Quasi-Periodic Solutions of (3+1) Generalized BKP Equation By Using Riemann Theta Functions

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Abstract

This paper is focused on quasi-periodic wave solutions of (3+1) generalized BKP equation. Because of some difficulties in calculations of $N = 3$ periodic solutions, hardly ever has there been a study on these solutions by using Rieamann theta function. In this study, we obtain one and two periodic wave solutions as well as three periodic wave solutions for (3+1) generalized BKP equation. Moreover we analyse the asymptotic behavior of the periodic wave solutions tend to the known soliton solutions under a small amplitude limit.

Keywords: Hirota's Bilinear Method, Quasi-Periodic Wave Solutions, Riemann Theta Functions, (3+1) generalized BKP Equation

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1 Introduction

In recent years, the problem of finding exact solutions of partial differential equations (PDE) is very popular for both mathematicians and physcists. Because if we know the exact solutions of PDE's, they can help us to understand

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complicated physical models. So, there are some successful methods to obtain exact solutions such as Hirota's direct method [1], Lie symmetry method [2], Bäcklund transformation method [3] and algebro geometric method [4].

In the late 1970's Novikov, Dubrovin, Mckean, Lax, Its, and Matveev et al. developed the algebro geometric method to obtain quasi-periodic or algebro-geometric solutions for many soliton equations [5, 8]. However this method involves complicated calculation. On the other hand, Hirota's direct method is rather useful and direct approach to construct multisoliton solutions.

In the 1980, Nakamura obtained the periodic wave solutions of the KdV and the Boussinesq equations by means of Hirota's bilinear method [9, 10]. Indeed this method has some advantages over algebro-geometric methods. We can get explicit periodic wave solutions directly.

Recently, Fan and his collaborators have extended this method to investigate the discrete Toda lattice [11] , Cheng Z.,Hao X. studied on periodic solution of (2+1) AKNS equation [12], Tian and Zhang obtained periodic wave solutions by Riemann theta functions of some nonlinear differential equations and super-symmetric equations [13, 14], Lu and Zhang studied on quasi periodic solutions of Jimbo-Miwa equation [15]

Soliton equations possess nice mathematical features, e.g., elastic interactions of solutions. Such equations contain the KdV equation, the Boussinesq equation, the KP equation and the BKP equation, and they all have multi-soliton solutions. Let us consider (3+1) dimensional generalized BKP equation [16].

$$u_{ty} - u_{xxxy} - 3(u_x u_y)_x + 3u_{xz} = 0 \tag{1.1}$$

Now, in this paper we briefly introduce a Hirota bilinear form and the Riemann theta function.Then after we apply the Hirota's bilinear method to construct one, two and three periodic wave solutions to (3+1) generalized BKP equation, respectively. We further use a limiting procedure to analyse the asymptotic behavior of the periodic wave solutions in the last section. It is rigorously shown that the periodic solutions tend to the well-known soliton solutions under a certain limit.

2 The Bilinear Form and The Riemann Theta Functions

In this section we introduce briefly bilinear form and some main points on the Riemann theta functions. The Hirota bilinear method is powerful when constructing exact solutions for nonlinear equations. Through the dependent variable transformation $u = 2(\ln f)_x$, eq. (1.1) is written bilinear form

$$(D_y D_t - D_x^3 D_y + 3D_x D_z)f.f = 0 \ . \tag{2.1}$$

Here D is differential bilinear operator defined by

$$\begin{aligned} &D_x^m D_y^n D_t^k f(x,y,t).g(x,y,t) = \\ &(\partial_x - \partial_{x'})^m (\partial_y - \partial_{y'})^n (\partial_t - \partial_{t'})^k f(x,y,t) g(x',y',t') \mid_{x'=x, y'=y, t'=t} \end{aligned} \tag{2.2}$$

and the operator has property for exponential functions namely

$$D_x^m D_y^n D_t^k e^{\xi_1} e^{\xi_2} = (\alpha_1 - \alpha_2)^m (\rho_1 - \rho_2)^n (\omega_1 - \omega_2)^k e^{\xi_1 + \xi_2} \tag{2.3}$$

where $\xi_i = \alpha_i x + \rho_i y + \omega_i t + \delta_i, \ i = 1, 2$. More general we can write following formula

$$G(D_x, D_y, D_t) e^{\xi_1} e^{\xi_2} = G(\alpha_1 - \alpha_2, \rho_1 - \rho_2, \omega_1 - \omega_2) e^{\xi_1 + \xi_2} \tag{2.4}$$

where $G(D_x, D_y, D_t)$ is a polinomial about D_x, D_y and D_t. According to the

Hirota bilinear theory, eq. (1.1) admits one-soliton solution

$$u_1 = 2\partial_x (\ln(1 + e^{\eta})) \tag{2.5}$$

where phase variable $\eta = \mu x + \nu y + \kappa z + \varpi t + \gamma$, dispersion relation $\varpi = -3\frac{\mu\kappa}{\rho} + \mu^3$, μ, ν, κ and γ are constants.

Two-soliton solution

$$u_2 = 2\partial_x (\ln(1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_1 + \eta_2 + A_{12}})) \tag{2.6}$$

with

$$e^{A_{12}} = -\frac{(\nu_1 - \nu_2)(\varpi_1 - \varpi_2) - (\mu_1 - \mu_2)^3 (\nu_1 - \nu_2) + 3(\mu_1 - \mu_2)(\kappa_1 - \kappa_2)}{(\nu_1 + \nu_2)(\varpi_1 + \varpi_2) - (\mu_1 + \mu_2)^3 (\nu_1 + \nu_2) + 3(\mu_1 + \mu_2)(\kappa_1 + \kappa_2)} \tag{2.7}$$

$$\begin{aligned} &\eta_j = \mu_j x + \nu_j y + \kappa_j z + \varpi_j t + \gamma_j \ , \qquad j = 1, 2 \\ &\varpi_1 = -3\frac{\mu_1 \kappa_1}{\rho_1} + \mu_1^3, \ \varpi_2 = -3\frac{\mu_2 \kappa_2}{\rho_2} + \mu_2^3 \end{aligned} \tag{2.8}$$

where μ_j, ν_j, κ_j and γ_j are arbitrary constants.

Three-soliton solution

$$u_3 = 2\partial_x (\ln(f)) \tag{2.9}$$

f is written as

$$\begin{aligned} f = 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_3} + e^{\eta_1+\eta_2+A_{12}} \\ + e^{\eta_1+\eta_3+A_{13}} + e^{\eta_2+\eta_3+A_{23}} + e^{\eta_1+\eta_2+\eta_3+A_{12}+A_{13}+A_{23}} \end{aligned} \tag{2.10}$$

with

$$e^{A_{ij}} = -\frac{(\nu_i-\nu_j)(\varpi_i-\varpi_j)-(\mu_i-\mu_j)^3(\nu_i-\nu_j)+3(\mu_i-\mu_j)(\kappa_i-\kappa_j)}{(\nu_i+\nu_j)(\varpi_i+\varpi_j)-(\mu_i+\mu_j)^3(\nu_i+\nu_j)+3(\mu_i+\mu_j)(\kappa_i+\kappa_j)} \tag{2.11}$$

$$\eta_j = \mu_j x + \nu_j y + \kappa_j z + \varpi_j t + \gamma_j \ , \qquad i, \ j = 1, 2, 3 \ \ , i < j$$

$$\varpi_1 = -3\frac{\mu_1\kappa_1}{\rho_1} + \mu_1^3, \ \varpi_2 = -3\frac{\mu_2\kappa_2}{\rho_2} + \mu_2^3 \tag{2.12}$$

$$\varpi_3 = -3\frac{\mu_3\kappa_3}{\rho_3} + \mu_3^3$$

In order to apply the Hirota's bilinear method to constact multi-periodic wave solutions we consider a slightly generalized form of bilinear equation (2.1). We look for our solution in the form

$$u = u_0 y + 2(\ln\vartheta(\xi))_x \tag{2.13}$$

where $u_0 y$ is a solution of (1.1) and phase variable $\xi = (\xi_1, ..., \xi_N)^T$, $\xi_i = \alpha_i x + \rho_i y + k_i z + \omega_i t + \delta_i$, $i = 1, 2..N$.

Substituting (2.13) into (1.1) and integration once respect to x , we obtain

$$H(D_x, D_y, D_z, D_t,) = (D_y D_t + 3D_x D_z - D_x^3 D_y - 3u_0 D_x^2 + c)\vartheta(\xi).\vartheta(\xi) = 0 \tag{2.14}$$

where $c = c(y, z, t)$ is integration constant. For finding multiperiodic wave solutions of (2.14), we consider the following multidimensional Riemann theta function

$$\vartheta(\xi, \tau) = \sum_{n\in\mathbb{Z}^N} e^{\pi i<\tau n, n>+2\pi i<\xi, n>} \tag{2.15}$$

where the integer value vector $n = (n_1...n_N)^T \in \mathbb{Z}^N$ and complex phase variables $\xi = (\xi_1...\xi_N)^T \in \mathbb{C}^N$, for N dimensional two vectors their inner product is defined by $< u, v >= u_1v_1 + ... + u_N v_N$. Period matrix of theta function is -$i\tau = -i(\tau_{ij})$ which is positive definite and real-valued symmetric $N \times N$ matrix and can be considered as free parametres of theta function. So the Fourier series (2.15) converges to a real valued function and for make the theta function real valued in this paper we take τ imaginay matrix.

Proposition 1 *The theta function $\vartheta(\xi, \tau)$ has the periodic properties*

$$\vartheta(\xi + 1 + \tau) = e^{-\pi i\tau - 2\pi i\xi}\vartheta(\xi, \tau)$$

we regard the vectors 1 and τ as a periods of the theta function $\vartheta(\xi, \tau)$ with multipliers 1 and $e^{-\pi i\tau-2\pi i\xi}$. Here τ is not a period of theta function $\vartheta(\xi, \tau)$, but it is the period of the functions $\partial_\xi^2 \ln\vartheta(\xi, \tau)$, $\partial_\xi \ln[\vartheta(\xi+e, \tau)/\vartheta(\xi+h, \tau)]$ and $\vartheta(\xi + e, \tau)\vartheta(\xi - e, \tau)/\vartheta^2(\xi + h, \tau)$.

3 One-periodic waves and asymptotic properties

3.1 Construct one periodic waves

If we take $N = 1$, we obtain one-periodic solutions and our Riemann theta function reduces following Fourier series

$$\vartheta(\xi,\tau) = \sum_{-\infty}^{\infty} e^{\pi i n^2 \tau + 2\pi i n \xi} \tag{3.1.1}$$

where the phase variable $\xi = \alpha x + \rho y + kz + \omega t + \delta$ and $\mathrm{Im}(\tau) > 0$.

Theorem 1 *Assuming that $\vartheta(\xi,\tau)$ is a Riemann theta function as $N = 1$ with $\xi = \alpha x_1 + \rho x_2 + ... + \omega t + \delta$ and $\alpha, \rho, ..., \omega, \delta$ satisfy the following system*

$$\sum_{n=-\infty}^{\infty} H(4n\pi i\alpha, 4n\pi i\rho, ..., 4n\pi i\omega) e^{2n^2\pi i\tau} = 0 \tag{3.1.2}$$

$$\sum_{n=-\infty}^{\infty} H(2\pi i(2n-1)\alpha, ..., 2\pi i(2n-1)\omega) \tag{3.1.3}$$
$$\times e^{(2n^2-2n+1)\pi i\tau} = 0$$

and *the following expression*

$$u = u_0 y + 2(\ln \vartheta(\xi))_x \tag{3.1.4}$$

is the one periodic wave solution of eq. (1.1). For the proof [14].

According to the Theorem 1 α, ρ, k and ω should provide the following system with (2.15)

$$\begin{aligned}
\widetilde{H}(0) = &\sum_{n=-\infty}^{\infty} (-16\pi^2 n^2 \rho\omega - 48\pi^2 n^2 \alpha k - 256\pi^4 n^4 \rho\alpha^3 \\
&+48u_0\pi^2 n^2\alpha^2 + c) e^{2\pi i n^2 \tau} = 0 \\
\widetilde{H}(1) = &\sum_{n=-\infty}^{\infty} (-4\pi^2(2n-1)^2\rho\omega - 12\pi^2(2n-1)^2\alpha k - 16\pi^4(2n-1)^4\rho\alpha^3 \\
&+12\pi^2 u_0(2n-1)^2\alpha^2 + c) e^{(2n^2-2n+1)\pi i\tau} = 0 \quad .
\end{aligned} \tag{3.1.5}$$

Our aim is solving this system about frequency ω and integration constant c, namely

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \omega \\ c \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}. \tag{3.1.6}$$

By introducing the notations as

$$\lambda = e^{\pi i\tau} \quad a_{11} = \sum_{n=-\infty}^{\infty} -16\pi^2 n^2 \rho \lambda^{2n^2}$$

$$a_{12} = \sum_{n=-\infty}^{\infty} \lambda^{2n^2}$$

$$a_{21} = \sum_{n=-\infty}^{\infty} -4\pi^2(2n-1)^2\rho\lambda^{2n^2-2n+1}$$

$$a_{22} = \sum_{n=-\infty}^{\infty} \lambda^{2n^2-2n+1} \tag{3.1.7}$$

$$b_1 = \sum_{n=-\infty}^{\infty} (48\pi^2 n^2 \alpha k + 256\pi^4 n^4 \rho\alpha^3 - 48\pi^2 n^2 \alpha^2 u_0)\lambda^{2n^2}$$

$$b_2 = \sum_{n=-\infty}^{\infty} (12\pi^2(2n-1)^2\alpha k + 16\pi^4(2n-1)^4\rho\alpha^3$$

$$-12\pi^2(2n-1)^2\alpha^2 u_0)\lambda^{2n^2-2n+1}$$

we can easily solve this system and then we obtain a one-periodic wave solution of Eq. (1.1)

$$u = u_0 y + 2(\ln\vartheta(\xi))_x \tag{3.1.8}$$

where the parameters ω and c are given by (3.1.7) but the other parameters $\alpha, \rho, k, \delta, \tau, u_0$ are free.

3.2 Asymptotic property of one periodic waves

Theorem 2 *If the vector $(\omega, c)^T$ is a solution of the system (3.1.6) and for the one-periodic wave solution (3.1.8) we let*

$$u_0 = 0, \quad \alpha = \frac{\mu}{2\pi i}, \quad \rho = \frac{\nu}{2\pi i}, \quad k = \frac{\kappa}{2\pi i}, \quad \delta = \frac{\gamma - \pi i\tau}{2\pi i} \tag{3.2.1}$$

where μ, ν and γ are given (2.5). Then we have following asymtotic properties

$$c \to 0, \quad \xi \to \frac{\eta - \pi i\tau}{2\pi i}, \qquad \vartheta(\xi,\tau) \to 1 + e^{\eta} \ \ when \ \ \lambda \to 0 \tag{3.2.2}$$

It implies that the one-periodic solution tends to the one-soliton solution eq. (2.5) under a small amplitude limit

Proof

The one-periodic wave solution (3.1.8) has two fundamental periods 1 and τ in the phase variable ξ . It's actually a kind of one-dimensional cnoidal waves and speed parameter is given by

$$\omega = \frac{b_1 a_{22} - b_2 a_{12}}{a_{11}a_{22} - a_{12}a_{21}} \ . \tag{3.2.3}$$

It has only one wave pattern for all time, and it can be viewed as a parallel superposition of overlapping one-solitary waves, placed one period apart

For consider asymptotic properties we have to find solution of system (3.1.6) . Using eq. (3.1.7) coefficent matrix and the right-side vector of system (3.1.6) are power series about λ so its solution $(\omega, c)^T$ also should be a series about λ

$$a_{11} = -32\pi^2\rho\lambda^2 - 128\pi^2\rho\lambda^8 + ...$$

$$a_{12} = 1 + 2\lambda^2 + 2\lambda^8 + ...$$

$$a_{21} = -8\pi^2\rho\lambda - 72\pi^2\rho\lambda^5 + ...$$

$$a_{22} = 2\lambda + 2\lambda^5 + ...$$

$$b_1 = (96\pi^2\alpha k + 512\pi^4\alpha^3\rho - 96u_0\pi^2\alpha^2)\lambda^2 \\ +(384\pi^2\alpha k + 8192\pi^4\alpha^3\rho - 384u_0\pi^2\alpha^2)\lambda^8 + ...$$

$$b_2 = (24\pi^2\alpha k + 32\pi^4\alpha^3\rho - 24u_0\pi^2\alpha^2)\lambda \\ +(216\pi^2\alpha k + 2592\pi^4\alpha^3\rho - 216u_0\pi^2\alpha^2)\lambda^5 + ...$$

We can solve the system (3.1.6) via small parameter expansion method and we obtain

$$\begin{aligned} \omega &= (-3\tfrac{\alpha k}{\rho} - 4\pi^2\alpha^3 + 3u_0\tfrac{\alpha^2}{\rho}) + (96\pi^2\alpha^3)\lambda^2 + (288\pi^2\alpha^3)\lambda^4 + o(\lambda^4) \\ c &= (384\pi^4\rho\alpha^3)\lambda^2 + (2304\pi^4\rho\alpha^3)\lambda^4 + o(\lambda^4)\ . \end{aligned} \tag{3.2.4}$$

From Theorem 2 and (3.2.4), we have

$$c \to 0, \quad \omega = -3\frac{\alpha k}{\rho} - 4\pi^2\alpha^3 \ when \ \ \lambda \to 0 \tag{3.2.5}$$

and substituting the relation (3.2.1) into (3.2.5) we obtain

$$\varpi = 2\pi i\omega = -3\frac{\mu\kappa}{\nu} + \mu^3\ . \tag{3.2.6}$$

The one-soliton solution of the (3+1) generalized BKP equation can be obtained as a limit of the periodic solution (3.1.8). We can expand the periodic function $\vartheta(\xi)$ in the following form

$$\begin{aligned} \vartheta(\xi, \tau) &= \sum_{-\infty}^{\infty} e^{\pi i n^2\tau + 2\pi i n\xi} \\ &= 1 + e^{\pi i\tau + 2\pi i\xi} + e^{\pi i\tau - 2\pi i\xi} + e^{4\pi i\tau + 4\pi i\xi} + ... \end{aligned} \tag{3.2.7}$$

By using the transformation

$$\xi \rightarrow \frac{\breve{\xi}-\pi i\tau}{2\pi i}, \quad \lambda = e^{\pi i\tau}$$
$$\vartheta(\xi,\tau) = 1 + e^{\breve{\xi}} + \lambda^2(e^{-\breve{\xi}} + e^{2\xi}) + ... \tag{3.2.8}$$

and when $\lambda \rightarrow 0$ we can write

$$\vartheta(\xi,\tau) = 1 + e^{\breve{\xi}} \ . \tag{3.2.9}$$

According to one soliton solution $\breve{\xi} = \eta$, therefore proof is completed.

4 Two-periodic waves and asymptotic properties

4.1 Construct two-periodic waves

We consider two-periodic wave solutions of Eq. (1.1) which are two dimensional generalization of one-periodic wave solutions. Let's consider $N = 2$, and Riemann theta function takes the form

$$\vartheta(\xi,\tau) = \vartheta(\xi_1,\xi_2,\tau) = \sum_{n\in\mathbb{Z}^2} e^{\pi i<\tau n,n>+2\pi i<\xi,n>} \tag{4.1.1}$$

where $n = (n_1,n_2)^T \in \mathbb{Z}^2$, $\xi = (\xi_1,\xi_2) \in \mathbb{C}^2$, $\xi_i = \alpha_i x + \rho_i y + k_i z + \omega_i t + \delta_i$, $i = 1,2$ and $-i\tau$ is a positive definite and real-valued symmetric 2×2 matrix which can take the form of

$$\tau = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{12} & \tau_{22} \end{pmatrix}, \qquad \mathrm{Im}(\tau_{11}) > 0,\ \mathrm{Im}(\tau_{22}) > 0,\ \tau_{11}\tau_{22} - \tau_{12}^2 < 0 \tag{4.1.2}$$

Theorem 3 *Assuming that $\vartheta(\xi_1,\xi_2,\tau)$ is one Riemann theta function as $N = 2$ with $\xi_i = \alpha_i x + \rho_i y + k_i z + \omega_i t + \delta_i$ and $\alpha_i, \rho_i, k_i, \omega_i, \delta_i$,$i = 1,2$ satisfy the following system*

$$\sum_{n\in\mathbb{Z}^2} H(2\pi i < 2n - \theta_j, \alpha >, ...2\pi i < 2n - \theta_j, \omega >) \tag{4.1.3}$$

$$\times e^{\pi i[<\tau(n-\theta_j),n-\theta_j>+<\tau n,n>]} \quad = \quad 0$$

where $\theta_j = (\theta_j^1, \theta_j^2)^T$, $\theta_1 = (0,0)^T$, $\theta_2 = (1,0)^T$, $\theta_3 = (0,1)^T$, $\theta_4 = (1,1)^T$, $j = 1,2,3,4$ and the following expression

$$u = u_0 y + 2(\ln\vartheta(\xi_1,\xi_2,\tau))_x$$

is the two-periodic wave solution of Eq. (1.1). For the proof [14]

According to the Theorem 3 α_i, ρ_i, k_i and ω_i should provide the following system with (2.14)

$$\sum_{n\in\mathbb{Z}^2}[-4\pi^2 < 2n-\theta_j, \rho >< 2n-\theta_j, \omega > -12\pi^2 < 2n-\theta_j, \alpha >< 2n-\theta_j, k >$$

$$-16\pi^4 < 2n-\theta_j, \alpha >^3 < 2n-\theta_j, \rho > +12\pi^2 u_0 < 2n-\theta_j, \alpha >^2$$

$$+c] \times e^{\pi i[<\tau(n-\theta_j), n-\theta_j> + <\tau n, n>]} = 0 \tag{4.1.4}$$

where $j = 1, 2, 3, 4$. Our aim is solving this system namely

$$X \begin{pmatrix} \omega_1 \\ \omega_2 \\ u_0 \\ c \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} \tag{4.1.5}$$

where $X = (a_{ij})_{4\times 4}$ matrix.

By introducing the notation as

$$\varepsilon_j = \lambda_1^{n_1^2+(n_1-\theta_j^1)^2} \lambda_2^{n_2^2+(n_2-\theta_j^2)^2} \lambda_3^{n_1 n_2+(n_1-\theta_j^1)(n_2-\theta_j^2)} \tag{4.1.6}$$

where

$$\lambda_1 = e^{\pi i \tau_{11}}, \quad \lambda_2 = e^{\pi i \tau_{22}}, \quad \lambda_3 = e^{2\pi i \tau_{12}} \quad \text{and} \quad j = 1, 2, 3, 4 \tag{4.1.7}$$

and

$$\begin{aligned} a_{j4} &= \sum_{n_1, n_2\in\mathbb{Z}^2} \varepsilon_j \\ a_{j3} &= 12\pi^2 \sum_{n\in\mathbb{Z}^2} < 2n-\theta_j, \alpha >^2 \varepsilon_j \\ a_{j2} &= -4\pi^2 \sum_{n\in\mathbb{Z}^2} < 2n-\theta_j, \rho > (2n_2-\theta_j^2)\varepsilon_j \\ a_{j1} &= -4\pi^2 \sum_{n\in\mathbb{Z}^2} < 2n-\theta_j, \rho > (2n_1-\theta_j^1)\varepsilon_j \\ b_j &= \sum_{n\in\mathbb{Z}^2} 12\pi^2 < 2n-\theta_j, \alpha >< 2n-\theta_j, k > \\ &+16\pi^4 < 2n-\theta_j, \alpha >^3 < 2n-\theta_j, \rho > \varepsilon_j \end{aligned} \tag{4.1.8}$$

we can solve this system and we obtain two-periodic wave solution as

$$u = u_0 y + 2(\ln \vartheta(\xi_1, \xi_2, \tau))_x \tag{4.1.9}$$

where $\vartheta(\xi_1, \xi_2, \tau)$ and parameters $\omega_1, \omega_2, u_0, c$ are given by (4.1.1) and (4.1.5). The other $\alpha_1, \alpha_2, \rho_1, \rho_2, k_1, k_2, \tau_{11}, \tau_{12}$ and τ_{22} are arbitrary parameters .

We notice that the total number of unknown parameters u_0 integration constant c , nonlinear frequency $\alpha_i, \rho_i, k_i, \omega_i$ and the term $\tau_{jk} = \tau_{kj}$, $1 \leq j, k \leq N$ is

$$\frac{1}{2}N(N+1) + 4N + 2 \ .$$

4.2 Asymptotic property of two periodic waves

Teorem 4 *If $(\omega_1, \omega_2, u_0, c)^T$ is a solution of the system (4.1.5) and for the two-periodic wave solution we take*

$$\alpha_j = \frac{\mu_j}{2\pi i}, \quad \rho_j = \frac{\nu_j}{2\pi i}, \ k_j = \frac{\kappa_j}{2\pi i} \ , \delta_j = \frac{\gamma_j - \pi i \tau_{jj}}{2\pi i}, \quad \tau_{12} = \frac{A_{12}}{2\pi i}, \quad j = 1, 2 \tag{4.2.1}$$

where $\mu_j, \nu_j, \kappa_j, \delta_j$ and A_{12} are given in Eq. (2.7) and (2.8) . Then we have the following asymtotic relations

$$\begin{aligned} & u_0 \to 0, \quad c \to 0, \quad \xi_j \to \tfrac{\eta_j - \pi i \tau_{jj}}{2\pi i}, \quad j = 1, 2 \\ & \vartheta(\xi_1, \xi_2, \tau) \to 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_1 + \eta_2 + A_{12}} \quad \text{as} \quad \lambda_1, \lambda_2 \to 0 \end{aligned} \tag{4.2.2}$$

That means the two-periodic solution tends to the two-solion solution under a small amplitude limit.

Proof The Riemann theta function is

$$\vartheta(\xi_1, \xi_2, \tau) = \sum_{n \in \mathbb{Z}^2} e^{\pi i < \tau n, n > + 2\pi i < \xi, n >} \tag{4.2.3}$$

Let's expand this function

$$\begin{aligned} & \sum_{n_1, n_2 \in \mathbb{Z}^2} e^{2\pi i (\xi_1 n_1 + \xi_2 n_2) + \pi i [n_1(\tau_{11} n_1 + \tau_{12} n_2) + n_2(\tau_{12} n_1 + \tau_{22} n_2)]} \\ = \quad & 1 + e^{2\pi i \xi_1 + \pi i \tau_{11}} + e^{-2\pi i \xi_1 + \pi i \tau_{11}} + ... \end{aligned} \tag{4.2.4}$$

and if we take $\xi_j \to \frac{\tilde{\xi}_j - \pi i \tau_{jj}}{2\pi i}$ in Eq. (4.2.4) we have

$$\vartheta(\xi_1, \xi_2, \tau) = 1 + e^{\tilde{\xi}_1} + e^{\tilde{\xi}_2} + e^{\tilde{\xi}_1 + \xi_2 + \tilde{2}\pi i \tau_{12}} + \lambda_1^2 e^{-\tilde{\xi}_1} + \lambda_2^2 e^{-\tilde{\xi}_2} + ... \tag{4.2.5}$$

where $\lambda_1 = e^{\pi i \tau_{11}}$, $\quad \lambda_2 = e^{\pi i \tau_{22}}$ and $\lambda_1, \lambda_2 \to 0$

$$\vartheta(\xi_1, \xi_2, \tau) = 1 + e^{\tilde{\xi}_1} + e^{\tilde{\xi}_2} + e^{\tilde{\xi}_1 + \xi_2 + \tilde{2}\pi i \tau_{12}} \ . \tag{4.2.6}$$

According to the two soliton solution (2.6) we can write

$$\tau_{12} = \frac{A_{12}}{2\pi i} \tag{4.2.7}$$

For solving system (4.1.5) we can expand each funcion into a series with λ_1 and λ_2

$$\begin{aligned} X = X_0 + X_1\lambda_1 + X_2\lambda_2 + X_{11}\lambda_1^2 + X_{22}\lambda_2^2 \\ +X_{12}\lambda_1\lambda_2 + o(\lambda_1^k, \lambda_2^j) \quad , \quad k + l \geq 2. \end{aligned} \tag{4.2.8}$$

and

$$\begin{pmatrix}\omega_1\\ \omega_2\\ u_0\\ c\end{pmatrix} = \begin{pmatrix}\omega_1^0\\ \omega_2^0\\ u_0^0\\ c^0\end{pmatrix} + \begin{pmatrix}\omega_1^1\\ \omega_2^1\\ u_0^1\\ c^1\end{pmatrix}\lambda_1 + \begin{pmatrix}\omega_1^2\\ \omega_2^2\\ u_0^2\\ c^2\end{pmatrix}\lambda_2 + \begin{pmatrix}\omega_1^3\\ \omega_2^3\\ u_0^3\\ c^3\end{pmatrix}\lambda_1^2 \tag{4.2.9}$$

$$+ \begin{pmatrix}\omega_1^4\\ \omega_2^4\\ u_0^4\\ c^4\end{pmatrix}\lambda_2^2 + \begin{pmatrix}\omega_1^5\\ \omega_2^5\\ u_0^5\\ c^5\end{pmatrix}\lambda_1\lambda_2 + o(\lambda_1^k\lambda_2^l) \ , \quad k + l \geq 2$$

Substituting these equations into the (4.1.5), we obtain

$$\begin{aligned} c &= (384\pi^4\alpha_1^3\rho_1)\lambda_1^2 + (384\pi^4\alpha_2^3\rho_2)\lambda_2^2 + o(\lambda_1, \lambda_2) \\ \omega_1 &= (-3\tfrac{\alpha_1 k_1}{\rho_1} - 4\pi^2\alpha_1^3 + 3\tfrac{\alpha_1^2}{\rho_1}u_0^0) + (3\tfrac{\alpha_1^2}{\rho_1}u_0^1)\lambda_1 + (3\tfrac{\alpha_1^2}{\rho_1}u_0^2\ )\lambda_2 \\ &\quad +o(\lambda_1, \lambda_2) \\ \omega_2 &= (-3\tfrac{\alpha_2 k_2}{\rho_2} - 4\pi^2\alpha_2^3 + 3\tfrac{\alpha_2^2}{\rho_2}u_0^0) + (3\tfrac{\alpha_2^2}{\rho_2}u_0^1)\lambda_1 + (3\tfrac{\alpha_2^2}{\rho_2}u_0^2\ )\lambda_2 \\ &\quad +o(\lambda_1, \lambda_2). \end{aligned} \tag{4.2.10}$$

If we choose $u_0^0 = 0$, and $(\lambda_1, \lambda_2) \to (0, 0)$, we can find

$$\begin{aligned} u_0 &= o(\lambda_1, \lambda_2) \to 0 \quad , \quad c \to 0 \\ \omega_1 &= -3\frac{\alpha_1 k_1}{\rho_1} - 4\pi^2\alpha_1^3 \\ \omega_2 &= -3\frac{\alpha_2 k_2}{\rho_2} - 4\pi^2\alpha_2^3. \end{aligned} \tag{4.2.11}$$

According to the Theorem 4, we obtain

$$\begin{aligned} &\varpi_1 = -\tfrac{3\mu_1\kappa_1}{\nu_1} + \mu_1^3 \ \ , \ \ \varpi_2 = -\tfrac{3\mu_2\kappa_2}{\nu_2} + \mu_2^3 \ , \ c \to 0 \\ &\text{when} \ \ u_0 = o(\lambda_1, \lambda_2) \to 0 \ \ . \end{aligned} \tag{4.2.12}$$

and when solving the system we obtain

$$\lambda_3 = -\tfrac{(\nu_1 - \nu_2)(\varpi_1 - \varpi_2) - (\mu_1 - \mu_2)^3(\nu_1 - \nu_2) + 3(\mu_1 - \mu_2)(\kappa_1 - \kappa_2)}{(\nu_1 + \nu_2)(\varpi_1 + \varpi_2) - (\mu_1 + \mu_2)^3(\nu_1 + \nu_2) + 3(\mu_1 + \mu_2)(\kappa_1 + \kappa_2)} \tag{4.2.13}$$

That means just by solving system we can obtain $e^{A_{12}}$,this is alternative proof for $\tau_{12} = \frac{A_{12}}{2\pi i}$

From (4.2.12), we conclude that the two-periodic solution tends to the two soliton solution as $\lambda_1, \lambda_2 \to 0$.

5 Three-periodic waves and asymptotic properties

We consider three-periodic wave solutions of Eq. **??**. Let's consider $N = 3$, and Riemann theta function takes the form

$$\vartheta(\xi,\tau) = \vartheta(\xi_{1,}\xi_{2,}\xi_{3,}\tau) = \sum_{n\in\mathbb{Z}^3} e^{\pi i<\tau n,n>+2\pi i<\xi,n>} \tag{5.1.1}$$

where $n = (n_{1,}n_2, n_3)^T \in \mathbb{Z}^3$, $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{C}^3$, $\xi_i = \alpha_i x+\rho_i y+k_i z+\omega_i t+\delta_i$, $i = 1, 2, 3$ and $-i\tau$ is a positive definite and real-valued symmetric 3×3 matrix which can take the form of

$$\tau = \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{12} & \tau_{22} & \tau_{23} \\ \tau_{13} & \tau_{23} & \tau_{33} \end{pmatrix}, \qquad \mathrm{Im}(\tau_{jk}) > 0, \quad j = k = 1, 2, 3 \tag{5.1.2}$$

Theorem 5 *Assuming that $\vartheta(\xi_{1,}\xi_{2,}\xi_3, \tau)$ is one Riemann theta function as $N = 3$ with $\xi_i = \alpha_i x + \rho_i y + k_i z + \omega_i t + \delta_i$ and $\alpha_i, \rho_i, k_i, \omega_i, \delta_i$,$i = 1, 2, 3$ satisfy the following system*

$$\begin{aligned}\sum_{n\in\mathbb{Z}^3} H(2\pi i < 2n - \theta_j, \alpha >, ...2\pi i < 2n - \theta_j, \omega >)& \\ e^{\pi i[<\tau(n-\theta_j),n-\theta_j>+<\tau n,n>]} = 0&\end{aligned} \tag{5.1.3}$$

where $\theta_j = (\theta_j^1, \theta_j^2, \theta_j^3)^T$, $\theta_1 = (0, 0, 0)^T$, $\theta_2 = (0, 0, 1)^T$, $\theta_3 = (0, 1, 0)^T$, $\theta_4 = (0, 1, 1)^T$, $\theta_5 = (1, 0, 0)^T$, $\theta_6 = (1, 0, 1)^T$, $\theta_7 = (1, 1, 0)^T$, $\theta_8 = (1, 1, 1)^T$, $j = 1, .., 8$ and the following expression

$$u = u_0 y + 2(\ln \vartheta(\xi_1, \xi_2, \xi_3, \tau))_x \tag{5.1.4}$$

is the three-periodic wave solution.

Proof. Substituting (5.1.1) into bilinear equation $H(D_x, D_y, D_z, D_t)$ and using yhe property (2.4), we have following result

$$H(D_x, D_y, D_z, D_t)\vartheta(\xi_1, \xi_2, \xi_3, \tau).\vartheta(\xi_1, \xi_2, \xi_3, \tau)$$

$$= \sum_{m,n\in\mathbb{Z}^3} H(2\pi i < n-m, \alpha >, ...2\pi i < n-m, \omega >)$$

$$e^{2\pi i<\xi, m+n>+\pi i(<\tau m,m>+<\tau n,n>)}$$

$$= \sum_{m'\in\mathbb{Z}^3} \{ \sum_{n\in\mathbb{Z}^3} H(2\pi i < 2n-m', \alpha >, ...2\pi i < 2n-m', \omega >) \tag{5.1.5}$$

$$e^{\pi i(<\tau(n-m'),n-m'>+<\tau n,n>)} \} e^{2\pi i<\xi,m'>}$$

$$= \sum_{m'\in\mathbb{Z}^3} \hat{H}(m'_1, m'_2, m'_3) e^{2\pi i<\xi,m'>}$$

$$= \sum_{m'\in\mathbb{Z}^3} \hat{H}(m') e^{2\pi i<\xi,m'>},\ m' = m+n$$

Shifting index n as $n' = n - \delta_{ij},\ j = 1,2,3$ we can compute that

$$\hat{H}(m') = \hat{H}(m'_1, m'_2, m'_3)$$

$$= \sum_{n\in\mathbb{Z}^3} H(2\pi i < 2n-m', \alpha >, ...2\pi i < 2n-m', \omega >)$$

$$e^{\pi i(<\tau(n-m'),n-m'>+<\tau n,n>)}$$

$$= \sum_{n\in\mathbb{Z}^3} H(2\pi i \sum_{i=1}^{3} [2n'_i - (m'_i - 2\delta_{ij})]\alpha_i, ..., 2\pi i \sum_{i=1}^{3} [2n'_i - (m'_i - 2\delta_{ij})]\omega_i)$$

$$e^{\pi i \sum_{i,k=1}^{3} [(n'_i+\delta_{ij})(n'_k+\delta_{kj})+(m'_i-n'_i-\delta_{ij})(m'_k-n'_k-\delta_{kj})]\tau_{ik}}$$

$$= \begin{cases} \hat{H}(m'_1-2, m'_2, m'_3) e^{2\pi i(m'_1-1)\tau_{11}+2\pi i(m'_2\tau_{12}+m'_3\tau_{13})} & ,\ j=1 \\ \hat{H}(m'_1, m'_2-2, m'_3) e^{2\pi i(m'_2-1)\tau_{22}+2\pi i(m'_1\tau_{12}+m'_3\tau_{13})} & ,\ j=2 \\ \hat{H}(m'_1, m'_2, m'_3-2) e^{2\pi i(m'_3-1)\tau_{33}+2\pi i(m'_1\tau_{11}+m'_2\tau_{12})} & ,\ j=3 \end{cases} \tag{5.1.6}$$

which implies that if

$$\hat{H}(m'_1, m'_2, m'_3) = 0 \tag{5.1.7}$$

hold for all combinations of $m'_1 = 0,1,\ m'_2 = 0,1,\ m'_3 = 0,1$, then all $\hat{H}(m'_1, m'_2, m'_3) = 0,\ m'_i \in \mathbb{Z}^3\ (i = 1,2,3)$ and δ_{ij} representing Kronecker's delta. If we require

$$\hat{H}(m') = \sum_{n\in\mathbb{Z}^3} H(2\pi i < 2n-\theta_j, \alpha >, ...2\pi i < 2n-\theta_j, \omega >) \tag{5.1.8}$$

$$e^{\pi i(<\tau(n-\theta_j),n-\theta_j>+<\tau n,n>)}$$

where $\theta_j = (\theta_j^1, \theta_j^2, \theta_j^3)^T$ and $\theta_1 = (0,0,0)^T,\ \theta_2 = (0,0,1)^T,\ \theta_3 = (0,1,0)^T,\ \theta_4 = (0,1,1)^T,\ \theta_5 = (1,0,0)^T,\ \theta_6 = (1,0,1)^T,\ \theta_7 = (1,1,0)^T,\ \theta_8 = (1,1,1)^T,\ j = 1,..,8$, we can obtain three-periodic wave solutions.

According to the Theorem 5 α_i, ρ_i, k_i and ω_i should provide the following system with (2.14)

$$\sum_{(n_1,n_2,n_3)\in\mathbb{Z}^3} [-4\pi^2 < 2n-\theta_j, \rho >< 2n-\theta_j, \omega >$$
$$-12\pi^2 < 2n-\theta_j, \alpha >< 2n-\theta_j, k > -16\pi^4 < 2n-\theta_j, \alpha >^3 < 2n-\theta_j, \rho >$$
$$+12\pi^2 u_0 < 2n-\theta_j, \alpha >^2 +c] \times e^{\pi i[<\tau(n-\theta_j), n-\theta_j> + <\tau n, n>]} = 0 \tag{5.1.9}$$

where $j = 1, ..., 8$. Our aim is solving this system namely

$$X(\omega_1, \omega_2, \omega_3, k_1, k_2, k_3, u_0, c)^T = b \tag{5.1.10}$$

where $X = (a_{ij})_{8\times 8}$ matrix and $b = (b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8)$.

By introducing the notation as

$$\varepsilon_j = \sum_{(n_1,n_2,n_3)\in\mathbb{Z}^3} e^{\pi i[<\tau(n-\theta_j), n-\theta_j> + <\tau n, n>]}$$
$$= \lambda_1^{n_1^2 + (n_1-\theta_j^1)^2} \lambda_2^{n_2^2 + (n_2-\theta_j^2)^2} \lambda_3^{n_3^2 + (n_3-\theta_j^3)^2} \tag{5.1.11}$$
$$\lambda_{12}^{n_1 n_2 + (n_1-\theta_j^1)(n_2-\theta_j^2)} \lambda_{13}^{n_1 n_3 + (n_1-\theta_j^1)(n_3-\theta_j^3)} \lambda_{23}^{n_2 n_3 + (n_2-\theta_j^2)(n_3-\theta_j^3)}$$

where

$$\lambda_1 = e^{\pi i \tau_{11}},\ \lambda_2 = e^{\pi i \tau_{22}},\ \lambda_3 = e^{\pi i \tau_{33}}$$
$$\lambda_{12} = e^{2\pi i \tau_{12}},\ \lambda_{13} = e^{2\pi i \tau_{13}},\ \lambda_{23} = e^{2\pi i \tau_{23}} \tag{5.1.12}$$
$$j = 1, .., 8$$

and

$$a_{j8} = \sum_{n\in\mathbb{Z}^3} \varepsilon_j$$

$$a_{j7} = \sum_{n\in\mathbb{Z}^3} 12\pi^2 < 2n-\theta_j, \alpha >^2 \varepsilon_j$$

$$a_{j6} = \sum_{n\in\mathbb{Z}^3} -12\pi^2 < 2n-\theta_j, \alpha > (2n_3-\theta_j^3)\varepsilon_j$$

$$a_{j5} = \sum_{n\in\mathbb{Z}^3} -12\pi^2 < 2n-\theta_j, \alpha > (2n_2-\theta_j^2)\varepsilon_j$$

$$a_{j4} = \sum_{n\in\mathbb{Z}^3} -12\pi^2 < 2n-\theta_j, \alpha > (2n_1-\theta_j^1)\varepsilon_j \qquad (5.1.13)$$

$$a_{j3} = \sum_{n\in\mathbb{Z}^3} -4\pi^2 < 2n-\theta_j, \rho > (2n_3-\theta_j^3)\varepsilon_j$$

$$a_{j2} = \sum_{n\in\mathbb{Z}^3} -4\pi^2 < 2n-\theta_j, \rho > (2n_2-\theta_j^2)\varepsilon_j$$

$$a_{j1} = \sum_{n\in\mathbb{Z}^3} -4\pi^2 < 2n-\theta_j, \rho > (2n_1-\theta_j^1)\varepsilon_j$$

$$b_j = \sum_{n\in\mathbb{Z}^3} 16\pi^4 < 2n-\theta_j, \alpha >^3 < 2n-\theta_j, \rho > \varepsilon_j$$

we can solve this system and we obtain three-periodic wave solution as

$$u = u_0 y + 2(\ln \vartheta(\xi_1, \xi_2, \xi_3, \tau))_x$$

where $\vartheta(\xi_1, \xi_2, \xi_3, \tau)$ and parameters $\omega_1, \omega_2, \omega_3, k_1, k_2, k_3, u_0, c$ are given by (5.1.1) and (5.1.10). The other $\alpha_1, \alpha_2, \alpha_3, \rho_1, \rho_2, \rho_3, \tau_{11}, \tau_{22}, \tau_{33}, \tau_{12}, \tau_{13}$ and τ_{23} are arbitrary parameters .

5.1 Asymptotic property of three periodic waves

Teorem 6 *If* $(\omega_1, \omega_2, \omega_3, k_1, k_2, k_3, u_0, c)^T$ *is a solution of the system (5.1.10) and for the three-periodic wave solution we take*

$$\begin{aligned} &\alpha_j = \tfrac{\mu_j}{2\pi i}, \quad \rho_j = \tfrac{\nu_j}{2\pi i}, \ k_j = \tfrac{\kappa_j}{2\pi i} \quad ,\delta_j = \tfrac{\gamma_j - \pi i \tau_{jj}}{2\pi i}, \\ &\tau_{ij} = \tfrac{A_{ij}}{2\pi i}, \quad i,j = 1,2,3, \ i<j \end{aligned} \qquad (5.2.1)$$

where $\mu_j, \nu_{j,} \kappa_j, \delta_j$ and A_{ij} are given in Eq. (2.11) and (2.12) . Then we have the following asymtotic relations

$$u_0 \to 0, \quad c \to 0, \quad \xi_j \to \tfrac{\eta_j - \pi i \tau_{jj}}{2\pi i}, \quad j = 1, 2, 3$$

$$\begin{aligned} &\vartheta(\xi_1, \xi_2, \xi_{3,} \tau) \to 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_3} + e^{\eta_1+\eta_2+A_{12}} \\ &+ e^{\eta_1+\eta_3+A_{13}} + e^{\eta_2+\eta_3+A_{23}} + e^{\eta_1+\eta_2+\eta_3+A_{12}+A_{13}+A_{23}} \\ &\text{as} \quad \lambda_1, \lambda_2, \lambda_3 \to 0. \end{aligned} \tag{5.2.2}$$

That means the three-periodic solution tends to the three-solion solution under a small amplitude limit.

Proof The Riemann theta function is

$$\vartheta(\xi_1, \xi_2, \xi_3, \tau) = \sum_{n\in\mathbb{Z}^3} e^{\pi i < \tau n, n> + 2\pi i < \xi, n>} \tag{5.2.3}$$

Let's expand this function

$$\begin{aligned} &= \sum_{n_1,n_2,n_3\in\mathbb{Z}^3} e^{2\pi i(\xi_1 n_1+\xi_2 n_2+\xi_3 n_3)+\pi i[\tau_{11}n_1^2+\tau_{22}n_2^2+\tau_{33}n_3^2+2n_1n_2\tau_{12}+2n_1n_3\tau_{13}+2n_2n_3\tau_{23})]} \\ &= 1 + e^{2\pi i\xi_1+\pi i\tau_{11}} + e^{-2\pi i\xi_1+\pi i\tau_{11}} + e^{2\pi i\xi_2+\pi i\tau_{22}} + e^{-2\pi i\xi_2+\pi i\tau_{22}} \\ &+ e^{2\pi i\xi_3+\pi i\tau_{33}} + e^{-2\pi i\xi_3+\pi i\tau_{33}} + e^{\pi i\tau_{11}+\pi i\tau_{22}+2\tau_{12}+2\pi i\xi_1+2\pi i\xi_2} + \ldots \end{aligned} \tag{5.2.4}$$

and if we take $\xi_j \to \frac{\widetilde{\xi}_j - \pi i \tau_{jj}}{2\pi i}$ in Eq. (5.2.4) we have

$$\begin{aligned} &\vartheta(\xi_1, \xi_2, \tau) = 1 + e^{\widetilde{\xi}_1} + e^{\widetilde{\xi}_2} + e^{\widetilde{\xi}_3} + e^{\widetilde{\xi}_1+\widetilde{\xi}_2+2\pi i\tau_{12}} + e^{\widetilde{\xi}_1+\widetilde{\xi}_3+2\pi i\tau_{13}} \\ &+ e^{\widetilde{\xi}_2+\widetilde{\xi}_3+2\pi i\tau_{23}} + e^{\widetilde{\xi}_1+\widetilde{\xi}_2+\widetilde{\xi}_3+2\pi i\tau_{12}+2\pi i\tau_{13}+2\pi i\tau_{23}} + \lambda_1^2 e^{-\widetilde{\xi}_1} + \lambda_2^2 e^{-\widetilde{\xi}_2} \\ &+ \lambda_3^2 e^{-\widetilde{\xi}_2} + \lambda_1^2\lambda_2^2 e^{-\widetilde{\xi}_1-\widetilde{\xi}_2+2\pi i\tau_{12}} + \ldots \end{aligned} \tag{5.2.5}$$

where $\lambda_1 = e^{\pi i\tau_{11}}$, $\lambda_2 = e^{\pi i\tau_{22}}$, $\lambda_3 = e^{\pi i\tau_{33}}$ and $\lambda_1, \lambda_2, \lambda_3 \to 0$

$$\begin{aligned} &\vartheta(\xi_1, \xi_2, \xi_3, \tau) = 1 + e^{\widetilde{\xi}_1} + e^{\widetilde{\xi}_2} + e^{\widetilde{\xi}_3} + e^{\widetilde{\xi}_1+\widetilde{\xi}_2+2\pi i\tau_{12}} + e^{\widetilde{\xi}_1+\widetilde{\xi}_3+2\pi i\tau_{13}} \\ &+ e^{\widetilde{\xi}_2+\widetilde{\xi}_3+2\pi i\tau_{23}} + e^{\widetilde{\xi}_1+\widetilde{\xi}_2+\widetilde{\xi}_3+2\pi i\tau_{12}+2\pi i\tau_{13}+2\pi i\tau_{23}} \end{aligned} \tag{5.2.6}$$

According to the three-soliton solution (2.9) we can write

$$\tau_{12} = \frac{A_{12}}{2\pi i}, \ \tau_{13} = \frac{A_{13}}{2\pi i}, \ \tau_{23} = \frac{A_{23}}{2\pi i} \tag{5.2.7}$$

For solving system (5.1.10) we can expand each funcion into a series with λ_1, λ_2 and λ_3

$$\begin{aligned} X &= X_0 + X_1\lambda_1 + X_2\lambda_2 + X_3\lambda_3 + X_4\lambda_1^2 + X_5\lambda_2^2 + X_6\lambda_3^2 \\ &+X_7\lambda_1\lambda_2 + X_8\lambda_1\lambda_3 + X_9\lambda_2\lambda_3 + ... \end{aligned} \tag{5.2.8}$$

and we obtain

$$\begin{aligned} c &= (384\pi^4\alpha_1^3\rho_1)\lambda_1^2 + (384\pi^4\alpha_2^3\rho_2)\lambda_2^2 + (384\pi^4\alpha_3^3\rho_3)\lambda_3^2 + o(\lambda_1^i, \lambda_2^j, \lambda_3^k) \ \ , \ i+j+k \geq 3 \\ \omega_1 &= (-3\tfrac{\alpha_1 k_1^{(0)}}{\rho_1} - 4\pi^2\alpha_1^3 + 3\tfrac{\alpha_1^2}{\rho_1}u_0^{(0)}) + (-3\tfrac{\alpha_1 k_1^{(1)}}{\rho_1} + 3\tfrac{\alpha_1^2}{\rho_1}u_0^{(1)})\lambda_1 + (-3\tfrac{\alpha_1 k_1^{(2)}}{\rho_1} + 3\tfrac{\alpha_1^2}{\rho_1}u_0^{(2)})\lambda_2 \\ &+(-3\tfrac{\alpha_1 k_1^{(3)}}{\rho_1} + 3\tfrac{\alpha_1^2}{\rho_1}u_0^{(3)})\lambda_3 + ... \\ \omega_2 &= (-3\tfrac{\alpha_2 k_2^{(0)}}{\rho_2} - 4\pi^2\alpha_2^3 + 3\tfrac{\alpha_2^2}{\rho_2}u_0^{(0)}) + (-3\tfrac{\alpha_2 k_2^{(1)}}{\rho_2} + 3\tfrac{\alpha_2^2}{\rho_2}u_0^{(1)})\lambda_1 + (-3\tfrac{\alpha_2 k_2^{(2)}}{\rho_2} + 3\tfrac{\alpha_2^2}{\rho_2}u_0^{(2)})\lambda_2 \\ &+(-3\tfrac{\alpha_2 k_2^{(3)}}{\rho_2} + 3\tfrac{\alpha_2^2}{\rho_2}u_0^{(3)})\lambda_3 + ... \\ \omega_3 &= (-3\tfrac{\alpha_3 k_3^{(0)}}{\rho_3} - 4\pi^2\alpha_3^3 + 3\tfrac{\alpha_3^2}{\rho_3}u_0^{(0)}) + (-3\tfrac{\alpha_3 k_3^{(1)}}{\rho_3} + 3\tfrac{\alpha_3^2}{\rho_3}u_0^{(1)})\lambda_1 + (-3\tfrac{\alpha_3 k_3^{(2)}}{\rho_3} + 3\tfrac{\alpha_3^2}{\rho_3}u_0^{(2)})\lambda_2 \\ &+(-3\tfrac{\alpha_3 k_3^{(3)}}{\rho_3} + 3\tfrac{\alpha_3^2}{\rho_3}u_0^{(3)})\lambda_3 + ... \end{aligned} \tag{5.2.9}$$

where we expand the notations as follows

$$\begin{aligned} k_i &= k_i^{(0)} + k_i^{(1)}\lambda_1 + k_i^{(2)}\lambda_2 + k_i^{(3)}\lambda_3 + k_i^{(11)}\lambda_1^2 + k_i^{(22)}\lambda_2^2 \\ &+k_i^{(33)}\lambda_3^2 + k_i^{(12)}\lambda_1\lambda_2 + k_i^{(13)}\lambda_1\lambda_3 + k_i^{(23)}\lambda_2\lambda_3 + ..\ i = 1,2,3 \end{aligned} \tag{5.2.10}$$

and parameters ω_i, c and u_0 are similar to (5.2.10).

If we choose $u_0^0 = 0$, and $(\lambda_1, \lambda_2, \lambda_3) \to (0,0,0)$, we can find

$$\begin{aligned} &u_0 \to 0 \ \ , \quad c \to 0 \\ &\omega_1 = -3\tfrac{\alpha_1 k_1}{\rho_1} - 4\pi^2\alpha_1^3 \\ &\omega_2 = -3\tfrac{\alpha_2 k_2}{\rho_2} - 4\pi^2\alpha_2^3 \\ &\omega_3 = -3\tfrac{\alpha_3 k_3}{\rho_3} - 4\pi^2\alpha_3^3 \end{aligned} \tag{5.2.11}$$

According to the Theorem 6, we obtain

$$\varpi_1 = -\frac{3\mu_1\kappa_1}{\nu_1} + \mu_1^3 \ , \ \varpi_2 = -\frac{3\mu_2\kappa_2}{\nu_2} + \mu_2^3$$

$$\varpi_3 = -\frac{3\mu_3\kappa_3}{\nu_3} + \mu_3^3, \quad c \to 0 \tag{5.2.12}$$

$$\text{when} \ \ u_0 = o(\lambda_1, \lambda_2, \lambda_3) \to 0 \ \ .$$

From (5.2.12), we conclude that the three-periodic solution tends to the three soliton solution as $\lambda_1, \lambda_2, \lambda_3 \to 0$

5.2 Conclusion

In this paper, we have obtained the one, two and three periodic wave solutions of the (3+1) generalized BKP equation, by using Hirota's bilinear method and the Riemann theta functions. Moreover, we have shown that they can be reduced to classical solitons, under a small amplitude limit.

The results can be extended to the case $N \geq 4$ but when solving the system we need more unknown parameters so there is certain difficulties in the calculation and it is still open problem for us .

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