# Differential-recurrence properties of dual Bernstein polynomials 

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#### Abstract

New differential-recurrence properties of dual Bernstein polynomials are given which follow from relations between dual Bernstein and orthogonal Hahn and Jacobi polynomials. Using these results, a fourth-order differential equation satisfied by dual Bernstein polynomials has been constructed. Also, a fourth-order recurrence relation for these polynomials has been obtained; this result may be useful in the efficient solution of some computational problems.


Keywords: Differential equations; Recurrence relations; Bernstein basis polynomials; Dual Bernstein polynomials; Jacobi polynomials; Hahn polynomials; Generalized hypergeometric functions.

## 1. Introduction

Dual Bernstein polynomials associated with the Legendre inner product were introduced by Ciesielski in 1987 [4]. Their properties and generalizations were studied, e.g., by Jüttler [12], Rababah and Al-Natour [19, 20], as well as by Lewanowicz and Woźny [14, 15, 24]. It is worth noticing that dual Bernstein polynomials introduced in [14], which are associated with the shifted Jacobi inner product, have recently found many applications in numerical analysis and computer graphics (curve intersection using Bézier clipping, degree reduction and merging of Bézier curves, polynomial approximation of rational Bézier curves, etc.). Note that skillful use of these polynomials often results in less costly algorithms which solve some computational problems (see [2, 7, 18, 16, 17, 21, 23, 24]).

The main purpose of this article is to give new properties of dual Bernstein polynomials considered in [14]. Namely, we derive some differential-recurrence relations which allow us to construct a differential equation and a recurrence relation for these polynomials.

The paper is organized as follows. Section 2 contains definitions, notation and important properties of dual Bernstein polynomials obtained in [14]. Next, in Section 3, we present new results which imply: i) the fourth-order differential equation with polynomial coefficients (see $\left.\S_{4}\right)$; ii) the recurrence relation of order four (see $\$ 5$ ), both of which are satisfied by dual Bernstein polynomials. The latter result may be useful in finding the efficient solution of some computational tasks, e.g., fast evaluation of dual Bernstein polynomials and their linear combinations or integrals involving these dual polynomials (see $\sqrt[6]{6}$ ).

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## 2. Dual Bernstein polynomials

The generalized hypergeometric function (see, e.g., [1, §2.1]) is defined by

$$
{ }_{p} F_{q}\left(\left.\begin{array}{c}
a_{1}, \ldots, a_{p} \\
b_{1}, \ldots, b_{q}
\end{array} \right\rvert\, x\right):=\sum_{l=0}^{\infty} \frac{\left(a_{1}\right)_{l} \ldots\left(a_{p}\right)_{l}}{\left(b_{1}\right)_{l} \ldots\left(b_{q}\right)_{l}} \cdot \frac{x^{l}}{l!},
$$

where $p, q \in \mathbb{N}, a_{i} \in \mathbb{C}(i=1,2, \ldots, p), b_{j} \in \mathbb{C}(j=1,2, \ldots, q), x \in \mathbb{C}$, and $(c)_{l}(c \in \mathbb{C} ; l \in \mathbb{N})$ denotes the Pochhammer symbol,

$$
(c)_{0}:=1, \quad(c)_{l}:=c(c+1) \ldots(c+l-1) \quad(l \geq 1) .
$$

Notice that if one of the parameters $a_{i}$ is equal to $-k(k \in \mathbb{N})$ then the generalized hypergeometric function is a polynomial in $x$ of degree at most $k$.

For $\alpha, \beta>-1$, let us introduce the inner product $\langle\cdot, \cdot\rangle_{\alpha, \beta}$ by

$$
\begin{equation*}
\langle f, g\rangle_{\alpha, \beta}:=\int_{0}^{1}(1-x)^{\alpha} x^{\beta} f(x) g(x) \mathrm{d} x . \tag{2.1}
\end{equation*}
$$

Recall that shifted Jacobi polynomials $R_{k}^{(\alpha, \beta)}$ (cf., e.g., [13, §1.8]),

$$
R_{k}^{(\alpha, \beta)}(x):=\frac{(\alpha+1)_{k}}{k!}{ }_{2} F_{1}\left(\begin{array}{c|c}
-k, k+\alpha+\beta+1 & 1-x  \tag{2.2}\\
\alpha+1 & 1-x
\end{array}\right) \quad(k=0,1, \ldots),
$$

are orthogonal with respect to the inner product (2.1), i.e.,

$$
\left\langle R_{k}^{(\alpha, \beta)}, R_{l}^{(\alpha, \beta)}\right\rangle_{\alpha, \beta}=\delta_{k l} h_{k} \quad(k, l \in \mathbb{N}),
$$

where $\delta_{k l}$ is the Kronecker delta ( $\delta_{k l}=0$ for $k \neq l$ and $\delta_{k k}=1$ ) and

$$
h_{k}:=K \frac{(\alpha+1)_{k}(\beta+1)_{k}}{k!(2 k / \sigma+1)(\sigma)_{k}} \quad(k=0,1, \ldots)
$$

with $\sigma:=\alpha+\beta+1, K:=\Gamma(\alpha+1) \Gamma(\beta+1) / \Gamma(\sigma+1)$.
Shifted Jacobi polynomials satisfy the second-order differential equation with polynomial coefficients of the form (cf. [13, Eq. (1.8.5)])

$$
\begin{equation*}
\boldsymbol{L}^{(\alpha, \beta)} R_{k}^{(\alpha, \beta)}(x)=\lambda_{k}^{(\alpha, \beta)} R_{k}^{(\alpha, \beta)}(x) \quad(k=0,1, \ldots), \tag{2.3}
\end{equation*}
$$

where

$$
\boldsymbol{L}^{(\alpha, \beta)}:=x(x-1) \boldsymbol{D}^{2}+\frac{1}{2}(\alpha-\beta+(\sigma+1)(2 x-1)) \boldsymbol{D}, \quad \lambda_{k}^{(\alpha, \beta)}:=k(k+\sigma),
$$

and $\boldsymbol{D}:=\frac{\mathrm{d}}{\mathrm{d} x}$ is a differentiation operator with respect to the variable $x$.
It is well known that (cf. [1, p. 117])

$$
\begin{equation*}
R_{k}^{(\alpha, \beta)}(x)=(-1)^{k} R_{k}^{(\beta, \alpha)}(1-x) . \tag{2.4}
\end{equation*}
$$

Moreover, we also use the second family of orthogonal polynomials, namely Hahn polynomials,

$$
Q_{k}(x ; \alpha, \beta ; N):={ }_{3} F_{2}\left(\left.\begin{array}{c}
-k, k+\alpha+\beta+1,-x  \tag{2.5}\\
\alpha+1,-N
\end{array} \right\rvert\, 1\right) \quad(k=0,1, \ldots, N ; N \in \mathbb{N})
$$

(see, e.g., [13, §1.5]).
Hahn polynomials satisfy the second-order difference equation with polynomial coefficients of the form

$$
\begin{equation*}
\mathcal{L}_{x}^{(\alpha, \beta, N)} Q_{k}(x ; \alpha, \beta ; N)=\lambda_{k}^{(\alpha, \beta)} Q_{k}(x ; \alpha, \beta ; N) \quad(k=0,1, \ldots), \tag{2.6}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathcal{L}_{x}^{(\alpha, \beta, N)} f(x):=a(x) f(x+1)-c(x) f(x)+b(x) f(x-1), \tag{2.7}
\end{equation*}
$$

and

$$
a(x):=(x-N)(x+\alpha+1), \quad b(x):=x(x-\beta-N-1), \quad c(x):=a(x)+b(x) .
$$

See, e.g., [13, Eq. (1.5.5)].
Let $\Pi_{n}(n \in \mathbb{N})$ denote the set of polynomials of degree at most $n$. Bernstein basis polynomials $B_{i}^{n}$ are given by

$$
\begin{equation*}
B_{i}^{n}(x):=\binom{n}{i} x^{i}(1-x)^{n-i} \quad(i=0,1, \ldots, n ; n \in \mathbb{N}) \tag{2.8}
\end{equation*}
$$

One can easily check that polynomials $B_{0}^{n}, B_{1}^{n}, \ldots, B_{n}^{n}$ form a basis of the space $\Pi_{n}$.
Bernstein basis polynomials (2.8) have many applications in approximation theory, numerical analysis, as well as in computer aided geometric design (see, e.g., books [3], [6] and papers cited therein). In view of their applications in computer graphics and numerical analysis, the so-called dual Bernstein polynomials have become quite popular.

Definition $2.1([\mathbf{1 4}, \S 5])$. Dual Bernstein polynomials of degree $n$,

$$
D_{0}^{n}(x ; \alpha, \beta), D_{1}^{n}(x ; \alpha, \beta), \ldots, D_{n}^{n}(x ; \alpha, \beta) \in \Pi_{n},
$$

are defined so that the following conditions hold:

$$
\left\langle B_{i}^{n}, D_{j}^{n}(\cdot ; \alpha, \beta)\right\rangle_{\alpha, \beta}=\delta_{i j} \quad(i, j=0,1, \ldots, n)
$$

(cf. (2.1)).
For the properties and applications of dual Bernstein polynomials $D_{i}^{n}(x ; \alpha, \beta)$, see 2, 7, 8, 14, 16, 17, 21, 23, 24. Note that in the case $\alpha=\beta=0$ these polynomials were defined earlier by Ciesielski in [4].

Remark 2.2. We adopt the convention that $D_{i}^{n}(x ; \alpha, \beta):=0$ for $i<0$ or $i>n$.

Dual Bernstein polynomials, Hahn polynomials and shifted Jacobi polynomials are related in the following way [14, Theorem 5.2)]:

$$
\begin{equation*}
D_{i}^{n}(x ; \alpha, \beta)=K^{-1} \sum_{k=0}^{n}(-1)^{k} \frac{(2 k / \sigma+1)(\sigma)_{k}}{(\alpha+1)_{k}} Q_{k}(i ; \beta, \alpha ; n) R_{k}^{(\alpha, \beta)}(x) \quad(0 \leq i \leq n) \tag{2.9}
\end{equation*}
$$

Note that

$$
\begin{equation*}
D_{i}^{n}(x ; \alpha, \beta)=D_{n-i}^{n}(1-x ; \beta, \alpha) \quad(i=0,1, \ldots, n) \tag{2.10}
\end{equation*}
$$

(see 14, Corollary 5.3]).
The polynomial $D_{i}^{n}(x ; \alpha, \beta)$ can be expressed as a short linear combination of $\min (i, n-$ $i)+1$ shifted Jacobi polynomials with shifted parameters:

$$
\begin{align*}
D_{i}^{n}(x ; \alpha, \beta) & =\frac{(-1)^{n-i}(\sigma+1)_{n}}{K(\alpha+1)_{n-i}(\beta+1)_{i}} \sum_{k=0}^{i} \frac{(-i)_{k}}{(-n)_{k}} R_{n-k}^{(\alpha, \beta+k+1)}(x),  \tag{2.11}\\
D_{n-i}^{n}(x ; \alpha, \beta) & =\frac{(-1)^{i}(\sigma+1)_{n}}{K(\alpha+1)_{i}(\beta+1)_{n-i}} \sum_{k=0}^{i}(-1)^{k} \frac{(-i)_{k}}{(-n)_{k}} R_{n-k}^{(\alpha+k+1, \beta)}(x),
\end{align*}
$$

where $i=0,1, \ldots, n$. See [14, Corollary 5.4].

## 3. Differential-recurrence relations

Let us first find the representation of the polynomial $D_{i}^{n}(x ; \alpha, \beta)$ in the basis $(1-x)^{j}$ $(j=0,1, \ldots, n)$. By using (2.2) in (2.11) and doing some algebra, we obtain

$$
\begin{align*}
D_{i}^{n}(x ; \alpha, \beta) & =\frac{(-1)^{n-i}(\sigma+1)_{n}}{K(\alpha+1)_{n-i}(\beta+1)_{i}} \sum_{k=0}^{i} \frac{(-i)_{k}}{(-n)_{k}} \frac{(\alpha+1)_{n-k}}{(n-k)!}{ }_{2} F_{1}\left(\left.\begin{array}{c}
k-n, n+\sigma+1 \\
\alpha+1
\end{array} \right\rvert\, 1-x\right) \\
& =A_{n i}^{(\alpha, \beta)} \frac{(\alpha+1)_{n}}{(n+1)!} \sum_{j=0}^{n} B_{n j}^{(\alpha, \beta)}{ }_{3} F_{2}\left(\left.\begin{array}{c}
j-n,-i, 1 \\
-n,-n-\alpha
\end{array} \right\rvert\, 1\right) \cdot(1-x)^{j}, \tag{3.1}
\end{align*}
$$

where

$$
\begin{equation*}
A_{n i}^{(\alpha, \beta)}:=\frac{(-1)^{n-i}(n+1)(\sigma+1)_{n}}{K(\alpha+1)_{n-i}(\beta+1)_{i}}, \quad B_{n j}^{(\alpha, \beta)}:=\frac{(-n)_{j}(n+\sigma+1)_{j}}{j!(\alpha+1)_{j}} . \tag{3.2}
\end{equation*}
$$

Let us define

$$
F(i, j):={ }_{3} F_{2}\left(\left.\begin{array}{c}
j-n,-i, 1 \\
-n,-n-\alpha
\end{array} \right\rvert\, 1\right) \quad(i, j=0,1, \ldots, n) .
$$

Using the Zeilberger algorithm [18, §6], one can prove the following lemma.
Lemma 3.1. Quantities $F(i, j)$ satisfy the first-order non-homogeneous recurrence relation of the form

$$
\begin{equation*}
(i-n)(n-i+\alpha) F(i+1, j)-(i+1)(n+j-i+\alpha+1) F(i, j)=-(n+1)(n+\alpha+1), \tag{3.3}
\end{equation*}
$$

where $0 \leq i, j \leq n$ and we adopt the convention that $F(n+1, j):=0$.

Lemma 3.1 allows us to give the first of the mentioned differential-recurrence relations for dual Bernstein polynomials.

Theorem 3.2. For $i=0,1, \ldots, n$, the following formula holds:

$$
\begin{align*}
((1-x) \boldsymbol{D}- & (n-i+\alpha+1) \boldsymbol{I}) D_{i}^{n}(x ; \alpha, \beta) \\
& =\frac{(i-n)(i+\beta+1)}{i+1} D_{i+1}^{n}(x ; \alpha, \beta)-A_{n i}^{(\alpha, \beta)} \frac{n+\alpha+1}{i+1} R_{n}^{(\alpha, \beta+1)}(x), \tag{3.4}
\end{align*}
$$

where $\boldsymbol{D}:=\frac{d}{d x}$ (cf. $p$. 园), and $\boldsymbol{I}$ is the identity operator.
Proof. We add up the recurrence relation (3.3), multiplied by $B_{n j}^{(\alpha, \beta)}(1-x)^{j}$, over all $0 \leq j \leq n$ and take into account that

$$
\begin{aligned}
& R_{n}^{(\alpha, \beta+1)}(x)=\frac{(\alpha+1)_{n}}{n!} \sum_{j=0}^{n} B_{n j}^{(\alpha, \beta)}(1-x)^{j}, \\
& \boldsymbol{D} D_{i}^{n}(x ; \alpha, \beta)=-A_{n i}^{(\alpha, \beta)} \frac{(\alpha+1)_{n}}{(n+1)!} \sum_{j=1}^{n} B_{n j}^{(\alpha, \beta)} F(i, j) \cdot j(1-x)^{j-1}
\end{aligned}
$$

(cf. (3.2)).
Another relation for $D_{i}^{n}(x ; \alpha, \beta)$ can be found by applying symmetry relations (2.4) and (2.10) in (3.4).

Theorem 3.3. For $i=0,1, \ldots, n$, we have

$$
\begin{align*}
& (x \boldsymbol{D}+(i+\beta+1) \boldsymbol{I}) D_{i}^{n}(x ; \alpha, \beta) \\
& \quad=\frac{i(n-i+\alpha+1)}{n-i+1} D_{i-1}^{n}(x ; \alpha, \beta)+A_{n i}^{(\alpha, \beta)} \frac{n+\beta+1}{n-i+1} R_{n}^{(\alpha+1, \beta)}(x) . \tag{3.5}
\end{align*}
$$

The next differential-recurrence relation is more complicated. It relates the second and first derivative of $D_{i}^{n}(x ; \alpha, \beta)$ with the polynomials $D_{i-1}^{n}(x ; \alpha, \beta), D_{i}^{n}(x ; \alpha, \beta), D_{i+1}^{n}(x ; \alpha, \beta)$.

Theorem 3.4. The following relation holds:

$$
\begin{align*}
& \left(x(x-1) \boldsymbol{D}^{2}+\frac{1}{2}(\alpha-\beta+(\sigma+1)(2 x-1)) \boldsymbol{D}\right) D_{i}^{n}(x ; \alpha, \beta)  \tag{3.6}\\
& =(i-n)(i+\beta+1) D_{i+1}^{n}(x ; \alpha, \beta)+i(i-\alpha-n-1) D_{i-1}^{n}(x ; \alpha, \beta) \\
& \quad-(i(i-\alpha-n-1)+(i-n)(i+\beta+1)) D_{i}^{n}(x ; \alpha, \beta)
\end{align*}
$$

where $i=0,1, \ldots, n$.
Proof. We use the representation (2.9) of dual Bernstein polynomials, the differential equation (2.3) for shifted Jacobi polynomials, as well as the difference equation (2.6) satisfied by Hahn polynomials.

Observe that

$$
\begin{aligned}
\boldsymbol{L}^{(\alpha, \beta)} D_{i}^{n}(x ; \alpha, \beta) & =K^{-1} \sum_{k=0}^{n}(-1)^{k} \frac{(2 k / \sigma+1)(\sigma)_{k}}{(\alpha+1)_{k}} Q_{k}(i ; \beta, \alpha ; n) \cdot \lambda_{k}^{(\alpha, \beta)} R_{k}^{(\alpha, \beta)}(x) \\
& =K^{-1} \sum_{k=0}^{n}(-1)^{k} \frac{(2 k / \sigma+1)(\sigma)_{k}}{(\alpha+1)_{k}} R_{k}^{(\alpha, \beta)}(x) \cdot \lambda_{k}^{(\beta, \alpha)} Q_{k}(i ; \beta, \alpha ; n) \\
& =\mathcal{L}_{i}^{(\beta, \alpha, n)} D_{i}^{n}(x ; \alpha, \beta) .
\end{aligned}
$$

## 4. Differential equation

Using the new properties of dual Bernstein polynomials given in Section 3, one can construct the differential equation for $D_{i}^{n}(x ; \alpha, \beta)$.

Theorem 4.1. Dual Bernstein polynomials satisfy the second-order non-homogeneous differential equation with polynomial coefficients of the form

$$
\begin{equation*}
\boldsymbol{M}_{n i}^{(\alpha, \beta)} D_{i}^{n}(x ; \alpha, \beta)=(n+\sigma+1) A_{n i}^{(\alpha, \beta)} R_{n}^{(\alpha+1, \beta+1)}(x) \tag{4.1}
\end{equation*}
$$

where

$$
\boldsymbol{M}_{n i}^{(\alpha, \beta)}:=x(x-1) \boldsymbol{D}^{2}+((n+\sigma+3) x-i-\beta-2) \boldsymbol{D}+(n+\sigma+1) \boldsymbol{I}
$$

Proof. By substituting the expressions for $D_{i+1}^{n}(x ; \alpha, \beta)$ and $D_{i-1}^{n}(x ; \alpha, \beta)$ determined by (3.4) and (3.5), respectively, into equation (3.6), we obtain

$$
\boldsymbol{M}_{n i}^{(\alpha, \beta)} D_{i}^{n}(x ; \alpha, \beta)=A_{n i}^{(\alpha, \beta)}\left((n+\alpha+1) R_{n}^{(\alpha, \beta+1)}(x)+(n+\beta+1) R_{n}^{(\alpha+1, \beta)}(x)\right) .
$$

To complete the proof, observe that

$$
(n+\alpha+1) R_{n}^{(\alpha, \beta+1)}(x)+(n+\beta+1) R_{n}^{(\alpha+1, \beta)}(x)=(n+\sigma+1) R_{n}^{(\alpha+1, \beta+1)}(x)
$$

which follows from (2.2) after some algebra.
Notice that by applying the second-order differential operator

$$
\boldsymbol{N}_{n i}^{(\alpha, \beta)}:=\boldsymbol{L}^{(\alpha+1, \beta+1)}-\lambda_{n}^{(\alpha+1, \beta+1)} \boldsymbol{I}
$$

(cf. (2.3)) to both sides of Eq. (4.1), we obtain the homogeneous differential equation for dual Bernstein polynomials.

Corollary 4.2. Dual Bernstein polynomials $D_{i}^{n}(x ; \alpha, \beta)(i=0,1, \ldots, n)$ satisfy the fourthorder differential equation with polynomial coefficients of the form

$$
\begin{equation*}
\boldsymbol{Q}_{4} D_{i}^{n}(x ; \alpha, \beta) \equiv \boldsymbol{N}_{n i}^{(\alpha, \beta)} \boldsymbol{M}_{n i}^{(\alpha, \beta)} D_{i}^{n}(x ; \alpha, \beta)=0 \tag{4.2}
\end{equation*}
$$

Observe that the operator $\boldsymbol{Q}_{4}$ is a composition of two second-order differential operators. For the reader's convenience, we give also the explicit form of the differential equation (4.2):

$$
\sum_{j=0}^{4} w_{j}(x) \boldsymbol{D}^{j} D_{i}^{n}(x ; \alpha, \beta)=0
$$

where

$$
\begin{aligned}
w_{4}(x): & =x^{2}(x-1)^{2}, \quad w_{3}(x):=x(x-1)[(n+2 \sigma+10) x-i-2 \beta-6], \\
w_{2}(x): & =[(n+\sigma+3)(\sigma-n+7)+\sigma+3] x^{2} \\
& \quad+\left[(n-1)^{2}+\alpha n-2 \beta-(\sigma+3)(i+2 \beta+8)-5\right] x+(\beta+2)(i+\beta+3), \\
w_{1}(x): & =-(n+\sigma+2)\left[\left(n^{2}+(n-2)(\sigma+3)\right) x+(2-n)(i+\beta+2)-2 i\right], \\
w_{0}(x): & =-n(n+\sigma+1)_{2} .
\end{aligned}
$$

## 5. Recurrence relation

In 14, Theorem 5.1], the following recurrence relation, which connects dual Bernstein polynomials of degrees $n+1$ and $n$, as well as the shifted Jacobi polynomial of degree $n+1$, was given:

$$
\begin{equation*}
D_{i}^{n+1}(x ; \alpha, \beta)=\left(1-\frac{i}{n+1}\right) D_{i}^{n}(x ; \alpha, \beta)+\frac{i}{n+1} D_{i-1}^{n}(x ; \alpha, \beta)+C_{n i}^{(\alpha, \beta)} R_{n+1}^{(\alpha, \beta)}(x), \tag{5.1}
\end{equation*}
$$

where $0 \leq i \leq n+1$, and

$$
C_{n i}^{(\alpha, \beta)}:=(-1)^{n-i+1} \frac{(2 n+\sigma+2)(\sigma+1)_{n}}{K(\alpha+1)_{n-i+1}(\beta+1)_{i}} .
$$

Let us mention that the case $\alpha=\beta=0$ of this relation was found earlier by Ciesielski in (4).
Now, using the results given in Section 3, we show that it is possible to construct a homogeneous recurrence relation connecting five consecutive (with respect to $i$ ) dual Bernstein polynomials of the same degree $n$.

Let $\mathcal{E}^{m}$ be the $m$ th shift operator acting on the variable $i$ in the following way:

$$
\mathcal{E}^{m} z_{i}:=z_{i+m} \quad(m \in \mathbb{Z}) .
$$

For the sake of simplicity, we write $\mathcal{I}:=\mathcal{E}^{0}$ and $\mathcal{E}:=\mathcal{E}^{1}$.
For example, the operator $\mathcal{L}_{i}^{(\alpha, \beta, N)}$ (cf. (2.7) and the proof of Theorem 3.4) can be written as:

$$
\mathcal{L}_{i}^{(\alpha, \beta, N)}=a(i) \mathcal{E}-c(i) \mathcal{I}+b(i) \mathcal{E}^{-1} .
$$

The following theorem holds.
Theorem 5.1. Dual Bernstein polynomials satisfy the second-order non-homogeneous recurrence relation of the form

$$
\begin{equation*}
\mathcal{M}_{i}^{(\alpha, \beta, n)} D_{i}^{n}(x ; \alpha, \beta)=G_{n i}^{(\alpha, \beta)}(x), \tag{5.2}
\end{equation*}
$$

where $i=0,1, \ldots, n$, and

$$
\begin{aligned}
& \mathcal{M}_{i}^{(\alpha, \beta, n)}:=(i)_{2}(n-i+\alpha+1)(x-1) \mathcal{E}^{-1}-(n-i)_{2}(i+\beta+1) x \mathcal{E} \\
& \quad+(i+1)(n-i+1)[(i+\beta+1)(1-x)+(n-i+\alpha+1) x] \mathcal{I}, \\
& G_{n i}^{(\alpha, \beta)}(x):=A_{n i}^{(\alpha, \beta)}\left((i+1)(n+\beta+1)(1-x) R_{n}^{(\alpha+1, \beta)}(x)\right. \\
&\left.\quad+(n-i+1)(n+\alpha+1) x R_{n}^{(\alpha, \beta+1)}(x)\right) .
\end{aligned}
$$

Proof. The recurrence (5.2) can be obtained in the following way: we subtract the relation (3.4), multiplied by $x$, from the relation (3.5), multiplied by $1-x$.

Notice that the quantity $H(i):=\left(A_{n i}^{(\alpha, \beta)}\right)^{-1} G_{n i}^{(\alpha, \beta)}(x)$ is a polynomial of the first degree in variable $i$. Thus we have $(\mathcal{E}-\mathcal{I})^{2} H(i)=0$. By applying the operator

$$
\mathcal{N}_{i}^{(\alpha, \beta, n)}:=\mathcal{E}^{-1}(\mathcal{E}-\mathcal{I})^{2}\left(A_{n i}^{(\alpha, \beta)}\right)^{-1} \mathcal{I}
$$

to both sides of the equation (5.2), we obtain a fourth-order homogeneous recurrence relation for the dual Bernstein polynomials.

Corollary 5.2. Dual Bernstein polynomials satisfy the fourth-order recurrence relation of the form

$$
\begin{equation*}
\mathcal{Q}_{4} D_{i}^{n}(x ; \alpha, \beta) \equiv \mathcal{N}_{i}^{(\alpha, \beta, n)} \mathcal{M}_{i}^{(\alpha, \beta, n)} D_{i}^{n}(x ; \alpha, \beta)=0 \quad(0 \leq i \leq n) . \tag{5.3}
\end{equation*}
$$

Let us stress that the operator $\mathcal{Q}_{4}$ is a composition of two second-order difference operators. Below, we also give the explicit form of the simplified recurrence relation (5.3):

$$
\begin{equation*}
\sum_{j=-2}^{2} v_{j}(i) D_{i+j}^{n}(x ; \alpha, \beta)=0 \tag{5.4}
\end{equation*}
$$

where

$$
\begin{aligned}
v_{-2}(i): & (1-x)(i-1)_{2}(n-i+\alpha)_{3}, \\
v_{-1}(i): & =-i(n-i+\alpha)_{2}\{(i+\beta)(n-3 i) \\
\quad & \quad+[n(n-3 i+\alpha-\beta+4)+i(4 i-\alpha+3 \beta-4)+2(\alpha+2)] x\}, \\
v_{0}(i): & (i+\beta)(n-i+\alpha)[z(i) x+(i+1)(i+\beta+1)(3 i-2 n)], \\
v_{1}(i): & (i-n)(i+\beta)_{2}\{(i+2)(i+\beta+2) \\
& \quad-[n(2 n-5 i+2 \alpha)+i(4 i-3 \alpha+\beta+4)+2(\beta+2)] x\}, \\
v_{2}(i): & =x(i+\beta)(i+\beta+1)_{2}(n-i-1)_{2},
\end{aligned}
$$

and $z(i):=-6 i^{3}+3(3 n+\alpha-\beta) i^{2}-[n(5 n-6 \beta)+(4 n+3) \sigma+3] i+n[(n+1)(n+\alpha+1)+2 \beta+2]$.

## 6. Applications

Now, we point out some possible applications of the obtained recurrence relation. Let us consider the following task.

Problem 6.1. Let us fix numbers: $n \in \mathbb{N}, x \in \mathbb{C}$ and $\alpha, \beta>-1$. Consider the problem of computing the values

$$
D_{i}^{n}(x ; \alpha, \beta)
$$

for all $i=0,1, \ldots, n$.
An efficient solution of this problem gives us, e.g., the fast method of evaluating the polynomial

$$
\begin{equation*}
d(x):=\sum_{i=0}^{n} d_{i} D_{i}^{n}(x ; \alpha, \beta), \tag{6.1}
\end{equation*}
$$

where coefficients $d_{0}, d_{1}, \ldots, d_{n}$ are given. Notice that such representation plays a crucial role in the algorithm for merging of Bézier curves which has been recently proposed in [23].

On the other hand, in many applications, such as least-square approximation in Bézier form (cf. [15], [16]) or numerical solving of boundary value problems (cf., e.g., report [9]) or fractional partial differential equations (see [10], 11] and papers cited therein), it is necessary to compute the collection of integrals of the form

$$
I_{k}:=\int_{0}^{1}(1-x)^{\alpha} x^{\beta} f(x) D_{k}^{n}(x ; \alpha, \beta) \mathrm{d} x
$$

for all $k=0,1, \ldots, n$ and a given function $f$. Recall that the main reason is that a polynomial

$$
p_{n}^{*}(x):=\sum_{k=0}^{n} I_{k} B_{k}^{n}(x)
$$

minimizes the value of the least-square error

$$
\int_{0}^{1}(1-x)^{\alpha} x^{\beta}\left(f(x)-p_{n}(x)\right)^{2} \mathrm{~d} x \quad\left(p_{n} \in \Pi_{n}\right) .
$$

The numerical approximations of the integrals $I_{0}, I_{1}, \ldots, I_{n}$ involving the dual Bernstein polynomials can be computed, for example, by quadrature rules (see, e.g., [5, §5]). It also requires the fast evaluation of polynomials $D_{0}^{n}(x ; \alpha, \beta), D_{1}^{n}(x ; \alpha, \beta), \ldots, D_{n}^{n}(x ; \alpha, \beta)$ in many nodes.

The solutions of Problem 6.1) which use the representations (2.9), (2.11) or (3.1) of dual Bernstein polynomials, or the recurrence relation (5.1) satisfied by these polynomials, have too high computational complexity (notice that one has to compute also shifted Jacobi and/or Hahn polynomials, cf. (2.2) and (2.5)).

Observe that it is more efficient to use the recurrence relation (5.4) which is not explicitly related to shifted Jacobi and Hahn polynomials. This recurrence allows us to solve the problem with the computational complexity $O(n)$. For details, see [22, $\S 7$ and $\S 10.2]$.

Horner's rule (see, e.g., [5, Eq. (1.2.2)]) for evaluating the $n$th degree polynomial given in the power basis also has the computational complexity $O(n)$. Taking into account that the dual Bernstein basis is much more complicated than the power basis, the algorithms based on the recurrence (5.1) for evaluating $D_{i}^{n}(x ; \alpha, \beta)$ or a polynomial given in the form (6.1) seem to be interesting.

To show the efficiency of the new recurrence relation for dual Bernstein polynomials, let us present the following numerical example. The results have been obtained on a computer with Intel Core i5-661 3.33 Hz processor and 8 GB of RAM, using computer algebra system Maple ${ }^{\text {TM }} 8$.

|  |  | Recurrence (5.1) |  | Recurrence (5.4) |  |
| :--- | :--- | :---: | :---: | :---: | :---: |
|  |  | time | error | time | error |
| $n=10$ | $\alpha=\beta=0$ | 2.453 | $0.40 \cdot 10^{-31}$ | 0.688 | $0.40 \cdot 10^{-31}$ |
|  | $\alpha=\beta=-0.5$ | 2.750 | $0.20 \cdot 10^{-28}$ | 0.953 | $0.71 \cdot 10^{-26}$ |
|  | $\alpha=-0.33, \beta=5.66$ | 3.845 | $0.25 \cdot 10^{-24}$ | 1.563 | $0.33 \cdot 10^{-24}$ |
| $n=15$ | $\alpha=\beta=0$ | 5.984 | $0.35 \cdot 10^{-25}$ | 0.937 | $0.41 \cdot 10^{-23}$ |
|  | $\alpha=\beta=-0.5$ | 8.000 | $0.19 \cdot 10^{-22}$ | 1.485 | $0.11 \cdot 10^{-22}$ |
|  | $\alpha=-0.33, \beta=5.66$ | 12.391 | $0.13 \cdot 10^{-21}$ | 2.781 | $0.44 \cdot 10^{-20}$ |
| $n=20$ | $\alpha=\beta=0$ | 12.327 | $0.18 \cdot 10^{-19}$ | 1.329 | $0.26 \cdot 10^{-19}$ |
|  | $\alpha=\beta=-0.5$ | 17.734 | $0.72 \cdot 10^{-19}$ | 2.125 | $0.17 \cdot 10^{-18}$ |
|  | $\alpha=-0.33, \beta=5.66$ | 27.797 | $0.41 \cdot 10^{-19}$ | 4.735 | $0.90 \cdot 10^{-19}$ |

Table 1: Results of numerical experiments (total time in seconds and maximum error for $M=100$ ).

Example 6.2. For $n=10,15,20$ and $\alpha=\beta=0$ (Legendre's case), $\alpha=\beta=-0.5$ (Chebyshev's case) and $\alpha=-0.33, \beta=5.66$ (non-standard case), the values of dual polynomials $D_{i}^{n}\left(x_{k} ; \alpha, \beta\right)$ at all the points $x_{r}:=\frac{k}{M}(0 \leq k \leq M ; M=100)$ and for all $i=0,1, \ldots, n$ have been computed by recurrence relations (5.1) (computational complexity $O\left(M n^{2}\right)$ ) and (5.4) (computational complexity $O(M n)$ ). Both methods give results of similar numerical quality. However the algorithm using the new recurrence relation (5.4) is significantly faster. See Table 1

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