt{2|E|}}\right]. \tag{106}$$

Then, by a convenient choice of n, the constant A which appears in (82b) becomes less than -1 and so, by the Raabe test, the solution converges at $z=\infty$.

5. Concluding remarks

By inserting a suitable weight function $w(z,t)$ into the integral relation (13) we have found the kernel equation (15) where the differential operators L_z and L_t depend on z and t in same manner as the operator of the CHE (11); this fact allows to get transformations of the kernels by examining the known transformations of the solutions for the CHE. As mentioned, this is an extension of a similar correspondence found in 2011 for the general Heun equation (HE) [1].

Actually, in 1942 Erdélyi used the appropriate weight function for the HE but he could not infer how to transform the kernels because the transformations of the HE were fully established only in 2007 [3]. On the other side, transformations of confluent Heun equations are known since 1978 [25, 26] but have not been applied to transform kernels – see, for example, references [4, 6, 7, 27]. In the present study we have considered transformations of kernels of the CHE and limiting cases. The initial kernels (to be transformed) come from kernels of the HE by a process of confluence [1]; however, for the sake of completeness, in section 2 we have reobtained them by solving the kernel equation.

By separation of variables we have found two groups of kernels presenting an arbitrary constant of separation. One group, with products of two confluent hypergeometric functions, includes some particular kernels already known in the literature [27]; the other group, with products of confluent hypergeometric functions and Gauss hypergeometric functions, is new as far as we know. By ascribing particular values to the constant of separation we have obtained three groups given by product of elementary functions with one special function: this is represented by confluent hypergeometric functions (two groups) and by Gauss hypergeometric functions (one group).

In section 3 we have found some integral transformations among known solutions of the confluent Heun equations. We have used two singularities as endpoints of integration and supposed that the solutions to be transformed are convergent at both endpoints (this assures that the bilinear concomitants vanish there). If the solutions are modified by the rules (18),

the kernels must be modified by the rules (20). This emphasizes that the correspondence between the transformations of the Heun equations and of the respective kernels are important parts of the transformation theory.

The applications of section 3 simply interconnect known solutions without affording new solutions. In contrast, in section 4, by means of an integral transformation we have obtained a new solution for the spheroidal wave equation, which in turn leads to a group of new solutions for the CHE. We have seen that these solutions may be used to compute the radial part of the wavefunctions for bound states of hydrogen moleculelike ions and, by this reason, can play the role of the expansions in series of Laguerre polynomials proposed by Hylleraas in 1931 [39] and the Jaffé power-series solutions [22] which have been used from 1934 up to now [40].

It is possible to realize further properties of the solutions by considering other problems, as the Lorentzian model of a quantum two-state system given by Ishkhanyan and Gregoryan [11]. This is ruled by a CHE with $z = (1 + it)/2$ and $z_0 = 1$, where t denotes the time. According to the authors, for certain values of a parameter R, the problem admits finite-series solutions which are bounded for any admissible value of t and assure that the system returns to the initial state after the interaction. By using the solutions of section 4.2, we have verified that the previous statement is correct; it seems that no other known solution of the CHE permits to prove the statement. In addition, since in this case $|z| \geq 1/2$, it would be interesting to check if there are *infinite-series* solutions suitable for some range of the parameter R (the Hylleraas and Jaffé solutions do not converge for $|z| < 1$).

We have omitted details concerning the derivation of the new solutions of the CHE. In addition, the solutions must be improved as follows: (i) by considering also expansions in series of regular confluent hypergeometric functions, we will get solution valid in the neighborhood of $z = 0$ [38], (ii) by using the Whittaker-Ince limit as in Ref. [18], we can expect solutions in series of Bessel functions for the RCHE (2), (iii) by inserting a "characteristic" parameter ν and letting that the series summation runs from minus to plus infinite [38] (two-sided series), we can obtain solutions for a CHE without free parameters [18].

As mentioned, besides the RCHE, there are two other equations which are associated with the CHE by formal limits. These are the double-confluent Heun equation (DHE) and the reduced DHE (RDHE) which appear when we allow that $z_0 \to 0$ in the CHE and RCHE, respectively, that is,

$$
\begin{aligned}
\text{DHE}: \qquad & z^2 \frac{d^2 U}{dz^2} + (B_1 + B_2 z)\frac{dU}{dz} + \left(B_3 - 2\eta\omega z + \omega^2 z^2\right)U = 0, \qquad [B_1 \neq 0, \ \omega \neq 0] \\
\text{RDHE}: \qquad & z^2 \frac{d^2 U}{dz^2} + (B_1 + B_2 z)\frac{dU}{dz} + (B_3 + qz)U = 0, \qquad\qquad\quad [q \neq 0, \ B_1 \neq 0]
\end{aligned}
\tag{107}
$$

where now $z = 0$ and $z = \infty$ are irregular singularities. At $z = \infty$ the behaviour is again given by Eq. (5), that is,

$$
\lim_{z \to \infty} U(z) \sim e^{\pm i\omega z}\, z^{\mp i\eta - (B_2/2)} \ \text{ for the DHE (1)}, \qquad \lim_{z \to \infty} U(z) \sim e^{\pm 2i\sqrt{qz}}\, z^{(1/4)-(B_2/2)} \ \text{ for the RDHE (2)},
$$

while at $z = 0$ the normal Thomé solutions affords [18]

$$
\lim_{z \to 0} U(z) \sim 1 \qquad \text{or} \qquad \lim_{z \to 0} U(z) \sim e^{B_1/z} z^{2-B_2} \qquad \text{for DHE and RDCE.}
$$

Starting with kernels of the CHE, in appendices C, D and E we have found that the Whitakker-Ince limit (3) and the Leaver limit ($z_0 \to 0$) lead to new kernels for the RCHE, DHE and RDHE, in accordance with a previous conjecture [1]. However, by integrating the kernel equations we have also found kernels which are not connected with known kernels of the CHE: for the RCHE we have a group of kernels expressed by products of Bessel and hypergeometric functions, while for the DHE and RDHE we have kernels given in terms of elementary functions. Therefore, the limiting procedures do not exhaust the possibilities for generating kernels.

In the appendix C we have found that the usual kernels of the Mathieu equation turn out to be particular cases of kernels of the RCHE. Furthermore, in appendix D, we have noticed that for DHE and RDHE in general it is convenient to use integral relations with variable limits of integration; this fact leads to an additional term in the bilinear concomitant - see Eq. (D.7).

Appendix A. Hypergeometric functions

The regular and irregular confluent hypergeometric functions are denoted by $\Phi(a, c; u)$ and $\Psi(a, c; u)$, respectively. They satisfy the confluent hypergeometric equation [37]

$$
u\frac{d^2\varphi(u)}{du^2} + (c - u)\frac{d\varphi(u)}{du} - a\varphi(u) = 0
\tag{A.1}
$$

which admits the solutions

$$
\varphi^1(u) = \Phi(a, c; u), \quad \varphi^2(u) = e^u u^{1-c}\Phi(1 - a, 2 - c; -u), \quad \varphi^3(u) = \Psi(a, c; u), \quad \varphi^4(u) = e^u u^{1-c}\Psi(1 - a, 2 - c; -u). \ \text{(A.2)}
$$

All of them are defined and distinct only if c is not an integer. Alternative forms for these solutions follow from the relations

$$
\Phi(a, c; u) = e^u\Phi(c - a, c; -u), \qquad \Psi(a, c; u) = u^{1-c}\Psi(1 + a - c, 2 - c; u).
\tag{A.3}
$$

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On the other side, solutions for the (Gauss) hypergeometric equation [37],

$$u(1-u)\frac{d^2F}{du^2} + \left[c - (a+b+1)u\right]\frac{dF}{du} - abF = 0, \tag{A.4}$$

are given by hypergeometric functions $F(a,b;c;u) = F(b,a;c;u)$. In fact, in the vicinity of the singular points 0, 1 and ∞, the formal solutions for the hypergeometric equation (A.4) are, respectively,

$$F^1(u) = F(a,b;c;u), \qquad F^2(u) = u^{1-c}F(a+1-c,b+1-c;2-c;u); \tag{A.5}$$

$$F^3(u) = F(a,b;a+b+1-c;1-u), \qquad F^4(u) = (1-u)^{c-a-b}F(c-a,c-b;1+c-a-b;1-u); \tag{A.6}$$

$$F^5(u) = u^{-a}F\left(a,a+1-c;a+1-b;\tfrac{1}{u}\right), \qquad F^6(u) = u^{-b}F\left(b+1-c,b;b+1-a;\tfrac{1}{u}\right). \tag{A.7}$$

Each of these may be written in four forms by using the relations

$$F(a,b;c;u) = (1-u)^{c-a-b}F(c-a,c-b;c;u), \qquad F(a,b;c;u) = (1-u)^{-a}F\left[a,c-b;c;u/(u-1)\right]. \tag{A.8}$$

On the other side, the usual form for the Bessel equation is [5]

$$y^2\frac{d^2Z(y)}{dy^2} + y\frac{dZ(y)}{dy} + \left[y^2 - \alpha^2\right]Z(y) = 0. \tag{A.9}$$

The solutions for this equation are denoted by $Z_\alpha^{(j)}(y)$ according as [5, 34]

$$Z_\alpha^{(1)}(y) = J_\alpha(y), \qquad Z_\alpha^{(2)}(y) = Y_\alpha(y), \qquad Z_\alpha^{(3)}(y) = H_\alpha^{(1)}(y), \qquad Z_\alpha^{(4)}(y) = H_\alpha^{(2)}(y) \tag{A.10}$$

where $J_\alpha(y)$ and $Y_\alpha(y)$ are the Bessel functions of the first and second kind, respectively; $H_\alpha^{(1)}(y)$ and $H_\alpha^{(2)}(y)$ are the first and the second Hankel functions. There are formulas connecting these functions [5]. For example,

$$Y_\alpha = \frac{1}{2i}\left[H_\alpha^{(1)} - H_\alpha^{(2)}\right] = \frac{\cos(\alpha\pi)J_\alpha - J_{-\alpha}}{\sin(\alpha\pi)}. \tag{A.11}$$

Bessel and confluent hypergeometric functions are connected by [37]

$$\Phi\left(\alpha + \tfrac{1}{2}, 2\alpha + 1; -2iy\right) = \Gamma(\alpha+1)\,e^{-iy}\left(\tfrac{y}{2}\right)^{-\alpha}J_\alpha(y), \quad \Psi\left(\alpha + \tfrac{1}{2}, 2\alpha + 1; -2iy\right) = \tfrac{i\sqrt{\pi}}{2}e^{-i(y-\alpha\pi)}(2y)^{-\alpha}H_\alpha^{(1)}(y),$$
$$\Psi\left(\alpha + \tfrac{1}{2}, 2\alpha + 1; 2iy\right) = -\tfrac{i\sqrt{\pi}}{2}e^{i(y-\alpha\pi)}(2y)^{-\alpha}H_\alpha^{(2)}(y). \tag{A.12}$$

In addition, we have the relations [37]

$$\lim_{a\to\infty}\Phi\left(a,c;-\frac{y}{a}\right) = \Gamma(c)\,y^{(1-c)/2}J_{c-1}\left(2\sqrt{y}\right),$$

$$\lim_{a\to\infty}\left[\Gamma(a+1-c)\,\Psi\left(a,c;-\frac{y}{a}\right)\right] = \begin{cases} -i\pi e^{i\pi c}y^{(1-c)/2}H_{c-1}^{(1)}\left(2\sqrt{y}\right), & \text{Im } y > 0, \\ i\pi e^{-i\pi c}y^{(1-c)/2}H_{c-1}^{(2)}\left(2\sqrt{y}\right), & \text{Im } y < 0. \end{cases} \tag{A.13}$$

Appendix B. Wilson's asymptotic expansions for the CHE

Such solutions were considered in 1928 by Wilson [10]. Actually they are given by 8 asymptotic Thomé expansions [17] which we denote by $\mathcal{W}_i(z)$ $(i = 1, 2, 3, 4)$ and $T_3\mathcal{W}_i(z)$. For the CHE in the form (1) we find

$$\mathcal{W}_1(z) = e^{i\omega z}\,z^{-i\eta - \frac{B_2}{2}}\sum_{n=0}^{\infty}a_n^1 z^{-n} \tag{B.1}$$

where the coefficients a_n^1 satisfy the three-term recurrence relations $(a_{-1}^1 = 0)$

$$2i\omega(n+1)a_{n+1}^1 - \left[n\left(n+1+2i\eta+2i\omega z_0\right) + i\omega z_0\left(B_2 + \tfrac{B_1}{z_0}\right) + B_3 + \left(\tfrac{B_2}{2}+i\eta\right)\left(1+i\eta - \tfrac{B_2}{2}\right)\right]a_n^1$$
$$+ z_0\left(n+i\eta+\tfrac{B_1}{z_0}+\tfrac{B_2}{2}\right)\left(n+i\eta+\tfrac{B_2}{2}-1\right)a_{n-1}^1 = 0. \tag{B.2}$$

For the other solutions we take

$$\mathcal{W}_2(z) = T_2 \mathcal{W}_1(z), \qquad \mathcal{W}_3(z) = T_4 \mathcal{W}_1(z), \qquad \mathcal{W}_4(z) = T_4 \mathcal{W}_2(z) = T_1 \mathcal{W}_3(z) \tag{B.3}$$

and $\mathcal{W}_{i+4} = T_3 \mathcal{W}_i$ $(i = 1, \cdots 4)$. Thus, from the first solution we get

$$\mathcal{W}_2(z) = e^{i\omega z} \, (z - z_0)^{1 - B_2 - \frac{B_1}{z_0}} \, z^{-i\eta - 1 + \frac{B_1}{z_0} + \frac{B_2}{2}} \sum_{n=0}^{\infty} a_n^2 z^{-n}, \qquad \text{where} \tag{B.4}$$

$$2i\omega(n+1)a_{n+1}^2 - \left[n\left(n + 1 + 2i\eta + 2i\omega z_0\right) + i\omega z_0\left(2 - B_2 - \frac{B_1}{z_0}\right) + B_3 + \left(\frac{B_2}{2} + i\eta\right)\left(1 + i\eta - \frac{B_2}{2}\right) \right] a_n^2$$
$$+ z_0 \left(n + i\eta - \frac{B_1}{z_0} - \frac{B_2}{2}\right)\left(n + i\eta + 1 - \frac{B_2}{2}\right) a_{n-1}^2 = 0. \tag{B.5}$$

This $\mathcal{W}_2(z)$ is the only solution relevant for section 5. For this reason we omit the other solutions.

By the D'Alembert test the solutions $\mathcal{W}_1(z)$ and $\mathcal{W}_2(z)$ converge for $|z| > |z_0|$, whereas $\mathcal{W}_3(z)$ and $\mathcal{W}_4(z)$ converge for $|z - z_0| > |z_0|$. However, by the Raabe test they converge also at $|z| = |z_0|$ and $|z - z_0| = |z_0|$ provided that

$$|z| \geq |z_0| \text{ if } \begin{cases} \text{Re}\left[B_2 + \frac{B_1}{z_0}\right] < 1 \text{ in } \mathcal{W}_1(z), \\ \text{Re}\left[B_2 + \frac{B_1}{z_0}\right] > 1 \text{ in } \mathcal{W}_2(z); \end{cases} \qquad |z - z_0| \geq |z_0| \text{ if } \begin{cases} \text{Re}\left[\frac{B_1}{z_0}\right] > -1 \text{ in } \mathcal{W}_3(z), \\ \text{Re}\left[\frac{B_1}{z_0}\right] < -1 \text{ in } \mathcal{W}_4(z), \end{cases} \tag{B.6}$$

where the restrictions on parameters of the equation are necessary only to assure convergence at $|z| = |z_0|$ or $|z - z_0| = |z_0|$.

The above regions of convergence suppose the minimal solutions for the series coefficients [17]. In the following we consider only the series which appears in $W_1(z)$, the convergence for the other solutions being obtained by using the transformations as indicated above. Thus, when $n \to \infty$ in $W_1(z)$ we have

$$2i\omega \frac{a_{n+1}^1}{a_n^1} - (n + 1 + 2i\eta + 2i\omega z_0) + z_0\left(n + 2i\eta + B_2 + \frac{B_1}{z_0} - 1\right) \frac{a_{n-1}^1}{a_n^1} = 0$$

whose minimal solution for a_{n+1}^1/a_n^1 satisfies

$$\frac{a_{n+1}^1}{a_n^1} \sim z_0 \left[1 + \frac{1}{n}\left(B_2 + \frac{B_1}{z_0} - 2\right)\right] \Rightarrow \frac{a_{n-1}^1}{a_n^1} \sim \frac{1}{z_0}\left[1 - \frac{1}{n}\left(B_2 + \frac{B_1}{z_0} - 2\right)\right].$$

Thence, when $n \to \infty$

$$\frac{a_{n+1}^1 z^{-n-1}}{a_n^1 z^{-n}} \sim \frac{z_0}{z}\left[1 + \frac{1}{n}\left(B_2 + \frac{B_1}{z_0} - 2\right)\right] \Rightarrow \left|\frac{a_{n+1}^1 z^{-n-1}}{a_n^1 z^{-n}}\right| \sim \frac{|z_0|}{|z|}\left[1 + \frac{1}{n}\text{Re}\left(B_2 + \frac{B_1}{z_0} - 2\right)\right].$$

So, by the D'Alambert test the series converges absolutely for $|z| > |z_0|$. However, by the Raabe test, the series converges even for $|z| = |z_0|$ provided that $\text{Re}\left[B_2 + (B_1/z_0)\right] < 1$.

Appendice C. Kernels for the reduced confluent Heun equation (RCHE)

In this appendix C:

- initially we get the substitutions of variables which preserve the form of the equation for the kernels of the RCHE;
- in C.1 we find a group of kernels containing products of two Bessel functions having an arbitrary constant of separation λ; these kernels cannot be derived from known kernels of the CHE by using the Whittaker-Ince limit (3);
- in C.2 we construct a group of kernels containing products of Bessel and hypergeometric functions with an arbitrary constant of separation λ; these kernels may be derived as limits of kernels of the CHE;
- in C.3, C.4 and C.5, by taking suitable values for λ in the above cases, we get kernels given by products of elementary and special functions; thus, in C.3 and C.4 the kernels are given by products of elementary and Bessel functions, while in C.5 the kernels are given by products of elementary and Gauss hypergeometric functions;
- in C.6 we find an integral relation between a solution of a RCHE in power series and a solution given by series of Bessel functions of first kind.
- the above results are also valid for the Mathieu equations because these are particular instances of the RCHE.

In Whittaker-Ince limit (3), Eqs. (13) and (16) remain formally unchanged but the operator (10) now reads

$$L_z = z(z - z_0)\frac{\partial^2}{\partial z^2} + [B_1 + B_2 z]\frac{\partial}{\partial z} + qz, \tag{C.1}$$

and so the RCHE (2) and equation (15) for the corresponding kernels $G(z,t)$ take the forms

$$[L_z + B_3 - qz_0]\, U(z) = 0, \qquad [L_z - L_t]\, G(z,t) = 0. \tag{C.2}$$

On the other side, if $U(z) = U(B_1, B_2, B_3; z_0, q; z)$ is solution of the RCHE, new solutions may be generated by the transformations $\mathscr{T}_1$, $\mathscr{T}_2$ and $\mathscr{T}_3$ given by [18]

$$\mathscr{T}_1 U(z) = z^{1+\frac{B_1}{z_0}}\, U(C_1, C_2, C_3; z_0, q; z),$$

$$\mathscr{T}_2 U(z) = (z - z_0)^{1-B_2-\frac{B_1}{z_0}}\, U(B_1, D_2, D_3; z_0, q; z), \tag{C.3}$$

$$\mathscr{T}_3 U(z) = U(-B_1 - B_2 z_0, B_2, B_3 - qz_0; z_0, -q; z_0 - z),$$

where C_i and D_i are defined in (19). Similarly, we can check that, if $G(z,t) = G(B_1, B_2; z_0, q; z, t)$ is a kernel, new kernels may generated by the transformations

$$\mathscr{R}_1 G(z,t) = (zt)^{1+\frac{B_1}{z_0}}\, G(C_1, C_2; z_0, q; z, t),$$

$$\mathscr{R}_2(z) G(z,t) = [(z - z_0)(t - z_0)]^{1-B_2-\frac{B_1}{z_0}}\, G(B_1, D_2; z_0, q; z, t), \tag{C.4}$$

$$\mathscr{R}_3 G(z,t) = G(-B_1 - B_2 z_0, B_2; z_0, -q; z_0 - z, z_0 - t).$$

We will see that the kernels for the RCHE reproduce all the known kernels [34, 41] for the Mathieu equation. To this end, we write the last equation as

$$\frac{d^2 w}{du^2} + \sigma^2 \left[\mathrm{a} - 2k^2 \cos(2\sigma u)\right] w = 0, \tag{C.5}$$

where $\sigma = 1$ or $\sigma = i$ for the Mathieu or modified Mathieu equations, respectively. Then, by setting $z = \cos^2(\sigma u)$ and $w(u) = U(z)$, Eq. (C.5) is converted into RCHE (2) with the following parameters:

$$z_0 = 1, \qquad B_1 = -\frac{1}{2}, \qquad B_2 = 1, \qquad B_3 = \frac{k^2}{2} - \frac{\mathrm{a}}{4}, \qquad q = k^2. \tag{C.6}$$

Besides this, putting $t = \cos^2(\sigma v)$ the integral (13) reads

$$\mathcal{U}(z) = \int_{v_1}^{v_2} G[z(u), t(v)]\, U[t(v)]\, dv. \tag{C.7}$$

where $z(u) = \cos^2(\sigma u)$ and $t(v) = \cos^2(\sigma v)$.

C.1. First group of kernels: products of Bessel functions

We find the set of kernels $G_{(\pm,\pm)}^{(i,j)}$ given by products of Bessel functions, namely,

$$
\begin{aligned}
G_{(\pm,\pm)}^{(i,j)}(z,t) = {} & \left[\lambda\sqrt{\tfrac{(z-z_0)(t-z_0)}{z_0}}\right]^{1-B_2-\frac{B_1}{z_0}} \left[\sqrt{\tfrac{zt}{z_0}}\right]^{1+\frac{B_1}{z_0}} \\
& \times\ Z_{\pm\left(1-B_2-\frac{B_1}{z_0}\right)}^{(i)} \left[\lambda\sqrt{\tfrac{(z-z_0)(t-z_0)}{z_0}}\right] Z_{\pm\left(1+\frac{B_1}{z_0}\right)}^{(j)} \left[\sqrt{\tfrac{1}{z_0}(\lambda^2 + 4q)zt}\right],
\end{aligned}
\tag{C.8}
$$

where $Z_\alpha^{(i)}$ are the four Bessel functions given in Eq. (A.10) or linear combinations of them. In addition, we find that the transformations $\mathscr{R}_1$ and $\mathscr{R}_2$ and $\mathscr{R}_3$ do not produce new kernels.

In fact, the substitutions

$$\xi = \sqrt{\tfrac{(z-z_0)(t-z_0)}{z_0}}, \qquad \zeta = \sqrt{\tfrac{zt}{z_0}}, \qquad G(z,t) = H(\xi, \zeta), \tag{C.9}$$

transform Eq. (C.2) into

$$\frac{\partial^2 H}{\partial \xi^2} + \frac{2}{\xi}\left(B_2 + \frac{B_1}{z_0} - \frac{1}{2}\right)\frac{\partial H}{\partial \xi} - 4qH - \left[\frac{\partial^2 H}{\partial \zeta^2} - \frac{2}{\zeta}\left(\frac{B_1}{z_0} + \frac{1}{2}\right)\frac{\partial H}{\partial \zeta}\right] = 0. \tag{C.10}$$

Thence, by the separation of variable

$$H(\xi, \zeta) = \xi^{1-B_2-\frac{B_1}{z_0}} \zeta^{1+\frac{B_1}{z_0}}\, X(\xi)\, Y(\zeta), \tag{C.11}$$

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we find

$$\frac{1}{X}\left[\frac{d^2X}{d\xi^2} + \frac{1}{\xi}\frac{dX}{d\xi} - \frac{1}{\xi^2}\left(B_2 + \frac{B_1}{z_0} - 1\right)^2 X\right] - \frac{1}{Y}\left[\frac{d^2Y}{d\zeta^2} + \frac{1}{\zeta}\frac{dY}{d\zeta} - \frac{1}{\zeta^2}\left(1 + \frac{B_1}{z_0}\right)^2 Y\right] - 4q = 0,$$

which leads to the Bessel equations

$$\xi^2\frac{d^2X}{d\xi^2} + \xi\frac{dX}{d\xi} + \left[\lambda^2\xi^2 - \left(1 - B_2 - \frac{B_1}{z_0}\right)^2\right]X = 0, \qquad \zeta^2\frac{d^2Y}{d\zeta^2} + \zeta\frac{dY}{d\zeta} + \left[(\lambda^2 + 4q)\zeta^2 - \left(1 + \frac{B_1}{z_0}\right)^2\right]Y = 0, \quad \text{(C.12)}$$

where λ^2 is a constant of separation. Thus, by taking

$$y = \lambda\xi, \quad \alpha = \pm\left(1 - B_2 - \frac{B_1}{z_0}\right) \quad \text{and} \quad y = \sqrt{(\lambda^2 + 4q)}\zeta, \quad \alpha = \pm\left(1 + \frac{B_1}{z_0}\right)$$

in the first and second equations, respectively, we obtain

$$X(\xi) = Z^{(i)}_{\pm\left(1 - B_2 - \frac{B_1}{z_0}\right)}[\lambda\xi], \qquad Y(\zeta) = Z^{(j)}_{\pm\left(1 + \frac{B_1}{z_0}\right)}\left[\sqrt{(\lambda^2 + 4q)}\zeta\right]. \qquad \text{(C.13)}$$

Inserting these solutions into (C.11) we get the kernels (C.8).

Now we let that the transformations $\mathscr{R}_i$ transform the parameter λ into λ_i. Since λ and λ_i are arbitrary, we conclude that $\mathscr{R}_i$ do not change the kernels. For instance,

$$\mathscr{R}_3 G^{(i,j)}_{(\pm,\pm)}(z,t) = \left[\frac{(z-z_0)(t-z_0)}{z_0}\right]^{\frac{1}{2} - \frac{B_2}{2} - \frac{B_1}{2z_0}}\left[\sqrt{\frac{zt}{z_0}}\right]^{1 + \frac{B_1}{z_0}} Z^{(i)}_{\pm\left(1 + \frac{B_1}{z_0}\right)}\left[\lambda_3\sqrt{\frac{zt}{z_0}}\right] Z^{(j)}_{\pm\left(1 - B_2 - \frac{B_1}{z_0}\right)}\left[\sqrt{\frac{[\lambda_3^2 - 4q][z-z_0][t-z_0]}{z_0}}\right]$$

By setting $\lambda_3^2 = \lambda^2 + 4q$, we see that the right-hand side is $G^{(i,j)}_{(\pm,\pm)}(z,t)$.

For Mathieu equation, the kernels (C.8) become

$$G^{(i,j)}_{(\pm,\pm)}(z,t) = \sqrt{\sin(2\sigma u)\sin(2\sigma v)}Z^{(i)}_{\pm\frac{1}{2}}[\lambda\sin(\sigma u)\sin(\sigma v)] Z^{(j)}_{\pm\frac{1}{2}}\left[\sqrt{\lambda^2 + 4q}\ \cos(\sigma u)\cos(\sigma v)\right], \qquad \text{(C.14)}$$

where the Bessel functions can be expressed in terms of elementary functions since [31]

$$\begin{pmatrix} J_{1/2}(x) = Y_{-1/2}(x) \\ J_{-1/2}(x) = -Y_{1/2}(x) \end{pmatrix} = \sqrt{\frac{2}{\pi x}}\begin{pmatrix} \sin x \\ \cos x \end{pmatrix}, \qquad \begin{pmatrix} H^{(1)}_{1/2}(x) = -iH^{(1)}_{-1/2}(x) \\ H^{(2)}_{1/2}(x) = iH^{(2)}_{-1/2}(x) \end{pmatrix} = \sqrt{\frac{2}{\pi x}}\begin{pmatrix} -ie^{ix} \\ ie^{-ix} \end{pmatrix}. \qquad \text{(C.15)}$$

C.2. Second group: products of Bessel and hypergeometric functions

The kernels given by products of Bessel and hypergeometric functions are written as

$$G^{(i,j)}_{\pm}(z,t) = \left[2\sqrt{q(z+t-z_0)}\right]^{1-B_2} Z^{(i)}_{\pm(2\lambda+1-B_2)}\left[2\sqrt{q(z+t-z_0)}\right] F^j(\zeta), \qquad [i = 1, \cdots, 4;\ j = 1, \cdots, 6] \quad \text{(C.16)}$$

where F^j denote the hypergeometric functions written in Eqs. (41-46). We can show that the transformations $\mathscr{R}_i$ simply rearrange the previous kernels.

By using properties (A.13) of the confluent hypergeometric functions, the above kernels may be obtained by applying the Whittaker-Ince limit to the kernels (32) of the CHE. To derive the kernels directly, we note that the substitutions

$$\xi = 2\sqrt{q(z+t-z_0)}, \qquad \zeta = \frac{zt}{z_0(z+t-z_0)}, \qquad G(z,t) = \xi^{1-B_2} H(\xi,\zeta), \qquad \text{(C.17)}$$

transform the second Eq. (C.2) into

$$\xi^2\frac{\partial^2 H}{\partial\xi^2} + \xi\frac{\partial H}{\partial\xi} + \left[\xi^2 - (1 - B_2)^2\right]H + 4\zeta(1-\zeta)\frac{\partial^2 H}{\partial\zeta^2} + 4\left[-\frac{B_1}{z_0} - B_2\zeta\right]\frac{\partial H}{\partial\zeta} = 0. \qquad \text{(C.18)}$$

The separation of variables

$$H(\xi,\zeta) = X(\xi)\ Y(\zeta) \ \Rightarrow\ G(z,t) = \xi^{1-B_2}\ X(\xi)\ Y(\zeta), \qquad \text{(C.19)}$$

leads to the following Bessel and hypergeometric equations, respectively,

$$\xi^2\frac{d^2X}{d\xi^2} + \xi\frac{dX}{d\xi} + \left[\xi^2 - (2\lambda+1-B_2)^2\right]X = 0, \qquad \zeta(1-\zeta)\frac{d^2Y}{d\zeta^2} + \left[-\frac{B_1}{z_0} - B_2\zeta\right]\frac{dY}{d\zeta} - \lambda(B_2 - \lambda - 1)Y = 0, \quad \text{(C.20)}$$

where we have denoted the constant of separation by $4\lambda(\lambda+1-B_2)$. Using Eqs. (C.17-C.20) we obtain the kernels (C.16).

To show that the transformations $\mathscr{R}_i$ do not produce new kernels we use the fact that the constants of separation are arbitrary. For example, since

$$\mathscr{R}_1 G_{\pm}^{(i,1)}(z,t) \propto \left[\frac{zt}{z+t-z_0}\right]^{1+\frac{B_1}{z_0}} \left[2\sqrt{q(z+t-z_0)}\right]^{1-B_2} \times$$
$$\times \; Z_{\pm\left(2\lambda_1-1-2-\frac{2B_1}{z_0}\right)}^{(i)}\left[2\sqrt{q(z+t-z_0)}\right] F\left[\lambda_1, 1+B_2+\frac{2B_1}{z_0}-\lambda_1; 2+\frac{B_1}{z_0}; \frac{zt}{z_0(z+t-z_0)}\right],$$

by taking $\lambda_1 = \lambda+1+(B_1/z_0)$ we find that the right-hand side is $G_{\pm}^{(i,2)}(z,t)$. Analogously,

$$\mathscr{R}_2 G_{\pm}^{(i,1)}(z,t) \propto \; [(z-z_0)(t-z_0)]^{1-B_2-\frac{B_1}{z_0}} \left[2\sqrt{q(z+t-z_0)}\right]^{B_2+\frac{2B_1}{z_0}-1}$$
$$\times \; Z_{\pm\left(2\lambda_2-1+B_2+\frac{2B_1}{z_0}\right)}^{(i)}\left[2\sqrt{q(z+t-z_0)}\right] F\left[\lambda_2, 1-B_2-\frac{2B_1}{z_0}-\lambda_2; -\frac{B_1}{z_0}; \frac{zt}{z_0(z+t-z_0)}\right].$$

Putting $\lambda_2 = \lambda+1-B_2-(B_1/z_0)$ and using Eq. (A.8), we find that the right-hand side is proportional to $G_{\pm}^{(i,1)}$.

For the Mathieu equation, whenever appropriate we use the relations [31]

$$F\left(\frac{\nu}{2}, -\frac{\nu}{2}; \frac{1}{2}; y^2\right) = \cos(\nu \arcsin y), \qquad F\left(\frac{1+\nu}{2}, \frac{1-\nu}{2}; \frac{3}{2}; y^2\right) = \frac{\sin(\nu \arcsin y)}{\nu y}$$

with $\nu = 2\lambda$. Then, the kernels (C.16) are rewritten as

$$G_{\pm}^{(i,1)}(u,v) = \cos\left[2\lambda \arcsin \frac{\sqrt{2}\cos(\sigma u)\cos(\sigma v)}{\sqrt{\cos(2\sigma u)+\cos(2\sigma v)}}\right] Z_{\pm 2\lambda}^{(j)}\left[k\sqrt{2\cos(2\sigma u)+2\cos(2\sigma v)}\right], \qquad \text{(C.21)}$$

$$G_{\pm}^{(i,2)}(u,v) = \sin\left[2\lambda \arcsin \frac{\sqrt{2}\cos(\sigma u)\cos(\sigma v)}{\sqrt{\cos(2\sigma u)+\cos(2\sigma v)}}\right] Z_{\pm 2\lambda}^{(i)}\left[k\sqrt{2\cos(2\sigma u)+2\cos(2\sigma v)}\right], \qquad \text{(C.22)}$$

$$G_{\pm}^{(i,3)}(u,v) = \cos\left[2\lambda \arcsin \frac{i\sqrt{2}\sin(\sigma u)\sin(\sigma v)}{\sqrt{\cos(2\sigma u)+\cos(2\sigma v)}}\right] Z_{\pm 2\lambda}^{(i)}\left[k\sqrt{2\cos(2\sigma u)+2\cos(2\sigma v)}\right], \qquad \text{(C.23)}$$

$$G_{\pm}^{(i,4)}(u,v) = \sin\left[2\lambda \arcsin \frac{i\sqrt{2}\sin(\sigma u)\sin(\sigma v)}{\sqrt{\cos(2\sigma u)+\cos(2\sigma v)}}\right] Z_{\pm 2\lambda}^{(i)}\left[k\sqrt{2\cos(2\sigma u)+2\cos(2\sigma v)}\right], \qquad \text{(C.24)}$$

$$G_{\pm}^{(i,5)}(u,v) = \left[\frac{\cos^2(\sigma u)\cos^2(\sigma v)}{\cos(2\sigma u)+\cos(2\sigma v)}\right]^{-\lambda} F\left[\lambda, \frac{1}{2}+\lambda; 1+2\lambda; \frac{\cos(2\sigma u)+\cos(2\sigma v)}{2\cos^2(\sigma u)\cos^2(\sigma v)}\right] Z_{\pm 2\lambda}^{(i)}\left[k\sqrt{2\cos(2\sigma u)+2\cos(2\sigma v)}\right], \quad \text{(C.25)}$$

$$G_{\pm}^{(i,6)}(u,v) = \left[\frac{\cos^2(\sigma u)\cos^2(\sigma v)}{\cos(2\sigma u)+\cos(2\sigma v)}\right]^{\lambda} F\left[-\lambda, \frac{1}{2}-\lambda; 1-2\lambda; \frac{\cos(2\sigma u)+\cos(2\sigma v)}{2\cos^2(\sigma u)\cos^2(\sigma v)}\right] Z_{\pm 2\lambda}^{(i)}\left[k\sqrt{2\cos(2\sigma u)+2\cos(2\sigma v)}\right]. \quad \text{(C.26)}$$

The kernels (C.21-C.24) are equivalent to the ones given on pp. 190 and 191 of McLachlan [41], but we have not found $G_{\pm}^{(i,5)}$ and $G_{\pm}^{(i,6)}$ in the literature.

C.3. Third group: Bessel functions

Up to a multiplicative constant, a initial set of kernels given by Bessel functions is

$$G_{1,\pm}^{(i)}(z,t) = \left[\sqrt{zt}\right]^{1+\frac{B_1}{z_0}} Z_{\pm\left(1+\frac{B_1}{z_0}\right)}^{(i)}\left[2\sqrt{\frac{qzt}{z_0}}\right]. \qquad \text{(C.27)}$$

These kernels are obtained by supposing that $H(\xi,\zeta)$ depends only on ζ in Eq. (C.10). Then, the substitution

$$G(z,t) = H(\xi,\zeta) = \zeta^{1+\frac{B_1}{z_0}} Y(\zeta), \qquad \zeta = \sqrt{zt/z_0}$$

leads to

$$\zeta^2 \frac{d^2 Y}{d\zeta^2} + \zeta \frac{dY}{d\zeta} + \left[4q\zeta^2 - \left(1+\frac{B_1}{z_0}\right)^2\right] Y = 0,$$

which is the Bessel equation (A.9) with argument $y = 2\sqrt{q}\zeta = 2\sqrt{qzt/z_0}$ and order $\alpha = \pm[1 + (B_1/z_0)]$. In this manner, we find (C.27). The remaining sets are obtained by using the transformations (C.4) as

$$G_{2,\pm}^{(i)}(z,t) = \mathscr{R}_2 G_{1,\pm}^{(i)}(z,t), \qquad G_{3,\pm}^{(i)}(z,t) = \mathscr{R}_3 G_{2,\pm}^{(i)}(z,t), \qquad G_{4,\pm}^{(i)}(z,t) = \mathscr{R}_1 G_{3,\pm}^{(i)}(z,t).$$

Thence,

$$G_{2,\pm}^{(i)}(z,t) = \left[\sqrt{zt}\right]^{1+\frac{B_1}{z_0}} [(z-z_0)(t-z_0)]^{1-B_2-\frac{B_1}{z_0}} Z_{\pm\left(1+\frac{B_1}{z_0}\right)}^{(i)} \left[2\sqrt{\frac{qzt}{z_0}}\right], \tag{C.28}$$

$$G_{3,\pm}^{(i)}(z,t) = [zt]^{1+\frac{B_1}{z_0}} \left[\sqrt{(z-z_0)(t-z_0)}\right]^{1-B_2-\frac{B_1}{z_0}} Z_{\pm\left(1-B_2-\frac{B_1}{z_0}\right)}^{(i)} \left[2\sqrt{-\frac{q}{z_0}(z-z_0)(t-z_0)}\right], \tag{C.29}$$

$$G_{4,\pm}^{(i)}(z,t) = \left[\sqrt{(z-z_0)(t-z_0)}\right]^{1-B_2-\frac{B_1}{z_0}} Z_{\pm\left(1-B_2-\frac{B_1}{z_0}\right)}^{(i)} \left[2\sqrt{-\frac{q}{z_0}(z-z_0)(t-z_0)}\right]. \tag{C.30}$$

This group of kernels can as well be generated by applying the Whittaker-Ince limit to kernels of the CHE given by hypergeometric functions in section 2.4.

For the Mathieu equation, up to constant factors, we find

$$G_{1,\pm}^{(i)}(u,v) = \sqrt{\cos(\sigma u)\cos(\sigma v)}\; Z_{\pm 1/2}^{(i)}\left[2k\cos(\sigma u)\cos(\sigma v)\right], \tag{C.31}$$

$$G_{2,\pm}^{(i)}(u,v) = \sin(\sigma u)\sin(\sigma v)\sqrt{\cos(\sigma v)\cos(\sigma v)}\, Z_{\pm 1/2}^{(i)}\left[2k\cos(\sigma u)\cos(\sigma v)\right], \tag{C.32}$$

$$G_{3,\pm}^{(i)}(u,v) = \cos(\sigma u)\cos(\sigma v)\sqrt{\sin(\sigma u)\sin(\sigma v)}\, Z_{\pm 1/2}^{(i)}\left[2ik\sin(\sigma u)\sin(\sigma v)\right], \tag{C.33}$$

$$G_{4,\pm}^{(i)}(u,v) = \sqrt{\sin(\sigma u)\sin(\sigma v)}\; Z_{\pm 1/2}^{(i)}\left[2ik\sin(\sigma u)\sin(\sigma v)\right]. \tag{C.34}$$

Kernels of this type have been used to generate solutions in series of Bessel functions out of Fourier-like expansions [34, 41] (that is, from solutions in series of trigonometric or hyperbolic functions).

C.4. Fourth group: Bessel functions again

Another group, given by products of elementary and Bessel functions, is generated from the initial set of the kernels

$$G_{1,\pm}^{(i)}(z,t) = \left[2\sqrt{q(z+t-z_0)}\right]^{1-B_2} Z_{\pm(B_2-1)}^{(i)}\left[2\sqrt{q(z+t-z_0)}\right], \tag{C.35}$$

which result when $H(\xi,\zeta)$ depends only on ξ in Eq. (C.18), that is, when $H(\xi,\zeta) = X(\xi)$. In effect, in this case we find

$$G(z,t) \overset{(C.17)}{=} \xi^{1-B_2}\, X(\xi), \qquad \xi = 2\sqrt{q(z+t-z_0)}$$

where $X(\xi)$ satisfies the Bessel equation

$$\xi^2\frac{d^2 X}{d\xi^2} + \xi\frac{dX}{d\xi} + \left[\xi^2 - (B_2-1)^2\right] X = 0$$

Taking $X(\xi) = Z_{\pm(B_2-1)}^{(i)}(\xi)$ we obtain the kernels (C.35). Then, by using the transformations $\mathscr{R}_1$ and $\mathscr{R}_2$ as ($\mathscr{R}_3$ is ineffective)

$$G_{2,\pm}^{(i)}(z,t) = \mathscr{R}_1 G_{1,\pm}^{(i)}(z,t), \qquad G_{3,\pm}^{(i)}(z,t) = \mathscr{R}_2 G_{2,\pm}^{(i)}(z,t), \qquad G_{4,\pm}^{(i)}(z,t) = \mathscr{R}_1 G_{3,\pm}^{(i)}(z,t).$$

we find that the other sets are

$$G_{2,\pm}^{(i)}(z,t) = [zt]^{1+\frac{B_1}{z_0}} \left[2\sqrt{q(z+t-z_0)}\right]^{-1-B_2-\frac{2B_1}{z_0}} Z_{\pm\left(1+B_2+\frac{2B_1}{z_0}\right)}^{(i)} \left[2\sqrt{q(z+t-z_0)}\right], \tag{C.36}$$

$$G_{3,\pm}^{(i)}(z,t) = [zt]^{1+\frac{B_1}{z_0}} [(z-z_0)(t-z_0)]^{1-B_2-\frac{B_1}{z_0}} \left[2\sqrt{q(z+t-z_0)}\right]^{B_2-3} Z_{\pm(3-B_2)}^{(i)} \left[2\sqrt{q(z+t-z_0)}\right], \tag{C.37}$$

$$G_{4,\pm}^{(i)}(z,t) = [(z-z_0)(t-z_0)]^{1-B_2-\frac{B_1}{z_0}} \left[2\sqrt{q(z+t-z_0)}\right]^{B_2+\frac{2B_1}{z_0}-1} Z_{\pm\left(1-B_2-\frac{2B_1}{z_0}\right)}^{(i)} \left[2\sqrt{q(z+t-z_0)}\right]. \tag{C.38}$$

The above kernels can as well be generated by applying the Whittaker-Ince limit to kernels of the CHE given by confluent hypergeometric functions in section 2.5. On the other hand, these kernels are instances of the kernels $G^{(i,j)}_{\pm}(z,t)$ given in (C.16), corresponding to four choices of λ which permit to write the hypergeometric functions $F^{(j)}(\zeta)$ as elementary functions. Indeed, up to constant factors we find that: (i) the kernels $G^{(i)}_{1,\pm}$ correspond to $\lambda = 0$ in $G^{(i,1)}_{\pm}$, $G^{(i,3)}_{\pm}$ and $G^{(i,5)}_{\pm}$; $G^{(i)}_{2,\pm}$ correspond to $\lambda = -1 - (B_1/z_0)$ in $G^{(i,2)}_{\pm}$; $G^{(i)}_{3,\pm}$ correspond to $\lambda = B_2 - 2$ in $G^{(i,5)}_{\pm}$; $G^{(i)}_{4,\pm}$ correspond to $\lambda = B_2 + (B_1/z_0) - 1$ in $G^{(i,5)}_{\pm}$.

Notice that $Z^{(i)}_{-\ell}(x) = (-1)^\ell Z^{(i)}_\ell(x)$ if ℓ is integer. So, up to multiplicative constants, for the Mathieu equation the previous kernels read

$$G^{(i)}_1(u,v) = Z^{(i)}_0\left[k\sqrt{2\cos(2\sigma u) + 2\cos(2\sigma v)}\right], \tag{C.39}$$

$$G^{(i)}_2(u,v) = \frac{\cos(\sigma u)\cos(\sigma v)}{\sqrt{\cos(2\sigma u)+\cos(2\sigma v)}}Z^{(i)}_1\left[k\sqrt{2\cos(2\sigma u) + 2\cos(2\sigma v)}\right], \tag{C.40}$$

$$G^{(i)}_3(u,v) = \frac{\sin(2\sigma u)\sin(2\sigma v)}{\cos(2\sigma u)+\cos(2\sigma v)}Z^{(i)}_2\left[k\sqrt{2\cos(2\sigma u) + 2\cos(2\sigma v)}\right], \tag{C.41}$$

$$G^{(i)}_4(u,v) = \frac{\sin(\sigma u)\sin(\sigma v)}{\sqrt{\cos(2\sigma u) + \cos(2\sigma v)}}Z^{(i)}_1\left[k\sqrt{2\cos(2\sigma u) + 2\cos(2\sigma v)}\right]. \tag{C.42}$$

These kernels for the Mathieu equation are connected with particular values of λ in the kernels (C.21-C.26). In fact: (i) for $G^{(i)}_1$ we take $\lambda = 0$ in (C.21), (C.23), (C.25) or (C.26); (ii) for $G^{(i)}_2$, $\lambda = 1/2$ in (C.22); (iii) for $G^{(i)}_3$, $\lambda = -1$ in (C.25) or $\lambda = 1$ in (C.26); (iv) for $G^{(i)}_4$, $\lambda = 1/2$ in (C.24).

C.5. Fifth group: hypergeometric functions

By taking $2\lambda + 1 - B_2 = 1/2$ in (C.16), we find

$$G^{(i,j)}_{\pm}(z,t) = \left[2\sqrt{q(z+t-z_0)}\right]^{1-B_2} Z^{(i)}_{\pm\frac{1}{2}}\left(2\sqrt{q(z+t-z_0)}\right)\left[F^{(j)}(\zeta)\right]_{\lambda=\frac{B_2}{2}-\frac{1}{4}}, \qquad [i = 1,\cdots,4;\ j = 1,\cdots,6], \tag{C.43}$$

where $F^{(j)}$ denote the hypergeometric functions written in Eqs. (41-46), whereas $Z^{(i)}_{\pm 1/2}$ are given by the elementary functions (C.15). For the Mathieu equation, the explicit form of the kernels is obtained by putting $\lambda = 1/4$ in Eqs. (C.21-C.26).

C.6. Power series and series of Bessel functions for the RCHE

In the Whittaker-Ince limit the power series solution (75a) becomes

$$U^{\text{baber}}_1(z) = \sum_{n=0}^{\infty} a^1_n(z - z_0)^n, \qquad (|z| = \text{finite}) \tag{C.44}$$

where the series coefficients now satisfy $(a^1_{-1} = 0)$

$$z_0\left(n + B_2 + \frac{B_1}{z_0}\right)(n+1)a^1_{n+1} + \left[n(n + B_2 - 1) + B_3\right]a^1_n + qa^1_{n-1} = 0. \tag{C.45}$$

In the following we find that, if

$$\text{if } \mathrm{Re}\,(B_1/z_0) < 0 \quad \text{and} \quad \mathrm{Re}\,[n + B_2 + (B_1/z_0)] > 0 \tag{C.46}$$

then, by means of an integral transformation, $U^{\text{baber}}_1(z)$ generates a known expansion $U_1(z)$ in series of Bessel functions given by [18]

$$U_1(z) = \sum_{n=0}^{\infty} (-1)^n c^1_n\left(\sqrt{qz}\right)^{-(n+B_2-1)} J_{n+B_2-1}\left(2\sqrt{qz}\right), \tag{C.47}$$

where the recurrence relations for c^1_n are obtained by writing

$$c^1_n = C(z_0)^n \Gamma\left(n + B_2 + \frac{B_1}{z_0}\right) a^1_n,$$

C being a constant independent of n. Thus,

$$(n+1)\,c_{n+1}^1 + \left[n\big(n+B_2-1\big)+B_3\right]c_n^1 + qz_0\left(n+B_2+\tfrac{B_1}{z_0}-1\right)c_{n-1}^1 = 0. \qquad (C.48)$$

In fact, by inserting $U_1(t) = U_1^{\text{baber}}(t)$ and the kernel $G(z,t) = G_{1,-}^{(1)}(z,t)$ given in (C.27) into (13), and taking $t_1 = 0$ and $t_2 = z_0$, we find

$$\mathcal{U}(z) = z^{\frac{1}{2}+\frac{B_1}{2z_0}} \sum_{n=0}^{\infty} a_n^{(1)} \int_0^{z_0} dt\left[t^{-\frac{1}{2}-\frac{B_1}{2z_0}}(t-z_0)^{n+B_2+\frac{B_1}{z_0}-1} J_{-1-\frac{B_1}{z_0}}\left(2\sqrt{qzt/z_0}\;\right)\right].$$

By using the integral [42]

$$\int_0^y x^{\frac{1}{2}\nu}(y-x)^{\mu-1}J_\nu(a\sqrt{x}\,)dx = 2^\mu y^{\frac{1}{2}(\mu+\nu)}\Gamma(\mu)a^{-\mu}J_{\mu+\nu}(\sqrt{y}a), \qquad [\text{Re}(\mu)>1, \quad \text{Re}(\nu)>-1],$$

we get $\mathcal{U} = U_1$, where U_1 is given in (C.47). On the other side, since

$$\frac{d}{dy}\left[y^{-\nu}J_\nu(y)\right] = -y^{-\nu}J_{\nu+1}(y),$$

the bilinear concomitant (16) takes the form

$$\begin{aligned}
P_1(z,t) = & -z^{\frac{1}{2}+\frac{B_1}{2z_0}}\,t^{\frac{1}{2}-\frac{B_1}{2z_0}}\,(t-z_0)^{B_2+\frac{B_1}{z_0}} \\
& \times \left[\sqrt{\frac{qz}{tz_0}}J_{-\frac{B_1}{z_0}}\left(2\sqrt{qzt/z_0}\right)\sum_{n=0}^{\infty}a_n^1(t-z_0)^n + J_{-1-\frac{B_1}{z_0}}\left(2\sqrt{qzt/z_0}\right)\sum_{n=1}^{\infty}na_n^1(t-z_0)^{n-1}\right].
\end{aligned}$$

Thence, the conditions (C.46) assure that $P_1(z,0) = P_1(z,z_0) = 0$.

Appendice D. Kernels for the double-confluent Heun equation (DHE)

As $z_0 \to 0$ the CHE (1) reduces to the double-confluent Heun equation (DHE)

$$[L_z + B_3]\,U = z^2\frac{d^2U}{dz^2} + [B_1 + B_2 z]\frac{dU}{dz} + \left[\omega^2 z^2 - 2\eta\omega z + B_3\right]U = 0, \qquad (D.1)$$

where $z = 0$ and $z = \infty$ are both irregular points. In this appendix D:
- initially we get the substitutions of variables which preserve the form of the equation for the kernels of the DHE;
- in D.1 we find kernels containing products of two confluent hypergeometric functions and presenting an arbitrary constant of separation λ; they may be derived by applying the Leaver limit $(z_0 \to 0)$ to kernels of the CHE given by products of hypergeometric and confluent hypergeometric functions (section 2.3);
- in D.2, by taking appropriate values for λ, we get kernels given products of elementary and confluent hypergeometric functions;
- in D.3 we obtain kernels given by elementary functions; these kernels cannot be derived as limits of known kernels of the CHE.

Since $\lim_{x\to 0}(1+x)^{1/x} = e$, when $z_0 \to 0$ the integral (13) assumes the form

$$\mathcal{U}(z) = \int_{t_1}^{t_2} K(z,t)U(t)dt = \int_{t_1}^{t_2} t^{B_2-2}e^{-\frac{B_1}{t}}G(z,t)U(t)dt, \qquad K(z,t) = w(z,t)G(z,t) = t^{B_2-2}e^{-\frac{B_1}{t}}G(z,t), \quad (D.2)$$

where $G(z,t)$ is determined from the equation

$$\left[z^2\frac{\partial^2}{\partial z^2} + (B_1+B_2 z)\frac{\partial}{\partial z} + \left(\omega^2 z^2 - 2\eta\omega z\right)\right]G - \left[t^2\frac{\partial^2}{\partial t^2} + (B_1+B_2 t)\frac{\partial}{\partial t} + \left(\omega^2 t^2 - 2\eta\omega t\right)\right]G = 0. \qquad (D.3)$$

Similarly, the expression (16) for the bilinear concomitant now reads

$$P(z,t) = t^2\left[U(t)\frac{\partial K(z,t)}{\partial t} - K(z,t)\frac{dU(t)}{dt}\right] + [(2-B_2)\,t - B_1]\,U(t)K(z,t) = t^{B_2}\,e^{-\frac{B_1}{t}}\left[U(t)\frac{\partial G(z,t)}{\partial t} - G(z,t)\frac{dU(t)}{dt}\right], \quad (D.4)$$

In general the solutions $U(t)$ of the DHE and RDHE converge in a domain including only one of the singular points, 0 or ∞. For this reason we must avoid using intervals of integration extending from $t_1 = 0$ to $t_2 = \infty$. In reference [43] this requirement was satisfied by using endpoints t_i which depend on the variable z, that is,

$$\mathcal{U}(z) = \int_{t_1(z)}^{t_2(z)} K(z,t)U(t)dt = \int_{t_1(z)}^{t_2(z)} t^{B_2-2}e^{-\frac{B_1}{t}}G(z,t)U(t)dt. \qquad (D.5)$$

Then, the formula

$$\frac{d}{dz}\int_{t_1(z)}^{t_2(z)} F(z,t)dt = \int_{t_1(z)}^{t_2(z)} \frac{\partial F(z,t)}{\partial z}dt + F(z,t_2)\frac{dt_2}{dz} - F(z,t_1)\frac{dt_1}{dz} \tag{D.6}$$

implies that Eq. (17) must be replaced by

$$[L_z + B_3]\mathcal{U}(z) = P(z,t_2) + Q(z,t_1) - [P(z,t_1) + Q(z,t_2)], \tag{D.7}$$

where $(i = 1,2)$

$$\begin{aligned}
Q(z,t_i) =\ & \left[z^2\frac{d^2 t_i}{dz^2} + (B_1 + B_2 z)\frac{dt_i}{dz}\right] U(t_i)K(z,t_i) \\
& + z^2 U(t_i)\left[\frac{\partial K(z,t_i)}{\partial t_i}\left(\frac{dt_i}{dz}\right)^2 + 2\frac{\partial K(z,t_i)}{\partial z}\frac{dt_i}{dz}\right] + z^2\left(\frac{dt_i}{dz}\right)^2\frac{dU(t_i)}{dt_i}K(z,t_i).
\end{aligned} \tag{D.8}$$

From Eq. (D.7) we see that the condition $P(z,t_2) = P(z,t_1)$ must be replaced by $P(z,t_2) + Q(z,t_2) = P(z,t_1) + Q(z,t_1)$.

Since the differential operators in Eqs. (D.1) and (D.3) have the same functional form, from the transformations of the DHE (D.1) we get the transformations for its kernels. In fact, if $U(z) = U(B_1, B_2, B_3; \omega, \eta; z)$ denotes a solution of the DHE, the substitutions which preserve the form of the equation are represented by the transformations t_1, t_2 and t_3 [25, 44]

$$t_1 U(z) = e^{\frac{B_1}{z}} z^{2-B_2} U(-B_1, 4 - B_2, B_3 + 2 - B_2; \omega, \eta; z),$$

$$t_2 U(z) = e^{i\omega z + \frac{B_1}{2z}} z^{-i\eta - \frac{B_2}{2}} U\left(B_1', B_2', B_3'; \omega', \eta'; \vartheta = \frac{iB_1}{2z}\right), \tag{D.9}$$

$$t_3 U(z) = U(B_1, B_2, B_3; -\omega, -\eta; z),$$

where

$$B_1' = \omega B_1, \quad B_2' = 2 + 2i\eta, \quad B_3' = B_3 - \left(\frac{B_2}{2} + i\eta\right)\left(\frac{B_2}{2} - i\eta - 1\right), \quad \omega' = 1, \quad i\eta' = \frac{B_2}{2} - 1,$$

From (D.9) we obtain the transformations for the kernels of the DHE, namely,

$$r_1 G(z,t) = e^{\frac{B_1}{z} + \frac{B_1}{t}} (zt)^{2-B_2} U(-B_1, 4 - B_2, ; \omega, \eta; z, t),$$

$$r_2 G(z,t) = e^{i\omega(z+t) + \frac{B_1}{2z} + \frac{B_1}{2t}} (zt)^{-i\eta - \frac{B_2}{2}} U\left(B_1', B_2'; \omega', \eta'; \frac{iB_1}{2z}, \frac{iB_1}{2t}\right), \tag{D.10}$$

$$r_3 G(z,t) = U(B_1, B_2; -\omega, -\eta; z, t).$$

D.1. Kernels with products of two confluent hypergeometric functions

We write the kernels before explaining how they are obtained. A group of solutions for (D.3) is given by the 16 kernels

$$G^{(i,j)}(z,t) = e^{-i\omega(z+t)}(zt)^{-\lambda}\varphi^i(\xi)\,\bar{\varphi}^j(\zeta), \qquad [i,j = 1,2,3,4] \tag{D.11}$$

where λ is a constant of separation, and $\varphi^i(\xi)$ and $\bar{\varphi}^j(\zeta)$ are the confluent hypergeometric functions (A.2) with the following arguments and parameters :

$$\begin{aligned}
\varphi^i(\xi): & \quad \xi = 2i\omega(z+t), & a = \frac{B_2}{2} - i\eta - \lambda, & \quad c = B_2 - 2\lambda; \\
\bar{\varphi}^j(\zeta): & \quad \zeta = \frac{B_1(z+t)}{zt}, & a = \lambda, & \quad c = 2\lambda + 2 - B_2.
\end{aligned} \tag{D.12}$$

The kernels given by regular confluent hypergeometric functions are

$$G^{(1,1)}(z,t) = e^{-i\omega(z+t)}[zt]^{-\lambda}\Phi\left[\frac{B_2}{2} - i\eta - \lambda, B_2 - 2\lambda; 2i\omega(z+t)\right]\Phi\left[\lambda, 2\lambda + 2 - B_2; \frac{B_1(z+t)}{zt}\right], \tag{D.13}$$

$$\begin{aligned}
G^{(1,2)}(z,t) =\ & e^{-i\omega(z+t) + \frac{B_1}{z} + \frac{B_1}{t}}[zt]^{\lambda+1-B_2}[z+t]^{B_2-1-2\lambda}\Phi\left[\frac{B_2}{2} - i\eta - \lambda, B_2 - 2\lambda; 2i\omega(z+t)\right] \\
& \times\ \Phi\left[1 - \lambda, B_2 - 2\lambda; -\frac{B_1(z+t)}{zt}\right],
\end{aligned} \tag{D.14}$$

$$\begin{aligned}
G^{(2,1)}(z,t) =\ & e^{i\omega(z+t)}[zt]^{-\lambda}[z+t]^{1+2\lambda-B_2}\Phi\left[1 + \lambda + i\eta - \frac{B_2}{2} -, 2 + 2\lambda - B_2; -2i\omega(z+t)\right] \\
& \times\ \Phi\left[\lambda, 2\lambda + 2 - B_2; \frac{B_1(z+t)}{zt}\right],
\end{aligned} \tag{D.15}$$

$$G^{(2,2)}(z,t) = e^{i\omega(z+t)+\frac{B_1}{z}+\frac{B_1}{t}}[zt]^{\lambda+1-B_2}\,\Phi\left[1+i\eta+\lambda-\tfrac{B_2}{2},2+2\lambda-B_2;-2i\omega(z+t)\right]$$
$$\times\ \Phi\left[1-\lambda,B_2-2\lambda;-\tfrac{B_1(z+t)}{zt}\right]. \tag{D.16}$$

The full group is obtained by replacing one or both functions Φ by Ψ. By using these explicit forms of the kernels, we can show that the transformations r_1, r_2 and r_3 simply rearrange the kernels. For example, we find

$$r_1 G^{(i,j)}(z,t) = e^{-i\omega(z+t)+\frac{B_1}{z_0}+\frac{B_1}{z_0}}[zt]^{2-B_2-\lambda_1}\,\tilde{\varphi}^i(\xi)\,\tilde{\tilde{\varphi}}^j(\zeta), \qquad [i,j=1,2,3,4] \tag{D.17}$$

where we have transformed λ into λ_1, and now $\tilde{\varphi}^i(\xi)$ and $\tilde{\tilde{\varphi}}^j(\zeta)$ are the confluent hypergeometric functions (A.2) with the following arguments and parameters :

$$\tilde{\varphi}^i(\xi): \quad \xi=2i\omega(z+t), \qquad a=2-i\eta-\lambda_1-\tfrac{B_2}{2}, \qquad c=4-B_2-2\lambda_1;$$
$$\tilde{\tilde{\varphi}}^j(\zeta): \quad \zeta=-\tfrac{B_1(z+t)}{zt}, \qquad a=\lambda_1, \qquad\qquad\qquad c=2\lambda_1-2+B_2.$$

In particular,

$$r_1 G^{(1,1)}(z,t) = e^{-i\omega(z+t)+\frac{B_1}{z}+\frac{B_1}{t}}[zt]^{2-B_2-\lambda_1}\,\Phi\left[1-i\eta-\lambda_1-\tfrac{B_2}{2},4-B_2-2\lambda_1;2i\omega(z+t)\right]$$
$$\times\ \Phi\left[\lambda_1,2\lambda_1-2+B_2;-\tfrac{B_1(z+t)}{zt}\right].$$

By taking $\lambda_1=1-\lambda$ and using (A.3), we see that the right-hand side of the above equation is $G^{(2,2)}(z,t)$ given in (D.16). Thus, in fact r_1 simply rearranges the kernels (D.11). The same is true of r_2 and r_3.

The kernels (D.11) can be found by solving Eq. (D.3), or by applying the limit when $z_0\to 0$ to the kernels (32) of the CHE. The latter procedure transforms the Gauss hypergeometric functions given in Eqs. (41-46) into confluent hypergeometric functions due to the relations [37]

$$\lim_{c\to\infty} F\left(a,b;c;1-\frac{c}{u}\right)=\lim_{c\to\infty} F\left(a,b;c;-\frac{c}{u}\right)=u^a\Psi(a,a+1-b;u),$$
$$\lim_{b\to\infty} F\left(a,b;c;\frac{u}{b}\right)=\Phi(a,c;u), \qquad \lim_{y\to\infty}\left(1+\frac{x}{y}\right)^y=e^x. \tag{D.18}$$

Thus, up to a multiplicative constant, we find

$$\lim_{z_0\to 0} F^1=\lim_{z_0\to 0} F^3=[zt]^{-\lambda}[z+t]^\lambda\Psi\left(\lambda,2+2\lambda-B_2;\tfrac{B_1(z+t)}{zt}\right),$$
$$\lim_{z_0\to 0} F^2=\lim_{z_0\to 0} F^4=e^{\frac{B_1}{z}+\frac{B_1}{t}}[zt]^{\lambda-1-B_2}[z+t]^{B_2+1-\lambda}\Psi\left(1-\lambda,B_2-2\lambda;-\tfrac{B_1(z+t)}{zt}\right),$$
$$\lim_{z_0\to 0} F^5=[zt]^{-\lambda}[z+t]^\lambda\Phi\left(\lambda,2+2\lambda-B_2;\tfrac{B_1(z+t)}{zt}\right),$$
$$\lim_{z_0\to 0} F^6=e^{\frac{B_1}{z}+\frac{B_1}{t}}[zt]^{\lambda+1-B_2}[z+t]^{B_2-1-\lambda}\Phi\left(1-\lambda,B_2-2\lambda;-\tfrac{B_1(z+t)}{zt}\right). \tag{D.19}$$

To get these limits, in some cases we have to rewrite the functions (41-46) in a convenient form. For example, using (A.8) we find

$$F^2=\tfrac{zt}{z+t-z_0}\left[\tfrac{(z-z_0)(t-z_0)}{z+t-z_0}\right]^{1-B_2}\left[1-\tfrac{z_0}{zt}(z+t-z_0)\right]^{-\frac{B_1}{z_0}}F\left(1-\lambda,2+\lambda-B_2;2+\tfrac{B_1}{z_0};\tfrac{zt}{z_0(z+t-z_0)}\right)$$

by suppressing a multiplicative constant depending on z_0. After this, we use the limits (D.18).

D.2. Kernels with one confluent hypergeometric function

For particular values of λ, the kernels (D.11) present only one of the confluent hypergeometric functions (A.2). As an initial set we take

$$G_1^{(i)}(z,t)=G^{(i,1)}(z,t)\big|_{\lambda=0}=e^{-i\omega(z+t)}\varphi^i(\xi), \qquad \xi=2i\omega(z+t), \qquad a=\tfrac{B_2}{2}-i\eta, \qquad c=B_2. \tag{D.20}$$

This set may also obtained by setting $z_0=0$ in the kernels (58) for the CHE and in their partners in terms of $\Psi(a,c;\xi)$. The kernels in terms of the regular functions $\Phi(a,c;\xi)$ read

$$G_1^{(1)}(z,t)=e^{-i\omega(z+t)}\Phi\left[\tfrac{B_2}{2}-i\eta,B_2;2i\omega(z+t)\right],$$
$$G_1^{(2)}(z,t)=e^{i\omega(z+t)}[z+t]^{1-B_2}\Phi\left[1+i\eta-\tfrac{B_2}{2},2-B_2;-2i\omega(z+t)\right], \tag{D.21}$$

while two other kernels result by replacing $\Phi(a, c; \xi)$ by $\Psi(a, c; \xi)$. From the transformations (D.10) we obtain three additional sets generate as

$$G_2^{(i)}(z,t) = r_1 G_1^{(i)}(z,t), \qquad G_3^{(i)}(z,t) = r_2 G_1^{(i)}(z,t), \qquad G_4^{(i)}(z,t) = r_2 G_2^{(i)}(z,t). \tag{D.22}$$

The transformation r_3 does not produce new kernels. Thus, the kernels with $\Phi(a, c; \xi)$ are

$$G_2^{(1)}(z,t) = r_1 G_1^{(1)}(z,t) = e^{-i\omega(z+t)+\frac{B_1}{z}+\frac{B_1}{t}}[zt]^{2-B_2} \Phi\left[2 - i\eta - \tfrac{B_2}{2}, 4 - B_2; 2i\omega(z+t)\right],$$

$$G_2^{(2)}(z,t) = r_1 G_1^{(2)}(z,t) = e^{i\omega(z+t)+\frac{B_1}{z}+\frac{B_1}{t}}[zt]^{2-B_2}[z+t]^{B_2-3} \Phi\left[i\eta - 1 + \tfrac{B_2}{2}, B_2 - 2; -2i\omega(z+t)\right]; \tag{D.23}$$

$$G_3^{(1)} = r_2 G_1^{(1)} = G^{(2,2)}(z,t) = e^{i\omega(z+t)+\frac{B_1}{z}+\frac{B_1}{t}}[zt]^{-i\eta-\frac{B_2}{2}} \Phi\left[2 + i\eta - \tfrac{B_2}{2}, 2 + 2i\eta; -\tfrac{B_1(z+t)}{zt}\right],$$

$$G_3^{(2)} = r_2 G_1^{(2)} = G^{(2,1)}(z,t) = e^{i\omega(z+t)}[zt]^{1+i\eta-\frac{B_2}{2}}[z+t]^{-1-2i\eta} \Phi\left[\tfrac{B_2}{2} - i\eta - 1, -2i\eta; \tfrac{B_1(z+t)}{zt}\right]; \tag{D.24}$$

$$G_4^{(1)} = r_2 G_2^{(1)} = e^{-i\omega(z+t)+\frac{B_1}{z}+\frac{B_1}{t}}[zt]^{i\eta-\frac{B_2}{2}} \Phi\left[2 - i\eta - \tfrac{B_2}{2}, 2 - 2i\eta; -\tfrac{B_1(z+t)}{zt}\right],$$

$$G_4^{(2)} = r_2 G_2^{(2)} = e^{-i\omega(z+t)}[zt]^{1-i\eta-\frac{B_2}{2}}[z+t]^{2i\eta-1} \Phi\left[i\eta - 1 + \tfrac{B_2}{2}, 2i\eta; \tfrac{B_1(z+t)}{zt}\right]. \tag{D.25}$$

D.3. Kernels given by elementary functions

The kernel

$$G_1(z,t) = e^{-i\omega(z+t)}\left[1 + \tfrac{2i\omega zt}{B_1}\right]^{i\eta-\frac{B_2}{2}} \tag{D.26}$$

has the same form as a kernel found by Schmidt and Wolf [45] who have considered a DCHE with only four parameters. To obtain (D.26), we insert

$$G(z,t) = e^{-i\omega(z+t)}f(z,t),$$

into Eq. (D.3). This leads to

$$z^2\tfrac{\partial^2 f}{\partial z^2} + \left[B_1 + B_2 z - 2i\omega z^2\right]\tfrac{\partial f}{\partial z} - t^2\tfrac{\partial^2 f}{\partial t^2} - \left[B_1 + B_2 t - 2i\omega t^2\right]\tfrac{\partial f}{\partial t} - 2i\omega\left(\tfrac{B_2}{2} - i\eta\right)(z - t)f = 0.$$

By supposing that $f(z,t)$ depends on z and t through the product $2i\omega zt/B_1 = y$, the previous equation gives

$$[1 + y]\tfrac{df}{dy} - \left[i\eta - \tfrac{B_2}{2}\right]f = 0 \qquad \Rightarrow \qquad f(z,t) = \left[1 + \tfrac{2i\omega zt}{B_1}\right]^{i\eta-\frac{B_2}{2}}.$$

Hence we obtain the kernel (D.26). The transformations (D.10) generate three additional kernels given by

$$G_2(z,t) = r_1 G_1(z,t), \qquad G_3^{(i)}(z,t) = r_3 G_1(z,t), \qquad G_4^{(i)}(z,t) = r_3 G_2(z,t). \tag{D.27}$$

Thus, we have

$$G_2(z,t) = e^{-i\omega(z+t)+\frac{B_1}{z}+\frac{B_1}{t}}[zt]^{2-B_2}\left[1 - \tfrac{2i\omega zt}{B_1}\right]^{i\eta-2+\frac{B_2}{2}}, \tag{D.28}$$

while $G_3(z,t)$ and $G_4(z,t)$ are obtained by substituting (η, ω) for $(-\eta, -\omega)$ in $G_1(z,t)$ and $G_2(z,t)$.

Appendice E. Kernels for the reduced double-confluent Heun equation (RDHE)

For the reduced double-confluent Heun equation (RDHE),

$$[L_z + B_3] U = z^2\frac{d^2 U}{dz^2} + [B_1 + B_2 z]\frac{dU}{dz} + [qz + B_3] U = 0, \tag{E.1}$$

in this appendix E:

- initially we get a substitution of variables which preserve the form of the equation for the kernels;
- in E.1 we find kernels given by products of Bessel and confluent hypergeometric functions; they present an arbitrary constant of separation λ and may be derived as limits of kernels of the DHE ($z_0 \to 0$) or of the RCHE (Whitaker-Ince limit);
- in E.2, by putting $\lambda = 0$, we obtain kernels given by products of elementary and Bessel functions;
- in E.3, by choosing appropriate values for λ, we obtain kernels given by products of elementary and confluent hypergeometric functions;
- in E.4 we find two kernels given by products of elementary functions; these can be derived by applying the Whittaker-Ince limit to kernels of the DHE.

For the RDHE, the integral (D.2) remains formally unaltered, that is,

$$\mathcal{U}(z) = \int_{t_1}^{t_2} t^{B_2-2} e^{-B_1/t} G(z,t) U(t) dt, \tag{E.2}$$

while the equation (D.3) for the kernels becomes

$$\left[z^2 \frac{\partial^2}{\partial z^2} + (B_1 + B_2 z) \frac{\partial}{\partial z} + qz \right] G - \left[t^2 \frac{\partial^2}{\partial t^2} + (B_1 + B_2 t) \frac{\partial}{\partial t} + qt \right] G = 0. \tag{E.3}$$

For fixed endpoints of integrations the bilinear concomitant (D.4) is again

$$P(z,t) = t^{B_2} \, e^{-B_1/t} \left[U(t) \frac{\partial G(z,t)}{\partial t} - G(z,t) \frac{dU(t)}{dt} \right]. \tag{E.4}$$

If the endpoints depend on z, we proceed as in appendix D.

On the other side, if $U(z) = U(B_1, B_2, B_3; q; z)$ denotes a known solutions of RDHE, other solution is generated by the transformation T defined by

$$TU(z) = e^{\frac{B_1}{z}} z^{2-B_2} \, U(-B_1, 4 - B_2, B_3 + 2 - B_2; q; z), \tag{E.5}$$

as we can show by substitutions of variables. Similarly, if $G(z,t) = G(B_1, B_2; q; z, t)$ denotes a solution of Eq. (E.3), the corresponding transformation R for this kernel is

$$RG(z,t) = e^{\frac{B_1}{z} + \frac{B_1}{t}} (zt)^{2-B_2} \, G(-B_1, 4 - B_2; q; z, t), \tag{E.6}$$

E.1. Kernels with products of Bessel and confluent hypergeometric functions

We obtain the kernels given by products of Bessel and confluent hypergeometric functions by taking the limits when $z_0 \to 0$ of the kernels (C.16) for the RCHE. Thus, up to a multiplicative constant

$$G_{\pm}^{(i,j)}(z,t) = [z+t]^{\frac{1}{2}-\frac{B_2}{2}} Z_{\pm(2\lambda+1-B_2)}^{(i)} \left[2\sqrt{q(z+t)} \right] \lim_{z_0 \to 0} F^j(\zeta) \tag{E.7}$$

where the $\lim_{z_0 \to 0} F^j$ are given in Eqs. (D.19). Explicitly, from F^1, F^2, F^5 and F^6 we get, respectively,

$$
\begin{aligned}
G_{\pm}^{(i,1)}(z,t) &= [zt]^{-\lambda} [z+t]^{\lambda+\frac{1}{2}-\frac{B_2}{2}} Z_{\pm(2\lambda+1-B_2)}^{(i)} \left[2\sqrt{q(z+t)} \right] \Psi \left[\lambda, 2+2\lambda - B_2; \frac{B_1(z+t)}{zt} \right], \\
G_{\pm}^{(i,2)}(z,t) &= e^{\frac{B_1}{z}+\frac{B_1}{t}} [zt]^{\lambda+1-B_2} [z+t]^{-\lambda-\frac{1}{2}+\frac{B_2}{2}} Z_{\pm(2\lambda+1-B_2)}^{(i)} \left[2\sqrt{q(z+t)} \right] \Psi \left[1-\lambda, B_2 - 2\lambda; -\frac{B_1(z+t)}{zt} \right], \\
G_{\pm}^{(i,3)}(z,t) &= [zt]^{-\lambda} [z+t]^{\lambda+\frac{1}{2}-\frac{B_2}{2}} Z_{\pm(2\lambda+1-B_2)}^{(i)} \left[2\sqrt{q(z+t)} \right] \Phi \left[\lambda, 2+2\lambda - B_2; \frac{B_1(z+t)}{zt} \right], \\
G_{\pm}^{(i,4)}(z,t) &= e^{\frac{B_1}{z}+\frac{B_1}{t}} [zt]^{\lambda+1-B_2} [z+t]^{-\lambda-\frac{1}{2}+\frac{B_2}{2}} Z_{\pm(2\lambda+1-B_2)}^{(i)} \left[2\sqrt{q(z+t)} \right] \Phi \left[1-\lambda, B_2 - 2\lambda; -\frac{B_1(z+t)}{zt} \right].
\end{aligned}
\tag{E.8}
$$

The transformation R given in Eq. (E.6) simply rearranges the previous kernel provided that we transform the arbitrary constant λ into another arbitrary constant $\bar{\lambda}$. For example, we find

$$RG_{\pm}^{(i,1)}(z,t) = e^{\frac{B_1}{z}+\frac{B_1}{t}} [zt]^{2-B_2-\bar{\lambda}} [z+t]^{\bar{\lambda}-\frac{3}{2}+\frac{B_2}{2}} Z_{\pm(2\bar{\lambda}-3-B_2)}^{(i)} \left[2\sqrt{q(z+t)} \right] \Psi \left[\bar{\lambda}, 2\bar{\lambda} - 2 + B_2; -\frac{B_1(z+t)}{zt} \right].$$

Putting $\bar{\lambda} = \lambda + 2 - B_2$ and using the second relation in (A.3), we find that the right-hand side of the above equation is a constant multiple of $G_{\pm}^{(i,2)}(z,t)$.

E.2. Kernels given by products of elementary and Bessel functions

If $\lambda = 0$ in $G^{(i,1)}_{\pm}$ or $G^{(i,3)}_{\pm}$, we have the kernels

$$G^{(i)}_{\pm}(z,t) = [z+t]^{\frac{1}{2}-\frac{B_2}{2}}\, Z^{(i)}_{\pm(1-B_2)}\left[2\sqrt{q(z+t)}\right]. \tag{E.9}$$

If $\lambda = 1$ in $G^{(i,2)}_{\pm}$ or $G^{(i,4)}_{\pm}$, we have the kernels

$$\tilde{G}^{(i)}_{\pm}(z,t) = e^{\frac{B_1}{z}+\frac{B_1}{t}}\, [zt]^{2-B_2}\, [z+t]^{-\frac{3}{2}+\frac{B_2}{2}}\, Z^{(i)}_{\pm(3-B_2)}\left[2\sqrt{q(z+t)}\right]. \tag{E.10}$$

$G^{(i)}_{\pm}$ and $\tilde{G}^{(i)}_{\pm}$ are connected by the transformation R given in (E.6).

E.3. Kernels given by products of elementary and confluent hypergeometric functions

If $\lambda = (B_2/2) - (1/4)$, the order of the Bessel functions is $\pm 1/2$ in the kernels (E.8) and, according to (C.15), these functions reduce to elementary functions. Then we have the following kernels given by products of elementary and confluent hypergeometric functions:

$$
\begin{aligned}
G^{(i,1)}_{\pm}(z,t) &= [zt]^{\frac{1}{4}-\frac{B_2}{2}}\, [z+t]^{\frac{1}{4}}\, Z^{(i)}_{\pm\frac{1}{2}}\left[2\sqrt{q(z+t)}\right]\ \Psi\left[\frac{B_2}{2}-\frac{1}{4},\frac{3}{2};\frac{B_1(z+t)}{zt}\right],\\
G^{(i,2)}_{\pm}(z,t) &= e^{\frac{B_1}{z}+\frac{B_1}{t}}\, [zt]^{\frac{3}{4}-\frac{B_2}{2}}\, [z+t]^{-\frac{1}{4}}\, Z^{(i)}_{\pm\frac{1}{2}}\left[2\sqrt{q(z+t)}\right]\ \Psi\left[\frac{5}{4}-\frac{B_2}{2},\frac{1}{2};-\frac{B_1(z+t)}{zt}\right],\\
G^{(i,3)}_{\pm}(z,t) &= [zt]^{\frac{1}{4}-\frac{B_2}{2}}\, [z+t]^{\frac{1}{4}}\, Z^{(i)}_{\pm\frac{1}{2}}\left[2\sqrt{q(z+t)}\right]\ \Phi\left[\frac{B_2}{2}-\frac{1}{4},\frac{3}{2};\frac{B_1(z+t)}{zt}\right],\\
G^{(i,4)}_{\pm}(z,t) &= e^{\frac{B_1}{z}+\frac{B_1}{t}}\, [zt]^{\frac{3}{4}-\frac{B_2}{2}}\, [z+t]^{-\frac{1}{4}}\, Z^{(i)}_{\pm\frac{1}{2}}\left[2\sqrt{q(z+t)}\right]\ \Phi\left[\frac{5}{4}-\frac{B_2}{2},\frac{1}{2};-\frac{B_1(z+t)}{zt}\right].
\end{aligned}
\tag{E.11}
$$

The transformation (E.6) simply rearranges the above kernels.

E.4. Kernels given by elementary functions

We find two kernels given by elementary functions. Up to multiplicative constants, we have

$$G_1(z,t) = \exp\left[\frac{qzt}{B_1}\right], \qquad G_2(z,t) = rG_1(z,t) = [zt]^{2-B_2}\exp\left[-\frac{qzt}{B_1}+\frac{B_1}{z}+\frac{B_1}{t}\right]. \tag{E.12}$$

These kernels can be obtained by applying the Whittaker-Ince limit (3) to the kernels (D.26) and (D.28) of the RCHE. Alternatively, we can compute $G_1(z,t)$ by supposing that $G(z,t)$ depends on z and t through the product $qzt/B_1 = y$, in which case Eq.(E.3) becomes $dG/dy = G$, whose solution is the kernel G_1.

At last we mention that there is an equation called doubly reduced double-confluent Heun equation [7]. However, as we have found no relation of such equation with the equations discussed here, we do not consider its kernels.

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[1] L. J. El-Jaick and B.D.B. Figueiredo, "Transformations of Heun's equation and its integral relations," *J. Phys. A: Math. Theor.* **44**, 075204 (2011) (arXiv: 1002.4559v3).
[2] K. Heun, "Zur Theorie der Rieman'schen Functionen zweiter Ordnung mit vier Verzweigungspunkten," *Math. Ann.* **33**, 161 (1899).
[3] R. S. Maier, "The 192 solutions of the Heun equation," *Mathematics of Computation* **76**, 811 (2007) (also available in math.CA/0408317).
[4] A. Ronveaux (*editor*), *Heun's differential equations* (Oxford University Press, 1995).
[5] *NIST Handbook of Mathematical Functions*, edited by F. W. J. Olver, D. W. Lozier, R. F. Boisvert and C.W. Clark (National Institute of Standards and Tecnology, Cambridge University Press, 2010).
[6] A. Ya. Kazakov and S. Yu. Slavyanov, "Integral relations for Heun-class special functions," *Theoretical and Mathematical Physics* **107**, 733-739 (1996).
[7] S. Yu. Slavyanov and W. Lay, "Special Functions: A Unified Theory Based on Singularities" (Oxford University Press, 2000).
[8] A. H. Wilson, "The ionised hydrogen molecule" *Proc. Roy. Soc. London* **A118**, 635-647 (1928).
[9] E. W. Leaver,"Solutions to a generalized spheroidal wave equation: Teukolsky equations in general relativity, and the two-center problem in molecular quantum mechanics," *J. Math. Phys.* **27**, 1238-1265 (1986).
[10] A. H. Wilson, "A generalised spheroidal wave equation," *Proc. Roy. Soc. London* **A118**, 617-635 (1928).

[11] A. M. Ishkhanyan and A. E. Grigoryan, "Fifteen classes of solutions of the quantum two-state problem in terms of the confluent Heun function", J Phys A Math. Theor. **47**, 465205 (2014) .

[12] J. M. Lévy-Leblond , "Electron capture by polar molecules", *Phys. Rev.* **153** , 1-4 (1967).

[13] L. J. El-Jaick and B. D. B. Figueiredo, "A limit of the confluent Heun equation and the Schrödinger equation for an inverted potential and for an electric dipole", *J. Math. Phys.* **50**, 123511 (2009).

[14] P. K. Jha and Y. V. Rostovtsev, "Coherent excitation of a two-level atom driven by a far-off-resonant classical field: Analytical solutions", *Phys. Review A* **81**, 033827 (2010).

[15] M. Renardy, "On the eigenfunctions for Hookean and FENE dumbbell models," *Journal of Rheology*, **57**, 1311-1324 (2013).

[16] N. Barbosa-Cendejas , A. Herrera-Aguilar, K. Kanakoglou, U. Nucamendi and I. Quiros, "Mass hierarchy, mass gap and corrections to Newton's law on thick branes with Poincaré symmetry", *Gen. Relativ. Gravit.*, **46**, 1631 (2014).

[17] F. M. J. Olver, *Asymptotics and special functions* (Academic Press, 1974).

[18] L. J. El-Jaick and B.D.B. Figueiredo, "Solutions for confluent and double-confluent Heun equations," *J. Math. Phys.* **49**, 083508 (2008) (arXiv: 0807.2219v2).

[19] B. D. B. Figueiredo, "Ince's limits for confluent and double-confluent Heun equations," *J. Math. Phys.* **46**, 113503 (2005).

[20] L. J. El-Jaick and B.D.B. Figueiredo, "Confluent and Double-Confluent Heun Equations: Convergence of Solutions in Series of Coulomb Wavefunctions", arXiv:1209.4673 [math-ph].

[21] A. Erdélyi, "Expansions of Lamé functions into series of Legendre functions", *Proc. Roy. Soc. Edimb.*, **62**, 247-267 (1948).

[22] G. Jaffé, "Zur Theorie des Wasserstoffmolekülions," *Z. Phys.* **87**, 535-544 (1934).

[23] W. G. Baber and H.R. Hassé, "The two centre problem in wave mechanics," *Proc. Cambr. Philos. Soc.* **25**, 564-581 (1935).

[24] E. L. Ince, *Ordinary Differential Equations* (Dover, 1926).

[25] A. Decarreau, M. C. Dumont-Lepage, P. Maroni, A. Robert and A. Ronveaux, "Formes canoniques des équations confluentes de l'équation de Heun," *Ann. Soc. Sci. Brux.* **T92(I-II)**, 53-78 (1978).

[26] A. Decarreau, P. Maroni and A. Robert, "Sur les équations confluentes de l'équation de Heun," *Ann. Soc. Sci. Brux.* **T92(III)**, 151-189 (1978).

[27] T. Masuda and H. Suzuki, "Integral equations on a rotating black hole," *J. Math. Phys.* **38**, 3669-3678 (1997).

[28] C.G. Lambe and D.R. Ward, "Some differential equations and associated integral equations," *Q. J. Math.* (Oxford series), **5**, 81-97 (1934).

[29] W. Gautschi, "Computational aspects of three-term recurrence relations", *SIAM Rev.* **9**, 24-82 (1967) .

[30] W. Gautschi, "Minimal solutions of three-term recurrence relations and orthogonal polynomials", *Mathematics of Computation* **36**(154), 447-554 (1981).

[31] I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series and products* (Academic Press, 2007).

[32] D.B. Hodge, "Eigenvalues and eigenfunctions of the spheroidal wave equation", *J. Math. Phys.* **11**, 2308-2312 (1970).

[33] J. W . Liu, "Analytical solutions to the generalized spheroidal wave equation and the Green's function of one-electron diatomic molecules", *J. Math. Phys.* **33** , 4026-4036 (1992) .

[34] F. M. Arscott, *Periodic Differential Equations* (Pergamon Press, Oxford, 1964).

[35] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge University Press,1954).

[36] K. Knopp, *Infinite Sequences and Series* (Dover, 1956).

[37] A. Erdélyi (*editor*), *Higher Transcendental Functions* Vol. 1 (McGraw-Hill,1953).

[38] L. J. El-Jaick and B.D.B. Figueiredo, "New solutions for the confluent Heun equation and quasiexact solvability", arXiv: 1311.7677v1.

[39] E. A. Hylleraas, "Über die Elektronenterme des Wasserstoffmoleküls," *Z. Phys.* **71**, 739-763 (1931).

[40] Santos, M. G. and S. O. Kepler, "Theoretical study of the line profiles of the hydrogen perturbed by collisions with protons", *Monthly Notices of the Royal Astronomical Society*, **423**, 68-79 (2012).

[41] N. W. McLachlan, *Theory and Application of Mathieu Functions* (Dover, 1964).

[42] A. Erdélyi (*editor*), *Tables of integral transforms* Vol. 2 (McGraw-Hill, 1954).

[43] B. D. B. Figueiredo, "Integral relations and new solutions to the double-confluent Heun equation", arXiv:math-ph/0402071v1 (2004).

[44] B. D. B. Figueiredo, "On some solutions to generalized spheroidal wave equations and applications ", *J. Phys. A: Math. Gen.* **35**, 2877-2906 (2002) .

[45] D. Schmidt and G. Wolf, *Double confluent Heun equation*, Part C of [4].