# A weak finite element method for elliptic problems in one space dimension 

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#### Abstract

We present a weak finite element method for elliptic problems in one space dimension. Our analysis shows that this method has more advantages than the known weak Galerkin method proposed for multi-dimensional problems, for example, it has higher accuracy and the derived discrete equations can be solved locally, element by element. We derive the optimal error estimates in the discrete $H^{1}$-norm, the $L_{2}$-norm and the $L_{\infty}$-norm, respectively. Moreover, some superconvergence results are also given. Finally, numerical examples are provided to illustrate our theoretical analysis.


Keywords: Weak finite element method; stability; optimal error estimate; superconvergence; elliptic problem in one space dimension. 2000 MSC: 65N15, 65N30

## 1. Introduction

Recently, the weak Galerkin finite element method attracts much attention in the field of numerical partial differential equations [1, 2, 3, 4, 5, 6, 6, 8, 9$]$. This method is presented originally by Wang and Ye for solving elliptic problem in multi-dimensional domain [1]. Since then, some modified weak Galerkin methods have also been studied, for example, see [10, 11, 12, 13]. The weak Galerkin method can be considered as an extension of the standard finite element method where classical derivatives are replaced in the variational equation by the weak derivatives defined on weak finite element functions. The main feature of this method is that it allows the use of totally discontinuous finite

[^0]element function and the trace of finite element function on element boundary may be independent with its value in the interior of element. This feature makes this method possess the advantage of the usual discontinuous Galerkin (DG) finite element method [14, 15, 16] and it has higher flexibility than the DG method. The readers are referred to articles [2, 3, 15] for more detailed explanation of this method and its relation with other finite element methods.

In this paper, we present a weak finite element method for general second order elliptic problem in one space dimension:

$$
\left\{\begin{array}{l}
-\left(a_{2}(x) u^{\prime}\right)^{\prime}+a_{1}(x) u^{\prime}+a_{0}(x) u=f(x), x \in(a, b)  \tag{1.1}\\
u(a)=0, u^{\prime}(b)=0
\end{array}\right.
$$

where $a_{2}(x) \geq a_{\text {min }}>0, a_{0}(x) \geq 0$.
We first define the weak derivative and discrete weak derivative on discontinuous function in one dimensional domain. Then, we construct the weak finite element space $S_{h}$ and use it to give the weak finite element approximation to problem (1.1). Though, in some aspects, our method is similar to the original weak Galerkin finite element method proposed for multi-dimensional problem [1], it still has itself features. For example, we impose the single value condition on space $S_{h}$ (see (2.8) and Remark 2.1), this condition can reduce the size of the finite element discrete equations; Next, our space $S_{h}$ admits a weak embedding inequality (see Lemma 3.2), which can be used to derive the $L_{\infty^{-}}$ error estimate on mesh point set; Furthermore, the discrete finite element system of equations derived from our method can be solved locally, element by element, and this local solvability is not feasible for the weak Galerkin method in multi-dimensional space case. Except the usual optimal error estimates in various norms, we also give some surperconvergence results for the weak finite element solution. Numerical results show that our method possesses very high computation accuracy. For example, for finite element polynomial of order $k$, our computation shows that the numerical convergence rates are at least of order $k+2$ in the discrete $H^{1}$-norm, the $L_{2}$-norm and the discrete $L_{\infty}$-norm. Our method also can be applied to solve other partial differential equations in one space dimension.

This paper is organized as follows. In Section 2, we introduce the weak finite element
method for the elliptic problem. In Section 3, the stability of the weak finite element method is analyzed. Section 4 is devoted to the optimal error estimate and superconvergence estimate in various norms. In Section 5, the local solvability of the weak finite element system of equations is discussed and numerical experiments are provided to illustrate our theoretical analysis.

Throughout this paper, we adopt the notations $H^{m}(I)$ to indicate the usual Sobolev spaces on interval $I$ equipped with the norm $\|\cdot\|_{m}=\|\cdot\|_{H^{m}(I)}$. The notations ( $\left.\cdot, \cdot\right)$ and $\|\cdot\|$ denote the inner product and norm, respectively, in the space $L_{2}(I)$. We will use letter $C$ to represent a generic positive constant, independent of the mesh size $h$.

## 2. Problem and its weak finite element approximation

Consider elliptic problem (1.1). Multiplying equation (1.1) by the transformation function

$$
\rho(x)=\exp \left(-\int_{0}^{x} \frac{a_{1}(x)}{a_{2}(x)} d x\right),
$$

we see that problem (1.1) can be transformed into the following form:

$$
-\left(\rho a_{2} u^{\prime}\right)^{\prime}+\rho a_{0} u=\rho f(x), x \in(a, b), u(a)=0, u^{\prime}(b)=0 .
$$

Therefore, in what follows, we only consider elliptic problems in the form:

$$
\left\{\begin{array}{l}
-\left(a_{2}(x) u^{\prime}\right)^{\prime}+a_{0}(x) u=f(x), x \in(a, b)  \tag{2.1}\\
u(a)=0, u^{\prime}(b)=0
\end{array}\right.
$$

where $a_{2}(x) \geq a_{\text {min }}>0, a_{0}(x) \geq 0$ and $u^{\prime}=\frac{d u}{d x}$. We assume that $a_{2}(x) \in H^{1}(a, b), a_{0}(x) \in$ $L_{\infty}(a, b)$.

First, let us introduce the weak derivative concept. Let closed interval $\bar{I}_{a}=\left[x_{a}, x_{b}\right]$ and its interior $I_{a}=\left(x_{a}, x_{b}\right)$. A weak function on $\bar{I}_{a}$ refers to a function $v=\left\{v^{0}, v^{a}, v^{b}\right\}$, $v^{0}=\left.v\right|_{I_{a}} \in L_{2}\left(I_{a}\right)$, values $v^{a}=v\left(x_{a}\right)$ and $v^{b}=v\left(x_{b}\right)$ exist. Note that $v^{a}$ and $v^{b}$ may not be necessarily the trace of $v^{0}$ at the interval endpoints $x_{a}$ and $x_{b}$. Denote the weak function space by

$$
W\left(I_{a}\right)=\left\{v=\left\{v^{0}, v^{a}, v^{b}\right\}: v^{0} \in L_{2}\left(I_{a}\right),\left|v^{a}\right|+\left|v^{b}\right|<\infty\right\} .
$$

Definition 2.1 Let $v \in W\left(I_{a}\right)$. The weak derivative $d_{w} v$ of $v$ is defined as a linear functional in the dual space $H^{-1}\left(I_{a}\right)$ whose action on each $q \in H^{1}\left(I_{a}\right)$ is given by

$$
\begin{equation*}
<d_{w} v, q>\doteq-\int_{I_{a}} v^{0} q^{\prime} d x+v^{b} q^{b}-v^{a} q^{a}, \forall q \in H^{1}\left(I_{a}\right) \tag{2.2}
\end{equation*}
$$

where $q^{a}=q\left(x_{a}\right), q^{b}=q\left(x_{b}\right)$.
Obviously, as a bounded linear functional on $H^{1}\left(I_{a}\right), d_{w} v$ is well defined for any $v \in$ $W\left(I_{a}\right)$. Moreover, for $v \in H^{1}\left(I_{a}\right)$, if we consider $v$ as a weak function with components $v^{0}=\left.v\right|_{I_{a}}, v^{a}=v\left(x_{a}\right)$ and $v^{b}=v\left(x_{b}\right)$, then by integration by parts, we have for $q \in H^{1}\left(I_{a}\right)$ that

$$
\begin{equation*}
\int_{I_{a}} v^{\prime} q d x=-\int_{I_{a}} v q^{\prime} d x+v^{b} q^{b}-v^{a} q^{a}=-\int_{I_{a}} v^{0} q^{\prime} d x+v^{b} q^{b}-v^{a} q^{a}, \tag{2.3}
\end{equation*}
$$

which implies that $d_{w} v=v^{\prime}$ is the usual derivative of function $v$.
Next, we introduce the discrete weak derivative which is actually used in our analysis. For nonnegative integer $r \geq 0$, let $P_{r}\left(I_{a}\right)$ be the space composed of all polynomials on $I_{a}$ with degree no more than $r$. Then, $P_{r}\left(I_{a}\right)$ is a subspace space of $H^{1}\left(I_{a}\right)$.

Definition 2.2 For $v \in W\left(I_{a}\right)$, the discrete weak derivative $d_{w, r} v \in P_{r}\left(I_{a}\right)$ is defined as the unique solution of the following equation

$$
\begin{equation*}
\int_{I_{a}} d_{w, r} v q d x=-\int_{I_{a}} v^{0} q^{\prime} d x+v^{b} q^{b}-v^{a} q^{a}, \forall q \in P_{r}\left(I_{a}\right) . \tag{2.4}
\end{equation*}
$$

From (2.2) and (2.4), we have

$$
<d_{w} v, q>=\int_{I_{a}} d_{w, r} v q d x, \forall q \in P_{r}\left(I_{a}\right) .
$$

This shows that $d_{w, r} v$ is a discrete approximation of $d_{w} v$ in $P_{r}\left(I_{a}\right)$. In particular, if $v \in H^{1}\left(I_{a}\right)$, we have from (2.3) and (2.4) that

$$
\int_{I_{a}}\left(d_{w, r} v-v^{\prime}\right) q d x=0, \forall q \in P_{r}\left(I_{a}\right) .
$$

That is, $d_{w, r} v$ is the $L_{2}$ projection of $v^{\prime}$ in $P_{r}\left(I_{a}\right)$ if $v \in H^{1}\left(I_{a}\right)$.
Now, we consider the weak finite element approximation of problem (2.1). For interval $I=(a, b)$, let $I_{h}: a=x_{1}<x_{2}<\cdots<x_{N-1}<x_{N}=b$ be a partition of $I$ with elements $I_{i}=\left(x_{i}, x_{i+1}\right), i=1, \ldots, N-1$. Denote the mesh size by $h=\max h_{i}, h_{i}=x_{i+1}-x_{i}, i=$
$1, \ldots, N-1$. In the weak finite element analysis, we will use the discrete weak function space defined on partition $I_{h}$. Such space is denoted by

$$
\begin{align*}
W\left(I_{h}, k\right) & =\left\{v:\left.v\right|_{I_{i}} \in W\left(I_{i}, k\right), i=1,2, \cdots, N-1\right\},  \tag{2.5}\\
W\left(I_{i}, k\right) & =\left\{v=\left\{v^{0}, v^{i}, v^{i+1}\right\}: v^{0} \in P_{k}\left(I_{i}\right),\left|v^{i}\right|+\left|v^{i+1}\right|<\infty\right\} . \tag{2.6}
\end{align*}
$$

Note that for a weak function $v \in W\left(I_{i}, k\right)$, the endpoint values $v^{i}=v\left(x_{i}\right)$ and $v^{i+1}=$ $v\left(x_{i+1}\right)$ may be independent with the interior value $v^{0}$. Recall the discrete weak derivative definition (2.4), for $v \in W\left(I_{i}, k\right)$, its discrete weak derivative $d_{w, r} v \in P_{r}\left(I_{i}\right)$ is given by the following formula

$$
\begin{equation*}
\int_{I_{i}} d_{w, r} v q d x=-\int_{I_{i}} v^{0} q^{\prime} d x+v^{i+1} q^{i+1}-v^{i} q^{i}, \forall q \in P_{r}\left(I_{i}\right), \tag{2.7}
\end{equation*}
$$

where $q^{i}=q\left(x_{i}\right), q^{i+1}=q\left(x_{i+1}\right)$.
In our discussion, except for weak function $v=\left\{v^{0}, v^{i}, v^{i+1}\right\} \in W\left(I_{i}, k\right)$, the endpoint values of a smooth function $w$ on $I_{i}$ should be determined by its trace from the interior of $I_{i}$. For example, for $w \in H^{1}\left(I_{i}\right), w^{i}=w\left(x_{i}\right)=\lim _{x \rightarrow x_{i}} w(x), x \in I_{i}$.

Let $I_{L}=\left(x_{i-1}, x_{i}\right)$ and $I_{R}=\left(x_{i}, x_{i+1}\right)$ be two adjacent elements with the common endpoint $x_{i}$, weak function $\left.v\right|_{\bar{I}_{L}}=\left\{v_{L}^{0}, v_{L}^{i-1}, v_{L}^{i}\right\},\left.v\right|_{\bar{I}_{R}}=\left\{v_{R}^{0}, v_{R}^{i}, v_{R}^{i+1}\right\}$. We define the jump of weak function $v$ at point $x_{i}$ by

$$
[v]_{x_{i}}=v_{R}^{i}-v_{L}^{i}, v \in W\left(I_{h}, k\right) .
$$

Then, weak function $v$ is single value at point $x_{i}$ if and only if $[v]_{x_{i}}=0$. Introduce the weak finite element space

$$
\begin{equation*}
S_{h}=\left\{v: v \in W\left(I_{h}, k\right), v^{1}=0,[v]_{x_{i}}=0, i=2, \ldots, N-1\right\} . \tag{2.8}
\end{equation*}
$$

Denote the discrete $L_{2}$ inner product and norm by

$$
(u, v)_{h}=\sum_{i=1}^{N-1}(u, v)_{I_{i}}=\sum_{i=1}^{N-1} \int_{I_{i}} u v d x, \quad\|u\|_{h}^{2}=(u, u)_{h} .
$$

We now define the weak finite element approximation of problem (2.1) by finding $u_{h} \in S_{h}$ such that

$$
\begin{equation*}
\left(a_{2} d_{w, r} u_{h}, d_{w, r} v\right)_{h}+\left(a_{0} u_{h}^{0}, v^{0}\right)=\left(f, v^{0}\right), \forall v \in S_{h} . \tag{2.9}
\end{equation*}
$$

Remark 2.1. The single value condition $\left([v]_{x_{i}}=0\right)$ has been imposed on space $S_{h}$, it was not required in the original weak Galerkin method [1]. This condition can reduce the size of discrete system of equations (2.9).

## 3. The stability of weak finite element method

In this section, we will show the stability of the weak finite element method and give some lemmas which are very useful in our analysis.

Lemma 3.1. Let $v \in W\left(I_{i}, k\right)$ and $r>k$. Then, $d_{w, r} v=0$ if and only if $v=$ $\left\{v^{0}, v^{i}, v^{i+1}\right\}$ is constant on $\bar{I}_{i}$, that is, $v^{0}=v^{i}=v^{i+1}$ holds.

Proof. First, let $v^{0}=v^{i}=v^{i+1}$. From (2.7) we have

$$
\int_{I_{i}} d_{w, r} v q d x=v^{0}\left(-\int_{I_{i}} q^{\prime} d x+q^{i+1}-q^{i}\right)=0, \forall q \in P_{r}\left(I_{i}\right) .
$$

This implies $d_{w, r} v=0$. Next, let $d_{w, r} v=0$. Then we have from (2.7) that

$$
\begin{equation*}
-\int_{I_{i}} v^{0} q^{\prime} d x+v^{i+1} q^{i+1}-v^{i} q^{i}=0, \forall q \in P_{r}\left(I_{i}\right) \tag{3.1}
\end{equation*}
$$

Let $\bar{v}=\frac{1}{h_{i}} \int_{I_{i}} v^{0} d x$ is the mean value of $v$ on interval $I_{i}$. Consider the initial value problem:

$$
\left\{\begin{array}{l}
q^{\prime}(x)=\bar{v}-v^{0}, x_{i}<x<x_{i+1}  \tag{3.2}\\
q\left(x_{i}\right)=v^{i+1}-v^{i}
\end{array}\right.
$$

Obviously, problem (3.2) has a unique solution $q_{1} \in P_{r}\left(I_{i}\right)$. By integrating (3.2), we obtain

$$
q_{1}^{i+1}-q_{1}^{i}=\int_{I_{i}}\left(\bar{v}-v^{0}\right) d x=0, \quad q_{1}^{i+1}=q_{1}^{i}=v^{i+1}-v^{i}
$$

Hence, taking $q=q_{1}$ in (3.1), we arrive at

$$
-\int_{I_{i}} v^{0}\left(\bar{v}-v^{0}\right) d x+\left(v^{i+1}-v^{i}\right)^{2}=\int_{I_{i}}\left(\bar{v}-v^{0}\right)^{2} d x+\left(v^{i+1}-v^{i}\right)^{2}=0 .
$$

This implies $v^{0}=\bar{v}$ and $v^{i}=v^{i+1}$. Substituting this two equalities into (3.1), it yields

$$
\left(\bar{v}-v^{i}\right)\left(q^{i+1}-q^{i}\right)=0, \forall q \in P_{r}\left(I_{i}\right) .
$$

Hence $\bar{v}=v^{i}$, so that $v^{0}=v^{i}=v^{i+1}$ holds.

Lemma 3.1 shows that the discrete weak derivative $d_{w, r} v$ possesses the prominent feature of the classical derivative $v^{\prime}$. The following result is an analogy of the Sobolev embedding theory in space $H_{E}^{1}(I)=\left\{v: v \in H^{1}(I), v(a)=0\right\}$.

Lemma 3.2. Let $v \in S_{h}$ and $r>k$. Then, the following weak embedding inequalities hold

$$
\begin{align*}
& \left|v^{i}\right| \leq\left|x_{i}-a\right|^{\frac{1}{2}}\left\|d_{w, r} v\right\|_{h}, i=1, \ldots, N, v \in S_{h}  \tag{3.3}\\
& \left\|v^{0}\right\| \leq((b-a)+h)\left\|d_{w, r} v\right\|_{h}, v \in S_{h} \tag{3.4}
\end{align*}
$$

Proof. In definition (2.7) of $d_{w, r} v$, taking $q=1$ we have

$$
\begin{equation*}
\int_{I_{i}} d_{w, r} v d x=v^{i+1}-v^{i} . \tag{3.5}
\end{equation*}
$$

Summing and using $v^{1}=0$ to obtain

$$
v^{i+1}=\sum_{j=1}^{i} \int_{I_{j}} d_{w, r} v d x \leq \sum_{j=1}^{i} \sqrt{h_{j}}\left\|d_{w, r} v\right\|_{L_{2}\left(I_{j}\right)} \leq\left|x_{i+1}-a\right|^{\frac{1}{2}}\left\|d_{w, r} v\right\|_{h} .
$$

This gives estimate (3.3). To prove (3.4), let $q_{1} \in P_{r}\left(I_{i}\right)$ satisfies the initial problem:

$$
\left\{\begin{array}{l}
q_{1}^{\prime}(x)=-v^{0}, x_{i}<x<x_{i+1}  \tag{3.6}\\
q_{1}\left(x_{i}\right)=v^{i+1}-v^{i}
\end{array}\right.
$$

Taking $q=q_{1}$ in (2.7), we obtain

$$
\begin{equation*}
\int_{I_{i}}\left|v^{0}\right|^{2} d x=\int_{I_{i}} d_{w, r} v q_{1} d x+v^{i} q_{1}^{i}-v^{i+1} q_{1}^{i+1} . \tag{3.7}
\end{equation*}
$$

Integrating (3.6), it yields

$$
\begin{equation*}
q_{1}(x)=v^{i+1}-v^{i}-\int_{x_{i}}^{x} v^{0} d x, \quad q_{1}^{i+1}=v^{i+1}-v^{i}-\int_{I_{i}} v^{0} d x \tag{3.8}
\end{equation*}
$$

Substituting (3.8) into (3.7) and using (3.5), we obtain

$$
\begin{aligned}
\left\|v^{0}\right\|_{L_{2}\left(I_{i}\right)}^{2}= & \int_{I_{i}} d_{w, r} v d x\left(v^{i+1}-v^{i}\right)-\int_{I_{i}} d_{w, r} v \int_{x_{i}}^{x} v^{0}(y) d y d x \\
& +v^{i}\left(v^{i+1}-v^{i}\right)-v^{i+1}\left(v^{i+1}-v^{i}-\int_{I_{i}} v^{0} d x\right) \\
= & -\int_{I_{i}} d_{w, r} v \int_{x_{i}}^{x} v^{0}(y) d y d x+v^{i+1} \int_{I_{i}} v^{0} d x .
\end{aligned}
$$

Hence, it follows from estimate (3.3) and the Cauchy inequality that

$$
\left\|v^{0}\right\|^{2} \leq h\left\|d_{w, r} v\right\|_{h}\left\|v^{0}\right\|+(b-a)\left\|d_{w, r} v\right\|_{h}\left\|v^{0}\right\| .
$$

The proof is completed.
Now we can prove the stability of weak finite element equation (2.9).
Theorem 3.3. Let $r>k$. Then problem (2.9) has a unique solution $u_{h} \in S_{h}$ and $u_{h}$ satisfies the stability estimate

$$
\begin{equation*}
\left\|u_{h}^{0}\right\|+\left\|d_{w, r} u_{h}\right\|_{h} \leq \frac{2((b-a)+1)^{2}}{a_{\min }}\|f\| . \tag{3.9}
\end{equation*}
$$

Proof. First, consider the stability. Taking $v_{h}=u_{h}$ in (2.9), we have

$$
a_{\min }\left\|d_{w, r} u_{h}\right\|_{h}^{2} \leq\|f\|\left\|u_{h}^{0}\right\| .
$$

Together with (3.4), estimate (3.9) is derived. Next, consider the unique existence. Since problem (2.9) is a linear system composed of $(k+2) \times(N-1)$ equations with $(k+2) \times(N-1)$ unknowns, we only need to prove that $u_{h}=0$ if $f=0$. Let $f=0$, then it follows from (3.9) that $\left\|d_{w, r} u_{h}\right\|_{h}=\left\|u_{h}^{0}\right\|=0$ holds. Therefore, from Lemma 3.1, we can conclude that $u_{h}$ is piecewise constant on partition $I_{h}$ so that $u_{h}^{i}=u_{h}^{0}=0$.

## 4. Error analysis

In this section, we do the error analysis for the weak finite element method (2.9). We will see that the weak finite element method possesses the same or better theoretical convergence rate as that of the conventional finite element method.

We first show the approximation property of the weak finite element space $S_{h}$. In order to balance the approximation accuracy between space $S_{h}$ and space $P_{r}\left(I_{i}\right)$ used for $d_{w, r} v$, from mow on, we always set the index $r=k+1$ in the definition of discrete weak derivative $d_{w, r} v$, see (2.7).

For $l \geq 0$, let $P_{h}^{l}$ is the local $L_{2}$ projection operator, restricted on each element $I_{i}$, $P_{h}^{l}: u \in L_{2}\left(I_{i}\right) \rightarrow P_{h}^{l} u \in P_{l}\left(I_{i}\right)$ such that

$$
\begin{equation*}
\left(u-P_{h}^{l} u, q\right)_{I_{i}}=0, \forall q \in P_{l}\left(I_{i}\right), i=1,2, \ldots, N-1 \tag{4.1}
\end{equation*}
$$

By the Bramble-Hilbert lemma, it is easy to prove that (see [16])

$$
\begin{equation*}
\left\|u-P_{h}^{l} u\right\|_{L_{2}\left(I_{i}\right)} \leq C h_{i}^{s}\|u\|_{H^{s}\left(I_{i}\right)}, 0 \leq s \leq l+1 . \tag{4.2}
\end{equation*}
$$

We now define a projection operator $Q_{h}: u \in H^{1}(I) \rightarrow Q_{h} u \in W(I, k)$ such that

$$
\begin{equation*}
\left.Q_{h} u\right|_{\bar{I}_{i}}=\left\{Q_{h}^{0} u,\left(Q_{h} u\right)^{i},\left(Q_{h} u\right)^{i+1}\right\} \doteq\left\{P_{h}^{k} u, u\left(x_{i}\right), u\left(x_{i+1}\right)\right\}, i=1, \ldots, N-1 . \tag{4.3}
\end{equation*}
$$

Obviously, $Q_{h} u \in S_{h}$ if $u \in H_{E}^{1}(I)$. From (4.2), we have

$$
\begin{equation*}
\left\|Q_{h}^{0} u-u\right\|_{L_{2}\left(I_{i}\right)}=\left\|P_{h}^{k} u-u\right\|_{L_{2}\left(I_{i}\right)} \leq C h_{i}^{s}\|u\|_{H^{s}\left(I_{i}\right)}, 0 \leq s \leq k+1 \tag{4.4}
\end{equation*}
$$

Furthermore, since

$$
\begin{aligned}
\int_{I_{i}} d_{w, r} Q_{h} u q d x & =-\int_{I_{i}} Q_{h}^{0} u q^{\prime} d x+\left(Q_{h} u\right)^{i+1} q^{i+1}-\left(Q_{h} u\right)^{i} q^{i} \\
& =-\int_{I_{i}} u q^{\prime} d x+u^{i+1} q^{i+1}-u^{i} q^{i}=\int_{I_{i}} u^{\prime} q d x, \quad \forall q \in P_{r}\left(I_{i}\right),
\end{aligned}
$$

hence $d_{w, r} Q_{h} u=P_{h}^{r} u^{\prime}$ holds and (noting that $r=k+1$ )

$$
\begin{equation*}
\left\|d_{w, r} Q_{h} u-u^{\prime}\right\|_{L_{2}\left(I_{i}\right)}=\left\|P_{h}^{r} u^{\prime}-u^{\prime}\right\|_{L_{2}\left(I_{i}\right)} \leq C h_{i}^{s}\|u\|_{H^{s+1}\left(I_{i}\right)}, 0 \leq s \leq k+2 \tag{4.5}
\end{equation*}
$$

Estimates (4.4) and (4.5) show that $Q_{h} u \in S_{h}$ is a very good approximation for function $u \in H_{E}^{1}(I) \bigcap H^{m}(I), m \geq 1$.

In order to do the error analysis, we still need to construct another special projection function.

Lemma 4.1. For $u \in H^{1}(I)$, there exists a projection function $\pi_{h} u \in H^{1}(I)$, restricted on element $I_{i}, \pi_{h} u \in P_{k+1}\left(I_{i}\right)$ satisfies

$$
\begin{align*}
& \left(\left(\pi_{h} u\right)^{\prime}, q\right)_{I_{i}}=\left(u^{\prime}, q\right)_{I_{i}}, \forall q \in P_{k}\left(I_{i}\right), i=1, \ldots, N-1,  \tag{4.6}\\
& \pi_{h} u\left(x_{i}\right)=u\left(x_{i}\right), i=1, \ldots, N  \tag{4.7}\\
& \left\|u-\pi_{h} u\right\|_{L_{2}\left(I_{i}\right)}+h_{i}\left\|u-\pi_{h} u\right\|_{H^{1}\left(I_{i}\right)} \leq C h_{i}^{s+1}\|u\|_{H^{s+1}\left(I_{i}\right)}, 0 \leq s \leq k+1 \tag{4.8}
\end{align*}
$$

Proof. Let $u \in H^{1}(I)$. For any given element $I_{i}$, let $\pi_{h}^{(i)} u \in P_{k+1}\left(I_{i}\right)$ be the unique solution of the initial problem:

$$
\left\{\begin{array}{l}
\left(\pi_{h}^{(i)} u\right)^{\prime}(x)=P_{h}^{k} u^{\prime}, x_{i}<x<x_{i+1}  \tag{4.9}\\
\pi_{h}^{(i)} u\left(x_{i}\right)=u\left(x_{i}\right)
\end{array}\right.
$$

Then, by the property of operator $P_{h}^{k}$, we obtain

$$
\begin{align*}
& \left(\left(\pi_{h}^{(i)} u\right)^{\prime}, q\right)_{I_{i}}=\left(u^{\prime}, q\right)_{I_{i}}, \forall q \in P_{k}\left(I_{i}\right)  \tag{4.10}\\
& \left\|u^{\prime}-\left(\pi_{h}^{(i)} u\right)^{\prime}\right\|_{L_{2}\left(I_{i}\right)} \leq C h_{i}^{s}\|u\|_{H^{s+1}\left(I_{i}\right)}, 0 \leq s \leq k+1 \tag{4.11}
\end{align*}
$$

Since

$$
\left(\pi_{h}^{(i)} u-u\right)(x)=\int_{x_{i}}^{x}\left(\pi_{h}^{(i)} u-u\right)^{\prime}(x) d x, x \in I_{i}
$$

hence, it follows from (4.10) and the Cauchy inequality that

$$
\begin{equation*}
\pi_{h}^{(i)} u\left(x_{i+1}\right)=u\left(x_{i+1}\right), \quad\left\|u-\pi_{h}^{(i)} u\right\|_{L_{2}\left(I_{i}\right)} \leq h_{i}\left\|u^{\prime}-\left(\pi_{h}^{(i)} u\right)^{\prime}\right\|_{L_{2}\left(I_{i}\right)} . \tag{4.12}
\end{equation*}
$$

Now, we set $\left.\pi_{h} u\right|_{I_{i}}=\pi_{h}^{(i)} u$ for $1 \leq i \leq N-1$, then conclusions (4.6) $\sim$ (4.8) can be derived by using (4.10) ~(4.12). Furthermore, since $\pi_{h}^{(i)} u\left(x_{i+1}\right)=u\left(x_{i+1}\right)=\pi_{h}^{(i+1)} u\left(x_{i+1}\right)$, this shows that $\pi_{h} u$ is continuous across junction point $x_{i+1}$, so $\pi_{h} u \in H^{1}(I)$ holds.
Lemma 4.2. Let $u \in H_{E}^{1}(I) \bigcap H^{2}(I)$ be the solution of problem (2.1). Then, u satisfies the following equation

$$
\begin{equation*}
\left(\pi_{h}\left(a_{2} u^{\prime}\right), d_{w, r} v\right)_{h}+\left(a_{0} u, v^{0}\right)=\left(f, v^{0}\right), \forall v \in S_{h} . \tag{4.13}
\end{equation*}
$$

Proof. By (2.7) and Lemma 4.1, we have for $v \in S_{h}$ that

$$
\begin{aligned}
\left(\pi_{h}\left(a_{2} u^{\prime}\right), d_{w, r} v\right)_{I_{i}} & =-\left(\left(\pi_{h}\left(a_{2} u^{\prime}\right)\right)^{\prime}, v^{0}\right)_{I_{i}}+\left(\pi_{h}\left(a_{2} u^{\prime}\right)\right)^{i+1} v^{i+1}-\left(\pi_{h}\left(a_{2} u^{\prime}\right)\right)^{i} v^{i} \\
& =-\left(\left(a_{2} u^{\prime}\right)^{\prime}, v^{0}\right){I_{i}}+\left(a_{2} u^{\prime}\right)^{i+1} v^{i+1}-\left(a_{2} u^{\prime}\right)^{i} v^{i} .
\end{aligned}
$$

Summing and noting that $v^{1}=0$ and $u^{\prime}\left(x_{N}\right)=0$, it yields

$$
\left(\pi_{h}\left(a_{2} u^{\prime}\right), d_{w, r} v\right)_{h}=-\left(\left(a_{2} u^{\prime}\right)^{\prime}, v^{0}\right)=-\left(a_{0} u, v^{0}\right)+\left(f, v^{0}\right) .
$$

Hence, equation (4.13) holds.
Theorem 4.3. Let $u$ and $u_{h}$ be the solutions of problems (2.1) and (2.9), respectively, $u \in H_{E}^{1}(I) \bigcap H^{2}(I)$ and $r=k+1$. Then we have

$$
\begin{equation*}
a_{\min }\left\|d_{w, r} Q_{h} u-d_{w, r} u_{h}\right\|_{h} \leq\left\|a_{2} d_{w, r} Q_{h} u-\pi_{h}\left(a_{2} u^{\prime}\right)\right\|_{h}+((b-a)+1)\left\|a_{0}\left(Q_{h}^{0} u-u\right)\right\| . \tag{4.14}
\end{equation*}
$$

Proof. From Lemma 4.2, we have

$$
\begin{aligned}
& \left(a_{2} d_{w, r} Q_{h} u, d_{w, r} v\right)_{h}+\left(a_{0} Q_{h}^{0} u, v^{0}\right) \\
= & \left(f, v^{0}\right)+\left(a_{2} d_{w, r} Q_{h} u-\pi_{h}\left(a_{2} u^{\prime}\right), d_{w, r} v\right)_{h}+\left(a_{0}\left(Q_{h}^{0} u-u\right), v^{0}\right) .
\end{aligned}
$$

Combining this with equation (2.9), we obtain the error equation

$$
\begin{align*}
& \left(a_{2} d_{w, r}\left(Q_{h} u-u_{h}\right), d_{w, r} v\right)_{h}+\left(a_{0}\left(Q_{h}^{0} u-u_{h}^{0}\right), v^{0}\right) \\
= & \left(a_{2} d_{w, r} Q_{h} u-\pi_{h}\left(a_{2} u^{\prime}\right), d_{w, r} v\right)_{h}+\left(a_{0}\left(Q_{h}^{0} u-u\right), v^{0}\right), v \in S_{h} . \tag{4.15}
\end{align*}
$$

Taking $v=Q_{h} u-u_{h} \in S_{h}$ and using the weak embedding equality (3.4), we arrive at the conclusion of Theorem 4.3.

By means of Theorem 4.3, we can derive the following error estimates.
Theorem 4.4. Let $u$ and $u_{h}$ be the solutions of problems (2.1) and (2.9), respectively, $u \in H_{E}^{1}(I) \bigcap H^{2+s}(I), a_{2} \in H^{1+s}(I), s \geq 0$, and $r=k+1$. Then we have

$$
\begin{array}{r}
\left\|d_{w, r} u_{h}-u^{\prime}\right\|_{h} \leq C h^{s+1}\|u\|_{s+2}, 0 \leq s \leq k \\
\max _{1 \leq i \leq N}\left|u_{h}^{i}-u\left(x_{i}\right)\right| \leq C h^{s+1}\|u\|_{s+2}, 0 \leq s \leq k \tag{4.17}
\end{array}
$$

Furthermore, if $a_{0}(x)=0$ and $u$ is smooth enough, then we have the superconvergence estimates

$$
\begin{array}{r}
\left\|d_{w, r} u_{h}-u^{\prime}\right\|_{h} \leq C h^{k+2}\|u\|_{k+3}, k \geq 0, \\
\max _{1 \leq i \leq N}\left|u_{h}^{i}-u\left(x_{i}\right)\right| \leq C h^{k+2}\|u\|_{k+3}, k \geq 0 . \tag{4.19}
\end{array}
$$

Proof. By the triangle inequality, we have

$$
\begin{aligned}
& \left\|d_{w, r} u_{h}-u^{\prime}\right\|_{h} \leq\left\|d_{w, r} u_{h}-d_{w, r} Q_{h} u\right\|_{h}+\left\|d_{w, r} Q_{h} u-u^{\prime}\right\|_{h}, \\
& \left\|a_{2} d_{w, r} Q_{h} u-\pi_{h}\left(a_{2} u^{\prime}\right)\right\|_{h} \leq\left\|a_{2}\left(d_{w, r} Q_{h} u-u^{\prime}\right)\right\|_{h}+\left\|a_{2} u^{\prime}-\pi_{h}\left(a_{2} u^{\prime}\right)\right\| .
\end{aligned}
$$

Together with Theorem 4.3, it yields

$$
\begin{equation*}
\left\|d_{w, r} u_{h}-u^{\prime}\right\|_{h} \leq C\left(\left\|d_{w, r} Q_{h} u-u^{\prime}\right\|_{h}+\left\|a_{2} u^{\prime}-\pi_{h}\left(a_{2} u^{\prime}\right)\right\|+\left\|a_{0}\left(Q_{h}^{0} u-u\right)\right\|\right) \tag{4.20}
\end{equation*}
$$

Then, estimate (4.16) follows from the approximation properties (4.4), (4.5) and (4.8). Furthermore, by the weak embedding inequality (3.3), we have

$$
\left|u\left(x_{i}\right)-u_{h}^{i}\right|=\left|\left(Q_{h} u\right)^{i}-u_{h}^{i}\right| \leq\left(x_{i}-a\right)^{\frac{1}{2}}\left\|d_{w, r} Q_{h} u-d_{w, r} u_{h}\right\|_{h} .
$$

Hence, we can obtain estimate (4.17) by using Theorem 4.3 and the approximations properties. The superconvergence estimates (4.18)-(4.19) can be derived by a similar argument, noting that $\left\|a_{0}\left(Q_{h}^{0} u-u\right)\right\|=0$ in (4.14) and (4.20) if $a_{0}=0$.

From Theorem 4.3 and the weak embedding inequality, we immediately obtain

$$
\begin{equation*}
\left\|Q_{h}^{0} u-u_{h}^{0}\right\| \leq C\left\|d_{w, r} Q_{h} u-d_{w, r} u_{h}\right\|_{h} \leq C h^{s+1}\|u\|_{s+2}, 0 \leq s \leq k \tag{4.21}
\end{equation*}
$$

which results in the $L_{2}$ error estimate

$$
\left\|u-u_{h}^{0}\right\| \leq\left\|u-Q_{h}^{0} u\right\|+\left\|Q_{h}^{0} u-u_{h}^{0}\right\| \leq C h^{s+1}\|u\|_{s+2}, 0 \leq s \leq k
$$

Below we give a superclose estimate for error $Q_{h}^{0} u-u_{h}^{0}$. To this end, we introduce the auxiliary problem: Find $w \in H_{E}^{1}(I) \bigcap H^{2}(I)$ such that

$$
\left\{\begin{array}{l}
-\left(a_{2}(x) w^{\prime}\right)^{\prime}+a_{0}(x) w=Q_{h}^{0} u-u_{h}^{0}, x \in(a, b),  \tag{4.22}\\
w(a)=0, w^{\prime}(b)=0,\|w\|_{2} \leq C\left\|Q_{h}^{0} u-u_{h}^{0}\right\| .
\end{array}\right.
$$

From Lemma 4.2, we know that $w$ satisfies equation:

$$
\begin{equation*}
\left(\pi_{h}\left(a_{2} w^{\prime}\right), d_{w, r} v\right)_{h}+\left(a_{0} w, v^{0}\right)=\left(Q_{h}^{0} u-u_{h}^{0}, v^{0}\right), \forall v \in S_{h} . \tag{4.23}
\end{equation*}
$$

Theorem 4.5. Let $u$ and $u_{h}$ be the solutions of problems (2.1) and (2.9), respectively, $u \in H_{E}^{1}(I) \bigcap H^{2+s}(I), a_{2} \in H^{1+s}(I), a_{0} \in H^{1}(I), s \geq 0$, and $r=k+1$. Then we have the following superclose estimate

$$
\begin{equation*}
\left\|Q_{h}^{0} u-u_{h}^{0}\right\| \leq C h^{s+2}\|u\|_{s+2}, 0 \leq s \leq k \tag{4.24}
\end{equation*}
$$

Proof. Taking $v=Q_{h} u-u_{h}$ in (4.23) and using error equation (4.15), we have

$$
\begin{align*}
& \left\|Q_{h}^{0} u-u_{h}^{0}\right\|^{2} \\
= & \left(d_{w, r}\left(Q_{h} u-u_{h}\right), \pi_{h}\left(a_{2} w^{\prime}\right)\right)_{h}+\left(a_{0}\left(Q_{h}^{0} u-u_{h}^{0}\right), w\right) \\
= & \left(d_{w, r}\left(Q_{h} u-u_{h}\right), \pi_{h}\left(a_{2} w^{\prime}\right)-a_{2} d_{w, r} Q_{h} w\right)_{h}+\left(a_{0}\left(Q_{h}^{0} u-u_{h}^{0}\right), w-Q_{h}^{0} w\right) \\
& +\left(a_{2} d_{w, r}\left(Q_{h} u-u_{h}\right), d_{w, r} Q_{h} w\right)_{h}+\left(a_{0}\left(Q_{h}^{0} u-u_{h}^{0}\right), Q_{h}^{0} w\right) \\
= & \left(d_{w, r}\left(Q_{h} u-u_{h}\right), \pi_{h}\left(a_{2} w^{\prime}\right)-a_{2} d_{w, r} Q_{h} w\right)_{h}+\left(a_{0}\left(Q_{h}^{0} u-u_{h}^{0}\right), w-Q_{h}^{0} w\right) \\
& +\left(a_{2} d_{w, r} Q_{h} u-\pi_{h}\left(a_{2} u^{\prime}\right), d_{w, r} Q_{h} w\right)_{h}+\left(a_{0}\left(Q_{h}^{0} u-u\right), Q_{h}^{0} w\right) \\
= & \left\{\left(d_{w, r}\left(Q_{h} u-u_{h}\right), \pi_{h}\left(a_{2} w^{\prime}\right)-a_{2} d_{w, r} Q_{h} w\right)_{h}+\left(a_{0}\left(Q_{h}^{0} u-u_{h}^{0}\right), w-Q_{h}^{0} w\right)\right\} \\
& +\left\{\left(a_{2} d_{w, r} Q_{h} u-\pi_{h}\left(a_{2} u^{\prime}\right), d_{w, r} Q_{h} w-w^{\prime}\right)_{h}+\left(a_{0}\left(Q_{h}^{0} u-u\right), Q_{h}^{0} w-w\right)\right\} \\
& +\left\{\left(a_{2} d_{w, r} Q_{h} u-\pi_{h}\left(a_{2} u^{\prime}\right), w^{\prime}\right)_{h}+\left(a_{0}\left(Q_{h}^{0} u-u\right), w\right)\right\} \\
= & E_{1}+E_{2}+E_{3} . \tag{4.25}
\end{align*}
$$

Below we estimate $E_{1} \sim E_{3}$. Using (4.21) and the approximation properties of operators
$Q_{h}$ and $\pi_{h}$, we have

$$
\begin{aligned}
E_{1} & =\left(d_{w, r}\left(Q_{h} u-u_{h}\right), \pi_{h}\left(a_{2} w^{\prime}\right)-a_{2} d_{w, r} Q_{h} w\right)_{h}+\left(a_{0}\left(Q_{h}^{0} u-u_{h}^{0}\right), w-Q_{h}^{0} w\right) \\
& \leq C\left\|d_{w, r}\left(Q_{h} u-u_{h}\right)\right\|_{h}\left(\left\|\pi_{h}\left(a_{2} w^{\prime}\right)-a_{2} w^{\prime}+a_{2} w^{\prime}-a_{2} d_{w, r} Q_{h} w\right\|_{h}+\left\|w-Q_{h}^{0} w\right\|\right) \\
& \leq C h^{s+2}\|u\|_{s+2}\|w\|_{2} . \\
E_{2} & =\left(a_{2} d_{w, r} Q_{h} u-\pi_{h}\left(a_{2} u^{\prime}\right), d_{w, r} Q_{h} w-w^{\prime}\right)_{h}+\left(a_{0}\left(Q_{h}^{0} u-u\right), Q_{h}^{0} w-w\right) \\
& \leq C h\left(\left\|a_{2} d_{w, r} Q_{h} u-a_{2} u^{\prime}+a_{2} u^{\prime}-\pi_{h}\left(a_{2} u^{\prime}\right)\right\|_{h}\right)\|w\|_{2}+C h^{s+2}\|u\|_{s+1}\|w\|_{1} \\
& \leq C h^{s+2}\|u\|_{s+2}\|w\|_{2} .
\end{aligned}
$$

Next, we write

$$
\begin{aligned}
E_{3} & =\left(a_{2} d_{w, r} Q_{h} u-\pi_{h}\left(a_{2} u^{\prime}\right), w^{\prime}\right)_{h}+\left(a_{0}\left(Q_{h}^{0} u-u\right), w\right) \\
& =\left(a_{2} d_{w, r} Q_{h} u-a_{2} u^{\prime}, w^{\prime}\right)_{h}+\left(a_{2} u^{\prime}-\pi_{h}\left(a_{2} u^{\prime}\right), w^{\prime}\right)_{h}+\left(a_{0}\left(Q_{h}^{0} u-u\right), w\right) \\
& =E_{31}+E_{32}+E_{33} .
\end{aligned}
$$

Since $d_{w, r} Q_{h} u=P_{h}^{r} u^{\prime}, Q_{h}^{0} u=P_{h}^{k} u$, then we have

$$
\begin{aligned}
E_{31}+E_{33} & =\left(P_{h}^{r} u^{\prime}-u^{\prime}, a_{2} w^{\prime}-P_{h}^{k}\left(a_{2} w^{\prime}\right)\right)_{h}+\left(P_{h}^{k} u-u, a_{0} w-P_{h}^{k}\left(a_{0} w\right)\right) \\
& \leq C h^{s+2}\|u\|_{s+2}\|w\|_{2}
\end{aligned}
$$

Furthermore, from Lemma 4.1 and integration by parts, we also obtain

$$
\begin{aligned}
E_{32} & =-\sum_{i=1}^{N-1}\left(\left(a_{2} u^{\prime}-\pi_{h}\left(a_{2} u^{\prime}\right)\right)^{\prime}, w\right)_{I_{i}}=-\sum_{i=1}^{N-1}\left(\left(a_{2} u^{\prime}\right)^{\prime}-P_{h}^{k}\left(a_{2} u^{\prime}\right)^{\prime}, w-P_{h}^{k} w\right)_{I_{i}} \\
& \leq C h^{s+2}\|u\|_{s+2}\|w\|_{2}
\end{aligned}
$$

Hence, we have that $E_{3} \leq C h^{s+2}\|u\|_{s+2}\|w\|_{2}$. The proof is completed by substituting estimates $E_{1} \sim E_{3}$ into (4.25), noting that $\|w\|_{2} \leq C\left\|Q_{h}^{0} u-u_{h}^{0}\right\|$.

From Theorem 4.5 and the triangle inequality, we immediately obtain the following optimal $L_{2}$-norm error estimate

$$
\begin{equation*}
\left\|u-u_{h}^{0}\right\| \leq C h^{k+1}\|u\|_{k+1}, k \geq 1 \tag{4.26}
\end{equation*}
$$

In order to derive the optimal $L_{\infty}$-error estimate, we need to strengthen the partition condition. Partition $I_{h}$ is called quasi-uniform if there exists a positive constant $\sigma$ such that

$$
h / h_{i} \leq \sigma, i=1, \cdots, N .
$$

This condition assures that the inverse inequality holds in space $S_{h}$.
Theorem 4.6. Assume that partition $I_{h}$ is quasi-uniform, and $u$ and $u_{h}$ are the solution of problems (2.1) and (2.9), respectively, and conditions in Theorem 4.5 hold. Then, we have

$$
\begin{equation*}
\left\|u-u_{h}^{0}\right\|_{L_{\infty}(I)} \leq C h^{s+1}\|u\|_{s+2}, 0 \leq s \leq k . \tag{4.27}
\end{equation*}
$$

Proof. From Theorem 4.5 and the finite element inverse inequality, we have that

$$
\left\|Q_{h}^{0} u-u_{h}^{0}\right\|_{L_{\infty}(I)} \leq C h^{-\frac{1}{2}}\left\|Q_{h}^{0} u-u_{h}^{0}\right\| \leq C h^{s+\frac{3}{2}}\|u\|_{s+2} .
$$

Hence, by using the approximation property of $Q_{h}^{0} u=P_{h}^{k} u$, we obtain

$$
\begin{aligned}
\left\|u-u_{h}^{0}\right\|_{L_{\infty}(I)} & \leq\left\|u-Q_{h}^{0} u\right\|_{L_{\infty}(I)}+\left\|Q_{h}^{0} u-u_{h}^{0}\right\|_{L_{\infty}(I)} \\
& \leq C h^{s+1}\left(\|u\|_{s+1, \infty}+\|u\|_{s+2}\right) \leq C h^{s+1}\|u\|_{s+2},
\end{aligned}
$$

where we have used the Sobolev embedding inequality.

## 5. The local solvability and numerical example

In this section, we discuss how to solve the discrete system of equations (2.9). We will design a local solver so that this linear system can be solved locally, element by element. Then, we provide some numerical examples to illustrate our theoretical analysis.

### 5.1. The local solvability of the weak finite element equation

Consider the weak finite element equation: (see (2.9)):

$$
\begin{equation*}
\left(a_{2} d_{w, r} u_{h}, d_{w, r} v\right)_{h}+\left(a_{0} u_{h}^{0}, v^{0}\right)=\left(f, v^{0}\right), \forall v \in S_{h} . \tag{5.1}
\end{equation*}
$$

In order to form the discrete linear system of equations (5.1), we introduce the basis functions of space $W\left(I_{i}, k\right)$ or $S_{h}$. Let weak basis functions $\psi_{j}(x)=\left\{\psi_{j}^{0}, \psi_{j}^{i}, \psi_{j}^{i+1}\right\}=$ $\left\{x^{j-1}, 0,0\right\}, j=1, \ldots, k+1$, and further let $\delta_{i}(x)$ be the node basis function, that is, $\delta_{i}\left(x_{i}\right)=1, \delta_{i}(x)=0, x \neq x_{i}$. Then, we have $W\left(I_{i}, k\right)=\operatorname{span}\left\{\psi_{1}(x), \ldots, \psi_{k+1}(x), \delta_{i}(x)\right.$, $\left.\delta_{i+1}(x)\right\}$, and for any $v \in S_{h}$, restricted on $I_{i}, v=\left\{v^{0}, v^{i}, v^{i+1}\right\}$ can be written as

$$
v(x)=\sum_{j=1}^{k+1} c_{j} \psi_{j}(x)+v^{i} \delta_{i}(x)+v^{i+1} \delta_{i+1}(x), x_{i} \leq x \leq x_{i+1} .
$$

For $v \in S_{h}$, by the definition (2.7) of discrete weak derivative, we see that the support set of $d_{w, r} \psi_{j}(x)$ is in $I_{i}$ and the support set of $d_{w, r} \delta_{i}(x)$ is in $\bigcup I_{j}$, where $\bar{I}_{j} \bigcap x_{i} \neq \varnothing$. Then, equation (5.1) is equivalent to the following system of equations

$$
\begin{align*}
& \left(a_{2} d_{w, r} u_{h}, d_{w, r} v\right)_{I_{i}}+\left(a_{0} u_{h}^{0}, v^{0}\right)_{I_{i}}=\left(f, v^{0}\right)_{I_{i}}, v=\left\{\psi_{j}\right\}, i=1, \ldots, N-1,  \tag{5.2}\\
& \left(a_{2} d_{w, r} u_{h}, d_{w, r} v\right)_{I_{i} \cup I_{i+1}}=0, v=\delta_{i+1}, i=1, \ldots, N-2,  \tag{5.3}\\
& \left(a_{2} d_{w, r} u_{h}, d_{w, r} v\right)_{I_{N-1}}=0, v=\delta_{N} . \tag{5.4}
\end{align*}
$$

Equations (5.2) ~(5.4) form a linear system composed of $(k+2)(N-1)$ equations with $(k+2)(N-1)$ unknowns. To solve this system, we need to design a solver for the discrete weak derivative $d_{w, r} v$ or $d_{w, r} u_{h}$. According to (2.7), for given $v \in W\left(I_{i}, k\right)$, $d_{w, r} v \in P_{r}\left(I_{i}\right)$ can be computed by the following formula

$$
\begin{equation*}
M_{i} d_{w, r} V=A_{i} V^{0}+B_{i} V^{i} \tag{5.5}
\end{equation*}
$$

where $d_{w, r} V$ and $V^{0}$ are the vectors associated with functions $d_{w, r} v \in P_{r}\left(I_{i}\right)$ and $v^{0} \in$ $P_{k}\left(I_{i}\right)$, respectively, and $V^{i}=\left(v^{i}, v^{i+1}\right)^{T}$. The matrixes in (5.5) are as follows

$$
\begin{aligned}
& M_{i}=\left(m_{s t}\right)_{(r+1) \times(r+1)}, \quad A_{i}=\left(a_{s t}\right)_{(r+1) \times(k+1)}, \quad B_{i}=\left(b_{s t}\right)_{(r+1) \times 2}, \\
& m_{s t}=\left(x^{s-1}, x^{t-1}\right)_{I_{i}}, a_{s t}=-\left(d_{x}\left(x^{s-1}\right), x^{t-1}\right)_{I_{i}}, b_{s 1}=-x_{i}^{s-1}, b_{s 2}=x_{i+1}^{s-1} .
\end{aligned}
$$

Now, linear system of equations (5.2)~(5.4) can be solved in the following two ways.
Method One. We first use formula (5.5) to derive the linear representation $d_{w, r} u_{h}\left(I_{i}\right)$ $=L\left(u_{h}^{0}\left(I_{i}\right), u_{h}^{i}, u_{h}^{i+1}\right)$. Then, by substituting $d_{w, r} u_{h}\left(I_{i}\right)$ into equations (5.2) $\sim$ (5.4), we can obtain a linear system of equations that only concerns unknowns $\left\{u_{h}^{0}\left(I_{i}\right), u_{h}^{i}, u_{h}^{i+1}\right\}, i=$ $1, \ldots, N-1$. Now, this linear system can be solved by using a proper linear solver, in which $d_{w, r} v\left(I_{i}\right)$ is computed by formula (5.5).

Method Two. We observe that the unknowns in equations (5.2) ~(5.4) are coupled only by equation (5.3) which concerning unknowns on two adjacent elements. If we can independently solve the unknowns on some single element, then we are able to uncouple this simultaneous equations and solve the whole linear system of equations (5.2) ~(5.4) locally, element by element. To this end, integrating equation (2.1), we find that the exact solution $u$ satisfies

$$
u\left(x_{N}\right)-u\left(x_{N-1}\right)+\int_{I_{N-1}} \frac{1}{a_{2}(x)} \widetilde{u}(x) d x=\int_{I_{N-1}} \frac{1}{a_{2}(x)} \widetilde{f}(x) d x
$$

where

$$
\widetilde{u}(x)=\int_{x}^{x_{N}} a_{0}(y) u(y) d y, \quad \widetilde{f}(x)=\int_{x}^{x_{N}} f(y) d y .
$$

This provides an additional equation for $u_{h}$ on the last element. Now, we can solve linear system of equations (5.2) ~(5.4) locally in the following procedure.

First, on element $I_{N-1}$, solve $u_{h}=\left(u_{h}^{0}, u_{h}^{N-1}, u_{h}^{N}\right)$ by the equations:

$$
\begin{aligned}
& \left(a_{2} d_{w, r} u_{h}, d_{w, r} v\right)_{I_{N-1}}+\left(a_{0} u_{h}^{0}, v^{0}\right)_{I_{N-1}}=\left(f, v^{0}\right)_{I_{N-1}}, v=\psi_{j}, j=1, \ldots, k+1, \\
& \left(a_{2} d_{w, r} u_{h}, d_{w, r} v\right)_{I_{N-1}}=0, v=\delta_{N}, \\
& u_{h}^{N}-u_{h}^{N-1}+\int_{I_{N-1}} \frac{1}{a_{2}(x)} \widetilde{u}_{h}^{0}(x) d x=\int_{I_{N-1}} \frac{1}{a_{2}(x)} \widetilde{f}(x) d x .
\end{aligned}
$$

Then, on each element $I_{i}$, solve $u_{h}=\left(u_{h}^{0}, u_{h}^{i}, u_{h}^{i+1}\right)$ by the following equations in the order of $i=N-2, \ldots, 1$,

$$
\begin{aligned}
& \left(a_{2} d_{w, r} u_{h}, d_{w, r} v\right)_{I_{i}}+\left(a_{0} u_{h}^{0}, v^{0}\right)_{I_{i}}=\left(f, v^{0}\right)_{I_{i}}, v=\psi_{j}, j=1, \ldots, k+1, \\
& \left(a_{2} d_{w, r} u_{h}, d_{w, r} v\right)_{I_{i}}=-\left(a_{2} d_{w, r} u_{h}, d_{w, r} v\right)_{I_{i+1}}, v=\delta_{i+1}, \quad\left[u_{h}\right]_{x_{i+1}}=0 .
\end{aligned}
$$

In the above computation procedure, $d_{w, r} v$ and $d_{w, r} u_{h}$ are still determined by formula (5.5). It is easy to see that Method Two is more economical than Method One.

### 5.2. Numerical example

Let us consider problem (2.1) with the following data:

$$
\begin{equation*}
u(x)=2(1-x) \sin (\pi x), a_{2}(x)=1+x^{2}, a_{0}(x)=\sin (\pi x), \tag{5.6}
\end{equation*}
$$

and $(a, b)=(0,1)$, the corresponding source term $f=-\left(a_{2} u^{\prime}\right)^{\prime}+a_{0} u$.
In the numerical experiments, we always partition the interval $I=(0,1)$ uniformly with the mesh size $h=1 / N$. We examine the computation error in the discrete $H^{1}$ norm, the $L_{2}$-norm and the $L_{\infty}$-norm on the mesh point set. The numerical convergence rate is computed by using the formula $r=\ln \left(e_{h} / e_{\frac{h}{2}}\right) / \ln 2$, where $e_{h}$ is the computation error. Table 5.1~Table 5.3 give the numerical results with finite element polynomials of order $k=0,1,2$, in sequence. We observe that the errors vanish very quickly and the convergence rates are at least one order higher than that theoretically predicted, i.e., the superconvergence results are obtained even $a_{0}(x) \neq 0$. When taking $a_{0}(x)=0$, we
obtain the same superconvergence rate as that in case of $a_{0}(x) \neq 0$. We further examine problem (2.1) with different test solutions and data, the convergence rates still remain unchanged. In conclusion, this weak finite element method is a high accuracy numerical method in both theory and experiment.

Table 5.1 History of convergence for $k=0$

|  | $\left\\|d_{w, r} u_{h}-u^{\prime}\right\\|_{h}$ | $\left\\|u-u_{h}^{0}\right\\|$ |  | $\max \left\|u_{h}^{i}-u\left(x_{i}\right)\right\|$ |  |  |
| :---: | :---: | :---: | :--- | :--- | :--- | :--- |
| mesh $h$ | error | rate | error | rate | error | rate |
| $1 / 4$ | 0.2281 | - | 0.0501 | - | 0.1221 |  |
| $1 / 8$ | 0.0579 | 1.9769 | 0.0131 | 1.9361 | 0.0302 | 2.0162 |
| $1 / 16$ | 0.0145 | 1.9942 | 0.0033 | 1.9836 | 0.0075 | 2.0039 |
| $1 / 32$ | 0.0036 | 1.9986 | 0.0008 | 1.9959 | 0.0019 | 2.0010 |
| $1 / 64$ | 0.0009 | 1.9996 | 0.0002 | 1.9990 | 0.0005 | 2.0002 |
| $1 / 128$ | 0.0002 | 1.9999 | 0.0001 | 1.9997 | 0.0001 | 2.0001 |

Table 5.2 History of convergence for $k=1$

|  | $\left\\|d_{w, r} u_{h}-u^{\prime}\right\\|_{h}$ | $\left\\|u-u_{h}^{0}\right\\|$ |  | $\max \left\|u_{h}^{i}-u\left(x_{i}\right)\right\|$ |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| mesh $h$ |  | rate | error | rate | error | rate |
| $1 / 4$ | 0.0154 | - | 0.0009 | - | 0.0003 |  |
| $1 / 8$ | 0.0020 | 2.9797 | $5.5590 \mathrm{e}-5$ | 3.9842 | $1.7547 \mathrm{e}-5$ | 4.0690 |
| $1 / 16$ | $2.4534 \mathrm{e}-4$ | 2.9952 | $3.4831 \mathrm{e}-6$ | 3.9964 | $1.1189 \mathrm{e}-6$ | 3.9710 |
| $1 / 32$ | $3.0693 \mathrm{e}-5$ | 2.9988 | $2.1785 \mathrm{e}-7$ | 3.9989 | $6.9728 \mathrm{e}-8$ | 4.0043 |
| $1 / 64$ | $3.8374 \mathrm{e}-6$ | 2.9997 | $1.3651 \mathrm{e}-8$ | 3.9963 | $4.3549 \mathrm{e}-9$ | 4.0010 |

Table 5.3 History of convergence for $k=2$

|  | $\left\\|d_{w, r} u_{h}-u^{\prime}\right\\|_{h}$ |  | $\left\\|u-u_{h}^{0}\right\\|$ |  | $\max \left\|u_{h}^{i}-u\left(x_{i}\right)\right\|$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| mesh $h$ | error | rate | error | rate | error | rate |
| $1 / 4$ | 0.0008 | - | $2.0906 \mathrm{e}-5$ | - | $1.1846 \mathrm{e}-6$ |  |
| $1 / 8$ | $5.1694 \mathrm{e}-5$ | 3.9944 | $6.5039 \mathrm{e}-7$ | 5.0065 | $1.7776 \mathrm{e}-8$ | 6.0583 |
| $1 / 16$ | $3.2341 \mathrm{e}-6$ | 3.9986 | $2.0305 \mathrm{e}-8$ | 5.0014 | $2.7789 \mathrm{e}-10$ | 5.9993 |
| $1 / 32$ | $2.0214 \mathrm{e}-7$ | 3.9999 | $6.3690 \mathrm{e}-10$ | 4.9947 | $4.2230 \mathrm{e}-12$ | 6.0401 |
| $1 / 64$ | $1.2594 \mathrm{e}-8$ | 4.0045 | $1.9884 \mathrm{e}-11$ | 5.0040 | $6.5939 \mathrm{e}-14$ | 6.0001 |

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