

Edge-fault-tolerant edge-bipancyclicity of balanced hypercubes

Pingshan Li Min Xu*

Sch. Math. Sci. & Lab. Math. Com. Sys., Beijing Normal University, Beijing, 100875, China

Abstract

The balanced hypercube, BH_n , is a variant of hypercube Q_n . Hao et al. [Appl. Math. Comput. 244 (2014) 447-456] showed that there exists a fault-free Hamiltonian path between any two adjacent vertices in BH_n with $(2n - 2)$ faulty edges. Cheng et al. [Inform. Sci. 297 (2015) 140-153] proved that BH_n is 6-edge-bipancyclic after $(2n - 3)$ faulty edges occur for all $n \geq 2$. In this paper, we improve these two results by demonstrating that BH_n is 6-edge-bipancyclic even when there exist $(2n - 2)$ faulty edges for all $n \geq 2$. Our result is optimal with respect to the maximum number of tolerated edge faults.

Key words: Balanced hypercubes; Hypercubes; Edge-pancyclicity; Fault-tolerance.

1 Introduction

In the field of parallel and distributed systems, interconnection networks are an important research area. Typically, the topology of a network can be represented as a graph in which the vertices represent processors and the edges represent communication links.

The hypercube network has been proved to be one of the most popular interconnection networks as it possesses many excellent properties such as a recursive structure, regularity, and symmetry. It is well known that no network typically meets all the aspects of a given set of requirements. Thus, a number of hypercube variants have been proposed, such as folded hypercubes [6], crossed cubes [5], Möbius cubes [4], twisted cubes [8], and shuffle cubes [10] and so on (see [14]).

The balanced hypercube, proposed by Huang and Wu [9], is also a hypercube variant. Similar to hypercubes, balanced hypercubes are bipartite graphs [9] that are vertex-transitive [13] and edge-transitive [19]. Balanced hypercubes are superior to hypercubes in that they have a smaller diameter as compared to hypercubes.

Studies on balanced hypercubes can be found in [2, 3, 7, 9, 11–13, 15–19].

For graph definitions and notations, we follow [1]. A graph G consists of a vertex set $V(G)$ and an edge set $E(G)$, where an edge is an unordered pair of distinct vertices of G . A graph G is called bipartite if its vertex set can be partitioned into two parts V_1, V_2 such that every edge has one endpoint in V_1 and one in V_2 . A vertex v is a neighbor of u if (u, v) is an edge of G , and $N_G(u)$ denotes the set of all the neighbors of u in G . A path P of length ℓ from x to y , denoted by ℓ -path P , is a finite sequence of distinct vertices $\langle v_0, v_1, \dots, v_\ell \rangle$ such that $x = v_0, y = v_\ell$, and $(v_i, v_{i+1}) \in E$ for $0 \leq i \leq \ell - 1$. We also denote the path P as $\langle v_0, v_1, \dots, v_i, Q, v_j, v_{j+1}, \dots, v_\ell \rangle$, where Q is the path $\langle v_i, v_{i+1}, \dots, v_j \rangle$. A cycle C of length $\ell + 1$ is a closed path $\langle v_0, v_1, \dots, v_\ell, v_0 \rangle$, denoted by $(\ell + 1)$ -cycle C .

In an interconnection network, the problem of simulating one network by another is modeled as a graph embedding problem. In all embedding problems, the cycle embedding problem is one of the most common problem; it refers to finding a cycle of a given length in a graph. A graph G of order $|V(G)|$ is m -pancyclic, if it contains every ℓ -cycle for $m \leq \ell \leq |V(G)|$. A bipartite graph G is m -bipancyclic, if it contains every even ℓ -cycle for $m \leq \ell \leq |V(G)|$. A graph G is pancyclic (resp. bipancyclic) if it

*Corresponding author.

E-mail address: xum@bnu.edu.cn (M. Xu) .

is g -pancyclic (g -bipancyclic), where $g = g(G)$ is the girth of G . A graph G is vertex-pancyclic (resp. edge-pancyclic) if every vertex (resp. edge) lies on various ℓ -cycles for all $g \leq \ell \leq V(G)$. A path is called a Hamiltonian path if it contains all the vertices of G . A graph G is said to be Hamiltonian connected if there exists a Hamiltonian path between any two vertices of G . A bipartite graph is Hamiltonian laceable if there is a Hamiltonian path between any two vertices in different bipartite sets.

A bipartite graph G is k -fault-tolerant hamiltonian laceable (resp. bipancyclic, vertex-bipancyclic, and edge-bipancyclic) if $G - F$ remains Hamiltonian laceable (resp. bipancyclic, vertex-bipancyclic, and edge-bipancyclic) for $F \subseteq V(G) \cup E(G)$, $|F| \leq k$. A bipartite graph G is k -edge-fault-tolerant Hamiltonian laceable (resp. bipancyclic, vertex-bipancyclic, and edge-bipancyclic) if $G - F$ remains Hamiltonian *laceable* (resp. bipancyclic, vertex-bipancyclic, and edge-bipancyclic) for $F \subseteq E(G)$, $|F| \leq k$.

The balanced hypercube, BH_n , has been studied by many researchers. Xu et al. [15] proved that BH_n is edge-bipancyclic and Hamiltonian laceable. Yang [16] proved that BH_n is bipanconnected. Yang [17] also demonstrated that the super connectivity of BH_n is $(4n - 4)$ and the super edge-connectivity of BH_n is $(4n - 2)$ for $n \geq 2$. Lü et al. [12] proved that BH_n is hyper-Hamiltonian laceable. Cheng et al. [2] proved that BH_n is $(n - 1)$ -vertex-fault-tolerant edge-bipancyclic. Hao et al. [7] showed that there exists a fault-free Hamiltonian path between any two adjacent vertices in BH_n with $(2n - 2)$ faulty edges. Zhou et al. [18] proved that BH_n is $(2n - 2)$ -edge-fault-tolerant Hamiltonian laceable. Cheng et al. [3] proved that BH_n is $(2n - 3)$ edge-fault-tolerant 6-edge-bipancyclic for all $n \geq 2$. In this paper, we improve the results of Hao et al. [7] and Cheng et al. [3] by demonstrating that BH_n is $(2n - 2)$ edge-fault-tolerant 6-edge-bipancyclic for all $n \geq 2$. Our result is optimal with respect to the maximum number of tolerated edge faults.

The rest of this paper is organized as follows. In Section 2, we introduce two equivalent definitions of balanced hypercubes and discuss some of their properties. In Section 3, we investigate edge-bipancyclic of BH_n with faulty edges. Finally, we conclude this paper in Section 4.

2 Balanced hypercubes

Wu and Huang [9] presented two equivalent definitions of BH_n as follows:

Definition 2.1 An n -dimensional balanced hypercube BH_n has 2^{2n} vertices, each labeled by an n -bit string $(a_0, a_1, \dots, a_{n-1})$, where $a_i \in \{0, 1, 2, 3\}$ for all $0 \leq i \leq n - 1$. A arbitrary vertex $(a_0, a_1, \dots, a_{i-1}, a_i, a_{i+1}, \dots, a_{n-1})$ is adjacent to the following $2n$ vertices:

- (1) $((a_0 \pm 1) \bmod 4, a_1, \dots, a_{i-1}, a_i, a_{i+1}, \dots, a_{n-1})$ where $1 \leq i \leq n - 1$,
- (2) $((a_0 \pm 1) \bmod 4, a_1, \dots, a_{i-1}, (a_i + (-1)^{a_0}) \bmod 4, a_{i+1}, \dots, a_{n-1})$ where $1 \leq i \leq n - 1$.

In BH_n , the first coordinate a_0 of vertex $(a_0, a_1, \dots, a_{n-1})$ is called the inner index, and the second coordinate a_i ($1 \leq i \leq n - 1$) is called the i -dimension index. From the definition, we have that $N_{BH_n}((a_0, a_1, \dots, a_{n-1})) = N_{BH_n}((a_0 + 2, a_1, \dots, a_{n-1}))$. Figure 1 shows two balanced hypercubes of dimensional one and two.

Briefly, we assume that ‘+’, ‘-’ for the coordinate of a vertex is an operation with mod 4 in the remainder of the paper. Let $X_{j,i} = \{(a_0, a_1, \dots, a_{j-1}, a_j, a_{j+1}, \dots, a_{n-1}) \mid a_k \in \{0, 1, 2, 3\}, 0 \leq k \leq n - 1, a_j = i\}$ for $1 \leq j \leq n - 1$ and $i \in \{0, 1, 2, 3\}$ and let $BH_{n-1}^{j,i} = BH_n[X_{j,i}]$. Then, BH_n can be divided into four copies: $BH_{n-1}^{j,0}, BH_{n-1}^{j,1}, BH_{n-1}^{j,2}, BH_{n-1}^{j,3}$ where $BH_{n-1}^{j,i} \cong BH_{n-1}$ for $i = 0, 1, 2, 3$ [2]. We use BH_{n-1}^i to denote $BH_{n-1}^{n-1,i}$ for $i = 0, 1, 2, 3$.

Definition 2.2 The balanced hypercube BH_n can be constructed recursively as follows:

1. BH_1 is a 4-cycle with vertex-set $\{0, 1, 2, 3\}$.

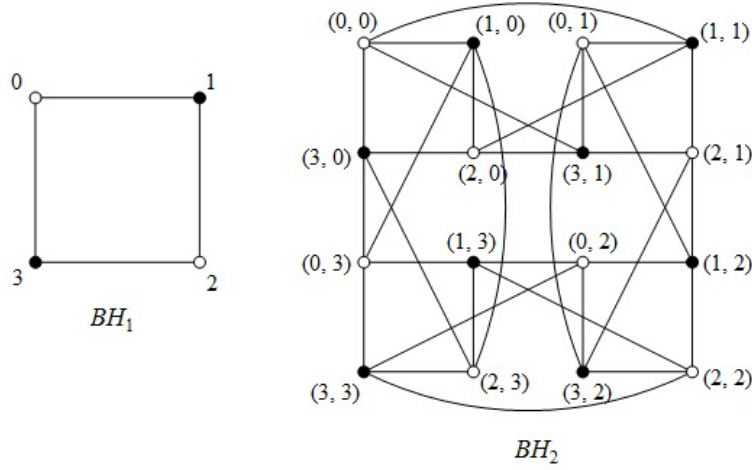


Figure 1: Illustration of BH_1 and BH_2

2. BH_n is a construct from four copies of BH_{n-1} : $BH_{n-1}^0, BH_{n-1}^1, BH_{n-1}^2, BH_{n-1}^3$. Each vertex $(a_0, a_1, \dots, a_{n-2}, i)$ has two extra adjacent vertices:

- (1) In BH_{n-1}^{i+1} : $(a_0 \pm 1, a_1, \dots, a_{n-2}, i+1)$ if a_0 is even.
- (2) In BH_{n-1}^{i-1} : $(a_0 \pm 1, a_1, \dots, a_{n-2}, i-1)$ if a_0 is odd.

Since BH_n is a bipartite graph, then $V(BH_n)$ can be divided into two disjoint parts. Obviously, the vertex-set $V_1 = \{a = (a_0, a_1, \dots, a_{n-1}) \mid a \in V(BH_n) \text{ and } a_0 \text{ is odd}\}$ and $V_2 = \{a = (a_0, a_1, \dots, a_{n-1}) \mid a \in V(BH_n) \text{ and } a_0 \text{ is even}\}$ form the desired partition. We use black nodes to denote the vertices in V_1 and white nodes to denote the vertices in V_2 .

Let (u, v) be an edge of BH_n , if u and v differ only with regard to the inner index, then (u, v) is said to be a 0-dimensional edge. If u and v differ not only in terms of the inner index but also with regard to the i -dimension index, then (u, v) is called the i -dimensional edge. We use $\partial D_d (0 \leq d \leq n-1)$ to denote the set of all d -dimensional edges.

There are some known properties about BH_n .

Lemma 2.3 ([13, 19]) *The balanced hypercube BH_n is vertex-transitive and edge-transitive.*

Lemma 2.4 ([18]) *The balanced hypercube BH_n is $(2n-2)$ -edge-fault-tolerant Hamiltonian laceable for $n \geq 2$.*

Lemma 2.5 ([15]) *The balanced hypercube BH_n is edge-bipancyclic for $n \geq 2$.*

Lemma 2.6 ([2]) *Let $e = (x, y)$ be an arbitrary edge in $BH_{n-1}^{j,0}$. Then, there exist two internal vertex-disjoint paths $\langle x, x_1, y_1, x_2, y_2, x_3, y_3, y \rangle$ and $\langle x, x'_1, y'_1, x'_2, y'_2, x'_3, y'_3, y \rangle$ in BH_n such that $(x_i, y_i), (x'_i, y'_i) \in E(BH_{n-1}^{j,i})$ where $1 \leq i \leq n-1$ and $i = 1, 2, 3$.*

Lemma 2.7 ([11]) *Let $n \geq 2$ be an integer. Then, $BH_n - \partial D_0$ has four components, and each component is isomorphic to BH_{n-1} .*

Remark. The above Lemma shows that one can divide BH_n into four BH_n s by deleting ∂D_d for any $d \in \{0, 1, \dots, n-1\}$. The four components of BH_n through the deletion of ∂D_j are $BH_{n-1}^{j,0}, BH_{n-1}^{j,1}, BH_{n-1}^{j,2}$, and $BH_{n-1}^{j,3}$ for $1 \leq j \leq n-1$. For convenience, we use $BH_{n-1}^{0,0}, BH_{n-1}^{0,1}, BH_{n-1}^{0,2}$, and $BH_{n-1}^{0,3}$ to denote the components of $BH_n - \partial D_0$ throughout this paper.

A graph G is hyper-Hamiltonian laceable if it is Hamiltonian laceable and, for an arbitrary vertex v in V_i where $i \in \{0, 1\}$, there exists a Hamiltonian path in $G - v$ joining any two different vertices in V_{1-i} . Lü et al. obtained the following result.

Lemma 2.8 ([12]) *The balanced hypercube BH_n is hyper-Hamiltonian laceable for $n \geq 1$.*

In the following, we discuss some properties that are used in the proof of our main results.

Lemma 2.9 *For an arbitrary vertex u in $BH_{n-1}^{j,i}$ where $0 \leq j \leq n-1$, $0 \leq i \leq 3$. Suppose that $F \subseteq E(BH_n)$, $|F| \leq 2n-2$ and $|F \cap BH_{n-1}^{j,i}| \leq 2n-3$. Then, there exists a 2-path $\langle u, v, w \rangle \subseteq BH_n \setminus F$ where $u, v \in BH_{n-1}^{j,i}$, $w \in BH_n \setminus BH_{n-1}^{j,i}$.*

Proof: Without loss of generality, we can assume that $u = (0, 0, \dots, 0) \in BH_{n-1}^0$. Note that $N_{BH_{n-1}^0}(u) = 2n-2$ and u is a white vertex, there exist $2(2n-2)$ different edges from $N_{BH_{n-1}^0}(u)$ to BH_{n-1}^3 . Suppose that $|F \cap BH_{n-1}^0| = k$, $|F \cap (BH_n \setminus BH_{n-1}^0)| = t$. We have

$$\begin{cases} k + t \leq 2n - 2; \\ k \leq 2n - 3. \end{cases}$$

Hence, there exists at least one 2-path $\langle u, v, w \rangle \subseteq BH_n \setminus F$ where $u, v \in BH_{n-1}^0$, $w \in BH_{n-1}^3$ owing to $2((2n-2) - k) - t \geq 2(2n-2) - (k+t) - k \geq 2n-2-k \geq 1$. See figure 2 for illustration.

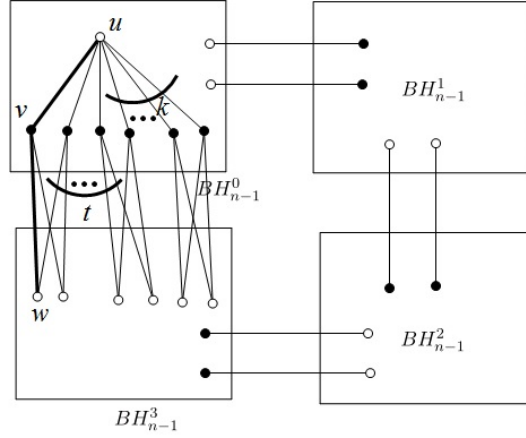


Figure 2: The fault-free path $\langle u, v, w \rangle$ of Lemma 2.9

Lemma 2.10 *Suppose that $e = (u, v)$ is an edge between $BH_{n-1}^{j,i}$ and $BH_{n-1}^{j,i+1}$ where $0 \leq j \leq n-1$, $0 \leq i \leq 3$ for $n \geq 2$. Then, there exists a cycle C of length 8 in $BH_n \setminus F$ where $F \subseteq E(BH_n)$, $|F| \leq 2n-2$ and $|F \cap \partial D_j| \geq 1$ such that $|E(C) \cap BH_{n-1}^{j,i}| = 1$.*

Proof: By Lemma 2.3, BH_n is edge-transitive, Without loss of generality, let $j = n-1$ and $u = (0, 0, \dots, 0)$, $v = (1, 0, \dots, 0, 1)$. There exist $4(n-1)$ edge disjoint paths of length 5 from $N_{BH_{n-1}^0}(u)$ to $N_{BH_{n-1}^1}(v)$ such that each path has an edge in BH_{n-1}^2 and BH_{n-1}^3 . We list them as follows (see figure 3):

$$\begin{aligned} P_{0,1} &= \langle (1, 0, \dots, 0), (2, 0, \dots, 0, 3), (3, 0, \dots, 0, 3), (0, 0, \dots, 0, 2), (1, 0, \dots, 2), (2, 0, \dots, 1) \rangle; \\ P_{0,2} &= \langle (1, 0, \dots, 0), (0, 0, \dots, 0, 3), (1, 0, \dots, 0, 3), (2, 0, \dots, 0, 2), (3, 0, \dots, 2), (0, 0, \dots, 1) \rangle; \\ P_{0,3} &= \langle (3, 0, \dots, 0), (2, 0, \dots, 0, 3), (1, 0, \dots, 0, 3), (0, 0, \dots, 0, 2), (3, 0, \dots, 2), (2, 0, \dots, 1) \rangle; \\ P_{0,4} &= \langle (3, 0, \dots, 0), (0, 0, \dots, 0, 3), (3, 0, \dots, 0, 3), (2, 0, \dots, 0, 2), (1, 0, \dots, 2), (0, 0, \dots, 1) \rangle; \\ P_{k,1} &= \langle (\overbrace{(1, 0, \dots, 0)}^{k-1}, \overbrace{1, 0, \dots, 0, 0}^{n-k-1}), (2, 0, \dots, 0, 1, 0, \dots, 0, 3), (3, 0, \dots, 0, 2, 0, \dots, 0, 3), \\ &\quad (0, 0, \dots, 0, 2, 0, \dots, 0, 2), (1, 0, \dots, 0, 3, 0, \dots, 0, 2), (2, 0, \dots, 0, 3, 0, \dots, 0, 1) \rangle; \\ P_{k,2} &= \langle (\overbrace{(1, 0, \dots, 0)}^{k-1}, \overbrace{1, 0, \dots, 0, 0}^{n-k-1}), (0, 0, \dots, 0, 1, 0, \dots, 0, 3), (1, 0, \dots, 0, 2, 0, \dots, 0, 3), \\ &\quad (2, 0, \dots, 0, 2, 0, \dots, 0, 2), (3, 0, \dots, 0, 3, 0, \dots, 0, 2), (0, 0, \dots, 0, 3, 0, \dots, 0, 1) \rangle; \end{aligned}$$

$$\begin{aligned}
P_{k,3} &= \langle (\overbrace{(3, 0, \dots, 0)}^{k-1}, \overbrace{1, 0, \dots, 0, 0}^{n-k-1}), (2, 0, \dots, 0, 1, 0, \dots, 0, 3), (1, 0, \dots, 0, 2, 0, \dots, 0, 3), \\
&\quad (0, 0, \dots, 0, 2, 0, \dots, 0, 2), (3, 0, \dots, 0, 3, 0, \dots, 0, 2), (2, 0, \dots, 0, 3, 0, \dots, 0, 1) \rangle; \\
P_{k,4} &= \langle (\overbrace{(3, 0, \dots, 0)}^{k-1}, \overbrace{1, 0, \dots, 0, 0}^{n-k-1}), (0, 0, \dots, 0, 1, 0, \dots, 0, 3), (3, 0, \dots, 0, 2, 0, \dots, 0, 3), \\
&\quad (2, 0, \dots, 0, 2, 0, \dots, 0, 2), (1, 0, \dots, 0, 3, 0, \dots, 0, 2), (0, 0, \dots, 0, 3, 0, \dots, 0, 1) \rangle \\
&\quad \text{where } 1 \leq k \leq n-1.
\end{aligned}$$

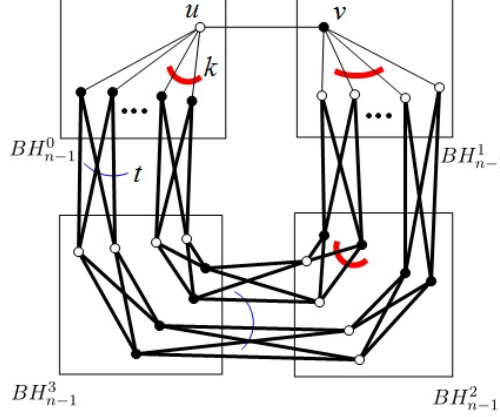


Figure 3: The 5-paths in Lemma 2.10

Suppose that $|F \cap (BH^n - \partial D_{n-1})| = k$, $|F \cap \partial D_{n-1}| = t$, then $k+t \leq 2n-2$ and $t \geq 1$. Hence, there exists at least one desired 8-cycle owing to $2((2n-2)-k)-t \geq 2(2n-2)-(k+t)-k \geq 2n-2-k \geq 1$. \square

3 Edge-bipancyclicity of BH_n under edge faults

In this section, we consider the edge-bipancyclicity of BH_n for at most $(2n-2)$ faulty edges.

Let $e = (x, y)$ be an edge between BH_{n-1}^0 and BH_{n-1}^1 and suppose that $x', y' \in BH_n$ such that $N_{BH_n}(x) = N_{BH_n}(x')$ and $N_{BH_n}(y) = N_{BH_n}(y')$. Let $F = \{(x, y'), (x', y)\}$. From the concluding remarks of [3], we have that there does not exist a cycle of length 4 in $BH_n \setminus F$ that contains e . Thus, in the following, we prove that BH_n is $(2n-2)$ edge-fault-tolerant 6-bipancyclic.

Lemma 3.1 *The balanced hypercube BH_2 is 2-edge-fault-tolerant 6-bipancyclic.*

Proof: The proof is rather long, and we therefore provide it in Appendix A.

Theorem 3.2 *The balanced hypercube BH_n is $(2n-2)$ -edge-fault-tolerant edge 6-bipancyclic for $n \geq 2$.*

Proof: We prove this theorem by induction on n . By Lemma 3.1, the theorem holds for $n = 2$. Assume that it is true for $2 \leq k < n$. Let F be any subset of $E(BH_n)$ with $|F| \leq 2n-2$ and $F_i = \partial D_i \cap F$ for $0 \leq i \leq n-1$. We get $|F| = \sum_{i=0}^{n-1} |F_i|$. Accordingly, without loss of generality, we can assume that $|F_{n-1}| \geq |F_{n-2}| \geq \dots \geq |F_0|$. Let $F^i = F \cap E(BH_{n-1}^i)$ for $0 \leq i \leq 3$. We obtain $F = F^0 \cup F^1 \cup F^2 \cup F^3 \cup F_{n-1}$ and $|F^0 \cup F^1 \cup F^2 \cup F^3| \leq 2n-4$. Let e be any edge in $BH_n \setminus F$ and ℓ be any even integer with $6 \leq \ell \leq 2^{2n}$. We need to construct an ℓ -cycle in $BH_n \setminus F$ containing e .

Case 1: $e = (u, v) \notin \partial D_{n-1}$.

Without loss of generality, we can assume that $e \in BH_{n-1}^0$.

Subcase 1.1: $6 \leq \ell \leq 2^{2n-2}$.

Since $|F^0| \leq |F^0 \cup F^1 \cup F^2 \cup F^3| \leq 2n - 4$, by induction hypothesis, it holds.

Subcase 1.2: $2^{2n-2} + 2 \leq \ell \leq 2^{2n-1} + 6$.

By induction hypothesis, there exists a fault-free Hamiltonian cycle C in BH_{n-1}^0 containing e , say $\langle c^1, c^2, \dots, c^{2^{2n-2}}, c^1 \rangle$ where $c^1 = u, c^{2^{2n-2}} = v$. We can observe that $C \setminus \{e\}$ is a $(2^{2n-2} - 1)$ -path. Then, $M = \{(c^1, c^{2^{2n-3}+6}), \dots, (c^i, c^{2^{2n-3}+i+5}), \dots, (c^{2^{2n-3}-5}, c^{2^{2n-2}})\}$ is a set with $2^{(2n-3)} - 5$ pairs of distinct vertices of BH_{n-1}^0 such that $d_{C \setminus \{e\}}(c^i, c^{2^{2n-3}+i+5}) = 2^{2n-3} + 5$ for all $1 \leq i \leq 2^{2n-3} - 5$. Thus, c^i and $c^{2^{2n-3}+i+5}$ are in different partite sets. There exists at least one pair $(c^t, c^{2^{2n-3}+t+5})$ in M such that

$$|F \cap \{e_1, e_2 \mid e_1, e_2 \text{ are two } (n-1)\text{-dimensional edges incident with } c^t\}| \leq 1 \text{ and}$$

$$|F \cap \{e_3, e_4 \mid e_3, e_4 \text{ are two } (n-1)\text{-dimensional edges incident with } c^{2^{2n-3}+t+5}\}| \leq 1$$

owing to $2 \cdot (2^{2n-3} - 5) > 2n - 2$ for all $n \geq 3$. Without loss of generality, let c^t be a white vertex and $c^{2^{2n-3}+t+5}$ be a black vertex. Then, there exist two fault-free $(n-1)$ -dimensional edges $(c^t, v^1), (c^{2^{2n-3}+t+5}, u^3)$ where $v^1 \in BH_{n-1}^1$ and $u^3 \in BH_{n-1}^3$. Let $P_0 = \langle c^{2^{2n-3}+t+5}, c^{2^{2n-3}+t+6}, \dots, c^{2^{2n-2}-1}, v, u, c^2, \dots, c^t \rangle$. Thus, P_0 is a $(2^{2n-3} - 5)$ -path that contains (u, v) . By Lemma 2.9, there exists a fault-free 2-path $\langle u^3, v^3, u^2 \rangle$ and a fault-free 2-path $\langle v^1, u^1, v^2 \rangle$ where $u^i, v^i \in BH_{n-1}^i$ for $1 \leq i \leq 3$. Since $|F^2| \leq 2n - 4$, by induction hypothesis, there exists a Hamiltonian cycle C_2 in $BH_{n-1}^2 \setminus F$. Thus, there exist two fault-free path P'_2, P''_2 in BH_{n-1}^2 joining u^2 and v^2 with length $|V(P'_2)|$ and $2^{2n-2} - |V(P'_2)|$, respectively, where $1 \leq |V(P'_2)| \leq 2^{2n-3} - 1$.

Subcase 1.2.1: $|V(P'_2)| = 2^{2n-3} - 1$.

We can represent $\ell = \ell_0 + \ell_1 + \ell_2 + \ell_3 + 4$, where ℓ_i satisfies one of the following conditions for $i = 0, 1, 2, 3$.

$$\begin{aligned} \ell_0 = 2^{2n-3} - 5, \quad \ell_1 = 1, \quad \ell_2 = 2^{2n-3} + 1, \quad \ell_3 = 1 & \quad \text{or} \\ \ell_0 = 2^{2n-3} - 5, \quad 5 \leq \ell_1 \leq 2^{2n-2} - 1, \quad \ell_2 = 2^{2n-3} - 1, \quad \ell_3 = 1 & \quad \text{or} \\ \ell_0 = 2^{2n-3} - 5, \quad 5 \leq \ell_1 \leq 2^{2n-2} - 1, \quad \ell_2 = 2^{2n-3} - 1, \quad 5 \leq \ell_3 \leq 2^{2n-2} - 1. \end{aligned}$$

Since $|F^i| \leq 2n - 4$ for $i = 1, 3$, by the induction hypothesis, there exists an $(\ell_i + 1)$ -cycle C_i in $BH_{n-1}^i \setminus F$ containing (u^i, v^i) if $5 \leq \ell_i \leq 2^{2n-2} - 1$ for $i = 1, 3$. Let

$$P_0 = \langle c^{2^{2n-3}+t+5}, c^{2^{2n-3}+t+6}, \dots, c^{2^{2n-2}-1}, v, u, c^2, \dots, c^t \rangle \text{ with length } \ell_0,$$

$$P_1 = \begin{cases} (v^1, u^1) & \text{if } \ell_1 = 1, \\ C_1 - (v^1, u^1) & \text{if } 5 \leq \ell_1 \leq 2^{2n-2} - 1, \end{cases}$$

$$P_2 = \begin{cases} P'_2 & \text{if } \ell_2 = 2^{2n-3} - 1, \\ P''_2 & \text{if } \ell_2 = 2^{2n-3} + 1, \end{cases}$$

$$P_3 = \begin{cases} (v^3, u^3) & \text{if } \ell_3 = 1, \\ C_3 - (v^3, u^3) & \text{if } 5 \leq \ell_3 \leq 2^{2n-2} - 1, \end{cases}$$

Then, $C = \langle c^t, v^1, P_1, u^1, v^2, P_2, u^2, v^3, P_3, u^3, c^{2^{2n-3}+t+5}, P_0, c^t \rangle$ (see figure 4) is the desired cycle.

Subcase 1.2.2: $1 \leq |V(P'_2)| \leq 2^{2n-3} - 3$.

We can represent $\ell = \ell_0 + \ell_1 + \ell_2 + \ell_3 + 4$, where ℓ_i satisfies one of the following conditions for $i = 0, 1, 2, 3$.

$$\begin{aligned} \ell_0 = 2^{2n-3} - 5, \quad 5 \leq \ell_1 \leq 2^{2n-2} - 1, \quad \ell_2 = |V(P'_2)|, \quad \ell_3 = 1 & \quad \text{or} \\ \ell_0 = 2^{2n-3} - 5, \quad 5 \leq \ell_1 \leq 2^{2n-2} - 1, \quad \ell_2 = |V(P'_2)|, \quad 5 \leq \ell_3 \leq 2^{2n-2} - 1. \end{aligned}$$

Since $|F^i| \leq 2n - 4$ for $i = 1, 3$, by the induction hypothesis, there exists an $(\ell_i + 1)$ -cycle C_i in $BH_{n-1}^i \setminus F$ containing (u^i, v^i) if $5 \leq \ell_i \leq 2^{2n-2} - 1$ for $i = 1, 3$. Let

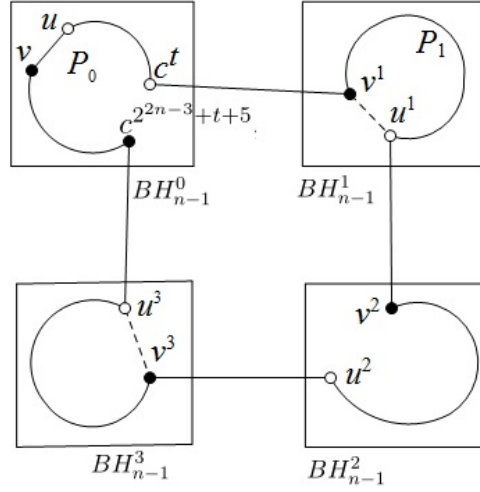


Figure 4: Illustration for the cycle C of subcase 1.2.1 and subcase 1.2.2 in theorem 3.2.

$P_0 = \langle c^{2^{2n-3}+t+5}, c^{2^{2n-3}+t+6}, \dots, c^{2^{2n-2}-1}, v, u, c^2, \dots, c^t \rangle$ with length ℓ_0 ,

$$P_1 = \begin{cases} (v^1, u^1) & \text{if } \ell_1 = 1, \\ C_1 - (v^1, u^1) & \text{if } 5 \leq \ell_1 \leq 2^{2n-2} - 1, \end{cases}$$

$$P_2 = P'_2,$$

$$P_3 = \begin{cases} (v^3, u^3) & \text{if } \ell_3 = 1, \\ C_3 - (v^3, u^3) & \text{if } 5 \leq \ell_3 \leq 2^{2n-2} - 1, \end{cases}$$

Then, $C = \langle c^t, v^1, P_1, u^1, v^2, P_2, u^2, v^3, P_3, u^3, c^{2^{2n-3}+t+5}, P_0, c^t \rangle$ (see figure 4) is the desired cycle.

Subcase 1.3: $2^{2n-1} + 8 \leq \ell \leq 2^{2n}$.

We can represent $\ell = \ell_0 + \ell_1 + \ell_2 + \ell_3 + 4$, where ℓ_i satisfies one of the following conditions for $i = 0, 1, 2, 3$.

$$\begin{aligned} \ell_0 = 2^{2n-2} - 1, \quad 5 \leq \ell_1 \leq 2^{2n-2} - 1, \quad \ell_2 = 2^{2n-2} - 1, \quad \ell_3 = 1 & \quad \text{or} \\ \ell_0 = 2^{2n-2} - 1, \quad 5 \leq \ell_1 \leq 2^{2n-2} - 1, \quad \ell_2 = 2^{2n-2} - 1, \quad 5 \leq \ell_3 \leq 2^{2n-2} - 1. \end{aligned}$$

By the induction hypothesis, there exists a fault-free Hamiltonian cycle C_0 in BH_{n-1}^0 containing e , say $\langle c^1, c^2, \dots, c^{2^{2n-2}}, c^1 \rangle$ with $c^1 = u, c^{2^{2n-2}} = v$. Let $M = \{(c^1, c^2), \dots, (c^{2^{2i-1}}, c^{2^i}), \dots, (c^{2^{2n-2}-1}, c^{2^{2n-2}})\}$, then M is a set with 2^{2n-3} mutually disjoint edges. There exists an edge $(c^{2^{2t-1}}, c^{2^{2t}})$ in M such that

$$|F \cap \{e_1, e_2 \mid e_1, e_2 \text{ are two } (n-1)\text{-dimensional edges incident with } c^{2^{2t-1}}\}| \leq 1 \text{ and}$$

$$|F \cap \{e_3, e_4 \mid e_3, e_4 \text{ are two } (n-1)\text{-dimensional edges incident with } c^{2^{2t}}\}| \leq 1$$

since $2 \cdot (2^{2n-3}) > 2n - 2$ for all $n \geq 3$. Let $(c^{2^{2t-1}}, v^1), (c^{2^{2t}}, u^3)$ be two fault-free $(n-1)$ -dimensional edges where $v^1 \in BH_{n-1}^1, u^3 \in BH_{n-1}^3$. By Lemma 2.9, there exists a fault-free 2-path $\langle u^3, v^3, u^2 \rangle$ and a fault-free 2-path $\langle v^1, u^1, v^2 \rangle$ where $v^i, u^i \in BH_{n-1}^i$ for $i = 1, 2, 3$. By Lemma 2.4, there exists a Hamiltonian path P_2 in $BH_{n-1}^2 \setminus F$ joining v^2 to u^2 . Note that $|F^i| \leq 2n - 4$, by the induction hypothesis, there exists an $(\ell_i + 1)$ -cycle C_i in $BH_{n-1}^i \setminus F$ containing (u^i, v^i) where $5 \leq \ell_i \leq 2^{2n-2} - 1$ for $i = 1, 3$. Let

$$P_0 = C_0 - (c^{2^{2t-1}}, c^{2^{2t}}),$$

$$P_1 = \begin{cases} (v^1, u^1) & \text{if } \ell_1 = 1, \\ C_1 - (v^1, u^1) & \text{if } 5 \leq \ell_1 \leq 2^{2n-2} - 1, \end{cases}$$

P_2 be the Hamiltonian path of BH_{n-1}^2 joining v^2 to u^2 ,

$$P_3 = \begin{cases} (v^3, u^3) & \text{if } \ell_3 = 1, \\ C_3 - (v^3, u^3) & \text{if } 5 \leq \ell_3 \leq 2^{2n-2} - 1, \end{cases}$$

Then, $C = \langle c^{2t-1}, v^1, P_1, u^1, v^2, P_2, u^2, v^3, P_3, u^3, c^{2t}, P_0, c^{2t-1} \rangle$ (see figure 5) is the desired cycle.

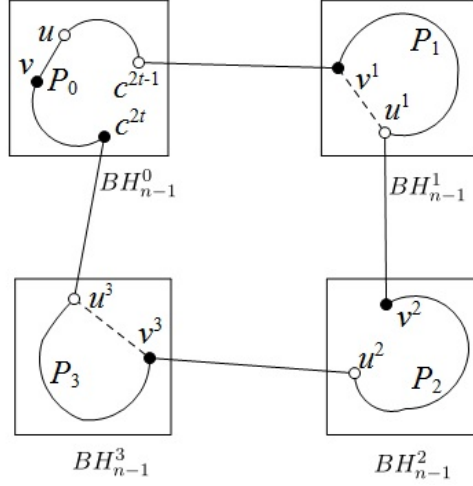


Figure 5: Illustration for the cycle C of subcase 1.3 in theorem 3.2.

Case 2: $e = (u, v) \in \partial D_{n-1}$.

Subcase 2.1: $|F_{n-1}| \leq 2n - 3$.

We divide BH_n into four parts, $BH_{n-1}^{n-2,0}$, $BH_{n-1}^{n-2,1}$, $BH_{n-1}^{n-2,2}$, and $BH_{n-1}^{n-2,3}$. If $|F| \leq 2n - 3$, then, $|F \cap (\cup_{i=0}^3 BH_{n-1}^{n-2,i})| \leq 2n - 3$. If $|F| = 2n - 2$, note that $|F_{n-1}| \geq |F_{n-2}| \geq \dots \geq |F_0|$, $|F_{n-1}| \leq 2n - 3$, we have $|F_{n-2}| = 1$. As a result, $|F \cap (\cup_{i=0}^3 BH_{n-1}^{n-2,i})| \leq 2n - 3$.

Subcase 2.1.1: $|F \cap BH_{n-1}^{n-2,i}| \leq 2n - 4$ for all $i = 0, 1, 2, 3$.

By a similar discussion as case 1, we obtain the result.

Subcase 2.1.2: There exists an $i \in \{0, 1, 2, 3\}$ such that $|F \cap BH_{n-1}^{n-2,i}| = 2n - 3$.

Without loss of generality, we can assume that $|F \cap BH_{n-1}^{n-2,0}| = 2n - 3$. Thus, $|F \cap (BH_n \setminus BH_{n-1}^{n-2,0})| \leq 1$ and $|F \cap BH_{n-1}^{n-2,i}| = 0$ for $i = 1, 2, 3$.

Subcase 2.1.2.1: $e \in BH_{n-1}^{n-2,0}$.

Subcase 2.1.2.1.1: $\ell = 6$.

Note that e is a fault-free edge and there are $(4n - 6)$ different 2-paths in $BH_{n-1}^{n-2,0}$ containing e . Since $4n - 6 - (2n - 3) = 2n - 3 \geq 1$, there exists at least one fault-free 2-path in $BH_{n-1}^{n-2,0}$ containing e , say $\langle u, v, w \rangle$. Without loss of generality, let v be a black vertex and u, w be two white vertices. Notice that $|F_{n-2}| \leq 1$, we obtain that there exists two fault-free $(n - 2)$ -dimensional edges $(u, u^1), (w, w^1)$ where $u^1, w^1 \in BH_{n-1}^{n-2,1}$. It is easy to verify that $d(u^1, w^1) = 2$. Suppose that v^1 is the vertex that is adjacent to both w^1 and u^1 . Since $|F \cap BH_{n-1}^{n-2,1}| = 0$, then $C = \langle u, v, w, w^1, v^1, u^1, u \rangle$ (see figure 6) is the desired cycle.

Subcase 2.1.2.1.2: $\ell = 8$.

By Lemma 2.6, there are two 8-cycles C_1, C_2 in BH_n containing e such that $E(C_1) \cap E(C_2) = e$ and $|E(C_i) \cap BH_{n-1}^{n-2,j}| = 1$ for $i = 1, 2, j = 0, 1, 2, 3$. Note that $|F \cap (BH_n \setminus BH_{n-1}^{n-2,0})| \leq 1$. There exists at least one fault-free 8-cycle that contains e , say $\langle u, v^1, u^1, v^2, u^2, v^3, u^3, v, u \rangle$ where $u^i, v^i \in BH_{n-1}^{n-2,i}$ for $1 \leq i \leq 3$.

Subcase 2.1.2.1.3: $10 \leq \ell \leq 3 \cdot 2^{2n-2} + 2$.

We can represent $\ell = \ell_0 + \ell_1 + \ell_2 + \ell_3 + 4$, where ℓ_i satisfies one of the following conditions for $i = 0, 1, 2, 3$.

$$\begin{array}{llll} \ell_0 = 1, 3 \leq \ell_1 \leq 2^{2n-2} - 1, & \ell_2 = 1, & \ell_3 = 1 & \text{or} \\ \ell_0 = 1, 3 \leq \ell_1 \leq 2^{2n-2} - 1, & 3 \leq \ell_2 \leq 2^{2n-2} - 1, & \ell_3 = 1 & \text{or} \\ \ell_0 = 1, 3 \leq \ell_1 \leq 2^{2n-2} - 1, & 3 \leq \ell_2 \leq 2^{2n-2} - 1, & 3 \leq \ell_3 \leq 2^{2n-2} - 1. & \end{array}$$

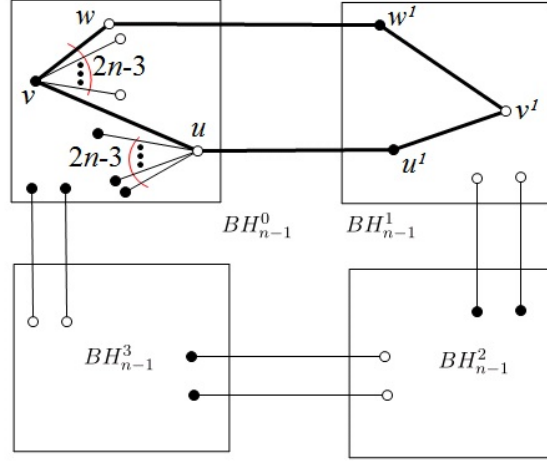


Figure 6: Illustration for the cycle C of subcase 2.1.2.1.1 in theorem 3.2

Let $\langle u, v^1, u^1, v^2, u^2, v^3, u^3, v, u \rangle$ be a fault-free 8-cycle where $u^i, v^i \in BH_{n-1}^{n-2,i}$ for $1 \leq i \leq 3$. Since $|F \cap BH_{n-1}^{n-2,i}| = 0$ for $i = 1, 2, 3$. By Lemma 2.5, there exists an $(\ell_i + 1)$ -cycle C_i in $BH_{n-1}^{n-2,i}$ containing (u^i, v^i) where $3 \leq \ell_i \leq 2^{2n-2} - 1$ for $1 \leq i \leq 3$. Let

$$P_1 = \begin{cases} (v^1, u^1) & \text{if } \ell_1 = 1, \\ C_1 - (v^1, u^1) & \text{if } 3 \leq \ell_1 \leq 2^{2n-2} - 1, \end{cases}$$

$$P_2 = \begin{cases} (v^2, u^2) & \text{if } \ell_2 = 1, \\ C_2 - (v^2, u^2) & \text{if } 3 \leq \ell_2 \leq 2^{2n-2} - 1, \end{cases}$$

$$P_3 = \begin{cases} (v^3, u^3) & \text{if } \ell_3 = 1, \\ C_3 - (v^3, u^3) & \text{if } 3 \leq \ell_3 \leq 2^{2n-2} - 1, \end{cases}$$

Then, $C = \langle u, v^1, P_1, u^1, v^2, P_2, u^2, v^3, P_3, u^3, v, u \rangle$ (see figure 7) forms the desired cycle.

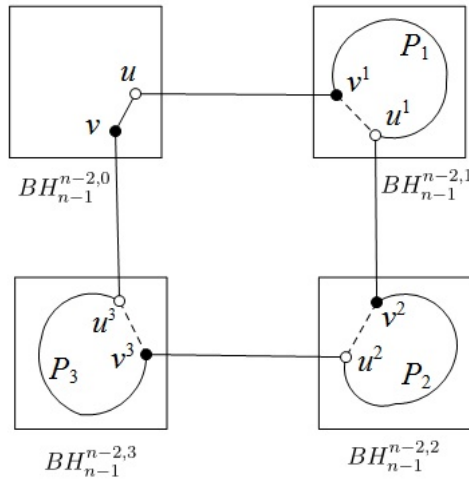


Figure 7: Illustration for the cycle C of subcase 2.1.2.1.3 in theorem 3.2

Subcase 2.1.2.1.4: $3 \cdot 2^{2n-2} + 4 \leq \ell \leq 2^{2n}$.

We can represent $\ell = \ell_0 + \ell_1 + \ell_2 + \ell_3 + 4$, where $\ell_0 = 2^{2n-2} - 1, 3 \leq \ell_i \leq 2^{2n-2} - 1$ for $i = 1, 2, 3$.

Let $\bar{e} = (u^0, v^0)$ be any faulty edge in $BH_{n-1}^{n-2,0}$. By the induction hypothesis, there exists a Hamiltonian cycle C_0 in $BH_{n-1}^{n-2,0} - F + \{\bar{e}\}$ containing e . Obviously, $|F \cap E(C_0)| \leq 1$. If $|F \cap E(C_0)| = 1$, then $\bar{e} \in E(C_0)$, we can assume that $(a^0, b^0) = \bar{e}$. If $|F \cap E(C_0)| = 0$, let (a^0, b^0) be any edge in $E(C_0) \setminus \{e\}$. Note that $|F \cap (BH_n \setminus BH_{n-1}^{n-2,0})| \leq 1$, by Lemma 2.6, there exists a fault-free

8-cycle $\langle a^0, b^1, a^1, b^2, a^2, b^3, a^3, b^0, a^0 \rangle$ in BH_n where $a^i, b^i \in BH_{n-1}^{n-2,i}$ for $i = 0, 1, 2, 3$. Note that $|F \cap BH_{n-1}^{n-2,i}| = 0$ for $1 \leq i \leq 3$, by Lemma 2.5, there exists an $(\ell_i + 1)$ -cycle C_i in $BH_{n-1}^{n-2,i}$ containing (a^i, b^i) where $3 \leq \ell_i \leq 2^{2n-2} - 1$ for $1 \leq i \leq 3$.

Let $P_i = C_i - (b^i, a^i)$ for $i = 0, 1, 2, 3$, then $C = \langle a^0, b^1, P_1, a^1, b^2, P_2, a^2, b^3, P_3, a^3, b^0, P_0, a^0 \rangle$ (see figure 8) forms the desired cycle.

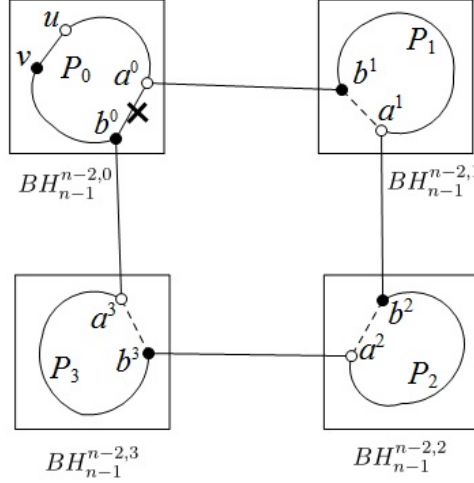


Figure 8: Illustration for the cycle C of subcase 2.1.2.1.4 in theorem 3.2

Subcase 2.1.2.2: $e \in BH_{n-1}^{n-2,i}$ where $i = 1, 2, 3$.

Without loss of generality, we assume that $e \in BH_{n-1}^{n-2,1}$.

Subcase 2.1.2.2.1: $6 \leq \ell \leq 2^{2n-2}$.

Since $|F \cap BH_{n-1}^{n-2,1}| = 0$, by the induction hypothesis, it holds.

Subcase 2.1.2.2.2: $2^{2n-2} + 2 \leq \ell \leq 2^{2n-1} - 2$.

We can represent $\ell = \ell_1 + \ell_2 + 2$, where $2 \leq \ell_1 \leq 2^{2n-2} - 2, \ell_2 = 2^{2n-2} - 2$.

Since $|F \cap BH_{n-1}^{n-2,1}| = 0$, by Lemma 2.5, there exists a Hamiltonian cycle C_1 in $BH_{n-1}^{n-2,1}$ containing e , say $\langle c^0, c^1, \dots, c^{2^{2n-2}-1}, c^0 \rangle$, where $c^0 = u, c^1 = v$. Let ℓ_1 be an even integer. Then, c^{ℓ_1} is a white vertex and $\langle u, c^1, c^2, \dots, c^{\ell_1} \rangle$ is an ℓ_1 -path in $BH_{n-1}^{n-2,1}$ containing e where $2 \leq \ell_1 \leq 2^{2n-2} - 2$. Notice that $|F_{n-2}| \leq 1$. We can assume that $(c^{\ell_1}, u^2), (u, v^2)$ are two fault-free $(n-2)$ -dimensional edges where $u^2, v^2 \in BH_{n-1}^{n-2,2}$ since every vertex has two extra neighbors. By Lemma 2.8, there exists a $(2^{2n-2} - 2)$ -path in $BH_{n-1}^{n-2,2}$ joining u^2 to v^2 . Let

$$P_1 = \langle u, v, c^2, c^3, \dots, c^{\ell_1} \rangle,$$

P_2 be the path of length $2^{2n-2} - 2$ in $BH_{n-1}^{n-2,2}$ joining u^2 and v^2 .

Then, the cycle $C = \langle u, P_1, c^{\ell_1}, u^2, P_2, v^2, u \rangle$ (see figure 9) forms the desired cycles.

Subcase 2.1.2.2.3: $2^{2n-1} \leq \ell \leq 2^{2n-1} + 8$.

We can represent $\ell = \ell_0 + \ell_1 + \ell_2 + \ell_3 + 4$, where $\ell_0 = 1, \ell_1 = 5, \ell_2 = 2^{2n-2} - 1, 2^{2n-2} - 9 \leq \ell_3 \leq 2^{2n-2} - 1$.

Let $\langle u, v, w^1, x^1, y^1, z^1 \rangle$ be a fault-free 5-path of $BH_{n-1}^{n-2,1}$ and $(z^1, u^0), (u, v^2)$ be two fault-free $(n-2)$ -dimensional edges where $u^0 \in BH_{n-1}^{n-2,0}, v^2 \in BH_{n-1}^{n-2,2}$. By Lemma 2.10, there exists a 2-path $\langle u^0, v^0, u^3 \rangle$ and a 2-path $\langle u^3, v^3, u^2 \rangle$ where $u^i, v^i \in BH_{n-1}^{n-2,i}$. By Lemma 2.5, there exists a $(\ell_3 + 1)$ -cycle of $BH_{n-1}^{n-2,3}$ containing (u^3, v^3) where $2^{2n-2} - 5 \leq \ell_3 \leq 2^{2n-2} - 1$.

By Lemma 2.4, there exists a Hamiltonian path P_2 in $BH_{n-1}^{n-2,2}$ joining u^2 and v^2 . Let $P_1 = \langle u, v, w^1, x^1, y^1, z^1 \rangle, P_3 = C_3 - (u_3, v_3)$. Then, $C = \langle u, P_1, z^1, u^0, v^0, u^3, P_3, v^3, u^2, P_2, v^2, u \rangle$ (see figure 10) is the desired cycle.

Subcase 2.1.2.2.4: $2^{2n-1} + 10 \leq \ell \leq 2^{2n}$.

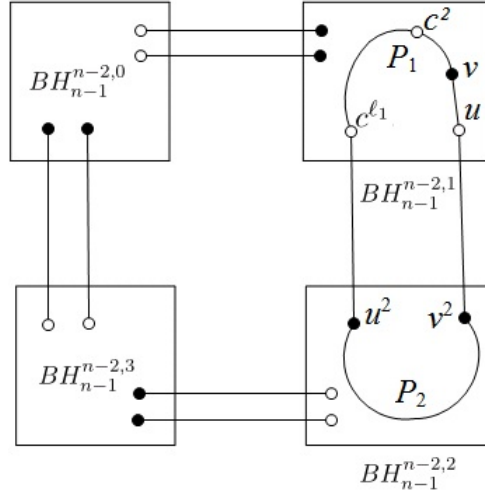


Figure 9: Illustration for the cycle C of subcase 2.1.2.2 in theorem 3.2

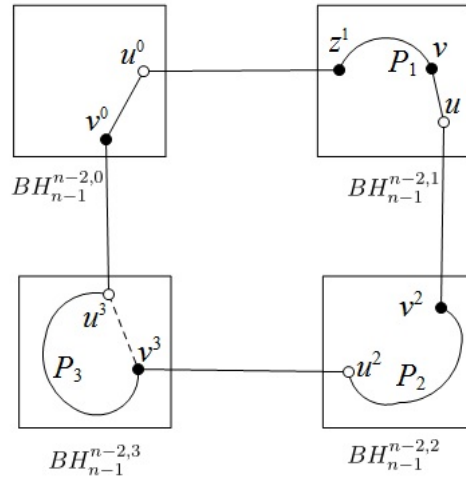


Figure 10: Illustration for the cycle C of subcase 2.1.2.3 in theorem 3.2

We can represent $\ell = \ell_0 + \ell_1 + \ell_2 + \ell_3 + 4$ where $5 \leq \ell_0 \leq 2^{2n-2} - 1, \ell_1 = 2^{2n-2} - 1, \ell_2 = 2^{2n-2} - 1, 3 \leq \ell_3 \leq 2^{2n-2} - 1$.

Let (a^0, b^0) be a faulty edge in $BH_{n-1}^{n-2,0}$, where a^0 is a white vertex. Assume that $(a^0, b^1), (b^0, a^3), (a^3, b^3)$, and (b^3, a^2) are fault-free edges where $a^i, b^i \in BH_{n-1}^{n-2,i}$ for $i = 0, 1, 2, 3$. Note that $|F \cap BH_{n-1}^{n-2,1}| = 0$, by Lemma 2.5, there exists a Hamiltonian cycle C_1 in $BH_{n-1}^{n-2,1}$ containing e . Suppose that $N_{C_1}(b^1) = \{a^1, c^1\}$. Thus, $(b^1, a^1) \neq e$ or $(b^1, c^1) \neq e$. Without loss of generality, assume that $(b^1, a^1) \neq e$. Note that $|N_{BH_{n-1}^{n-2,2}}(a^1)| = 2$ and $|F_{n-2}| \leq 1$. Suppose that (a^1, b^2) is a fault-free edge where $b^2 \in BH_{n-1}^{n-2,2}$. By Lemma 2.4, there exists a fault-free Hamiltonian path P_2 in $BH_{n-1}^{n-2,2}$ joining a^2 and b^2 . By the induction hypothesis, there exists an $(\ell_0 + 1)$ -cycle C_0 in $BH_{n-1}^{n-2,0} - F + (a^0, b^0)$ containing (a^0, b^0) where $5 \leq \ell_0 \leq 2^{2n-2} - 1$. By Lemma 2.5, there exists an $(\ell_3 + 1)$ -cycle C_3 in $BH_{n-1}^{n-2,3}$ containing a^3, b^3 where $3 \leq \ell_3 \leq 2^{2n-2} - 1$. Let

$$\begin{aligned} P_0 &= C_0 - (a^0, b^0), \\ P_1 &= C_1 - (b^1, a^1), \\ P_2 &\text{ be the Hamiltonian path joining } a^2 \text{ and } b^2, \\ P_3 &= C_3 - (a^3, b^3). \end{aligned}$$

Then, $C = \langle a^0, P_0, b^0, a^3, P_3, b^3, a^2, P_2, b^2, a^1, P_1, b^1, a^0 \rangle$ (see figure 11) is the desired cycle.

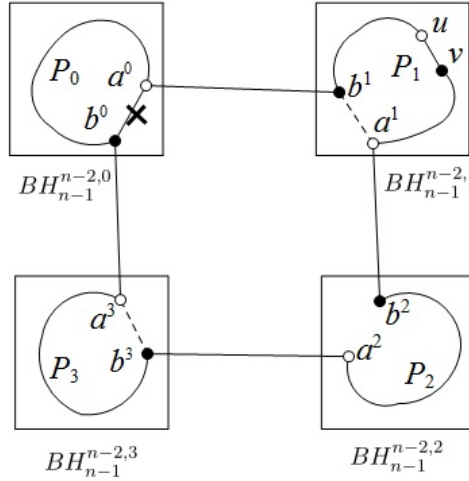


Figure 11: Illustration for the cycle C of subcase 2.1.2.2.4 in theorem 3.2

Subcase 2.2: $|F_{n-1}| = 2n - 2$.

Subcase 2.2.1: $\ell = 6$.

Without loss of generality, we can assume that $e \in BH_{n-1}^{n-2,0}$. If $|F \cap BH_{n-1}^{n-2,0}| \leq 2n - 4$, by the induction hypothesis, there exists a 6-cycle in $BH_{n-1}^{n-2,0}$. Thus, we assume that $|F \cap BH_{n-1}^{n-2,0}| = 2n - 3$ or $2n - 2$. Note that e is a fault-free edge and there are $(4n - 6)$ different 2-paths in $BH_{n-1}^{n-2,0}$ containing e . Since $4n - 6 - (2n - 2) = 2n - 4 \geq 1$, there exists at least one fault-free 2-path in $BH_{n-1}^{n-2,0}$ containing e , say $\langle u, v, w \rangle$. Without loss of generality, let v be a black vertex and u, w be two white vertices. Notice that $|F_{n-2}| = 0$, we can assume that $(u, u^1), (w, w^1)$ are two fault-free $(n - 2)$ -dimensional edges where $u^1, w^1 \in BH_{n-1}^{n-2,1}$. It is easy to check that $d(u^1, w^1) = 2$. Suppose that v^1 is the vertex that is adjacent to both w^1 and u^1 . Let \bar{v}^1 be the vertex such that v^1 and \bar{v}^1 differ in only the inner index. Then, $(w^1, \bar{v}^1), (u^1, \bar{v}^1) \in E(BH_n)$. Since $|F \cap BH_{n-1}^{n-2,1}| \leq 1$, then $\langle u, v, w, w^1, v^1, u^1, u \rangle$ or $\langle u, v, w, w^1, \bar{v}^1, u^1, u \rangle$ is the desired cycle.

Subcase 2.2.2: $\ell = 8$.

By Lemma 2.10, there exists a fault-free 8-cycle $\langle u = u^0, v = v^1, u^1, v^2, u^2, v^3, u^3, v^0, u \rangle$.

Subcase 2.2.3: $10 \leq \ell \leq 2^{2n}$.

We can represent $\ell = \ell_0 + \ell_1 + \ell_2 + \ell_3 + 4$, where ℓ_i satisfies one of the following conditions for

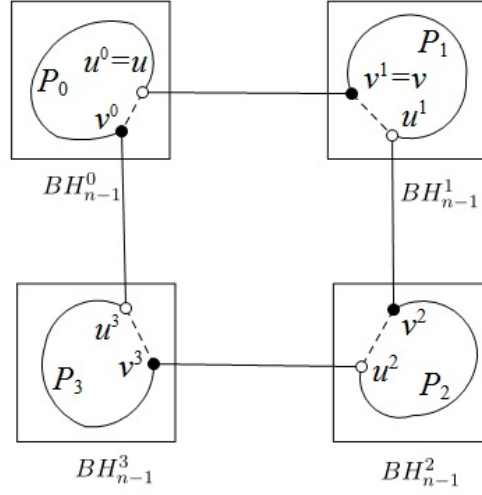


Figure 12: Illustration for the cycle C of subcase 2.2.3 in theorem 3.2

$i = 0, 1, 2, 3$.

$$\begin{array}{llllll}
3 \leq \ell_0 \leq 2^{2n-2} - 1, & \ell_1 = 1, & \ell_2 = 1, & \ell_3 = 1 & \text{or} \\
3 \leq \ell_0 \leq 2^{2n-2} - 1, & 3 \leq \ell_1 \leq 2^{2n-2} - 1, & \ell_2 = 1, & \ell_3 = 1 & \text{or} \\
3 \leq \ell_0 \leq 2^{2n-2} - 1, & 3 \leq \ell_1 \leq 2^{2n-2} - 1, & 3 \leq \ell_2 \leq 2^{2n-2} - 1, & \ell_3 = 1 & \text{or} \\
3 \leq \ell_0 \leq 2^{2n-2} - 1, & 3 \leq \ell_1 \leq 2^{2n-2} - 1, & 3 \leq \ell_2 \leq 2^{2n-2} - 1, & 3 \leq \ell_3 \leq 2^{2n-2} - 1. &
\end{array}$$

Note that $F_{n-1} = 2n - 2$, we have $F \cap BH_{n-1}^i = \emptyset$ for all $i = 0, 1, 2, 3$. By Lemma 2.5, there exists an $(\ell_i + 1)$ -cycle C_i in BH_{n-1}^i containing (u^i, v^i) where $3 \leq \ell_i \leq 2^{2n-2} - 1$ for $i = 0, 1, 2, 3$.

Let

$$\begin{aligned}
P_0 &= \begin{cases} (v^0, u^0) & \text{if } \ell_0 = 1, \\ C_0 - (v^0, u^0) & \text{if } 3 \leq \ell_0 \leq 2^{2n-2} - 1, \end{cases} \\
P_1 &= \begin{cases} (v^1, u^1) & \text{if } \ell_1 = 1, \\ C_1 - (v^1, u^1) & \text{if } 3 \leq \ell_1 \leq 2^{2n-2} - 1, \end{cases} \\
P_2 &= \begin{cases} (v^2, u^2) & \text{if } \ell_2 = 1, \\ C_2 - (v^2, u^2) & \text{if } 3 \leq \ell_2 \leq 2^{2n-2} - 1, \end{cases} \\
P_3 &= \begin{cases} (v^3, u^3) & \text{if } \ell_3 = 1, \\ C_3 - (v^3, u^3) & \text{if } 3 \leq \ell_3 \leq 2^{2n-2} - 1, \end{cases}
\end{aligned}$$

Then, $C = \langle v^0, P_0, u^0, v^1, P_1, u^1, v^2, P_2, u^2, v^3, P_3, u^3, v^0 \rangle$ (see figure 12) forms the desired cycle. \square

Appendix A. Proof of Lemma 3.1

Lemma 3.1 The balanced hypercube BH_2 is 2-edge-fault-tolerant 6-bipancyclic.

By Lemma 2.4, for an arbitrary fault-free edge (u, v) , there exists a fault-free Hamiltonian path P that joins u and v , then $\langle u, P, v, u \rangle$ is the fault-free 16-cycle. Hence, we only need to construct a fault-free ℓ -cycle in BH_2 containing (u, v) where $6 \leq \ell \leq 14$. Suppose that $|F| = 2$, without loss of generality, we can assume that $|F \cap \partial D_1| \geq |F \cap \partial D_0|$.

Case 1: $|F \cap \partial D_1| = 2, |F \cap \partial D_0| = 0$.

Subcase 1.1: $e = (u, v) \in \partial D_0$.

Without loss of generality, we can assume that $e = (u, v) \in BH_1^0$. Suppose that $u = (a_0, 0)$ is a white vertex, $v = (b_0, 0)$ is a black vertex.

Subcase 1.1.1: $6 \leq \ell \leq 14$.

There are three ℓ -cycles C_1, C_2, C_3 in BH_n containing e where $6 \leq \ell \leq 14$, such that $(C_i \cap \partial D_1) \cap (C_j \cap \partial D_1) = \emptyset$ for all $1 \leq i \neq j \leq 3$. We list them as follows:

Three 6-cycles:

$$\begin{cases} \langle (a_0, 0), (b_0, 0), (a_0, 3), (b_0 + 2, 3), (a_0 + 2, 3), (b_0 + 2, 0), (a_0, 0) \rangle; \\ \langle (a_0, 0), (b_0, 0), (a_0 + 2, 3), (b_0, 3), (a_0, 3), (b_0 + 2, 0), (a_0, 0) \rangle; \\ \langle (a_0, 0), (b_0, 0), (a_0 + 2, 0), (b_0, 1), (a_0, 1), (b_0 + 2, 1), (a_0, 0) \rangle. \end{cases}$$

Three 8-cycles:

$$\begin{cases} \langle (a_0, 0), (b_0, 0), (a_0, 3), (b_0, 3), (a_0, 2), (b_0 + 2, 3), (a_0 + 2, 3), (b_0 + 2, 0), (a_0, 0) \rangle; \\ \langle (a_0, 0), (b_0, 0), (a_0 + 2, 3), (b_0, 3), (a_0 + 2, 2), (b_0 + 2, 3), (a_0, 3), (b_0 + 2, 0), (a_0, 0) \rangle; \\ \langle (a_0, 0), (b_0, 0), (a_0 + 2, 0), (b_0, 1), (a_0 + 2, 1), (b_0, 2), (a_0, 1), (b_0 + 2, 1), (a_0, 0) \rangle. \end{cases}$$

Three 10-cycles:

$$\begin{cases} \langle (a_0, 0), (b_0, 0), (a_0, 3), (b_0, 3), (a_0, 2), (b_0, 2), (a_0 + 2, 2), (b_0 + 2, 3), (a_0 + 2, 3), (b_0 + 2, 0), (a_0, 0) \rangle; \\ \langle (a_0, 0), (b_0, 0), (a_0 + 2, 3), (b_0, 3), (a_0 + 2, 2), (b_0, 2), (a_0, 2), (b_0 + 2, 3), (a_0, 3), (b_0 + 2, 0), (a_0, 0) \rangle; \\ \langle (a_0, 0), (b_0, 0), (a_0 + 2, 0), (b_0, 1), (a_0 + 2, 1), (b_0 + 2, 2), (a_0 + 2, 2), (b_0, 2), (a_0, 1), (b_0 + 2, 1), (a_0, 0) \rangle. \end{cases}$$

Three 12-cycles:

$$\begin{cases} \langle (a_0, 0), (b_0, 0), (a_0, 3), (b_0, 3), (a_0 + 2, 3), (b_0 + 2, 3), (a_0 + 2, 2), (b_0, 2), (a_0, 2), (b_0 + 2, 2), (a_0, 1), \\ (b_0, 1), (a_0, 0) \rangle; \\ \langle (a_0, 0), (b_0, 0), (a_0 + 2, 3), (b_0 + 2, 3), (a_0, 3), (b_0, 3), (a_0, 2), (b_0 + 2, 2), (a_0 + 2, 2), (b_0, 2), (a_0 + 2, 1), \\ (b_0 + 2, 1), (a_0, 0) \rangle; \\ \langle (a_0, 0), (b_0, 0), (a_0 + 2, 0), (b_0, 1), (a_0 + 2, 1), (b_0 + 2, 2), (a_0 + 2, 2), (b_0, 2), (a_0, 2), (b_0 + 2, 3), (a_0, 3), \\ (b_0 + 2, 0), (a_0, 0) \rangle. \end{cases}$$

Three 14-cycles:

$$\begin{cases} \langle (a_0, 0), (b_0, 0), (a_0, 3), (b_0, 3), (a_0 + 2, 3), (b_0 + 2, 3), (a_0 + 2, 2), (b_0, 2), (a_0, 2), (b_0 + 2, 2), (a_0 + 2, 1), \\ (b_0, 1), (a_0, 1), (b_0 + 2, 1), (a_0, 0) \rangle; \\ \langle (a_0, 0), (b_0, 0), (a_0 + 2, 3), (b_0 + 2, 3), (a_0, 3), (b_0, 3), (a_0 + 2, 2), (b_0 + 2, 2), (a_0, 2), (b_0, 2), (a_0 + 2, 1), \\ (b_0 + 2, 1), (a_0, 1), (b_0, 1), (a_0, 0) \rangle; \\ \langle (a_0, 0), (b_0, 0), (a_0 + 2, 0), (b_0, 1), (a_0, 1), (b_0 + 2, 2), (a_0 + 2, 2), (b_0, 2), (a_0, 2), (b_0 + 2, 3), (a_0, 3), \\ (b_0, 3), (a_0 + 2, 3), (b_0 + 2, 0), (a_0, 0) \rangle. \end{cases}$$

Notice that $|F \cap \partial D_1| = 2, |F \cap \partial D_0| = 0$, there exists at least one fault-free ℓ -cycle in BH_2 containing e where $6 \leq \ell \leq 14$.

Subcase 1.2: $e = (u, v) \in \partial D_1$.

Without loss of generality, we can assume that $e = (u, v) = (u^0, v^1)$ is an edge between BH_1^0 and BH_1^1 where $u^0 = (a_0, 0), v^1 = (b_0, 1)$.

Subcase 1.2.1: $\ell = 6, 8$.

There exist three ℓ -cycles C_1, C_2, C_3 in BH_2 containing e where $\ell = 6$ or 8 , such that $(C_i \cap \partial D_1) \cap (C_j \cap \partial D_1) = \{e\}$ for $1 \leq i \neq j \leq 3$. We list them as follows:

Three 6-cycles:

$$\begin{cases} \langle (a_0, 0), (b_0, 1), (a_0, 1), (b_0 + 2, 1), (a_0 + 2, 0), (b_0, 0), (a_0, 0) \rangle; \\ \langle (a_0, 0), (b_0, 1), (a_0, 1), (b_0 + 2, 2), (a_0 + 2, 1), (b_0 + 2, 1), (a_0, 0) \rangle; \\ \langle (a_0, 0), (b_0, 1), (a_0 + 2, 0), (b_0 + 2, 0), (a_0, 3), (b_0, 0), (a_0, 0) \rangle. \end{cases}$$

Three 8-cycles:

$$\begin{cases} \langle (a_0, 0), (b_0, 1), (a_0, 1), (b_0, 2), (a_0, 2), (b_0, 3), (a_0, 3), (b_0, 0), (a_0, 0) \rangle; \\ \langle (a_0, 0), (b_0, 1), (a_0 + 2, 1), (b_0 + 2, 2), (a_0 + 2, 2), (b_0 + 2, 3), (a_0 + 2, 3), (b_0 + 2, 0), (a_0, 0) \rangle; \\ \langle (a_0, 0), (b_0, 1), (a_0, 1), (b_0 + 2, 2), (a_0, 2), (b_0 + 2, 3), (a_0, 3), (b_0 + 2, 0), (a_0, 0) \rangle. \end{cases}$$

Since $|F \cap \partial D_1| = 2, |F \cap \partial D_0| = 0$, and e is a fault-free edge, then there exists at least one fault-free 6-cycle and one fault-free 8-cycle in BH_2 containing e .

Subcase 1.2.2: $10 \leq \ell \leq 14$.

By the proof of subcase 1.2.1, there exists a fault-free 8-cycle C that contains e such that $|C \cap BH_1^i| = 1$ for $0 \leq i \leq 3$, say $\langle u^0, v^0, u^3, v^3, u^2, v^2, u^1, v^1, u^0 \rangle$ where $u^i, v^i \in BH_{n-1}^i$ for $i = 0, 1, 2, 3$. Since $|F \cap \partial D_0| = 0$. It is easy to check that there exists an ℓ_i -path P_i in BH_1^i joining u^i to v^i where $\ell_i = 1$ or 3 for $i = 1, 2, 3$. Then, the cycle $\langle u^0, v^0, u^3, P_3, v^3, u^2, P_2, v^2, u^1, P_1, v^1, u^0 \rangle$ with length $\ell = 5 + \ell_1 + \ell_2 + \ell_3$ forms the desired cycle.

Case 2: $|F \cap \partial D_1| = 1, |F \cap \partial D_0| = 1$.

Subcase 2.1: $e = (u, v) \in \partial D_1$.

Without loss of generality, we can assume that $e = (u, v) = (u^0, v^1)$ is an edge between BH_1^0 and BH_1^1 where $u^0 = (a_0, 0)$, $v = (b_0, 1)$.

Subcase 2.1.1: $\ell = 6$.

If $((a_0 + 2, 0), (b_0 + 2, 1))$ is a fault-free edge. Let

$$\begin{aligned} C_1 &= \langle u^0, v^1, (a_0, 1), (b_0 + 2, 1), (a_0 + 2, 0), (b_0, 0), u^0 \rangle; \\ C_2 &= \langle u^0, v^1, (a_0 + 2, 1), (b_0 + 2, 1), (a_0 + 2, 0), (b_0 + 2, 0), u^0 \rangle. \end{aligned}$$

Then, C_1, C_2 are two cycles in BH_2 containing e and $C_1 \cap C_2 = \{e, ((a_0 + 2, 0), (b_0 + 2, 1))\}$ is the fault-free edge set. Thus, C^1 or C^2 is a fault-free 6-cycle.

If $((a_0 + 2, 0), (b_0 + 2, 1))$ is a faulty edge. Then, $(u^0, (b_0 + 2, 1))$ is a fault-free edge. Let

$$\begin{aligned} C_3 &= \langle u^0, v^1, (a_0, 1), (b_0, 2), (a_0 + 2, 1), (b_0 + 2, 1), u^0 \rangle; \\ C_4 &= \langle u^0, v^1, (a_0 + 2, 1), (b_0 + 2, 2), (a_0, 1), (b_0 + 2, 1), u^0 \rangle. \end{aligned}$$

Then, C_3, C_4 are two cycles in BH_2 containing e and $C_3 \cap C_4 = \{e, (u^0, (b_0 + 2, 1))\}$ is the fault-free edge set. Thus, C^3 or C^4 is a fault-free 6-cycle.

Subcase 2.1.2: $\ell = 8$.

By Lemma 2.10, it holds.

Subcase 2.1.3: $10 \leq \ell \leq 14$.

By the proof of subcase 2.1.2, there exists a fault-free 8-cycle C that contains e such that $|C \cap BH_1^i| = 1$ for $0 \leq i \leq 3$, say $\langle u^0, v^0, u^3, v^3, u^2, v^2, u^1, v^1, u^0 \rangle$ where $u^i, v^i \in BH_{n-1}^i$ for $i = 0, 1, 2, 3$. Note that $|F \cap \partial D_0| = 1$. Without loss of generality, let $|F \cap BH_1^0| = 1$. It is easy to check that there exists an ℓ_i -path in BH_1^i joining u^i to v^i where $\ell_i = 1$ or 3 for $i = 1, 2, 3$. Then, $\langle u, v^0, u^3, P_3, v^3, u^2, P_2, v^2, u^1, P_1, v, u \rangle$ with length $\ell = 5 + \ell_1 + \ell_2 + \ell_3$ forms the desired cycle.

Subcase 2.2: $e = (u, v) \in \partial D_0$.

We divide BH_2 into four BH_1 s, denoted by $\overline{BH_1^0}, \overline{BH_1^1}, \overline{BH_1^2}, \text{ and } \overline{BH_1^3}$, by deleting all 1-dimensional edges. Then, e is an edge between $\overline{BH_1^i}$ and $\overline{BH_1^{i+1}}$ for $0 \leq i \leq 3$. By a similar discussion for subcase 2.1, we obtain the result. \square

4 Conclusion

In this paper, we consider the edge-bipancyclicity of BH_n for at most $(2n - 2)$ faulty edges and prove that each fault-free edge lies on a fault-free cycle of any even length from 6 to 2^{2n} . Our result improves the results of Hao et al. [7] and Cheng et al. [3] and it is optimal with respect to the maximum number of tolerated edge faults. In addition, it is of interest to consider the problem of fault-tolerant embedding cycles with each vertex incident to at least two non-faulty edges.

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