

Logarithmic Quasi-distance Proximal Point Scalarization Method for Multi-Objective Programming*

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Abstract

Recently, Gregório and Oliveira developed a proximal point scalarization method (applied to multi-objective optimization problems) for an abstract strict scalar representation with a variant of the logarithmic-quadratic function of Auslender et al. as regularization. In this study, a variation of this method is proposed, using the regularization with logarithm and quasi-distance, which entails losing important properties, such as convexity and differentiability. However, proceeding differently, it is shown that any sequence $\{(x^k, z^k)\} \subset R^n \times R_{++}^m$ generated by the method satisfies: $\{z^k\}$ is convergent and $\{x^k\}$ is bounded and its accumulation points are weak pareto solutions of the unconstrained multi-objective optimization problem

Keywords: Proximal point algorithm, Scalar representation, Multi-Objective programming, Quasi-distance.

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1 Introduction

This study considers the unconstrained multi-objective optimization problem

$$\min \{F(x) : x \in R^n\} \quad (1)$$

where $F = (F_1, F_2, \dots, F_m)^T : R^n \rightarrow R^m$ is a convex mapping related to the lexicographic order generated by the cone R_+^m , i.e., for all $x, y \in R^n$ and $\lambda \in (0, 1)$, $F_i(\lambda x + (1 - \lambda)y) \leq \lambda F_i(x) + (1 - \lambda)F_i(y)$, $\forall i = 1, \dots, m$. Moreover, it will be required that one of the objective functions must be coercive, i.e., there is $r \in \{1, \dots, m\}$ such that $\lim_{\|x\| \rightarrow \infty} F_r(x) = \infty$.

The importance of multi-objective optimization can be seen from the large variety of applications presented in the literature. White [28] offers a bibliography of 504 papers describing various different applications addressing, for example, problems concerning agriculture, banking, health services, energy, industry and water. More information with regard to multi-objective optimization can be found, for example, in section 3 and Miettinen [19].

There is a more general class of problems, known as vector optimization, that contains multi-objective optimization. See, for example, Luc [17]. On the other hand, the methods developed for this class of problem can be classified into two types: scalarization methods and extensions of nonlinear algorithms to vector optimization. Some global optimization techniques are discussed in Chinchuluun and Pardalos [8].

The classic proximal point method to minimize a scalar convex function $f : R^n \rightarrow R$ generates a sequence $\{x^k\}$ via the iterative scheme: given a starting point $x^0 \in R^n$, then

$$x^{k+1} \in \operatorname{argmin}\{f(x) + \lambda_k \|x - x^k\|^2 : x \in R^n\}, \quad (2)$$

where λ_k is a sequence of real positive numbers and $\|\cdot\|$ is the usual norm. This method was originally introduced by Martinet [18] and developed and studied by Rockafellar [24]. In recent decades the convergence analysis of the sequence $\{x^k\}$ has been extensively studied, and several extensions of the method have been developed in order to consider cases in which the function f is not convex and/or cases where the usual quadratic term in (2) is replaced by a generalized distance, e.g., Bregman distances, φ -divergences, proximal distances and quasi-distances. The papers containing these generalizations include: Chen and Teboulle [5], Iusem and Teboulle [13], Pennanen [23], Hamdi [12], Chen and Pan [6], Papa Quiroz and Oliveira [22], Moreno et al. [20] and Langenberg and Tichatschke [15].

This class of proximal point algorithms has been extended to vector optimization. The first method in this direction was the multi-objective proximal bundle method (see, Miettinen [19]). Göpfert et al. [10] have presented a proximal point method for the scalar representation $\langle F(x), z \rangle$ with a regularization based on Bregman functions on finite dimensional spaces. Bonnel et al. [2] and Ceng and Yao [4] present a proximal algorithm with a quadratic regularization in vector form. Villacorta and Oliveira [29] also present a proximal algorithm in vector form with the regularization being a proximal distance. Gregório and Oliveira [11] present a proximal algorithm in multi-objective optimization for an abstract strict scalar representation with a variant of the logarithmic-quadratic functions of Auslender et al. [1] as regularization.

We will present a brief description of the method of Gregório and Oliveira [11]. Let $F : R^n \rightarrow R^m$ be a convex application. Given the starting points $x^0 \in R^n$ and $z^0 \in R_{++}^m$ and sequences $\beta_k, \mu_k > 0, k = 0, 1, \dots$, the method generates a sequence $\{(x^k, z^k)\} \subset R^n \times R_{++}^m$ via the iterative scheme:

$$(x^{k+1}, z^{k+1}) \in \operatorname{argmin} \{f(x, z) + \beta_k H(z) + \frac{\alpha_k}{2} \|x - x^k\|^2 : x \in \Omega^k, z \in R_{++}^n\} \quad (3)$$

where $\Omega^k = \{x \in R^n : F_i(x) \leq F_i(x^k)\}$, $f : R^n \times R_+^m \rightarrow R$ satisfies the properties (P1) to (P4) (see section 4) and $H : R_{++}^m \rightarrow R$ is such that $H(z) = \langle z/z^k - \log(z/z^k) - e, e \rangle$ where $e = (1, \dots, 1) \in R^m$, $z/z^k = (z_1/z_1^k, \dots, z_m/z_m^k)$ and $\log(z/z^k) = (\log(z_1/z_1^k), \dots, \log(z_m/z_m^k))$.

We are proposing a generalization of this method considering (3) with the quasi-distance $q : R^n \times R^n \rightarrow R_+$ (see definition 2.1) in place of the Euclidean norm $\|\cdot\|$, i.e.,

$$(x^{k+1}, z^{k+1}) \in \operatorname{argmin} \{f(x, z) + \beta_k H(z) + \frac{\alpha_k}{2} q^2(x, x^k) : x \in \Omega^k, z \in R_{++}^n\}. \quad (4)$$

As quasi-distances are not necessarily symmetric (see definition 2.1), they generalize the distances. Therefore, our algorithm generalizes Gregório and Oliveira's algorithm [11]. A quasi-distance is not necessarily a convex function, nor continuously differentiable, nor even a coercive function in any of its arguments. Supposing that the quasi-distance satisfies the condition (5) (see section 2.1), then the coercivity and Lipschitz properties are recovered (see Propositions 2.1 and 2.2). However, important properties such as the convexity and differentiability will be lost. Accordingly, we had to proceed differently to guarantee the convergence of our method. What is more, we found a new example of a function $f : R^n \times R_+^m \rightarrow R$ which satisfies the properties (P1) to (P4) (see proposition 4.1), which were fundamental to the convergence of our method. Gregório and Oliveira [11], drawing on the

work of Fliege and Svaiter [9], supposed that the set Ω^0 is limited and established the convergence of their method. In our case, a condition of coercivity was imposed on us in only one of the objective functions, i.e., suppose that there is $r \in \{1, \dots, m\}$ such that $\lim_{\|x\| \rightarrow \infty} F_r(x) = \infty$, and that it has as a consequence the limitation of Ω^0 (see Lemma 4.1). The importance of the limitation of the set Ω^0 is that it guarantees that the sequence $\{x^k\}$ generated by our algorithm is limited (see proposition proof 4.4 (i)).

Quasi-distances can be applied not only to computer theory (see, for example, Brattka [3] and Kunzi et al. [14]), but also to economy, for example and, more directly, to consumer choice and to utility functions (see, for example, Romanguera and Sanchis [26] and Moreno et al. [20]). Note that Moreno et al. [20] developed a proximal algorithm with quasi-distance regularization applied to non-convex and non-differentiable scalar functions, satisfying the Kurdyka-Lojasiewics inequality. And because the quasi-distance is not necessarily symmetric, they derived an economic interpretation of this algorithm, applied to habit formation. In this respect, the work of Moreno et al. encourages us, in future investigations, to seek an economic interpretation of our algorithm applied to economy-related multi-objective problems.

One important point is that our proximal algorithm and the proximal algorithm developed by Gregório and Oliveira [11] were developed in multi-objective optimization and belong to the class of proximal point scalarization methods. Meanwhile, the algorithms developed by Bonnel et al. [2], Ceng and Yao [4], and Villacorta and Oliveira [29] were developed in vector optimization and belong to the class of proximal methods in vector form. This means that the subproblems of our algorithm and Gregório and Oliveira's are problems relating to the minimization of scalar functions, while the subproblems of the other studies' algorithms are problems relating to the minimization of vector functions.

Section 2 presents concepts and results relating to quasi-distance and subdifferential theory. In section 3, concepts and results of general multi-objective optimization theory are presented. Section 4 presents the authors' own method, where we assure the existence of the iterations, stop criterion and convergence. In section 5, a variation of that method is considered. Finally, in section 6, the method is tested and numerical examples are offered.

2 Quasi-distance and Subdifferential Theory

In this section, the quasi-distance application is defined, with examples, and some of its properties that are fundamental to the course of our work are presented. The concepts of Fréchet subdifferential and limiting subdifferential are also revisited, along with some of their properties.

2.1 Quasi-Distance

Definition 2.1 ([27]) *Let X be a set. A mapping $q : X \times X \rightarrow R_+$ is called a quasi-distance if for all $x, y, z \in X$,*

1. $q(x, y) = q(y, x) = 0 \iff x = y$
2. $q(x, z) \leq q(x, y) + q(y, z)$.

Notice that if q satisfies the property of symmetry, i.e., if for all $x, y \in X$, $q(x, y) = q(y, x)$, then q is a distance. A quasi-distance is not necessarily a convex function and coercive in the first argument, nor in the second (see [20], Example 3.1 and Remark 3). Moreno et al. [20] presented the following example of quasi-distance.

Example 2.1 *For each $i = 1, \dots, n$, consider $c_i^-, c_i^+ > 0$ and $q_i : R \times R \rightarrow R_+$ defined by*

$$q_i(x_i, y_i) = \begin{cases} c_i^+(y_i - x_i) & \text{if } y_i - x_i > 0 \\ c_i^-(x_i - y_i) & \text{if } y_i - x_i \leq 0 \end{cases}$$

is a quasi-distance on R , therefore $q(x, y) = \sum_{i=1}^n q_i(x_i, y_i)$ is a quasi-distance on R^n . On the other hand, for each $\bar{z} \in R^n$,

$$q(x, \bar{z}) = \sum_{i=1}^n q_i(x_i, \bar{z}_i) = \sum_{i=1}^n \max\{c_i^+(\bar{z}_i - x_i), c_i^-(x_i - \bar{z}_i)\}, \quad x \in R^n,$$

thus $q(\cdot, \bar{z})$ is a convex function. By the same reasoning, $q(\bar{z}, \cdot)$ is convex.

Moreno et al. [20] took into account the following condition with regard to the quasi-distance q : There are positive constants α and β such that

$$\alpha\|x - y\| \leq q(x, y) \leq \beta\|x - y\|, \quad \forall x, y \in R^n \quad (5)$$

Notice that the quasi-distance in the example 2.1 exhibits the property (5). With this condition, Moreno et al. [20] showed that, in each one of the arguments, the quasi-distance exhibits important properties, such as Lipschitz and coercivity. The results follow below:

Proposition 2.1 ([20], Propositions 3.6 and 3.7) *Let $q : R^n \times R^n \rightarrow R_+$ be a quasi-distance that exhibits (5). Then for each $\bar{z} \in R^n$ the functions $q(\bar{z}, \cdot)$ and $q(\cdot, \bar{z})$ are Lipschitz continuous and the functions $q^2(\bar{z}, \cdot)$ and $q^2(\cdot, \bar{z})$ are locally Lipschitz continuous functions on R^n .*

Proposition 2.2 ([20], Remark 5) *Let $q : R^n \times R^n \rightarrow R_+$ be a quasi-distance that exhibits (5). Then for each $\bar{z} \in R^n$ the functions $q(\bar{z}, \cdot)$, $q(\cdot, \bar{z})$, $q^2(\bar{z}, \cdot)$ and $q^2(\cdot, \bar{z})$ are coercive.*

2.2 Subdifferential Theory

Here Frechet's concepts of subdifferential and limiting subdifferential are recalled. Only the results fundamental to our study are presented. For more details, see [25].

Definition 2.2 *Let $h : R^n \rightarrow R \cup \{\infty\}$ be a proper lower semi-continuous function and $x \in R^n$.*

1. *The Fréchet subdifferential of h at x , $\hat{\partial}h(x)$, is defined as follows*

$$\hat{\partial}h(x) := \begin{cases} \left\{ x^* \in R^n : \liminf_{y \neq x, y \rightarrow x} \frac{h(y) - h(x) - \langle x^*, y - x \rangle}{\|x - y\|} \geq 0 \right\}, & \text{if } x \in \text{dom}(h) \\ \emptyset, & \text{if } x \notin \text{dom}(h) \end{cases}$$

2. *The limiting-subdifferential of h at $x \in R^n$, $\partial h(x)$, is defined as follows*

$$\partial h(x) := \left\{ x^* \in R^n : \exists x_n \rightarrow x, \quad h(x_n) \rightarrow h(x), \quad x_n^* \in \hat{\partial}h(x_n) \rightarrow x^* \right\}$$

Proposition 2.3 (Optimality condition - [25], Theorem 10.1)

If a proper function $h : R^n \rightarrow R \cup \{+\infty\}$ has a local minimum at \bar{x} , then $0 \in \hat{\partial}h(\bar{x})$, $0 \in \partial h(\bar{x})$.

Remark 2.1 *Let $C \subset R^n$. If a proper function $h : C \rightarrow R \cup \{\infty\}$ has a local minimum at $\bar{x} \in C$, then $0 \in \hat{\partial}(h + \delta_C)(\bar{x})$, $0 \in \partial(h + \delta_C)(\bar{x})$, where δ_C is the indicator function of the set C , defined as $\delta_C(x) = 0$ if x belongs to C and $\delta_C(x) = \infty$ otherwise.*

Proposition 2.4 ([25], Exercise 10.10) *If f_1 is locally Lipschitz continuous at \bar{x} , f_2 is lower semi-continuous and proper with $f_2(\bar{x})$ finite, then*

$$\partial(f_1 + f_2)(\bar{x}) \subset \partial f_1(\bar{x}) + \partial f_2(\bar{x}).$$

Proposition 2.5 ([21], Theorem 7.1) *Let $f_i : R^n \rightarrow R, i = 1, 2$, be Lipschitz continuous around \bar{x} . If $f_i \geq 0, i = 1, 2$, then one has a product rule of the equality form*

$$\partial(f_1 \cdot f_2)(\bar{x}) = \partial(f_2(\bar{x})f_1 + f_1(\bar{x})f_2)(\bar{x}).$$

Proposition 2.6 ([25], Proposition 5.15) *A mapping $S : R^n \rightarrow P(R^m)$ is locally bounded if and only if $S(B)$ is bounded for every bounded set B .*

Proposition 2.7 ([25], Theorem 9.13) *Suppose $h : R^n \rightarrow R \cup \{\pm\infty\}$ is locally lower semi-continuous at \bar{x} with $h(\bar{x})$ finite. Then the following conditions are equivalent:*

- (a) *h is locally Lipschitz continuous at \bar{x} ,*
- (b) *the mapping $\hat{\partial}h : x \mapsto \hat{\partial}h(x)$ is locally bounded at \bar{x} ,*
- (c) *the mapping $\partial h : x \mapsto \partial h(x)$ is locally bounded at \bar{x} .*

Moreover, when these conditions hold, $\partial h(\bar{x})$ is non-empty and compact.

3 Multi-objective programming - preliminary concepts

Only the concepts and results that are fundamental to the course of our work are presented. For more details, see, for example, Miettinen [19] and Chinchuluun et al. [7].

Definition 3.1 *$a \in R^n$ is said to be a **local pareto solution** to the problem (1) if there is a disc $B_\delta(a) \subset R^n$, with $\delta > 0$, such that there is no $x \in B_\delta(a)$ that satisfies $F_i(x) \leq F_i(a)$ for all $i = 1, \dots, m$ and $F_j(x) < F_j(a)$ for at least one index $j \in \{1, \dots, m\}$.*

Definition 3.2 *$a \in R^n$ is known as a **weak local pareto solution** if there is a disc $B_\delta(a) \subset R^n$, with $\delta > 0$, such that there is no $x \in B_\delta(a)$ that satisfies $F_i(x) < F_i(a)$ for all $i = 1, \dots, m$.*

Generally, if a constrained or unconstrained multi-objective optimization problem is a convex problem, that is, if an objective function $F : R^n \rightarrow R^m$ is a convex function, then any (weak) local pareto solution is also a

(weak) global pareto solution. This result is discussed in Theorem 2.2.3, in Miettinen [19].

Let $\text{argmin}\{F(x)|x \in R^n\}$ and $\text{argmin}_w\{F(x)|x \in R^n\}$ denote the local pareto solution set and the local weak pareto solution set to the problem (1). It is easy to see that $\text{argmin}\{F(x)|x \in R^n\} \subset \text{argmin}_w\{F(x)|x \in R^n\}$.

Definition 3.3 *A real-valued function $f : R^n \rightarrow R$ is said to be a **strict scalar representation** of a map $F = (F_1, \dots, F_m) : R^n \rightarrow R^m$ when given $x, \bar{x} \in R^n$*

$$F_i(x) \leq F_i(\bar{x}), \forall i = 1, \dots, m \implies f(x) \leq f(\bar{x})$$

and

$$F_i(x) < F_i(\bar{x}), \forall i = 1, \dots, m \implies f(x) < f(\bar{x}).$$

*Futhermore, f is said to be a **weak scalar representation** of F if*

$$F_i(x) < F_i(\bar{x}), \forall i = 1, \dots, m \implies f(x) < f(\bar{x}).$$

It is obvious that all strict scalar representations are weak scalar representations. The next result demonstrates an interesting way to obtain scalar representation for applications. To demonstrate, see [11], proposition 1.

Proposition 3.1 *Let $f : R^n \rightarrow R$ be a function. f is a strict scalar representation of F if, and only if f is a composition of F with a strictly increasing function $g : F(R^n) \rightarrow R$.*

According to Proposition 2.9 in Luc [17], to obtain the convexity of the scalar problem, it is necessary to choose a convex increasing function g from $F(R^n)$ to R . The next result establishes an important relation between the sets $\text{argmin}\{f(x)|x \in R^n\}$ and $\text{argmin}_w\{F(x)|x \in R^n\}$. The Proof follows immediately from the Definition 3.3.

Proposition 3.2 *Let $f : R^n \rightarrow R$ be a weak scalar representation of a map $F : R^n \rightarrow R^m$ and $\text{argmin}\{f(x)|x \in R^n\}$ the local minimizer set of f . Therefore:*

$$\text{argmin}\{f(x)|x \in R^n\} \subset \text{argmin}_w\{F(x)|x \in R^n\}.$$

4 Logarithmic Quasi-Distance Proximal point Scalarization (LQDPS) Method

Gregório and Oliveira [11] showed the existence of a function $f : R^n \times R_+^m \longrightarrow R$ that exhibits the following properties:

(P1) f is bounded below for any $\alpha \in R$, i.e., $f(x, z) \geq \alpha$ for every $(x, z) \in R^n \times R_+^m$;

(P2) f is convex in $R^n \times R_+^m$, i.e., given $(x_1, z_1), (x_2, z_2) \in R^n \times R_+^m$ and $\lambda \in (0, 1)$

$$f(\lambda(x_1, z_1) + (1 - \lambda)(x_2, z_2)) \leq \lambda f(x_1, z_1) + (1 - \lambda)f(x_2, z_2);$$

(P3) f is a strict scalar representation of F , with respect to x , i.e.,

$$F_i(x) \leq F_i(y) \quad \forall i = 1, \dots, m \Rightarrow f(x, z) \leq f(y, z)$$

and

$$F_i(x) < F_i(y) \quad \forall i = 1, \dots, m \Rightarrow f(x, z) < f(y, z)$$

for every $x, y \in R^n$ and $z \in R_+^m$;

(P4) f is differentiable, with respect to z and

$$\frac{\partial}{\partial z} f(x, z) = h(x, z),$$

where $h(x, z) = (h_1(x, z), \dots, h_m(x, z))^T$ is a continuous map from $R^n \times R^m$ to R_+^m , i.e., $h_i(x, z) \geq 0$ for all $i = 1, \dots, m$.

More precisely, they showed that the function $f : R^n \times R_+^m \longrightarrow R$ such that

$$f(x, z) = \sum_{i=1}^m \exp(z_i + F_i(x)) \quad (6)$$

satisfies the properties (P1) to (P4). As another example, consider the following proposition:

Proposition 4.1 *Let $F = (F_1, F_2, \dots, F_m) : R^n \rightarrow R^m$ be a convex application, then $f : R^n \times R_+^m \rightarrow R$ such that $f(x, z) = \sum_{i=1}^m [z_i + h(F_i(x))]$ where*

$$h(F_i(x)) = \begin{cases} \frac{1}{2 - F_i(x)} & \text{if } F_i(x) \leq 1 \\ (F_i(x))^2 & \text{if } F_i(x) > 1 \end{cases} \text{ satisfies the properties (P1) to (P4).}$$

Proof. It is easy to see that $h : R \rightarrow R$ given by $h(x) = \begin{cases} \frac{1}{2-x} & \text{if } x \leq 1 \\ x^2 & \text{if } x > 1 \end{cases}$ is positive ($h > 0$), convex and strictly increasing. It will now be shown that $f(x, z) = \sum_{i=1}^m [z_i + h(F_i(x))]$ satisfies the properties (P1) to (P4). **(P1):**

$h(x) > 0 \forall x \in R$ and $z_i \geq 0 \forall i = 1, \dots, m$ implies $f(x, z) > 0 \forall (x, z) \in R^n \times R_+^m$. **(P2):** Be $(x, z), (\bar{x}, \bar{z}) \in R^n \times R_+^m$ and $\alpha \in [0, 1]$. Then, as F_i is convex $\forall i = 1, \dots, m$ and h is strictly increasing and convex:

$$\begin{aligned}
f(\alpha(x, z) + (1 - \alpha)(\bar{x}, \bar{z})) &= \sum_{i=1}^m [\alpha z_i + (1 - \alpha)\bar{z}_i + h(F_i(\alpha x + (1 - \alpha)\bar{x}))] \\
&\leq \sum_{i=1}^m [\alpha z_i + (1 - \alpha)\bar{z}_i + h(\alpha F_i(x) + (1 - \alpha)F_i(\bar{x}))] \\
&\leq \sum_{i=1}^m [\alpha z_i + (1 - \alpha)\bar{z}_i + \alpha h(F_i(x)) + (1 - \alpha)h(F_i(\bar{x}))] \\
&= \alpha \sum_{i=1}^m (z_i + h(F_i(x))) + (1 - \alpha) \sum_{i=1}^m (\bar{z}_i + h(F_i(\bar{x}))) \\
&= \alpha f(x, z) + (1 - \alpha)f(\bar{x}, \bar{z}).
\end{aligned}$$

(P3): Consider $\bar{z} \in R_+^m$ fixed. As h is strictly increasing, $F_i(x) \leq F_i(y) \forall i = 1, \dots, m$ implies $\bar{z}_i + h(F_i(x)) \leq \bar{z}_i + h(F_i(y))$, $\forall i = 1, \dots, m$ and therefore, $\sum_{i=1}^m [\bar{z}_i + h(F_i(x))] \leq \sum_{i=1}^m [\bar{z}_i + h(F_i(y))]$, i.e., $f(x, \bar{z}) \leq f(y, \bar{z})$. Similarly $F_i(x) < F_i(y) \forall i = 1, \dots, m$ implies $f(x, \bar{z}) < f(y, \bar{z})$. **(P4)** It can be easily shown that $\frac{\partial}{\partial z} f(x, z) = (1, 1, \dots, 1)$. ■

Notation: Let $y, \bar{y} \in R^m$, then $y \leq \bar{y} \iff y_i \leq \bar{y}_i \forall i = 1, \dots, m$ and $y \ll \bar{y} \iff y_i < \bar{y}_i \forall i = 1, \dots, m$.

The (LQDPS) Method:

Let $F : R^n \rightarrow R^m$ be convex and $q : R^n \times R^n \rightarrow R_+$ a quasi-distance application, satisfying (5). Given the initial points $x^0 \in R^n$, $z^0 \in R_{++}^m$ and sequences $\beta^k > 0, k = 0, 1, \dots$ and $0 < l < \mu^k < L \forall k = 1, 2, \dots$, the logarithmic quasi-distance proximal point scalarization (LQDPS) method generates sequences $\{x^k\}_{k \in N} \subset R^n$ and $\{z^k\}_{k \in N} \subset R_{++}^m$ with the iterates x^{k+1} and z^{k+1} defined as the solution of the (LQDPS) problem

$$\min \varphi^k(x, z) = f(x, z) + \beta^k \langle \frac{z}{z^k} - \log \frac{z}{z^k} - e, e \rangle + \frac{\mu^k}{2} q^2(x, x^k), \quad (7)$$

$$x \in \Omega^k, z \in R_{++}^m,$$

where $f : R^n \times R_+^m \rightarrow R$ satisfies the properties (P1) to (P4), $\frac{z}{z^k}$ and $\log \frac{z}{z^k}$ which are the vectors whose i th components are given by $\frac{z_i}{z_i^k}$ and $\frac{z_i}{z_i^k}$

$\log \frac{z_i}{z_i^k}$, respectively, $e \in R^m$ is the vector with all components equal to 1 and $\Omega^k = \{x \in R^n | F(x) \leq F(x^k)\}$.

4.1 Well-posedness

The function $\varphi^k : R^n \times R_{++}^m \rightarrow R$ in (7), was considered by Gregório and Oliveira [11] with a variant of the logarithm-quadratic function of Auslender et al. [1] as regularization and, in that case, due to the strict convexity of the function φ^k , they showed that the method's iterations are unique and within the constraints. As, in the present problem, the quasi-distance is not necessarily a convex function, we will not be able to demonstrate the uniqueness of the iterations, nor that the iterations x^{k+1} are within the constraints Ω^k . Therefore, we will have to act differently to assure that the sequences are well-defined and also to find their characterizations.

Lemma 4.1 *Let $F : R^n \rightarrow R^m$ be a convex map such that there exists $r \in \{1, \dots, m\}$ satisfying $\lim_{\|x\| \rightarrow \infty} F_r(x) = \infty$. Then $\Omega^k, \forall k \in N$ is a convex and compact set. Particularly, $\Omega^k \times R_+^m$ is a convex and closed set.*

Proof. Suppose, for contradiction that $\Omega^0 = \{x \in R^n | F(x) \leq F(x^0)\}$ is unbounded. Then there is $\{x_n\}_{n \in N} \subset \Omega^0$ such that $\|x_n\| \rightarrow \infty$ when $n \rightarrow \infty$. As $\{x_n\}_{n \in N} \subset \Omega^0$, then $F(x_n) \leq F(x^0) \forall n \in N$, and then, $F_i(x_n) \leq F_i(x^0), \forall i = 1, \dots, m$ and $n \in N$. Therefore, in particular, $F_r(x_n) \leq F_r(x^0) \forall n \in N$. Since F_r is coercive and $\|x_n\| \rightarrow \infty$ when $n \rightarrow \infty$, then " $\infty \leq F_r(x^0) < \infty$ ", which is a contradiction. So Ω^0 is limited. As $\Omega^{k+1} \subseteq \Omega^k, k \geq 0$, it follows that $\Omega^k \subseteq \Omega^0, k \geq 1$ and therefore Ω^k is limited $\forall k \geq 0$. The convexity of F implies its continuity and the convexity of $\Omega^k, \forall k$. It follows from the continuity of F that $\Omega^k, \forall k$ is closed. Therefore, $\Omega^k \forall k$ is a compact convex set. ■

Remark 4.1 *Since $\Omega^{k+1} \subseteq \Omega^k, \forall k \in N$ and $\Omega^k, \forall k \in N$ is a compact set, then: $\bigcap_{k=0}^{\infty} \Omega^k \neq \emptyset$.*

Lemma 4.2 *The function $H : R_{++}^m \rightarrow R$, such that*

$$H(z) = \left\langle \frac{z}{z^k} - \log \frac{z}{z^k} - e, e \right\rangle = \left\| \frac{z}{z^k} - \log \frac{z}{z^k} - e \right\|_1$$

where $\|\bullet\|_1$ is the 1-norm on R^m defined by $\|z\|_1 = \sum_{i=1}^m |z_i|$, is strictly convex, non-negative and coercive.

Proof. See [11], demonstration of Lemma 1. ■

Proposition 4.2 (Well-posedness) *Let $F : R^n \rightarrow R^m$ be a convex map such that there exists $r \in \{1, \dots, m\}$ satisfying $\lim_{\|x\| \rightarrow \infty} F_r(x) = \infty$, $q : R^n \times R^n \rightarrow R_+$ a quasi-distance map satisfying (5) and $f : R^n \times R_+^m \rightarrow R$ be a function satisfying the properties (P1) to (P4). Then, for every $k \in N$, there is one solution (x^{k+1}, z^{k+1}) for the (LQDPS) problem.*

Proof. The function $\varphi^k : \Omega^k \times R_{++}^m \rightarrow R$ is coercive. In fact, for (P1) we have:

$$\begin{aligned} \varphi^k(x, z) &= f(x, z) + \beta^k \left\langle \frac{z}{z^k} - \log \frac{z}{z^k} - e, e \right\rangle + \frac{\mu^k}{2} q^2(x, x^k) \\ &\geq \alpha + \beta^k (\| \frac{z}{z^k} - \log \frac{z}{z^k} - e \|_1) + \frac{\mu^k}{2} q^2(x, x^k). \end{aligned} \quad (8)$$

Let us define $\|(x, z)\| = \|x\| + \|z\|$ and suppose that $\|(x, z)\| \rightarrow \infty$. This is the same as $\|x\| \rightarrow \infty$ or $\|z\| \rightarrow \infty$. As Ω^k is compact (see lemma 4.1) and the function $\| \frac{z}{z^k} - \log \frac{z}{z^k} - e \|_1$ is coercive in R_{++}^m (see lemma 4.2), it follows from (8) that φ^k is coercive in $\Omega^k \times R_{++}^m$.

The function $\varphi^k : R^n \times R_{++}^m \rightarrow R$ is continuous in $R^n \times R_{++}^m$. In fact, (P2) implies f is continuous in $R^n \times R_{++}^m$. The lemma 4.2 implies $H(z) = \langle \frac{z}{z^k} - \log \frac{z}{z^k} - e, e \rangle$ continuous in R_{++}^m . As a consequence of proposition 2.1, $q^2(\cdot, x^k) : R^n \rightarrow R$ is a continuous application in R^n . Therefore, the function $\varphi^k : R^n \times R_{++}^m \rightarrow R \cup \{+\infty\}$ is continuous in $R^n \times R_{++}^m$.

As $\varphi^k : \Omega^k \times R_{++}^m \rightarrow R$ is continuous, coercive and proper in $\Omega^k \times R_{++}^m$, we have that the set $\operatorname{argmin}\{\varphi^k(x, z) / (x, z) \in \Omega^k \times R_{++}^m\}$ is not empty, i.e., for every k , there is a solution (x^{k+1}, z^{k+1}) to the (LQDPS) problem. ■

Definition 4.1 *Let $C \subset R^n$ be a convex set and $\bar{x} \in C$. The normal cone (cone of normal directions) at the point \bar{x} related to the set C is given by*

$$N_C(\bar{x}) = \{v \in R^n \mid \langle v, x - \bar{x} \rangle \leq 0 \quad \forall x \in C\}.$$

Corollary 4.1 (Characterization)

The solutions (x^{k+1}, z^{k+1}) of the (LQDPS) problem are characterized by:

- (i) There are $\xi^{k+1} \in \partial f(\cdot, z^{k+1})(x^{k+1})$, $\zeta^{k+1} \in \partial(q(\cdot, x^k))(x^{k+1})$ and $v^{k+1} \in N_{\Omega^k}(x^{k+1})$ such that

$$\xi^{k+1} = -\mu^k q(x^{k+1}, x^k) \zeta^{k+1} - v^{k+1} \quad (9)$$

and

(ii)

$$\frac{1}{z_i^{k+1}} - \frac{1}{z_i^k} = \frac{h_i(x^{k+1}, z^{k+1})}{\beta^k}, \quad i = 1, \dots, m. \quad (10)$$

$$x^{k+1} \in \Omega^k, z^{k+1} \in R_{++}^m$$

Proof.

From observation 2.1 we have

$$0 \in \partial \left(f(\cdot, z^{k+1}) + \beta^k \left\langle \frac{z^{k+1}}{z^k} - \log \frac{z^{k+1}}{z^k} - e, e \right\rangle + \frac{\mu^k}{2} q^2(\cdot, x^k) + \delta_{\Omega^k} \right) (x^{k+1}). \quad (11)$$

From (P2), $f(\cdot, z^{k+1}) + \beta^k \left\langle \frac{z^{k+1}}{z^k} - \log \frac{z^{k+1}}{z^k} - e, e \right\rangle$ is continuous in x^{k+1} ; from proposition 2.1, $\frac{\mu^k}{2} q^2(\cdot, x^k)$ is locally Lipschitz in x^{k+1} ; the convexity of Ω^k implies the convexity of δ_{Ω^k} and therefore that δ_{Ω^k} is locally Lipschitz. Therefore, using the proposition 2.4 in (11) and remembering that

$$\beta^k \left\langle \frac{z^{k+1}}{z^k} - \log \frac{z^{k+1}}{z^k} - e, e \right\rangle$$

is constant in relation to Ω^k , we obtain

$$0 \in \partial \left(f(\cdot, z^{k+1}) \right) (x^{k+1}) + \partial \left(\frac{\mu^k}{2} q^2(\cdot, x^k) \right) (x^{k+1}) + \partial (\delta_{\Omega^k}) (x^{k+1}). \quad (12)$$

As Ω^k is closed and convex, it follows $\partial (\delta_{\Omega^k}(\cdot)) (x^{k+1}) = N_{\Omega^k}(x^{k+1})$, where $N_{\Omega^k}(x^{k+1})$ denotes the normal cone at the point x^{k+1} in relation to the set Ω^k (see def. 4.1). From the proposition 2.1, $q(\cdot, x^k)$ is Lipschitz continuous in R^n . Therefore, taking $f_1 = f_2 = q$ in the proposition 2.5, we have from (12) that

$$0 \in \partial (f(\cdot, z^{k+1})) (x^{k+1}) + \mu^k q(x^{k+1}, x^k) \partial (q(\cdot, x^k)) (x^{k+1}) + N_{\Omega^k}(x^{k+1}),$$

i.e., there are $\xi^{k+1} \in \partial f(\cdot, z^{k+1})(x^{k+1})$, $\zeta^{k+1} \in \partial (q(\cdot, x^k))(x^{k+1})$ and $v^{k+1} \in N_{\Omega^k}(x^{k+1})$ such that

$$\xi^{k+1} = -\mu^k q(x^{k+1}, x^k) \zeta^{k+1} - v^{k+1}.$$

To end the demonstration, observe (see [11], Lemma 1) that

$$\frac{1}{z_i^{k+1}} - \frac{1}{z_i^k} = \frac{h_i(x^{k+1}, z^{k+1})}{\beta^k}, \quad i = 1, \dots, m.$$

$$x^{k+1} \in \Omega^k, z^{k+1} \in R_{++}^m$$

■

4.2 STOP CRITERION

As in Gregório and Oliveira [11], we will establish the same stopping rule as was used by Bonnel et al in [2].

Proposition 4.3 (Stop criterion) *Let $\{(x^k, z^k)\}_{k \in N}$ be the sequence generated by the (LQDPS) method. If $(x^{k+1}, z^{k+1}) = (x^k, z^k)$ for any integer k then x^k is a weak pareto solution for the unconstrained multi-objective optimization problem (1).*

Proof. Now, suppose that the stopping rule is satisfied in the k th iteration. By contradiction, admit that x^k is not a weak pareto solution. Then, there is $\bar{x} \in R^n$ such that $F(\bar{x}) \ll F(x^k)$. From (P3) we have

$$f(\bar{x}, z^k) < f(x^k, z^k).$$

This implies that there exists $\alpha > 0$ such that $f(\bar{x}, z^k) = f(x^k, z^k) - \alpha$. Define $x_\lambda = \lambda x^k + (1 - \lambda)\bar{x}$, $\lambda \in (0, 1)$. Then we have that

$$(x_\lambda, z^k) = \lambda(x^k, z^k) + (1 - \lambda)(\bar{x}, z^k).$$

Since (x^{k+1}, z^{k+1}) solves the (LQDPS) problem, $(x^{k+1}, z^{k+1}) = (x^k, z^k)$, $q^2(x^k, x^k) = 0$ and $x_\lambda \in \Omega^k$, $\forall \lambda \in (0, 1)$, we obtain,

$$f(x^k, z^k) \leq f(x_\lambda, z^k) + \frac{\mu^k}{2} q^2(x_\lambda, x^k), \quad \forall \lambda \in (0, 1).$$

From (5), we have,

$$f(x^k, z^k) \leq f(x_\lambda, z^k) + \frac{\mu^k}{2} \beta^2 \|x_\lambda - x^k\|^2, \quad \forall \lambda \in (0, 1). \quad (13)$$

As $x_\lambda - x^k = (1 - \lambda)(\bar{x} - x^k)$, of (13), then

$$f(x^k, z^k) \leq f(x_\lambda, z^k) + \frac{\mu^k}{2} \beta^2 (1 - \lambda)^2 \|\bar{x} - x^k\|^2, \quad \forall \lambda \in (0, 1). \quad (14)$$

On the other hand, the convexity of f implies that

$$\begin{aligned} f(x_\lambda, z^k) &\leq \lambda f(x^k, z^k) + (1 - \lambda) f(\bar{x}, z^k) \\ &= \lambda f(x^k, z^k) + (1 - \lambda) (f(x^k, z^k) - \alpha) \\ &= f(x^k, z^k) - (1 - \lambda) \alpha. \end{aligned} \quad (15)$$

From (14) and (15), $f(x^k, z^k) \leq f(x^k, z^k) - (1 - \lambda) \alpha + \frac{\mu^k}{2} \beta^2 (1 - \lambda)^2 \|\bar{x} - x^k\|^2$. So

$$\alpha \leq (1 - \lambda) \frac{\mu^k}{2} \beta^2 \|\bar{x} - x^k\|^2, \quad \forall \lambda \in (0, 1).$$

Hence, $\alpha \leq \lim_{\lambda \rightarrow 1^-} (1 - \lambda) \frac{\mu^k}{2} \beta^2 \|\bar{x} - x^k\|^2$, and therefore, $\alpha \leq 0$, which is a contradiction. Therefore, x^k is a weak pareto solution for the unconstrained multi-objective optimization problem (1). ■

4.3 CONVERGENCE

Based on Fliege and Svaiter [9], Gregório and Oliveira [11], assuming that Ω^0 is limited, established the convergence of the log-quadratic proximal scalarization method. In this study, we assume that one of the objective functions is coercive. Consequently, Ω^0 is limited (see the proof of lemma 4.1).

Proposition 4.4 *Let $\{(x^k, z^k)\}_{k \in N}$ be a sequence generated by the (LQDPS) Method. Then (i) $\{x^k\}_{k \in N}$ is bounded; (ii) $\{z^k\}_{k \in N}$ is convergent; (iii) $\{f(x^k, z^k)\}_{k \in N}$ is noincreasing and convergent.*

Proof. (i) Since $\Omega^k \supseteq \Omega^{k+1}$, $k = 0, 1, \dots$, we have $x^k \in \Omega^{k-1} \subseteq \Omega^0 \quad \forall k \geq 1$. As Ω^0 is limited, it follows that $\{x^k\}$ is limited.
(ii) Since $h_i(x, z) \geq 0$, $\beta^k > 0$ and $\{z_i^k\}_{k \in N}$ is bounded below, Equation (10) implies $\{z^k\}_{k \in N}$ is convergent (see [11], proof of theorem 1).
(iii) $\varphi^k(x^{k+1}, z^{k+1}) \leq \varphi^k(x^k, z^k)$, $\forall k \in N$, i.e, for every $k \in N$,

$$f(x^{k+1}, z^{k+1}) + \beta^k \left\langle \frac{z^{k+1}}{z^k} - \log \frac{z^{k+1}}{z^k} - e, e \right\rangle + \frac{\mu^k}{2} q^2(x^{k+1}, x^k) \leq f(x^k, z^k). \quad (16)$$

As $\beta^k \left\langle \frac{z^{k+1}}{z^k} - \log \frac{z^{k+1}}{z^k} - e, e \right\rangle + \frac{\mu^k}{2} q^2(x^{k+1}, x^k) \geq 0 \quad \forall k \in N$, we have,

$$f(x^{k+1}, z^{k+1}) \leq f(x^k, z^k) \quad \forall k \in N,$$

i.e., $\{f(x^k, z^k)\}_{k \in N}$ is a noincreasing sequence. For (P1), $\{f(x^k, z^k)\}$ is bounded below, and therefore convergent. ■

Proposition 4.5 *Let $\{x^k\}_{k \in N}$ be a sequence generated by (LQDPS) Method. Then*

- (i) $\sum_{k=0}^{\infty} q^2(x^{k+1}, x^k) < \infty$. In particular $\lim_{k \rightarrow \infty} q^2(x^{k+1}, x^k) = 0$.
- (ii) $\lim_{k \rightarrow \infty} \|x^k - x^{k+1}\| = 0$.

Proof. (i) As $\beta^k \left\langle \frac{z^{k+1}}{z^k} - \log \frac{z^{k+1}}{z^k} - e, e \right\rangle \geq 0$, of (16) we have:

$$f(x^{k+1}, z^{k+1}) + \frac{\mu^k}{2} q^2(x^{k+1}, x^k) \leq f(x^k, z^k), \quad \forall k \in N.$$

Hence,

$$\begin{aligned} q^2(x^{k+1}, x^k) &\leq \frac{2}{\mu^k} \left(f(x^k, z^k) - f(x^{k+1}, z^{k+1}) \right), \quad \forall k \in N \\ &\leq \frac{2}{l} \left(f(x^k, z^k) - f(x^{k+1}, z^{k+1}) \right), \quad \forall k \in N. \end{aligned}$$

Therefore, as long as $\{f(x^k, z^k)\}_{k \in N}$ is nonincreasing and convergent,

$$\sum_{k=0}^n q^2(x^{k+1}, x^k) \leq \frac{2}{l} \left(f(x^0, z^0) - \lim_{k \rightarrow \infty} f(x^{k+1}, z^{k+1}) \right) < \infty \quad \forall n \in N$$

(ii) (5) implies $\alpha^2 \|x^k - x^{k+1}\|^2 \leq q^2(x^{k+1}, x^k)$, $\forall k \in N$. So from (i), $\lim_{k \rightarrow \infty} \|x^k - x^{k+1}\| = 0$. ■

Proposition 4.6 *If $\{x^k\}_{k \in N}$ is bounded, then the set $\partial(q(\cdot, x^k))(x^{k+1})$ is bounded for every $k \in N$.*

Proof. It follows from the propositions 2.6 and 2.7, see [20] Lemma 5.1. ■

Now, we can prove the convergence of our method if the stopping rule never applies.

Theorem 4.1 (convergence) *Let $F : R^n \rightarrow R^m$ be a convex map such that $\lim_{\|x\| \rightarrow \infty} F_r(x) = \infty$ for some $r \in \{1, \dots, m\}$, $f : R^n \times R_+^m \rightarrow R$ be a function satisfying the properties (P1) to (P4), and $q : R^n \times R^n \rightarrow R_+$ be a quasi-distance function that satisfies (5). If $\{\mu^k\}_{k \in N}$ and $\{\beta^k\}_{k \in N}$ are sequences of real positive numbers, with $0 < l < \mu^k < L, \forall k \in N$, then the sequence $\{(x^k, z^k)\}_{k \in N}$ generated by the logarithmic quasi-distance proximal point scalarization method is bounded and each cluster point of $\{x^k\}_{k \in N}$ is a weak pareto solution for the unconstrained multi-objective optimization problem (1).*

Proof. From Proposition 4.4, there are $x^* \in R^n$, $z^* \in R_+^m$ and $\{x^{k_j}\}_{j \in N}$, a subsequence of $\{x^k\}_{k \in N}$, such that $\lim_{j \rightarrow \infty} x^{k_j} = x^*$ and $\lim_{k \rightarrow \infty} z^k = z^*$. From (P2) and (P4) f is continuous in $R^n \times R_+^m$, so $f(x^*, z^*) = \lim_{k \rightarrow \infty} f(x^{k_j}, z^{k_j}) = \inf_{k \in N} \{f(x^k, z^k)\}$. From corollary 4.1(i), there are $\zeta^{k+1} \in \partial(q(\cdot, x^k))(x^{k+1})$ and $v^{k+1} \in N_{\Omega^k}(x^{k+1})$ such that

$$-\mu^k q(x^{k+1}, x^k) \zeta^{k+1} - v^{k+1} \in \partial f(., z^{k+1})(x^{k+1}).$$

Hence, from subgradient inequality for the convex function $f(., z^{k+1})$ we have: $\forall x \in R^n$,

$$\begin{aligned} f(x, z^{k+1}) &\geq f(x^{k+1}, z^{k+1}) - \mu^{k+1} q(x^{k+1}, x^k) < \zeta^{k+1}, x - x^{k+1} > \\ &- < v^{k+1}, x - x^{k+1} > \end{aligned} \quad (17)$$

As $v^{k+1} \in N_{\Omega^{k+1}}(x^{k+1})$ we have $- < v^{k+1}, x - x^{k+1} > \geq 0 \quad \forall x \in \Omega^{k+1}$ (See definition 4.1). By remark 4.1, $\Omega = \bigcap_{k=0}^{\infty} \Omega^k \neq \emptyset$. Therefore, in particular, from (17): $\forall x \in \Omega$,

$$f(x, z^{k+1}) \geq f(x^{k+1}, z^{k+1}) - \mu^{k+1} q(x^{k+1}, x^k) < \zeta^{k+1}, x - x^{k+1} > \quad (18)$$

From Propositions 4.5 and 4.6, $\lim_{k \rightarrow \infty} \|x^{k+1} - x^{k+1}\| = 0$ and $\|\zeta^{k+1}\| \leq M$ respectively. As $0 < l < \mu^k < L, \forall k \in N$, using (5) and inequality of Cauchy-Swartz, $|\mu^{k+1} q(x^{k+1}, x^k) < \zeta^{k+1}, x - x^{k+1} >| \rightarrow 0$ when $j \rightarrow \infty$. Therefore from (18),

$$f(x, z^*) \geq f(x^*, z^*), \quad \forall x \in \Omega. \quad (19)$$

We will now show that $x^* \in \operatorname{argmin}_w \{F(x)/x \in R^n\}$. Suppose, by contradiction, that there is $\bar{x} \in R^n$ such that $F(\bar{x}) \ll F(x^*)$. As $z^* \in R_+^m$, for (P3),

$$f(\bar{x}, z^*) < f(x^*, z^*). \quad (20)$$

As $\Omega^{k+1} \subseteq \Omega^k, \forall k \geq 0$ and $x^{k+1} \in \Omega^{k+1}, \forall j$ with $x^{k+1} \rightarrow x^*; j \rightarrow \infty$, then $x^* \in \Omega$, i.e., $F(x^*) \leq F(x^k), \forall k \in N$. Hence, $F(\bar{x}) \ll F(x^k), \forall k \in N$, i.e., $\bar{x} \in \Omega$, which contradicts (19) and (20). ■

5 Regularization with the quasi-distance, q

In this section it will be shown that, using the (*LQDPS*) method with the quasi-distance q as regularization in place of q^2 , the result as to the existence of the iterations remains valid and the sequence generated by the algorithm is limited. To ensure convergence, we assume that the sequence of parameters $\{\mu^k\}$ satisfies: $\mu^k > 0 \quad \forall k = 1, 2, \dots$ and $\mu^k \rightarrow 0$ when $k \rightarrow \infty$, thus confirming the importance of regularization with the quasi-distance q^2 . A different stop criterion will be used.

Well-posedness: Let $\bar{x} \in R^n$ fixed. As $q(., \bar{x}) : R^n \rightarrow R$ is continuous and coercive (see Propositions 2.1 and 2.2), the proposition 4.2 is still valid if we replace q^2 by q in (7).

Characterization: The item (i) in the corollary 4.1 will be replaced by:

- (i-1) There are $\xi^{k+1} \in \partial f(., z^{k+1})(x^{k+1})$, $\zeta^{k+1} \in \partial(q(., x^k))(x^{k+1})$ and $v^{k+1} \in N_{\Omega^k}(x^{k+1})$ such that

$$\xi^{k+1} = -\frac{\mu^k}{2}\zeta^{k+1} - v^{k+1}. \quad (21)$$

Proof: As, from Proposition 2.1, $\frac{\mu^k}{2}q(., x^k)$ is Lipschitz continuous in x^{k+1} , the proof of (i-1) is similar to the proof of (i) in the corollary 4.1. The item (ii) in the corollary 4.1 will be the same.

Stop Criterion: We will use the following stop criterion:

Proposition 5.1 (Stop Criterion - Case q) *Let $\{(x^k, z^k)\}_{k \in N}$ be the sequence generated by the (LQDPS) method with q instead of q^2 . If there is $\tilde{z} \in R_+^m$ such that*

$$x^{k+1} \in \operatorname{argmin}\{f(., \tilde{z}); x \in \Omega^k\} \text{ for any integer } k \geq 0,$$

then x^{k+1} is a weak pareto solution for the unconstrained multi-objective optimization problem (1).

Proof. As $x^{k+1} \in \operatorname{argmin}\{f(., \tilde{z}); x \in \Omega^k\}$, then

$$f(x^{k+1}, \tilde{z}) \leq f(x, \tilde{z}) \quad \forall x \in \Omega^k. \quad (22)$$

Suppose, for contradiction, that there is $\bar{x} \in R^n$ such that $F(\bar{x}) \ll F(x^{k+1})$. Then, $\bar{x} \in \Omega^k$ ($\Omega^{k+1} \subseteq \Omega^k \forall k \geq 0$) and for (P3), $f(\bar{x}, \tilde{z}) < f(x^{k+1}, \tilde{z})$, which contradicts (22). So $x^{k+1} \in \operatorname{argmin}_w \{F(x)/x \in R^n\}$. ■

Convergence: Let $\{(x^k, z^k)\}_{k \in N}$ be a sequence generated by (LQDPS) Method with q instead of q^2 . Proposition 4.4 is still valid if we replace q^2 by q in (7). In fact, clearly Proposition 4.4(i) and Proposition 4.4(ii) are still valid. As $q(x^k, x^k) = 0$ and $q(x, y) \geq 0 \forall (x, y) \in R^n \times R^n$, if we replace q^2 by q in (7) the proof of (iii) is similar to the case q^2 . Then (i)', $\{x^k\}_{k \in N}$ is bounded; (ii)', $\{z^k\}_{k \in N}$ is convergent; (iii)', $\{f(x^k, z^k)\}_{k \in N}$ is non-increasing and convergent.

Suppose that the sequence of parameters $\{\mu^k\}$ satisfies: $\mu^k > 0 \forall k = 1, 2, \dots$

and $\mu^k \rightarrow 0$ when $k \rightarrow \infty$. We will show that the Theorem 4.1 (convergence) is still valid if we replace q^2 by q in (7). Let $\{x^{k_j}\}_{j \in N}$ be a subsequence of $\{x^k\}_{k \in N}$ that satisfies $\lim_{j \rightarrow \infty} x^{k_j} = x^*$ e $z^* \in R_+^m$ such that $\lim_{j \rightarrow \infty} z^{k_j} = z^*$. It is easy to see that, if (21) is true, the inequality (18) becomes

$$f(x, z^{k_j+1}) \geq f(x^{k_j+1}, z^{k_j+1}) - \frac{\mu^k}{2} < \zeta^{k_j+1}, x - x^{k_j+1} >, \forall x \in \Omega. \quad (23)$$

From (P2) and (P4), f is continuous in $R^n \times R_+^m$, so that $f(x^*, z^*) = \lim_{k \rightarrow \infty} f(x^{k_j}, z^{k_j}) = \inf_{k \in N} \{f(x^k, z^k)\}$. Therefore from (iii), $\lim_{k \rightarrow \infty} f(x^{k_j+1}, z^{k_j+1}) = f(x^*, z^*)$. As $\|\zeta^k\| \leq M$ (see Prop. 4.6), $\lim_{j \rightarrow \infty} \mu^k = 0$ and $\|x - x^{k+1}\| \leq M_1$, using Cauchy-Schwarz inequality, we conclude that $|\frac{\mu^k}{2} < \zeta^{k+1}, x - x^{k+1} >| \rightarrow 0$ when $k \rightarrow \infty$. Therefore, from (23), $f(x, z^*) \geq f(x^*, z^*)$, $\forall x \in \Omega^k$, and then, similarly to the end of the demonstration of Theorem 4.1, we conclude that $x^* \in \operatorname{argmin}_w \{F(x)/x \in R^n\}$.

6 Numerical examples

In this section we will implement the (*LQDPS*) method given in section 4. All numerical experiments were performed using an Intel(R) Core(TM) 2 Duo with Windows 7. The source code is written in Matlab. We tested our method taking into account three multi-objective test functions presented by Li and Zhang in [16], that is, we took into account the following functions:

(a) ([16], **function F1, pg. 287**): $F_a = (F_a^1, F_a^2) : R^3 \rightarrow R^2$ given by $F_a^1 = x_1 + 2(x_3 - x_1^2)^2$, $F_a^2 = 1 - \sqrt{x_1} + 2(x_2 - x_1^{0.5})^2$ and $x_i \in [0, 1]$, $i = 1, 2, 3$ with the set of all Pareto optimal points (PS) given by $x_2 = x_1^{0.5}$ and $x_3 = x_1^2$, $x_1 \in [0, 1]$.

(b) ([16], **function F4, pg. 287**): $F_b = (F_b^1, F_b^2) : R^3 \rightarrow R^2$ given by $F_b^1 = x_1 + 2(x_3 - 0.8x_1 \cos((6\pi x_1 + \pi)/3))^2$, $F_b^2 = 1 - \sqrt{x_1} + 2(x_2 - 0.8x_1 \sin(6\pi x_1 + 2\pi/3))^2$ and $(x_1, x_2, x_3) \in [0, 1] \times [-1, 1] \times [-1, 1]$ with the set (PS) given by $x_2 = 0.8x_1 \sin(6\pi x_1 + 2\pi/3)$ and $x_3 = 0.8x_1 \cos((6\pi x_1 + \pi)/3)$, $x_1 \in [0, 1]$.

(c) ([16], **function F6, pg. 287**): $F_c = (F_c^1, F_c^2, F_c^3) : R^3 \rightarrow R^3$ given by: $F_c^1 = \cos(0, 5x_1\pi)\cos(0, 5x_2\pi)$, $F_c^2 = \cos(0, 5x_1\pi)\sin(0, 5x_2\pi)$, $F_c^3 = \sin(0, 5x_1\pi) + 2(x_3 - 2x_2 \sin(2\pi x_1 + \pi))^2$ and $(x_1, x_2, x_3) \in [0, 1] \times [0, 1] \times [-2, 2]$ with the set (PS) given by $x_3 = 2x_2 \sin(2\pi x_1 + \pi)$, $(x_1, x_2) \in [0, 1] \times [0, 1]$.

The tests will be performed using the scalarization function proposed in this study (see prop. 4.1) and the scalarization function proposed by Gregório and Oliveira in [11] (see function given by (6)). All tests will consider the quasi-distance application $q : R^n \times R^n \rightarrow R_+$ presented by Moreno et al. in [20]; more specifically, we will consider $q : R^3 \times R^3 \rightarrow R_+$ given by $q(x, y) = \sum_{i=1}^3 q_i(x_i, y_i)$ where $q_i(x_i, y_i) = 3(y_i - x_i)$ se $y_i - x_i > 0$ or $q_i(x_i, y_i) = 2(x_i - y_i)$ if $y_i - x_i \leq 0$.

In the tables, tol denotes the stop criterion tolerance ($\|x^k - x^{k+1}\|_\infty \leq tol$); μ_k, β_k are the parameters of the ($LQDPS$) method; k_i^* , $i = 1, 2$ the number of iterations of the algorithm using the scalarization function $f_i : R^n \times R_+^m \rightarrow R$, $i = 1, 2$ where f_1 is given by Proposition 4.1 and f_2 is given by (6) and $\|x_{k_i^*}^* - x^*\|_\infty$ is the distance between the approximate solution from f_i and the exact solution, i.e., the error committed with the scalarization function f_i . The maximum number of iterations is 100.

Example 6.1 *In this example, we will consider the multi-objective function $F_a : R^3 \rightarrow R^2$ given above, and the initial iterations $x_0 = (0.5, 0.5, 0.5) \in R^3$ and $z_0 = (1, 1) \in R_{++}^2$. The numeric results are shown in the table below.*

No.	tol	μ_k	β_k	k_1^*	$\ x_{k_1^*}^* - x^*\ _\infty$	k_2^*	$\ x_{k_2^*}^* - x^*\ _\infty$
1	10^{-2}	$1 + 1/k$	$1 + 1/k$	9	5.545339e-003	10	5.118762e-002
2	10^{-3}	$1 + 1/k$	$1 + 1/k$	28	6.247045e-009	23	6.979995e-003
3	10^{-4}	$1 + 1/k$	$1 + 1/k$	87	7.960987e-009	62	8.279647e-009
4	10^{-2}	$1 + 1/k$	k	7	1.701151e-002	9	5.726488e-002
5	10^{-3}	$1 + 1/k$	k	28	7.351215e-009	24	6.281176e-003
6	10^{-4}	$1 + 1/k$	k	100	3.576296e-009	41	8.260435e-009
7	10^{-2}	$2 - 1/k$	$1/k$	7	2.273775e-002	8	9.389888e-002
8	10^{-3}	$2 - 1/k$	$1/k$	15	2.790977e-003	32	1.040779e-002
9	10^{-4}	$2 - 1/k$	$1/k$	28	1.071720e-008	100	9.213105e-009
10	10^{-2}	$2 - 1/k$	k	7	1.674806e-002	8	9.413130e-002
11	10^{-3}	$2 - 1/k$	k	27	8.168611e-009	32	1.039107e-002
12	10^{-4}	$2 - 1/k$	k	100	8.096220e-009	65	7.790086e-009
13	10^{-2}	1	1	8	6.966285e-003	9	5.000950e-002
14	10^{-3}	1	1	26	1.906054e-009	23	6.138829e-003
15	10^{-4}	1	1	83	8.254353e-009	39	1.546241e-005

Example 6.2 *In this example we consider the multi-objective function $F_b : R^3 \rightarrow R^2$ given above, and the initial iterations $x_0 = (0.5, 0.5, 0.5) \in R^3$ and $z_0 = (1, 1) \in R_{++}^2$. the numeric results are shown in the table below.*

<i>No.</i>	<i>tol</i>	μ_k	β_k	k_1^*	$\ x_{k_1^*}^* - x^*\ $	k_2^*	$\ x_{k_2^*}^* - x^*\ _\infty$
1	10^{-2}	$1 + 1/k$	$1 + 1/k$	10	4.419117e-003	10	3.800596e-002
2	10^{-3}	$1 + 1/k$	$1 + 1/k$	29	7.617346e-009	20	5.872760e-003
3	10^{-4}	$1 + 1/k$	$1 + 1/k$	92	7.831306e-009	100	8.102059e-009
4	10^{-2}	$1 + 1/k$	k	7	1.423293e-002	10	3.771631e-002
5	10^{-3}	$1 + 1/k$	k	20	4.560126e-009	21	5.533943e-003
6	10^{-4}	$1 + 1/k$	k	98	6.872232e-009	38	1.000619e-007
7	10^{-2}	$2 - 1/k$	$1/k$	7	2.265857e-002	9	6.038495e-002
8	10^{-3}	$2 - 1/k$	$1/k$	15	3.304754e-003	25	8.176106e-003
9	10^{-4}	$2 - 1/k$	$1/k$	28	7.814512e-009	100	7.497307e-009
10	10^{-2}	$2 - 1/k$	k	7	1.251182e-002	7	6.525365e-002
11	10^{-3}	$2 - 1/k$	k	30	8.735791e-009	23	8.117802e-003
12	10^{-4}	$2 - 1/k$	k	100	5.561728e-009	52	7.547563e-009
13	10^{-2}	1	1	8	5.099261e-003	10	4.231940e-002
14	10^{-3}	1	1	27	5.045036e-009	22	5.051759e-003
15	10^{-4}	1	1	88	8.499673e-009	82	9.235203e-010

Example 6.3 In this example we consider the multi-objective function $F_c : R^3 \rightarrow R^3$ given above, and the initial iterations $x_0 = (0.5, 0.5, 0.5) \in R^3$ and $z_0 = (1, 1, 1) \in R_{++}^3$. The numeric results are shown in the table below.

<i>No.</i>	<i>tol</i>	μ_k	β_k	k_1^*	$\ x_{k_1^*}^* - x^*\ $	k_2^*	$\ x_{k_2^*}^* - x^*\ _\infty$
1	10^{-2}	$1 + 1/k$	$1 + 1/k$	10	1.066481e-002	18	5.068830e-002
2	10^{-3}	$1 + 1/k$	$1 + 1/k$	31	1.698174e-008	33	5.315241e-003
3	10^{-4}	$1 + 1/k$	$1 + 1/k$	100	5.432795e-009	100	1.130028e-008
4	10^{-2}	$1 + 1/k$	k	10	9.733382e-003	19	5.908485e-002
5	10^{-3}	$1 + 1/k$	k	28	4.176586e-010	34	2.307318e-007
6	10^{-4}	$1 + 1/k$	k	100	7.086278e-011	35	7.450585e-009
7	10^{-2}	$2 - 1/k$	$1/k$	11	2.653977e-002	20	9.899806e-002
8	10^{-3}	$2 - 1/k$	$1/k$	18	2.046561e-007	47	1.059293e-002
9	10^{-4}	$2 - 1/k$	$1/k$	33	1.161832e-008	100	9.253656e-009
10	10^{-2}	$2 - 1/k$	k	11	1.835990e-002	22	8.862843e-002
11	10^{-3}	$2 - 1/k$	k	28	5.441347e-010	48	1.047200e-002
12	10^{-4}	$2 - 1/k$	k	100	9.476497e-010	75	2.793537e-009
13	10^{-2}	1	1	9	8.326796e-003	17	5.182457e-002
14	10^{-3}	1	1	29	1.343175e-008	32	4.799238e-003
15	10^{-4}	1	1	96	6.882171e-009	100	3.961009e-009

7 Conclusions

Gregório and Oliveira [11] presented an example of a function that satisfies the properties (P1) to (P4). Based on Fliege and Svaiter [9], Gregório and

Oliveira assumed that Ω^0 is bounded, and they established the convergence of the logarithm-quadratic proximal scalarization method.

In this study, we propose adding a condition to one of the objective functions, which limits Ω^0 . We also propose another example of a function that satisfies the properties (P1) to (P4). As a variation of the logarithm-quadratic proximal scalarization method of Gregório and Oliveira [11], we replaced the quadratic term with a quasi-distance, which entails losing important properties, such as convexity and differentiability. Proceeding differently, however, the convergence of the method was proved. Finally, some numerical examples of the (*LQDPS*) method were presented.

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