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The p-restricted edge-connectivity of Kneser graphs

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ABSTRACT

Given a connected graph G and an integer $1 \le p \le \lfloor |V(G)|/2 \rfloor$, a p-restricted edge-cut of G is any set of edges $S \subset E(G)$, if any, such that $G - S$ is not connected and each component of $G - S$ has at least p vertices; and the p-restricted edge-connectivity of G, denoted $\lambda_p(G)$, is the minimum cardinality of such a p-restricted edge-cut. When p-restricted edge-cuts exist, G is said to be super-λ_p if the deletion from G of any p-restricted edge-cut S of cardinality $\lambda_p(G)$ yields a graph $G - S$ that has at least one component with exactly p vertices. In this work, we prove that Kneser graphs $K(n, k)$ are λ_p-connected for a wide range of values of p. Moreover, we obtain the values of $\lambda_p(G)$ for all possible p and all $n \ge 5$ when $G = K(n, 2)$. Also, we discuss in which cases $\lambda_p(G)$ attains its maximum possible value, and determine for which values of p graph $G = K(n, 2)$ is super-λ_p.

1. Introduction

For other terminology and notation not defined here, we refer the reader to the book by Chartrand and Lesniak [9].

All *graphs* are considered hereafter as finite and simple, that is, with a finite number of vertices and without loops or multiple edg es. If G is such a graph, its sets of vertices and edges are denoted as $V(G)$, $E(G)$, respectively. For a nonempty subset of vertices $X \subset V(G)$, $G[X]$ stands for the subgraph of G *induced by* X. The *clique number of* G is the maximum cardinality of $X \subset V(G)$ such that $G[X]$ is a complete graph. The *connectivity* (or vertex-connectivity) of G is written $\kappa(G)$, and the *edge-connectivity* of G is denoted as $\lambda(G)$. For nonempty disjoint sets X, $Y \subset V(G)$ let $[X, Y]$ be the set of edges with one end in X and the other end in Y. Clearly, $[X, V(G) \backslash X]$ is an edge-cut of G. Denote $\omega_G(X) = [X, V(G) \setminus X]$. The *degree* of a vertex $x \in V(G)$ is $\deg_G(x) = |\omega_G(\{x\})|$, and $\delta(G)$ stands for the *minimum degree of* G.

In [12,13] Fàbrega and Fiol proposed the concept of p-restricted edge-connectivity. Given a connected graph G and an integer $1 \le p \le \lfloor |V(G)|/2 \rfloor$, a *$p$-restricted edge-cut* of G is any set of edges $S \subset E(G)$, if any, such that $G - S$ is not connected and all components of $G - S$ have at least p vertices. If p-restricted edge-cuts of G exist, then G is said to be *λ_p-connected*. When G is λ_p-connected, the *p-restricted edge-connectivity* of G, $\lambda_p(G)$, is defined as follows:

$$\lambda_p(G) = \min_{S \subset E(G)} \{|S| \ : \ S \text{ is a } p\text{-restricted edge-cut of } G\}.$$

If G is λ_q-connected for some $q > p$, note that G is λ_p-connected and $\lambda_p(G) \le \lambda_q(G)$ holds. When $p = 1$, $\lambda_p(G) = \lambda_1(G)$ is the standard edge-connectivity $\lambda(G)$; and for the case $p = 2$, $\lambda_2(G)$ is usually known as the *edge-superconnectivity of* G (also denoted $\lambda'(G)$). A p-restricted edge-cut of cardinality $\lambda_p(G)$ is called a *λ_p-cut*. When p-restricted edge-cuts of G exist, G is

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said to be *super-*λ_p if the deletion from G of any λ_p-cut S yields a graph $G-S$ that has at least one component with exactly p vertices. If G is super-λ_p and also λ_q-connected for some $q>p$, observe that $\lambda_p(G)<\lambda_q(G)$ necessarily. For the case $p=1$, saying that G is super-λ_1 and that G is *edge-superconnected* are synonyms.

The optimization of $\lambda_p(G)$ requires an upper bound. Let

$$\xi_p(G)=\min_{X\subset V(G)}\{|\omega_G(X)|:|X|=p,G[X]\text{ is connected}\}.$$

It has been shown that $\lambda_p(G)\le\xi_p(G)$ for many graphs [4,6,16,21,28,30] and sufficient conditions to establish that $\lambda_p(G)=\xi_p(G)$ have been given in [4,18,26] among others.

It is worth noting that attaining super-λ_p property implies minimizing the number of minimum p-restricted edge-cuts (see [23] for the case $p=1$). In general, to determine whether a graph is super-λ_p is a hard problem, and only some special graphs have been shown to possess the super-λ_p property.

Fàbrega and Fiol also proposed the concept of p-restricted (vertex-)connectivity κ_p and some results for this kind of connectivity have been obtained in [2,3,27,29]. Other kind of connectivity measures involving both vertices and edges are studied in [11,19], for instance. Hellwig and Volkmann [17] provide a comprehensive survey of sufficient conditions for a graph to achieve lower bounds on other index of connectivities.

In this paper, we are interested in studying the p-restricted edge-connectivity of *Kneser graphs*, which are a class of graphs introduced by Lovász [20] to prove *Kneser's conjecture*. Given integers $n\ge k\ge 1$, the Kneser graph $K(n, k)$ is the graph whose vertices are the k-subsets of the set $\{1,\dots,n\}$, two vertices being adjacent if and only if they correspond to disjoint subsets. Therefore, $K(n, k)$ has $\binom{n}{k}$ vertices, and has no edges in case that $n<2k$. When $n\ge 2k$, $K(n, k)$ is $\binom{n-k}{k}$-regular, then it has $\binom{n}{k}\binom{n-k}{k}/2$ edges; hence for the case $n=2k$, $K(n, k)$ consists of a set of $\binom{n}{k}/2$ independent edges. Note that $K(n, 1)$ is the complete graph on n vertices and also that $K(5, 2)$ is the Petersen graph.

A number of structural properties are known for $K(n, k)$. Chen and Lih [10] showed that Kneser graphs are *vertex-* and *edge-transitive*. Valencia-Pavon and Vera [25] showed that the *diameter* of $K(n, k)$ is equal to $\lceil (k-1)/(n-2k)\rceil+1$. When $n\ge 2k$, Lovász [20] proved that the *chromatic number* of $K(n, k)$ is $n-2k+2$. Many of these results can be checked in the book by Aigner and Ziegler [1]; for instance, the clique number of $K(n, k)$ is $\lfloor n/k\rfloor$, and its *independence number* is $\binom{n-1}{k-1}$. It has long been conjectured that $K(n, k)$ is Hamiltonian (with the exception of $K(5, 2)$) for $n>2k$, and this was verified by Shields and Savage [22] for $n\le 27$. It is also worth noting that the Kneser graph $K(n, 2)$ is *distance-regular* with intersection array $\{(n-2)(n-3)/2, 2n-8; 1, (n-3)(n-4)/2\}$ (see [24], p. 86). Brouwer and Haemers proved in [8] that distance-regular graphs are edge-superconnected, then $K(n, 2)$ is edge-superconnected.

Concerning the connectedness of Kneser graphs the following results were obtained in [7]. Note that $K(n, k)$ is connected whenever $n\ge 2k+1$, since it has a finite diameter (see again [25]).

Theorem 1.1 *([7]). Let n, k be two integers, $n\ge 2k+1\ge 5$. The following statements hold:*

(i) *the graph $K(n, k)$ is maximally connected; that is, its (vertex-)connectivity is equal to $\binom{n-k}{k}$;*
(ii) *the graph $K(n, 2)$ is (vertex-)superconnected;*
(iii) *the (vertex-)superconnectivity of $K(n, 2)$ is equal to $\binom{n}{2}-6$.*

The paper (Section 2) is organized into two subsections as follows. Section 2.1 is devoted to prove for $G=K(n,k)$ that there exists some $n_0\ge 2k+1$ such that G is λ_p-connected and satisfies $\lambda_p\le\xi_p$ for all $n\ge n_0$ and all $1\le p\le\lfloor |V(G)|/2\rfloor$; moreover, we prove that $n_0=5$ when $k=2$. In Section 2.2 we focus on $G=K(n,2)$, approaching the problem of finding for which values of $1\le p\le\lfloor |V(G)|/2\rfloor$ the optimal result $\lambda_p=\xi_p$ holds, and we study if G is super-λ_p in the affirmative case. This is done by computing the exact values of ξ_p for all $1\le p\le\lfloor |V(G)|/2\rfloor$, from where all the values of λ_p will follow.

For the sake of simplicity, most of quantities defined for a graph G will be written from now on without any explicit reference to G, unless it is necessary; for instance, κ, λ, $\omega(X)$ will be written instead of $\kappa(G)$, $\lambda(G)$, $\omega_G(X)$, respectively.

2. Results

2.1. $\lambda_p\le\xi_p$ for $K(n, k)$

Let $G_1,\dots,G_s$ be s copies of a complete graph K_t. The graph denoted as $G^*_{s,t}$ is obtained by adding a new vertex u and joining u to every vertex in $V(G_i)$, $i=1,\dots,s$. In [30] it is proved the following result.

Theorem 2.1 *([30]). Let G be a connected graph with order at least $2(\delta(G)+1)$ which is not isomorphic to any $G^*_{s,t}$ with $t=\delta(G)$. Then for any $p\le\delta(G)+1$, G has p-restricted edge-cuts and $\lambda_p(G)\le\xi_p(G)$.*

In the following statement we prove a similar result for graphs of order less than $2(\delta(G)+1)$.

Lemma 2.1. *Let G be a connected graph with vertex connectivity κ and order $\nu\le 2\kappa-1$. Then G is λ_p-connected and $\lambda_p\le\xi_p$ for all integer p such that $1\le p\le\lfloor\nu/2\rfloor$.*

Proof. Let $X\subset V(G)$ satisfying $|X|=p$, $G[X]$ connected and $\omega(X)=\xi_p$. Then $G-X$ is connected because $|X|=p\le\lfloor\nu/2\rfloor\le\lfloor(2\kappa-1)/2\rfloor=\kappa-1$. Moreover, $|V(G)\setminus X|=\nu-p\ge\nu-\lfloor\nu/2\rfloor=\lceil\nu/2\rceil\ge p$. Hence, $\omega(X)=[X,V(G)\setminus X]$ is a p-restricted edge-cut yielding that G is λ_p-connected and $\lambda_p\le\xi_p$. □

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We now apply the above results to Kneser graphs $K(n, k)$.

Theorem 2.2. *Let n, k be two integers, $n \geq 2k+1 \geq 5$, $G = K(n,k)$, and p be an integer. Then G is λ_p-connected and $\lambda_p \leq \xi_p$ if*

(i) $\binom{n}{k} \geq 2\binom{n-k}{k} + 2$ *for* $1 \leq p \leq \binom{n-k}{k} + 1$.
(ii) $\binom{n}{k} \leq 2\binom{n-k}{k} + 1$ *for* $1 \leq p \leq \lfloor |V(G)|/2 \rfloor$.

Proof. Since $n \geq 2k+1$, $G = K(n,k)$ is connected. Let $\nu = \binom{n}{k}$ and $d = \binom{n-k}{k}$ be the order and degree of G, respectively. If $\nu \geq 2d+2$, then G is λ_p-connected and $\lambda_p \leq \xi_p$ for $p \leq d+1$ by Theorem 2.1 because clearly G is not isomorphic to $G^*_{s,t}$. Hence item (i) holds. If $\nu \leq 2d-1$, then G is λ_p-connected and $\lambda_p \leq \xi_p$ by Lemma 2.1, as $\kappa = d$ by Theorem 1.1. Therefore it remains to study for item (ii) the case when either $\nu = 2d$ or $\nu = 2d+1$. The former case $\binom{n}{k} = 2\binom{n-k}{k}$ is not possible because $\binom{n}{k} = \sum_{i=1}^{k}\binom{n-i}{k-1} + \binom{n-k}{k}$ and $\sum_{i=1}^{k}\binom{n-i}{k-1} \neq \binom{n-k}{k}$. The latter case $\binom{n}{k} = 2\binom{n-k}{k} + 1$ only holds when $n = 7$ and $k = 2$; for the rest of values of n, k we also have $\sum_{i=1}^{k}\binom{n-i}{k-1} \neq \binom{n-k}{k} + 1$. When $n = 7$ and $k = 2$ let us take the following set of vertices:

$$X = \{x_1 = \{1,2\}, x_2 = \{3,4\}, x_3 = \{5,6\}, x_4 = \{1,7\}, x_5 = \{2,4\},$$
$$x_6 = \{3,5\}, x_7 = \{6,7\}, x_8 = \{2,7\}, x_9 = \{1,6\}, x_{10} = \{4,5\}\}.$$

It is not difficult to check that for all $p = 1, \ldots, \lfloor |V(G)|/2 \rfloor = 10$, both $X_p = \{x_1, \ldots, x_p\} \subseteq X$ and $G - X_p$ induce connected subgraphs of G, with $|\omega(X_p)| = \xi_p$. Hence, item(ii) holds, and the proof is complete. □

Observe from the above theorem that for all $k \geq 2$ there exists an integer $n_0 \geq 2k+1$ such that for all $n \geq n_0$, $G = K(n,k)$ is λ_p-connected and $\lambda_p \leq \xi_p$ for all p with $1 \leq p \leq \lfloor |V(G)|/2 \rfloor$. In the following corollary we prove that $n_0 = 5$ when $k = 2$.

Corollary 2.1. *Let $n \geq 5$ be an integer, $G = K(n,2)$, and p be an integer such that $1 \leq p \leq \lfloor |V(G)|/2 \rfloor$. Then G is λ_p-connected and $\lambda_p \leq \xi_p$.*

Proof. The result follows from Theorem 2.2 (ii) when $n \geq 7$. When $n = 5, 6$, from Theorem 2.2 (i) we have that G is λ_p-connected and $\lambda_p \leq \xi_p$ for $p \leq \binom{n-2}{2} + 1$. This implies that the result is valid for $1 \leq p \leq \lfloor |V(G)|/2 \rfloor$ when $n = 6$; and when $n = 5$ the result holds for $1 \leq p \leq 4$. Thus, the only remaining case is $n = p = 5 = |V(G)|/2$. The graph $G = K(5,2)$ is isomorphic to Petersen graph and it can be described as two disjoint cycles of length 5 joined by a matching. Hence G is λ_5-connected and $\lambda_5 \leq \xi_5 = 5$, ending the proof. □

2.2. λ_p-optimality and super-λ_p in $K(n, 2)$

Let G be a λ_p-connected graph and let $X \subset V(G)$ with $|X| \geq p$ such that $\omega_G(X)$ is a λ_p-cut. Then, X is called a *λ_p-fragment of G*. Define

$$r_p(G) = \min_{X \subset V(G)} \{|X| : X \text{ is a } \lambda_p\text{-fragment of } G\}.$$

Clearly, $p \leq r_p(G) \leq \lfloor |V(G)|/2 \rfloor$. A λ_p-fragment X is called a *λ_p-atom of G* when $|X| = r_p(G)$. Next, we recall a result obtained by Wang et al. [28], where λ_p-connected $(q+1)$-clique-free graphs were nicely addressed. Then a first result for the equality of $\lambda_p(K(n, 2))$ and $\xi_p(K(n, 2))$ will follow quite straightforwardly for some values of p.

Theorem 2.3. *([28]) Let G be a λ_p-connected and $(q+1)$-clique-free graph. If $\lambda_p(G) < \xi_p(G)$, then $r_p(G) \geq \max\{p+1, \frac{q}{q-1}\delta(G) - p - 1\}$.*

Proposition 2.1. *Let $n \geq 7$ be an integer and $G = K(n,2)$. Then $\lambda_p = \xi_p$ if*

$$p \leq \begin{cases} \dfrac{n(n-5)}{4} - 2, & \text{if } n \text{ is even} \\ \text{[1ex]}\dfrac{(n-1)(n-4)}{4} - 2, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. We know that G is a $(q+1)$-clique-free graph, where $q = \lfloor n/2 \rfloor$. First, suppose that n is even. Suppose $p \leq \frac{n(n-5)}{4} - 2$ and $\lambda_p < \xi_p$. From Theorem 2.3 it follows that $r_p(G) \geq \max\{p+1, \frac{q}{q-1}\binom{n-2}{2} - p - 1\}$, yielding that $r_p(G) \geq \frac{q}{q-1}\binom{n-2}{2} - p - 1 \geq \frac{n}{n-2}\binom{n-2}{2} - \frac{n(n-5)}{4} + 1 = \frac{1}{2}\binom{n}{2} + 1 = \frac{|V(G)|}{2} + 1$, an absurdity. Similarly, when n is odd and $p \leq \frac{(n-1)(n-4)}{4} - 2$ we have $r_p(G) \geq \frac{q}{q-1}\binom{n-2}{2} - p - 1 \geq \frac{n-1}{n-3}\binom{n-2}{2} - \frac{(n-1)(n-4)}{4} + 1 = \frac{1}{2}\binom{n}{2} + 1$ which is again a contradiction. Hence, $\lambda_p \geq \xi_p$, and by Corollary 2.1 we can conclude that $\lambda_p = \xi_p$. □

For $K(n, 2)$, our objectives now are to compute λ_p for all $1 \leq p \leq \lfloor |V(K(n, 2))|/2 \rfloor$ (extending the result in Proposition 2.1), and to study when $K(n, 2)$ is super-λ_p. As we show in the following lemma for a general graph G, these objectives can be reached provided that the values of $\xi_p(G)$ are known for all $1 \leq p \leq \lfloor |V(G)|/2 \rfloor$. In the rest of the paper, by $\binom{V(G)}{p}$ we denote the set of those subsets of $V(G)$ having cardinality p.

Lemma 2.2. *Let G be a λ_p-connected graph with $\lambda_p \le \xi_p$ for all $1 \le p \le \lfloor |V(G)|/2 \rfloor$. The following statements hold:*

(i) $\lambda_p = \min\{\xi_q : p \le q \le \lfloor |V(G)|/2 \rfloor\}$.
(ii) *For $p = \lfloor |V(G)|/2 \rfloor$ it follows that $\lambda_p = \xi_p$ and G is super-λ_p.*
(iii) *For $p \le \lfloor |V(G)|/2 \rfloor - 1$ it follows that:*
  1) *$\lambda_p = \xi_p$ if and only if $\xi_p \le \xi_q$ for all q such that $p < q \le \lfloor |V(G)|/2 \rfloor$.*
  2) *$\lambda_p = \xi_p$ and G is super-λ_p if and only if $\xi_p < \xi_q$ for all q such that $p < q \le \lfloor |V(G)|/2 \rfloor$.*

Proof. (i) Let $t = r_p(G)$ be the cardinality of a λ_p-atom of G. Clearly $p \le t \le \lfloor |V(G)|/2 \rfloor$. Let $X \in \binom{V(G)}{t}$ be such that $\omega(X)$ is a λ_p-cut (note that $|V(G) \setminus X| = |V(G)| - t \ge |V(G)| - \lfloor |V(G)|/2 \rfloor \ge \lfloor |V(G)|/2 \rfloor \ge p$), then $\lambda_p = |\omega(X)| \ge \xi_t$. But $\lambda_p \le \lambda_t \le \xi_t$, hence $\lambda_p = \xi_t$. Suppose next that there exists some integer q such that $p \le q \le \lfloor |V(G)|/2 \rfloor$ and $\xi_q < \xi_t$. Then

$$\xi_q < \xi_t = \lambda_p \le \lambda_q \le \xi_q,$$

that is, $\xi_q < \xi_q$, an absurdity. As a consequence, $\xi_t \le \xi_q$ for all $p \le q \le \lfloor |V(G)|/2 \rfloor$ and therefore

$$\lambda_p = \xi_t = \min\{\xi_q(G) : p \le q \le \lfloor |V(G)|/2 \rfloor\},$$

as claimed in (i).

When $p = \lfloor |V(G)|/2 \rfloor$ we have $\lambda_p = \xi_p$ by (i), and note that every p-restricted edge-cut $\omega(Y)$ is such that $|Y| = p$ or $|V(G) \setminus Y| = p$. As a consequence, G is super-λ_p. This proves item (ii).

Item $(iii.1)$ follows directly from (i). For $(iii.2)$, if $\lambda_p = \xi_p$ and G is super-λ_p then $\xi_p = \lambda_p < \lambda_q \le \xi_q$ for all $q > p$, hence $\xi_p < \xi_q$. Conversely, suppose that $\xi_p < \xi_q$ for all $q > p$. Then $\lambda_p = \xi_p$ follows from (i). Moreover, if G is not super-λ_p, we can consider some $Y \subset V(G)$ such that $|Y| \ge p+1$, $|V(G) \setminus Y| \ge p+1$, $G[Y]$ and $G - Y$ are both connected and $|\omega(Y)| = \lambda_p$. Setting $m = \min\{|Y|, |V(G) \setminus Y|\}$ it follows that

$$\xi_p = \lambda_p \ge \xi_m > \xi_p,$$

again an absurdity. Then G must be super-λ_p, ending the proof of $(iii.2)$. □

As $G = K(n, 2)$ is a regular graph, minimizing the cardinality of $\omega(X)$ among all sets $X \subset V(G)$ on p vertices that induce a connected subgraph is equivalent to finding such a set X which maximizes $|E(G[X])|$. In the following result we present a set X_p^* of p vertices (for each $1 \le p \le \lfloor |V(G)|/2 \rfloor$) with large $|E(G[X_p^*])|$, for which we will finally prove that $\xi_p(G) = \omega(X_p^*)$.

Proposition 2.2. *Let $n \ge 5$ be an integer, and let $G = K(n, 2)$. For all integers $1 \le p \le \lfloor |V(G)|/2 \rfloor$ there exists a set $X_p^* \in \binom{V(G)}{p}$ such that $G[X_p^*]$ is connected and*

$$|E(G[X_p^*])| = \frac{1}{2}\left(p^2 + \lfloor 2p/n \rfloor (1 + \lfloor 2p/n \rfloor) n - p(1 + 4\lfloor 2p/n \rfloor)\right).$$

Proof. Suppose first that $n \ge 6$ is even. The following partition of $V(K(n, 2))$ is direct from some related known results, see for instance Baranyai's Theorem ([5]):

$$\begin{array}{l} V(K(n,2)) = \mathcal{E}_1 \cup \cdots \cup \mathcal{E}_{n-1}, \\ \qquad \text{where the following statements hold for all } i = 1, \ldots, n-1: \\ \qquad \bullet\ \mathcal{E}_i \cap \mathcal{E}_j = \emptyset, \quad \text{for all} \quad j \ne i; \\ \qquad \bullet\ |\mathcal{E}_i| = n/2; \\ \qquad \bullet\ e_k \cap e_l = \emptyset, \quad \text{for all distinct} \quad e_k, e_l \in \mathcal{E}_i; \\ \qquad \bullet\ \text{the union of all elements of } \mathcal{E}_i \text{ is equal to } \{1, \ldots, n\}\ . \end{array} \tag{1}$$

Let p be an integer, $1 \le p \le \lfloor |V(G)|/2 \rfloor$, and set $c = \lfloor p/(n/2) \rfloor = \lfloor 2p/n \rfloor$, for which $0 \le c \le n/2 - 1 < n - 1$. Hence we write $p = c\frac{n}{2} + r$, where $0 \le r \le \frac{n}{2} - 1$. Suppose $c \ge 1$ and consider the set X_p^* of p vertices defined as

$$X_p^* = \mathcal{E}_1 \cup \cdots \cup \mathcal{E}_c \cup R,$$

where $R \subset \mathcal{E}_{n-1}$ is any subset of cardinality r. Observe that each $\mathcal{E}_i$ induces a clique in G of cardinality $\frac{n}{2}$, and R (if nonempty) induces a complete graph on r vertices. Hence

$$|E(G[\mathcal{E}_i])| = \frac{1}{2}\frac{n}{2}\left(\frac{n}{2} - 1\right), \quad |E(G[R])| = \frac{1}{2} r(r-1).$$

As the union of all elements of $\mathcal{E}_i$ is equal to $\{1, \ldots, n\}$, note that each vertex in $\mathcal{E}_j$ is adjacent to exactly $\frac{n}{2} - 2$ vertices in $\mathcal{E}_i$, for $i \ne j$; and analogously, each vertex in R is adjacent to exactly $\frac{n}{2} - 2$ vertices in $\mathcal{E}_i$. As a consequence we have:

$$\begin{aligned} |E(G[X_p^*])| &= \sum_{i=1}^{c} |E(G[\mathcal{E}_i])| + |E(G[R])| + \sum_{i=1}^{c} |[R, \mathcal{E}_i]| \\ &\quad + \sum_{1 \le j < i \le c} |[\mathcal{E}_i, \mathcal{E}_j]| \end{aligned}$$

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$$
\begin{aligned}
&= c\frac{1}{2}\frac{n}{2}\left(\frac{n}{2}-1\right)+\frac{1}{2}r(r-1)+rc\left(\frac{n}{2}-2\right)+\frac{1}{2}c(c-1)\frac{n}{2}\left(\frac{n}{2}-2\right)\\
&= \frac{1}{2}\left(p^2+\lfloor 2p/n\rfloor(1+\lfloor 2p/n\rfloor)n-p(1+4\lfloor 2p/n\rfloor)\right),
\end{aligned}
$$

last expression obtained after replacing c with $\lfloor 2p/n\rfloor$ and r with $p-\lfloor 2p/n\rfloor\frac{n}{2}$. Note that this expression for $|E(G[X_p^*])|$ still holds when $c=0$, where $X_p^*=R$ is taken. Observe that $G[X_p^*]$ is connected by construction. Hence the proof is complete when n is even.

Next we consider the case when $n\geq 5$ is odd. Note that (1) can be applied to $V(K(n+1,2))$ yielding $V(K(n+1,2))=\mathcal{E}_1\cup\cdots\cup\mathcal{E}_n$. Observe that after a suitable relabeling of the elements of $\{1,\ldots,n+1\}$ and, if necessary, a reordering of sets $\mathcal{E}_1,\ldots,\mathcal{E}_n$, we can assume that

$$\{i,n+1\}\in\mathcal{E}_i \text{ for all } i=1,\ldots,n; \text{ and } \mathcal{E}_n=\{t_j=\{2j-1,2j\}\ :\ j=1,\ldots,(n+1)/2\}.$$

By defining $\mathcal{O}_i=\mathcal{E}_i\setminus\{i,n+1\}$ for all $i=1,\ldots,n$, from (1) we can write

$$
\begin{aligned}
&V(K(n,2))=\mathcal{O}_1\cup\cdots\cup\mathcal{O}_n,\\
&\qquad\text{where } \mathcal{O}_n=\{t_j=\{2j-1,2j\}\ :\ j=1,\ldots,(n-1)/2\}\\
&\qquad\text{and where the following statements hold for all } i=1,\ldots,n:\\
&\qquad\bullet\ \mathcal{O}_i\cap\mathcal{O}_j=\emptyset,\quad \text{for all } j\neq i;\\
&\qquad\bullet\ |\mathcal{O}_i|=(n-1)/2;\\
&\qquad\bullet\ e_k\cap e_l=\emptyset,\quad \text{for all distinct } e_k,e_l\in\mathcal{O}_i;\\
&\qquad\bullet\ \text{the union of all elements of } \mathcal{O}_i \text{ is equal to } \{1,\ldots,n\}\setminus\{i\}.
\end{aligned}
\tag{2}
$$

Let p be an integer, $1\leq p\leq\lfloor|V(G)|/2\rfloor$, set $c=\lfloor\frac{p}{(n-1)/2}\rfloor=\lfloor 2p/(n-1)\rfloor$, $0\leq c\leq(n-1)/2<n$, and write $p=c\frac{n-1}{2}+r$, where $0\leq r\leq\frac{n-1}{2}-1$. Suppose $c\geq 1$ and consider the set of p vertices X_p^* defined as

$$X_p^*=\mathcal{O}_1\cup\cdots\cup\mathcal{O}_c\cup R,$$

where $R=\{t_j=\{2j-1,2j\}\ :\ j=1,\ldots,r\}\subset\mathcal{O}_n$. Again

$$|E(G[\mathcal{O}_i])|=\frac{1}{2}\frac{n-1}{2}\left(\frac{n-1}{2}-1\right),\quad |E(G[R])|=\frac{1}{2}r(r-1)$$

because the respective induced subgraphs are complete. Furthermore, all but one vertices in $\mathcal{O}_j$ are adjacent to exactly $\frac{n-1}{2}-2$ vertices in $\mathcal{O}_i=\mathcal{E}_i\setminus\{i,n+1\}$, for $i\neq j$; and one only vertex in $\mathcal{O}_j$ is adjacent to $\frac{n-1}{2}-1$ vertices in $\mathcal{O}_i$, precisely a vertex of the kind $\{i,\alpha\}$, with $\alpha\in\{1,\ldots,n\}\setminus\{i,j\}$. Then, for all $1\leq j<i\leq c$ we have

$$|[\mathcal{O}_i,\mathcal{O}_j]|=1+\frac{n-1}{2}\left(\frac{n-1}{2}-2\right).$$

Notice now that vertex $t_j=\{2j-1,2j\}\in R$ is adjacent to exactly $\frac{n-1}{2}-2$ vertices in $\mathcal{O}_i$ for all $i\in\{1,\ldots,c\}\setminus\{2j-1,2j\}$, and $t_j=\{2j-1,2j\}\in R$ is adjacent to $\frac{n-1}{2}-1$ vertices in $\mathcal{O}_{2j-1}$ and to $\frac{n-1}{2}-1$ vertices in $\mathcal{O}_{2j}$ whenever $2j-1\leq c$ or $2j\leq c$ respectively. Therefore,

$$
\begin{aligned}
\textstyle\sum_{i=1}^{c}|[R,\mathcal{O}_i]| &= \begin{cases} rc\left(\frac{n-1}{2}-2\right)+2r, & \text{if } c>2r\\ rc\left(\frac{n-1}{2}-2\right)+c, & \text{if } c\leq 2r\end{cases}\\
&= rc\left(\tfrac{n-1}{2}-2\right)+\min\{2r,c\}.
\end{aligned}
$$

Then we have:

$$
\begin{aligned}
|E(G[X_p^*])| &= \sum_{i=1}^{c}|E(G[\mathcal{O}_i])|+|E(G[R])|+\sum_{i=1}^{c}|[R,\mathcal{O}_i]|\\
&\quad+\textstyle\sum_{1\leq j<i\leq c}|[\mathcal{O}_i,\mathcal{O}_j]|\\
&= c\frac{1}{2}\frac{n-1}{2}\left(\frac{n-1}{2}-1\right)+\frac{1}{2}r(r-1)+rc\left(\frac{n-1}{2}-2\right)+\min\{2r,c\}\\
&\quad+\frac{1}{2}c(c-1)\left(1+\frac{n-1}{2}\left(\frac{n-1}{2}-2\right)\right)\\
&= \frac{1}{2}\left(p^2+\lfloor 2p/(n-1)\rfloor(1+\lfloor 2p/(n-1)\rfloor)n-p(1+4\lfloor 2p/(n-1)\rfloor)\right)\\
&\quad -c+\min\{2r,c\}.
\end{aligned}
\tag{3}
$$

Observe again that this expression for $|E(G[X_p^*])|$ still holds when $c=0$, where $X_p^*=R$. Note that $G[X_p^*]$ is connected by construction.

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We continue by discussing on the sign of $c-2r$. When $c-2r\le 0$ we have $\lfloor\frac{2p}{n-1}\rfloor n-2p\le 0$, that is, $\lfloor\frac{2p}{n-1}\rfloor\le\frac{2p}{n}$. As $\frac{2p}{n}<\frac{2p}{n-1}$ it follows that $\lfloor\frac{2p}{n}\rfloor=\lfloor\frac{2p}{n-1}\rfloor$. Since $\min\{2r,c\}=c$, by replacing $\lfloor\frac{2p}{n}\rfloor$ with $\lfloor\frac{2p}{n-1}\rfloor$ in (3) we get

$$|E(G[X_p^*])|=\frac{1}{2}\left(p^2+\lfloor 2p/n\rfloor(1+\lfloor 2p/n\rfloor)n-p(1+4\lfloor 2p/n\rfloor)\right),$$

and the result follows in this case.

When $c-2r>0$ we have $\lfloor\frac{2p}{n-1}\rfloor n-2p>0$, hence $\lfloor\frac{2p}{n-1}\rfloor>\frac{2p}{n}\ge\lfloor\frac{2p}{n}\rfloor$. Since we have $0<\frac{2p}{n-1}-\frac{2p}{n}=\frac{p}{n(n-1)/2}\le\frac{1}{2}<1$ it follows that $\lfloor\frac{2p}{n-1}\rfloor=1+\lfloor\frac{2p}{n}\rfloor$. As $-c+\min\{2r,c\}=2r-c=2p-\lfloor\frac{2p}{n-1}\rfloor n$, replacing $\lfloor\frac{2p}{n-1}\rfloor$ with $1+\lfloor\frac{2p}{n}\rfloor$ in (3) we obtain

$$\begin{aligned}|E(G[X_p^*])|&=\tfrac{1}{2}\left(p^2+(1+\lfloor 2p/n\rfloor)(2+\lfloor 2p/n\rfloor)n-p(1+4(1+\lfloor 2p/n\rfloor)\right)\\&\quad+2p-(1+\lfloor 2p/n\rfloor)n\\&=\tfrac{1}{2}\left(p^2+\lfloor 2p/n\rfloor(1+\lfloor 2p/n\rfloor)n-p(1+4\lfloor 2p/n\rfloor)\right),\end{aligned}$$

proving the result also in this case. The proof is so complete. □

The following theorem makes use of the adjacency matrix of $K(n,2)$, and its proof follows similar lines of reasoning as those used for this topic in the literature (see, for instance, [14,15]). With this theorem we deduce the exact value of $\xi_p(K(n,2))$ for all possible p.

Theorem 2.4. *Let $n\ge 5$ be an integer, $G=K(n,2)$, and let p be an integer such that $1\le p\le\lfloor|V(G)|/2\rfloor$. Then it follows that*

$$\max\left\{|E(G[X])|\ :\ X\in\binom{V(G)}{p}\right\}=|E(G[X_p^*])|,$$

where $X_p^*\in\binom{V(G)}{p}$ is the set of vertices given in Proposition 2.2. As a consequence,

$$\xi_p=p\binom{n-2}{2}-p^2-\lfloor 2p/n\rfloor(1+\lfloor 2p/n\rfloor)n+p(1+4\lfloor 2p/n\rfloor).$$

Proof. Note that G is connected because $n\ge 5$. Set $V(G)=\{v_1,\dots,v_N\}$, with $N=|V(G)|=n(n-1)/2$, and consider some $X\in\binom{V(G)}{p}$. Let us represent set X of cardinality p as

$$Z_X=[\begin{matrix}t_1 & t_2 & \cdots & t_N\end{matrix}]^T,\ \text{with } t_j=\begin{cases}1, & \text{if } v_j\in X\\ 0, & \text{if } v_j\notin X\end{cases}\quad\text{for all } j=1,\dots,N.$$

If A is the adjacency matrix of G, it is known (see [15] for a proof) that its eigenvalues are

$$\lambda_1=\binom{n-2}{2}>\lambda_2=\cdots=\lambda_{m+1}=1>\lambda_{m+2}=\cdots=\lambda_N=-(n-3),$$

where $m=n(n-3)/2$. Then, we can write

$$Z_X=Z_1+Z_2+Z_3,\quad\text{with}\ \begin{cases}Z_i^TZ_j=0, & \text{for all } i\ne j\\ Z_1=\frac{p}{N}\mathbf{1} & \\ AZ_1=\binom{n-2}{2}Z_1,\quad AZ_2=Z_2,\quad AZ_3=-(n-3)Z_3,\end{cases}\tag{4}$$

where $\mathbf{1}$ is a column matrix full of ones, with N rows. Notice that

$$p=Z_X^TZ_X=Z_1^TZ_1+Z_2^TZ_2+Z_3^TZ_3,\quad\text{hence}\quad Z_2^TZ_2=p-Z_1^TZ_1-Z_3^TZ_3.$$

Since $AZ_X=\binom{n-2}{2}Z_1+Z_2-(n-3)Z_3$, it turns out that

$$\begin{aligned}2|E(G[X])|&=Z_X^tAZ_X=\tbinom{n-2}{2}Z_1^TZ_1+Z_2^TZ_2-(n-3)Z_3^TZ_3\\&=\tbinom{n-2}{2}Z_1^TZ_1+\left(p-Z_1^TZ_1-Z_3^TZ_3\right)-(n-3)Z_3Z_3^T\\&=p+\left(\tbinom{n-2}{2}-1\right)Z_1^TZ_1-(n-2)Z_3^TZ_3,\end{aligned}$$

once replaced $Z_2^TZ_2$ with $p-Z_1^TZ_1-Z_3^TZ_3$. As $Z_1^TZ_1=\frac{p}{N}\mathbf{1}^T\cdot\frac{p}{N}\mathbf{1}=\frac{p^2}{N}$, we get

$$\begin{aligned}2|E(G[X])|&=p+\left(\tbinom{n-2}{2}-1\right)\tfrac{p^2}{N}-(n-2)Z_3^TZ_3\\&=p+(1-4/n)p^2-(n-2)Z_3^TZ_3.\end{aligned}\tag{5}$$

Let us next compute $Z_3^TZ_3$ in a more useful manner. To this end, for all $j\in\{1,\dots,n\}$, let Y_j be a column matrix on N rows, with i-row entry equal to one if $j\in v_i$ (that is, when $v_i=\{j,\ell\}$ for some $\ell\ne j$), and zero otherwise (note that Y_j has exactly $n-1$ ones). Since for all $j\in\{2,\dots n\}$ we have

$$(Y_j-Y_1)^T\cdot\mathbf{1}=Y_j^T\cdot\mathbf{1}-Y_1^T\cdot\mathbf{1}=(n-1)-(n-1)=0,$$

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following [14] (p. 34) we conclude that

$$\{Y_j - Y_1 \ : \ j = 2, \dots, n\} \ \text{ is a basis of the eigenspace associated to eigenvalue } -(n-3).$$

Therefore, there must exist some $\mu_2, \dots, \mu_n \in \mathbb{R}$ such that

$$Z_3 = \sum_{j=2}^{n} \mu_j (Y_j - Y_1).$$

Since $(Y_i - Y_1)^T (Y_j - Y_1) = \begin{cases} 2(n-2), & \text{if } i = j \\ n-2, & \text{if } i \neq j \end{cases}$, we can write

$$Z_3^T Z_3 = \sum_{i=2}^{n} \sum_{j=2}^{n} \mu_i \mu_j (Y_i - Y_1)^T (Y_j - Y_1) = (n-2) \, [\mu_2 \quad \cdots \quad \mu_n] (\mathbf{I} + \mathbf{J}) \begin{bmatrix} \mu_2 \\ \vdots \\ \mu_n \end{bmatrix},$$

where $\mathbf{I}$ is the identity matrix of order $n-1$, and $\mathbf{J}$ is a square matrix of order $n-1$ full of ones. In order to obtain the values of $\mu_2, \dots, \mu_n$, we compute next $(Y_i - Y_1)^T Z_3$ in two different ways, for all $i = 2, \dots, n$. First,

$$(Y_i - Y_1)^T Z_3 = \sum_{j=2}^{n} \mu_j (Y_i - Y_1)^T (Y_j - Y_1) = (n-2) \left(2\mu_i + \sum_{j=2,\ j \neq i}^{n} \mu_j \right).$$

Secondly, taking into account that $(Y_i - Y_1)^T Z_1 = (Y_i - Y_1)^T Z_2 = 0$ by (4):

$$(Y_i - Y_1)^T Z_3 = (Y_i - Y_1)^T Z_X = \sigma_i - \sigma_1,$$

where σ_j (for all $j \in \{1, \dots, n\}$) is the number of elements in $X \in \binom{V(G)}{p}$ of the kind $\{j, h\}$, with $h \neq j$. Note then that $\sum_{j=1}^{n} \sigma_j = 2p$. By combining these two expressions of $(Y_i - Y_1)^T Z_3$ we get

$$2\mu_i + \sum_{j=2,\ j \neq i}^{n} \mu_j = \frac{\sigma_i - \sigma_1}{n-2} \quad \text{for all} \quad i = 2, \dots, n;$$

or, in matrix form,

$$(\mathbf{I} + \mathbf{J}) \begin{bmatrix} \mu_2 \\ \vdots \\ \mu_n \end{bmatrix} = \frac{1}{n-2} \begin{bmatrix} \sigma_2 - \sigma_1 \\ \vdots \\ \sigma_n - \sigma_1 \end{bmatrix}.$$

As $(\mathbf{I} + \mathbf{J})^T = \mathbf{I} + \mathbf{J}$ and $(\mathbf{I} + \mathbf{J})^{-1} = \mathbf{I} - \frac{1}{n}\mathbf{J}$ it follows for $Z_3^T Z_3$ t hat:

$$\begin{aligned} Z_3^T Z_3 &= (n-2) \, [\ \mu_2 \quad \cdots \quad \mu_n\ ] (\mathbf{I} + \mathbf{J})^T \left(\mathbf{I} - \tfrac{1}{n}\mathbf{J} \right) (\mathbf{I} + \mathbf{J}) \begin{bmatrix} \mu_2 \\ \vdots \\ \mu_n \end{bmatrix} \\ &= \tfrac{1}{n-2} \left(\sum_{j=2}^{n} (\sigma_j - \sigma_1)^2 - \tfrac{1}{n} \left(\sum_{j=2}^{n} (\sigma_j - \sigma_1) \right)^2 \right), \end{aligned}$$

which, after some algebra, can be written as

$$Z_3^T Z_3 = \frac{1}{n(n-2)} \sum_{1 \le i < j \le n} (\sigma_i - \sigma_j)^2.$$

The minimum possible value of $\sum_{1 \le i < j \le n} (\sigma_i - \sigma_j)^2$ occurs for the most possible balanced distribution of σ_j's: when $(\lfloor 2p/n \rfloor + 1)n - 2p$ elements in $\{\sigma_1, \dots, \sigma_n\}$ are equal to $\lfloor 2p/n \rfloor$, the remaining $2p - \lfloor 2p/n \rfloor n$ elements in $\{\sigma_1, \dots, \sigma_n\}$ being equal to $\lfloor 2p/n \rfloor + 1$. That is,

$$\sum_{1 \le i < j \le n} (\sigma_i - \sigma_j)^2 \ge ((\lfloor 2p/n \rfloor + 1)n - 2p)(2p - \lfloor 2p/n \rfloor n).$$

Hence, coming back to expression (5):

$$2|E(G[X])| \le p + (1 - -4/n)p^2 - \frac{1}{n} ((\lfloor 2p/n \rfloor + 1)n - 2p)(2p - \lfloor 2p/n \rfloor n).$$

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It takes a few calculations to see that the right hand side of this inequality is precisely equal to $2|E(G[X_p^*])|$. As a consequence,

$$\max\left\{2|E(G[X])| \;:\; X \in \binom{V(G)}{p}\right\} = 2|E(G[X_p^*])|.$$

Since $G[X_p^*]$ is connected and G is $\binom{n-2}{2}$-regular we finally obtain

$$\xi_p = |\omega(X_p^*)| = p\binom{n-2}{2} - 2|E(G[X_p^*])|,$$

and the proof ends by replacing $|E(G[X_p^*])|$ with the value given by Proposition 2.2. □

From both Lemma 2.2 and Theorem 2.4 we get the following theorem, which constitutes the main result of this work.

Theorem 2.5. *Let $n \geq 5$ be an integer, $G = K(n, 2)$, and p be any integer such that $1 \leq p \leq \lfloor |V(G)|/2 \rfloor$. Then, the following statements hold:*

(i) $\lambda_p = \xi_{p+1} = \xi_p - 1 < \xi_p$ *when* $n \equiv 1 \,(mod\, 4)$ *and* $p = \lfloor |V(G)|/2 \rfloor - 1$.
(ii) $\lambda_p = \xi_p$ *but G is not super-λ_p in the following cases:* $n = 6$ *and* $p = 5$; $n \equiv 1 \,(mod\, 4)$ *and* $p = \lfloor |V(G)|/2 \rfloor - 2$; $n \equiv 3 \,(mod\, 4)$ *and* $p = \lfloor |V(G)|/2 \rfloor - 1$.
(iii) $\lambda_p = \xi_p$ *and G is super-λ_p for all values of n, p not considered in (i), (ii).*

Proof. By Lemma 2.2 (*ii*), when $p = \lfloor |V(G)|/2 \rfloor$ it turns out that $\lambda_p = \xi_p$ and G is super-λ_p, so the statement holds for this value of p. Suppose then $1 \leq p \leq \lfloor |V(G)|/2 \rfloor - 1$ from now on. By Corollary 2.1, G is λ_p-connected and $\lambda_p \leq \xi_p$.

Let us consider $n = 5, 6, 7$, for which we get all possible values of ξ_p from Theorem 2.4. When $n = 5 \equiv 1$ (*mod* 4) and $1 \leq p \leq \lfloor |V(G)|/2 \rfloor = 5$:

p	1	2	3	4	5
ξ_p	3	4	5	6	5

From Lemma 2.2 (*i*) we get $\lambda_1 = \xi_1, \lambda_2 = \xi_2, \lambda_3 = \xi_3$ and $\lambda_4 = \xi_5 = \xi_4 - 1 < \xi_4$; and by Lemma 2.2 (*iii*.2), G is super-λ_p only when $p = 1, 2$. Hence the result holds. For $n = 6$ and $1 \leq p \leq \lfloor |V(G)|/2 \rfloor = 7$:

p	1	2	3	4	5	6	7
ξ_p	6	10	12	16	18	18	20

Therefore, again from Lemma 2.2 (*iii*) it turns out that $\lambda_p = \xi_p$ for all $1 \leq p \leq 6$, and G is super-λ_p for all those values of p except for $p = 5$. And when $n = 7 \equiv 3$ (*mod* 4) and $1 \leq p \leq \lfloor |V(G)|/2 \rfloor = 10$ we obtain

p	1	2	3	4	5	6	7	8	9	10
ξ_p	10	18	24	30	36	40	42	46	48	48

Then, $\lambda_p = \xi_p$ for all $1 \leq p \leq 9$, and G is super-λ_p for all $1 \leq p \leq 8$.

So the statement holds for $n = 5, 6, 7$. Take $n \geq 8$ from now on, and let us next study the sign of $\xi_{p+1} - \xi_p$ for all $1 \leq p \leq \lfloor |V(G)|/2 \rfloor - 1$.

<u>n even</u>:

Let us write $p = c\frac{n}{2} + r$, where $c = \left\lfloor \frac{2p}{n} \right\rfloor$ and $\begin{cases} 0 \leq r \leq \frac{n}{2} - 1, & \text{if } 0 \leq c \leq \frac{n-4}{2}; \\ 0 \leq r \leq \left\lfloor \frac{n}{4} \right\rfloor - 1, & \text{if } c = \frac{n-2}{2}. \end{cases}$

Suppose first that $\left\lfloor \frac{2(p+1)}{n} \right\rfloor = \left\lfloor \frac{2p}{n} \right\rfloor = c$. Hence from Theorem 2.4 we obtain:

$$\xi_{p+1} - \xi_p = \binom{n-2}{2} - c(n-4) - 2r. \tag{6}$$

Observe that $\left\lfloor \frac{2(p+1)}{n} \right\rfloor = \left\lfloor \frac{2p}{n} \right\rfloor$ implies $r \leq \frac{n}{2} - 2$ when $c \leq \frac{n-4}{2}$. Then, for all $c \leq \frac{n-2}{2}$ it follows easily from (6) that $\xi_{p+1} - \xi_p > 0$.

Suppose next that $\left\lfloor \frac{2(p+1)}{n} \right\rfloor = c + 1 > c = \left\lfloor \frac{2p}{n} \right\rfloor$, then $c \leq \frac{n-4}{2}$ and $r = \frac{n}{2} - 1$. Theorem 2.4 yields in this case:

$$\xi_{p+1} - \xi_p = \binom{n-2}{2} - (n-4)c - (n-2) \geq \frac{n-6}{2} > 0.$$

Having obtained $\xi_{p+1} - \xi_p > 0$ for all p when $n \geq 8$ is even, we get

$$\xi_1 < \cdots < \xi_{\lfloor |V(G)|/2 \rfloor - 1} < \xi_{\lfloor |V(G)|/2 \rfloor}.$$

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Then Lemma 2.2 (*iii*.2) allows us to assure that $\lambda_p = \xi_p$ and G is super-λ_p for all p, and we are done for the case that n is even.

n odd:

We write $p = c\frac{n-1}{2} + r$, with $c = \left\lfloor \frac{2p}{n-1} \right\rfloor$, $\begin{cases} 0 \le r \le \frac{n-1}{2} - 1, & \text{if } 0 \le c \le \frac{n-3}{2}; \\ 0 \le r \le \left\lfloor \frac{n-1}{4} \right\rfloor - 1, & \text{if } c = \frac{n-1}{2}. \end{cases}$

In this case, it is more convenient to use expression (3) for obtaining ξ_p, instead of applying Theorem 2.4 directly. That is, from $\xi_p = p\binom{n-2}{2} - 2|E(G[X_p^*])|$ and expression (3) we write

$$\xi_p = p\binom{n-2}{2} - p^2 - c(c+1)n + p(1+4c) + 2c - 2\min\{2r, c\}. \tag{7}$$

Suppose first that $\left\lfloor \frac{2(p+1)}{n-1} \right\rfloor = \left\lfloor \frac{2p}{n-1} \right\rfloor = c$. Hence from (7) it follows that:

$$\xi_{p+1} - \xi_p = \binom{n-2}{2} - c(n-5) - 2r + 2\min\{2r, c\} - 2\min\{2r+2, c\}. \tag{8}$$

Observe that $\left\lfloor \frac{2(p+1)}{n-1} \right\rfloor = \left\lfloor \frac{2p}{n-1} \right\rfloor$ implies $r \le \frac{n-1}{2} - 2$ when $c \le \frac{n-3}{2}$. Then, for $c \le \frac{n-1}{2}$ it follows easily from (8) that $\xi_{p+1} - \xi_p > 0$, except for the following cases (for which $2\min\{2r, c\} - 2\min\{2r+2, c\} = -4$):

$$\begin{array}{ll} \xi_{p+1} - \xi_p = -1, & \text{when } n \equiv 1 \ (mod\ 4),\ c = \frac{n-1}{2},\ \text{and } r = \frac{n-1}{4} - 1; \\ \xi_{p+1} - \xi_p = 0, & \text{when } n \equiv 3 \ (mod\ 4),\ c = \frac{n-1}{2},\ \text{and } r = \frac{n-3}{4} - 1. \end{array}$$

Indeed, for the former case we have

$$\xi_{p+1} - \xi_p = \binom{n-2}{2} - \frac{(n-1)(n-5)}{2} - \frac{(n-1)}{2} - 2 = -1 < 0;$$

and for the latter,

$$\xi_{p+1} - \xi_p = \binom{n-2}{2} - \frac{(n-1)(n-5)}{2} - \frac{(n-3)}{2} - 2 = 0.$$

Suppose next that $\left\lfloor \frac{2(p+1)}{n-1} \right\rfloor = c + 1 > c = \left\lfloor \frac{2p}{n-1} \right\rfloor$, then $c \le \frac{n-3}{2}$ and $r = \frac{n-1}{2} - 1$. In this case expression (7) yields

$$\xi_{p+1} - \xi_p = \binom{n-2}{2} - (c+1)(n-3) - 2 \ge \frac{n-7}{2} > 0$$

because $n \ge 9$ in the odd case.

Let us gather together all these deductions for $n \ge 9$ odd. Firstly, when $n \equiv 1\,(mod\,4)$ we have obtained $\xi_{p+1} - \xi_p > 0$ for all p except for the case $p = \lfloor |V(G)|/2 \rfloor - 1$, where $\xi_{p+1} - \xi_p = \xi_{\lfloor |V(G)|/2 \rfloor} - \xi_{\lfloor |V(G)|/2 \rfloor - 1} = -1$. As it is easy to compute from (3), $\xi_{\lfloor |V(G)|/2 \rfloor} - \xi_{\lfloor |V(G)|/2 \rfloor - 2} = 0$, that is,

$$\xi_1 < \cdots < \xi_{\lfloor |V(G)|/2 \rfloor - 2} < \xi_{\lfloor |V(G)|/2 \rfloor - 1} > \xi_{\lfloor |V(G)|/2 \rfloor} = \xi_{\lfloor |V(G)|/2 \rfloor - 2}.$$

Then from Lemma 2.2 (*i*) we have that $\lambda_p = \xi_p$ for all $p \ne \lfloor |V(G)|/2 \rfloor - 1$, and among these values of p graph G is super-λ_p for all $p \ne \lfloor |V(G)|/2 \rfloor - 2$, so the statement holds. Finally, when $n \equiv 3\,(mod\,4)$ we have obtained $\xi_{p+1} - \xi_p > 0$ for all p except for the case $p = \lfloor |V(G)|/2 \rfloor - 1$, where $\xi_{p+1} - \xi_p = \xi_{\lfloor |V(G)|/2 \rfloor} - \xi_{\lfloor |V(G)|/2 \rfloor - 1} = 0$. Therefore,

$$\xi_1 < \cdots < \xi_{\lfloor |V(G)|/2 \rfloor - 2} < \xi_{\lfloor |V(G)|/2 \rfloor - 1} = \xi_{\lfloor |V(G)|/2 \rfloor},$$

and Lemma 2.2 states that $\lambda_p = \xi_p$ holds for all p, G being super-λ_p for all those values of p except for $p = \lfloor |V(G)|/2 \rfloor - 1$. The proof is so complete. □

References

[1] M. Aigner, G.M. Ziegler, Proofs from the Book, second ed., Springer-Verlag, New York, 2000.
[2] C. Balbuena, A. Carmona, J. Fàbrega, M.A. Fiol, Extraconnectivity of graphs with large minimum degree and girth, in: Proceedings of the Fifteenth British Combinatorial Conference (Stirling, 1995), Discr. Math., 167/168, 1997, pp. 85–100.
[3] C. Balbuena, M. Cera, A. Diánez, P.G. a Vázquez, X. Marcote, Diameter-girth sufficient conditions for optimal extraconnectivity graphs, Discr. Math. 308 (16) (2008) 3526–3536.
[4] C. Balbuena, X. Marcote, The *k*-restricted edge-connectivity of a product of graphs, Discr. Appl. Math. 161 (2013) 52–59.
[5] Z. Baranyai, On the factorization of the complete uniform hypergraph, Colloquia Math. Soc. János Bolya 10 (1975) 91–108.
[6] P. Bonsma, N. Ueffing, L. Volkmann, Edge-cuts leaving components of order at least three, Discr. Math. 256 (2002) 431–439.
[7] G. Boruzanli, J.B. Gauci, The super-connectivity of Kneser graphs, Discuss. Math. Graph Theory, (2018). 10.7151/dmgt.2051. In press.
[8] A.E. Brouwer, W.H. Haemers, Eigenvalues and perfect matchings, Linear Algebr. Appl. 395 (2005) 155–162.
[9] G. Chartrand, L. Lesniak, Graphs and Digraphs, third ed., Chapman and Hall, London, 1996.
[10] B.L. Chen, K.W. Lih, Hamiltonian uniform subset graphs, J. Combin. Th. Ser. B. 42 (1987) 257–263.

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[11] Y. Egawa, A. Kaneko, M. Matsumoto, A mixed version of Mengerâ s theorem, Combinatorica 11 (1991) 71–74.
[12] J. Fàbrega, M.A. Fiol, Extraconnectivity of graphs with large girth, Discr. Math. 127 (1994) 163–170.
[13] J. Fàbrega, M.A. Fiol, On the extraconnectivity of graphs, Discr. Math. 155 (1996) 49–57.
[14] C. Godsil, K. Meagher, Erdös-Ko-Rado Theorems: Algebraic Approaches, Cambridge University Press, Cambridge, UK, 2016.
[15] C.D. Godsil, G.F. Royle, Algebraic Graph Theory, Springer-Verlag, New York, 2001.
[16] C. Guo, W. Yang, X. Guo, On a kind of realiability analysis of networks, Appl. Math. Comput. 218 (2011) 2711–2715.
[17] A. Hellwig, L. Volkmann, Maximally edge-connected and vertex-connected graphs and digraphs: a survey, Discr. Math. 308 (15) (2008) 3265–3296.
[18] A. Holtkamp, D. Meierling, L.P. Montejano, k-restricted edge-connectivity in triangle-free graphs, Discr. Appl. Math. 160 (2012) 1345–1355.
[19] R. Gu, Y. Shi, N. Fan, Mixed connectivity of random graphs, in: X. Gao, H. Du, M. Han (Eds.), Combinatorial Optimization and Applications, COCOA, 10627, 2017, pp. 133–140. LNCS 2017
[20] L. Lovász, Kneser's conjecture, chromatic numbers and homotopy, J. Comb. Th. A 25 (1978) 314–324.
[21] J. Ou, A bound on 4-restricted edge connectivity of graphs, Discr. Math. 307 (2007) 2429–2437.
[22] I. Shields, C.D. Savage, A note on hamilton cycles in Kneser graphs, Bull. Inst. Combin. Appl. 40 (2004) 13–22.
[23] T. Soneoka, Super edge-connectivity of dense digraphs and graphs, Discr. Appl. Math. 37/38 (1992) 511–523.
[24] Z. Stanić, in: Regular Graphs: A Spectral Approach, De Gruyter, Berlin, Germany, 2017.
[25] M. Valencia-Pabon, J.C. Vera, On the diameter of Kneser graphs, Discr. Math. 305 (2005) 353–385.
[26] S. Wang, S. Lin, C. Li, Sufficient conditions for super k-restricted edge connectivity in graphs of diameter 2, Discr. Math. 309 (2009) 908–919.
[27] S. Wang, X. Ma, The g-extra connectivity and diagnosability of crossed cubes, Appl. Math. Comput. 336 (2018) 60–66.
[28] S. Wang, L. Zhang, S. Lin, k-restricted edge connectivity in $(p+1)$-clique-free graphs, Discr. Appl. Math. 181 (2015) 255–259.
[29] D.W. Yang, Y.Q. Feng, J. Lee, J.X. Zhou, On extra connectivity and extra edge-connectivity of balanced hypercubes, Appl. Math. Comput. 320 (2018) 464–473.
[30] Z. Zhang, J. Yuan, A proof of an inequality concerning k-restricted edge-connectivity, Discr. Math. 304 (2005) 128–134.

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