COUNTERING VIOLENT EXTREMISM: A MATHEMATICAL MODEL

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Abstract. The term radicalization refers to the process of developing extremist religious political or social beliefs and ideologies. Radicalization becomes a threat to national security when it leads to violence. Prevention and de-radicalization initiatives are part of a set of strategies used to combat violent extremism, which taken together are known as Countering Violent Extremism (CVE). Prevention programs aim to stop the radicalization process before it starts. De-radicalization programs attempt to reform convicted extremists with the ultimate goal of social reintegration. We describe prevention and de-radicalization programs mathematically using a compartmental model. The prevention initiatives are modeled by including a vaccination compartment, while the de-radicalization process is modeled by including a treatment compartment. The model exhibits a threshold dynamics characterized by the basic reproduction number R_0. When $R_0 < 1$ the system has a unique equilibrium that is asymptotically stable. When $R_0 > 1$ the system has another equilibrium called "endemic equilibrium", which is globally asymptotically stable. These results are established by using Lyapunov functions and LaSalle's invariance principle. We perform numerical simulations to confirm our theoretical results.

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1. Introduction

The term *radicalization* refers to the process of developing extremist religious political or social beliefs and ideologies. While radical thinking is by no means problematic in itself, it becomes a threat to national security when it leads to violence. Because of this fact, radicalization is of particular concern for governments, law enforcement and security agencies.

A conventional, but arguably antiquated, approach to national security is based on counterterrorism. Counterterrorism strategies consist of a law enforcement component (terrorists are arrested, tried, and convicted) and a military component (terrorists lose their life or are captured on the battleground).

Practitioners of counterterrorism, however, agree that these approaches alone cannot break the cycle of violence [23]. In light of this, governments use an additional set of initiatives collectively known as countering violent extremism (CVE). CVE programs can be classified into three categories[23, 13, 5]

(1) *Prevention programs*, which aim to stop the radicalization process before it starts;
(2) *Disengagement programs*, which endeavor to block radicalization while it is taking place.

(3) *De-radicalization programs*, which aim to reform convicted extremists with the ultimate goal of social reintegration.

The development of viable intervention strategies to mitigate radicalization and violence requires a thorough understanding of the radicalization process, prevention, disengagement and deradicalization programs. Mathematical models can provide a first step in this direction. The aim of this paper is to use a compartmental epidemiological model to analyze CVE programs, focusing on prevention and deradicalization initiatives.

The use of differential equations to describe social science problems dates back, at least, to the work of Lewis F. Richardson [20] who pioneered the application of mathematical techniques by studying the causes of war, and the relationship between arms race and the eruption of war. A summary of his research was published posthumously in the book [19].

Modern applications of compartmental models to the social sciences range from models of political party growth, to models of the spread of crime (see for instance [6, 8, 10, 15, 16, 21, 25, 26, 27]). In recent years compartmental models have also been used to study terrorism, the spread of fanatic behavior, and radicalization [2, 1, 7, 14, 22, 17]. Furthermore, an age-structure model of radicalization was considered by Chuang, Chou and D'Orsogna [3], a bi-stable model of radicalization within sectarian conflict was studied Chuang, D'Orsogna and Chou [4], and a game theoretic model of radicalization was analyzed by Short, McCalla and d'Orsogna [24].

The model we study here extends the one considered in [22] by including a vaccinated class. The purpose of this model is to analyze two of the CVE strategies, namely prevention programs and de-radicalization programs. As in [14, 22] we use Lyapunov functions to study the global stability of the equilibria of the model and the basic reproduction number $\mathcal{R}_0$ to assess initiatives for combating terrorism.

Although the literature indicates some degree of success of CVE programs, according to [13], there is little consensus regarding the validity of CVE prevention programs or disengagement/de-radicalization programs, largely due to the lack of empirical data. Furthermore, it is very difficult to evaluate these programs since indicators of success and measures of efficacy remain elusive [18]. These are key issues, since the degree of government support for these programs depends, to a large extent, on demonstrating their effectiveness. The results we present in this paper are theoretical in nature and are fairly independent from the specific choices of the parameter values. Our model can, in principle, be used to evaluate the efficacy of CVE programs in combating terrorism whenever empirical data are known.

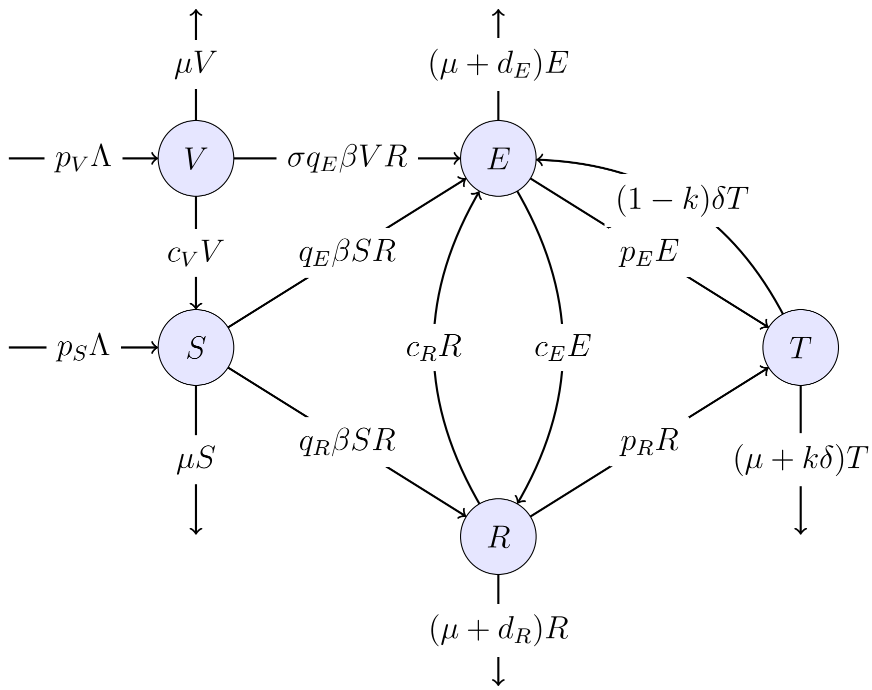
2. Equations

We use a compartmental model to describe the dynamics. We divide the population at risk of adopting an extreme ideology into five compartments

(1) (S) Susceptible
(2) (E) Extremists
(3) (R) Recruiters

(4) (T) Treated
(5) (V) Vaccinated.

Our model is extends the treatment model studied in [22] by adding a vaccinated compartment (V). This allows us to describe individuals in prevention programs. Our transfer diagram is similar, but different, to the one proposed by Yang, et al. [29] to model the spread of tuberculosis with vaccination and treatment. The differences in the models are enough to create some complications in the construction of Lyapunov functions for our problem. The transfer diagram for our system is given below.



Recruitment occurs in the system with rate constant $\Lambda > 0$. Of these individuals, a fraction p_V enters the vaccination compartment, while a fraction $p_S = 1 - p_V$ enters the susceptible population. The rate at which susceptibles are recruited is βSR. A fraction q_E of the newly recruited individuals are assumed to transfer to the extremist class, while the remainder $q_R = 1 - q_E \ll 1$ transfer to the recruiter class. Vaccinated individuals are recruited at a reduced rate $\sigma q_E \beta V R$, with $0 \leq \sigma \leq 1$. The rate constant at which an individual leaves the extremist compartment to became a recruiter is c_E, while c_R is the rate constant at which a recruiter abandons the recruiter class to become an extremist. The natural death rate constant is μ, d_E and d_R are supplementary death rate constants for individuals in compartments E and R, respectively. The additional rates d_E and d_R take into consideration individuals that die or are sentenced to lifelong incarceration as a result of police or military action. The rate constants of extremists transferred to the treatment compartment is p_E, while p_R is the rate constant of extremists moving to compartment T.

Treated individuals exit T at a rate δ. We remove a fraction $k \in [0,1]$ of treated individuals, since effectively treated individuals are de-radicalized forever. For a fraction $1-k$ of treated individuals the de-radicalization program is unsuccesful. These individuals get into the E compartment after treatment.

Based on the above assumptions we obtain the following model:

$$\begin{aligned} S' &= p_S\Lambda + c_V V - \mu S - \beta SR \\ E' &= q_E\beta SR + \sigma q_E \beta VR - (\mu + d_E + c_E + p_E)E + c_R R + (1-k)\,\delta T \\ R' &= q_R\beta SR + c_E E - (\mu + d_R + c_R + p_R)R \\ T' &= p_E E + p_R R - (\mu+\delta)T \\ V' &= p_V\Lambda - c_V V - \mu V - \sigma q_E \beta VR \end{aligned} \tag{2.1}$$

where $q_E + q_R = 1$, $q_E, q_R \in [0,1]$. To simplify system (2.1) we introduce the following parameters $b_E = \mu + d_E + c_E + p_E$, $b_R = \mu + d_R + c_R + p_R$, $b_T = \mu + \delta$ and $b_V = c_V + \mu$. Using these new constants in (2.1) yields:

$$\begin{aligned} S' &= p_S\Lambda + c_V V - \mu S - \beta SR \\ E' &= q_E\beta(S+\sigma V)R - b_E E + c_R R + (1-k)\,\delta T \\ R' &= q_R\beta SR + c_E E - b_R R \\ T' &= p_E E + p_R R - b_T T \\ V' &= p_V\Lambda - b_V V - \sigma q_E \beta VR. \end{aligned} \tag{2.2}$$

It is not difficult to show that the region

$$\Delta = \left\{ (S,E,R,T,V) \in \mathbb{R}^5_{\geq 0} : S+E+R+T+V \leq \frac{\Lambda}{\mu} \right\}$$

is a compact positively invariant and attracting set that attracts all solutions of (2.2) with initial conditions in $\mathbb{R}^5_{\geq 0}$. See Proposition 2.1 in [22] for a proof of a similar statement.

3. Radicalization-free equilibrium and basic reproduction number $\mathcal{R}_0$

There is a unique equilibrium with $E = R = T = 0$ given by $x_0 = (S_0, 0, 0, 0, V_0)$, where

$$S_0 = \frac{\left(p_S + \frac{c_V}{b_V} p_V\right)\Lambda}{\mu} = \frac{\left(p_S + \frac{c_V}{\mu}\right)\Lambda}{b_V} \qquad V_0 = \frac{p_V \Lambda}{b_V} \tag{3.1}$$

We denote by $\mathcal{R}_0$ the spectral radius of the matrix G evaluated at x_0. $\mathcal{R}_0$ is called the *basic reproduction number* and can be obtained as outlined by Van Den Driessche and Watmough [28]. First we identify the infected classes, that in this example turn out to be E, R, T.

Suppose $\mathcal{F}_E$, $\mathcal{F}_R$ and $\mathcal{F}_T$ are the rates of arrival of newly radicalized individuals in the compartment E, R, and T, respectively. Let $\mathcal{V}_j = \mathcal{V}_j^- - \mathcal{V}_j^+$, with $\mathcal{V}_j^+$ be the rate of transmission of individuals into compartment $j \in \{E, R, T\}$ by all remining methods, and $\mathcal{V}_j^-$ the rate of removal of individuals from compartment j, where j is one of E, R, and T. In our problem

$$\mathcal{F} = \begin{bmatrix} \mathcal{F}_E \\ \mathcal{F}_R \\ \mathcal{F}_T \end{bmatrix} = \beta \begin{bmatrix} q_E(SR + \sigma VR) \\ q_R SR \\ 0 \end{bmatrix}$$

and

$$\mathcal{V} = \begin{bmatrix} \mathcal{V}_E \\ \mathcal{V}_R \\ \mathcal{V}_T \end{bmatrix} = \begin{bmatrix} b_E E - c_R R - (1-k)\delta T \\ b_R R - c_E E \\ b_T T - (p_E E + p_R R) \end{bmatrix}.$$

Then consider the matrices

$$F = \begin{bmatrix} \frac{\partial \mathcal{F}_E}{\partial E} & \frac{\partial \mathcal{F}_E}{\partial R} & \frac{\partial \mathcal{F}_E}{\partial T} \\ \frac{\partial \mathcal{F}_R}{\partial E} & \frac{\partial \mathcal{F}_R}{\partial R} & \frac{\partial \mathcal{F}_R}{\partial T} \\ \frac{\partial \mathcal{F}_T}{\partial E} & \frac{\partial \mathcal{F}_T}{\partial R} & \frac{\partial \mathcal{F}_T}{\partial T} \end{bmatrix} (x_0) \quad \text{and} \quad V = \begin{bmatrix} \frac{\partial \mathcal{V}_E}{\partial E} & \frac{\partial \mathcal{V}_E}{\partial R} & \frac{\partial \mathcal{V}_E}{\partial T} \\ \frac{\partial \mathcal{V}_R}{\partial E} & \frac{\partial \mathcal{V}_R}{\partial R} & \frac{\partial \mathcal{V}_R}{\partial T} \\ \frac{\partial \mathcal{V}_T}{\partial E} & \frac{\partial \mathcal{V}_T}{\partial R} & \frac{\partial \mathcal{V}_T}{\partial T} \end{bmatrix} (x_0),$$

which in our problem take the form

$$F = \beta \begin{bmatrix} 0 & q_E(S_0 + \sigma V_0) & 0 \\ 0 & q_R S_0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} b_E & -c_R & -\alpha_E \\ -c_E & b_R & 0 \\ -p_E & -p_R & b_T \end{bmatrix},$$

where $\alpha_E = (1-k)\delta$.

Finally, we can compute the next generation matrix $G = FV^{-1}$:

$$\begin{aligned} G &= -\frac{\beta}{b_T D} \begin{bmatrix} 0 & q_E(S_0 + \sigma V_0) & 0 \\ 0 & q_R S_0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -b_R b_T & -(\alpha_E p_R + c_R b_T) & \alpha_E b_R \\ -c_E b_T & \alpha_E p_E - b_E b_T & -\alpha_E c_E \\ -b_R p_E - c_E p_R & -b_E p_R - c_R p_E & -b_E b_R + c_R c_E \end{bmatrix} \\ &= -\frac{\beta}{b_T D} \begin{bmatrix} -q_E c_E b_T (S_0 + \sigma V_0) & q_E(\alpha_E p_E - b_E b_T)(S_0 + \sigma V_0) & -q_E \alpha_E c_E S_0 \\ -q_R c_E b_T S_0 & q_R(\alpha_E p_E - b_E b_T) S_0 & -q_R \alpha_E c_E S_0 \\ 0 & 0 & 0 \end{bmatrix}, \end{aligned}$$

where $D = b_E b_R - c_E c_R - \frac{\alpha_E}{b_T}(b_R p_E + c_E p_R) > 0$. Since the matrix G has only one non-zero eigenvalue, its spectral radius is:

$$\mathcal{R}_0 = \frac{\beta S_0 (c_E q_E + b_E q_R - \frac{\alpha_E p_E}{b_T} q_R) + \beta \sigma V_0 q_E c_E}{b_E b_R - c_E c_R - \frac{\alpha_E}{b_T}(c_E p_R + b_R p_E)}. \tag{3.2}$$

4. Global Asymptotic Stability of x_0 for $\mathcal{R}_0 < 1$

In this section we prove the global asymptotic stability of the equilibrium x_0. For this purpose we introduce the following Lyapunov function

$$U = A \frac{(S - S_0)^2}{2S_0} + \frac{(V - V_0)^2}{2V_0} + \frac{H}{b_T c_E},$$

where H is the function defined by

$$H(E,R,T) = b_T c_E E + (b_T b_E - \alpha_E p_E)R + \alpha_E c_E T.$$

Let

$$Q = b_T b_E - \alpha_E p_E = (\mu+\delta)(\mu + d_E + c_E) + \mu p_E + k\delta p_E > 0,$$

then

$$A = \beta \frac{c_E q_E + b_E q_R - \frac{\alpha_E p_E}{b_T} q_R}{c_E}$$

is a positive constant since

$$c_E q_E + \frac{q_R}{b_T}(b_E b_T - \alpha_E p_E) = c_E q_E + \frac{q_R}{b_T} Q > 0.$$

We can now prove the following theorem

Theorem 4.1. *Suppose* $Ap_V < 4$, *and* $\mathcal{R}_0 \leq 1$ *then* x_0 *is globally asymptotically stable on* $\mathbb{R}^4_{\geq 0}$.

Proof. We study the stability of x_0 by taking the Lyapunov function

$$U = A\,\frac{(S-S_0)^2}{2S_0} + \frac{(V-V_0)^2}{2V_0} + \frac{H}{b_T c_E}$$

where H is the function defined by

$$H(E,R,T) = b_T c_E E + (b_T b_E - \alpha_E p_E)R + \alpha_E c_E T,$$

with

$$Q = b_T b_E - \alpha_E p_E = (\mu+\delta)(\mu + d_E + c_E) + \mu p_E + k\delta p_E > 0,$$

and

$$A = \beta \frac{c_E q_E + b_E q_R - \frac{\alpha_E p_E}{b_T} q_R}{c_E}$$

is a positive constant since

$$c_E q_E + \frac{q_R}{b_T}(b_E b_T - \alpha_E p_E) = c_E q_E + \frac{q_R}{b_T} Q > 0.$$

Using (3.1) we obtain

$$\begin{aligned} p_V \Lambda &= b_V V_0 \\ p_S \Lambda &= \mu S_0 - c_V V_0. \end{aligned} \tag{4.1}$$

Differentiating U with respect to t along the trajectories of the system (2.2), and using both equations in (4.1) to rewrite $p_V\Lambda$ and $p_S\Lambda$ yields

$$\begin{aligned}
U' =& A\frac{(S-S_0)}{S_0}S' + \frac{(V-V_0)}{V_0}V' + \frac{b_Tc_EE' + (b_Tb_E - \alpha_Ep_E)R' + \alpha_Ec_ET'}{b_Tc_E}\\
=& f(S,V) - A\beta\frac{(S-S_0)^2}{S_0}R - \sigma\beta q_E\frac{(V-V_0)^2}{V_0}R - A\beta(S-S_0)R\\
&- \sigma q_E\beta(V-V_0)R + A\beta SR + \sigma q_E\beta VR + \frac{D}{b_Tc_E}\,R\\
\le& f(S,V) - A\beta(S-S_0)R - \sigma q_E\beta(V-V_0)R + A\beta SR + \sigma q_E\beta VR + \frac{D}{b_Tc_E}\,R\\
=& f(S,V) + \left[A\beta S_0 + \sigma q_E\beta V_0 + \frac{D}{b_Tc_E}\right]\,R\\
=& f(S,V) + \frac{D}{b_Tc_E}\left[1 + b_Tc_E\frac{A\beta S_0 + \sigma q_E\beta V_0}{D}\right]\,R\\
=& f(S,V) + \frac{D}{b_Tc_E}\left[1-\mathcal{R}_0\right]\,R
\end{aligned}$$

where $f(S,V) = -A\,\mu\frac{(S-S_0)^2}{S_0} - b_V\frac{(V-V_0)^2}{V_0} + A\,c_V\frac{(S-S_0)(V-V_0)}{S_0}$. It remains to show that $f(S,V)\le 0$ and $f(S,V)=0$ if and only if $S=S_0$ and $V=V_0$.

Note that the Hessian matrix of $f(S,V)$ is

$$\begin{bmatrix} -\frac{2A\mu}{S_0} & \frac{Ac_V}{S_0}\\ \frac{Ac_V}{S_0} & -\frac{2b_V}{V_0}\end{bmatrix}$$

and its determinant is $-\frac{A(AV_0c_V^2 - 4S_0b_V\mu)}{S_0^2V_0}$. By the second derivative test, $(S,V)=(S_0,V_0)$ is a maximum when the determinant is positive, and $-\frac{2A\mu}{S_0}<0$. The latter inequality holds, hence, it only remains to study the determinant. Since $A>0$ we must have $-AV_0c_V^2 + 4S_0b_V\mu > 0$. Substituting S_0 and V_0 in this expression we obtain

$$-AV_0c_V^2 + 4S_0b_V\mu = \frac{\Lambda}{b_V}\left[-Ap_Vc_V^2 + 4b_Vc_V + 4\mu p_Sb_V\right]$$

Since $b_V = c_V+\mu$ the expression inside the bracket reduces to

$$(4-Ap_V)c_V^2 + 4\mu c_V + 4\mu p_Sp_V,$$

which proves that the determinant is always positive if $p_VA<4$.

In this case, for $\mathcal{R}_0\le 1$, we have $U'\le 0$, with equality if and only if $S=S_0$ and $V=V_0$, and $R=0$. The largest invariant set for which $U'=0$, then consists of just the equilibrium x_0. The theorem then follows from LaSalle's Invariance Principle. □

Note that, Theorem 4.1 holds when the parameters satisfy the inequality $Ap_V<4$. This does not seem to pose a substantial restriction in the allowable value of the constants, since

Ap_V is a small number for any reasonable choice. For instance if we choose the parameters as in Figure 3, then $Ap_V = 6.633301503 \times 10^{-7}$. It may be of interest to see if it is possible to remove the restriction $Ap_V < 4$ by using a different Lyapunov function.

5. Endemic Equilibrium

We now look for equilibria of (2.2) for which at least one of the populations E^*, R^*, T^* and V^* is different from zero. We call such point an *endemic equilibrium* and we denote it by $x^* = (S^*, E^*, R^*, T^*, V^*) \in \mathbb{R}^5_{>0}$. The endemic equilibria of (2.2) are given by the following system of equations

$$\begin{aligned}
&p_S\Lambda + c_V V - \mu S - \beta S R = 0\\
&q_E\beta(S+\sigma V)R - b_E E + c_R R + (1-k)\,\delta T = 0\\
&q_R\beta S R + c_E E - b_R R = 0\\
&p_E E + p_R R - b_T T = 0\\
&p_V\Lambda - b_V V - \sigma q_E \beta V R = 0.
\end{aligned} \tag{5.1}$$

Solving the first, third, fourth and fifth equation in (5.1) and treating R^* as a parameter we obtain the following

$$\begin{aligned}
S^* &= \frac{\Lambda}{\mu+\beta R^*}\left(p_S + \frac{c_V p_V}{b_V + \sigma q_E \beta R^*}\right)\\
E^* &= \omega R^*\\
T^* &= \frac{p_E\omega + p_R}{b_T} R^*\\
V^* &= \frac{p_V\Lambda}{b_V + \sigma q_E\beta R^*},
\end{aligned} \tag{5.2}$$

where

$$\omega = \frac{b_R - q_R\beta S^*}{c_E} = \frac{b_R b_T q_E + b_T c_R q_R + p_R q_R \alpha_E}{q_R(b_E b_T - \alpha_E p_E) + c_E q_E b_T} > 0,$$

since $(b_E b_T - \alpha_E p_E) > 0$. Substituting the expressions in 5.2 in the second equation of (5.1) yields the following equation for R^*

$$R^*\left(\alpha_2 (R^*)^2 + \alpha_1 R^* + \alpha_0\right) = 0$$

where

$$\begin{aligned}
\alpha_2 &= -\beta^2 q_E \sigma D b_T\\
\alpha_1 &= -\beta b_T(b_V + \mu\sigma q_E)D + \sigma(((b_E q_R + c_E q_E)b_T + (k-1)\delta q_R p_E p_S + b_T c_E p_V))\beta^2 q_E \Lambda\\
\alpha_0 &= b_T b_V \mu D\,(\mathcal{R}_0 - 1)
\end{aligned}$$

This equation has solution if either $R^* = 0$ or

$$\Phi(R^*) = \alpha_2 (R^*)^2 + \alpha_1 R^* + \alpha_0 = 0. \tag{5.3}$$

The case $R^* = 0$ must be excluded since it yields a solution with $E^* = R^* = T^* = 0$, which was already known. Consider equation (5.3). Clearly, $\alpha_2 < 0$ (since $D > 0$) and $\alpha_0 > 0$ whenever $\mathcal{R}_0 > 1$. It follows that (5.3) has a unique positive root R^* if $\mathcal{R}_0 > 1$. Since Δ is attracting within $\mathbb{R}^5_{\geq 0}$ we have that $R^* \in (0, \Lambda/\mu)$.

Note that we can also find an useful formula for S^* as a function of V^*. This formula can be obtained by substituting the equations for E^* and T^* given in (5.2) into the second line of (5.1), and solving for S^*. This gives

$$S^* = \frac{b_E b_R - c_E c_R - \frac{\alpha_E}{b_T}(p_R c_E + p_E b_R) - q_E \beta \sigma c_E V^*}{\beta(c_E q_E + b_E q_R - \frac{\alpha_E p_E}{b_T} q_R)}. \tag{5.4}$$

Theorem 5.1. *The endemic equilibrium x^* of (2.2) is globally asymptotically stable on $\mathbb{R}^5_{>0}$, whenever $\mathcal{R}_0 > 1$.*

Proof. Consider the Lyapunov function

$$W = S^* g\left(\frac{S}{S^*}\right) + a_1 E^* g\left(\frac{E}{E^*}\right) + a_2 R^* g\left(\frac{R}{R^*}\right) + a_3 T^* g\left(\frac{T}{T^*}\right) + a_4 V^* g\left(\frac{V}{V^*}\right)$$

with $g(x) = x - 1 - \ln x$, and $a_i > 0$ ($i = 1, \ldots, 4$), where the a_is are constants to be determined. Clearly, W is C^1, $W(x^*) = 0$, and $W > 0$ for any $p \in \mathbb{R}^5_{>0}$ such that $p \neq x^*$. Computing the derivative of W along the solutions of (2.2) gives

$$\begin{aligned}
W' =& \left(1 - \frac{S^*}{S}\right) S' + a_1 \left(1 - \frac{E^*}{E}\right) E' + a_2 \left(1 - \frac{R^*}{R}\right) R' + a_3 \left(1 - \frac{T^*}{T}\right) T' + a_4 \left(1 - \frac{V^*}{V}\right) V' \\
=& \left(1 - \frac{S^*}{S}\right) [p_S \Lambda + c_V V - \mu S - \beta S R] \\
&+ a_1 \left(1 - \frac{E^*}{E}\right) [q_E \beta (S + \sigma V) R - b_E E + c_R R + (1-k)\delta T] \\
&+ a_2 \left(1 - \frac{R^*}{R}\right) [q_R \beta S R + c_E E - b_R R] + a_3 \left(1 - \frac{T^*}{T}\right) [p_E E + p_R R - b_T T] \\
&+ a_4 \left(1 - \frac{V^*}{V}\right) [p_V \Lambda - b_V V - \sigma q_E \beta V R] \\
=& C - (\mu + a_2 \beta q_R R^*) S + (a_1 q_E + a_2 q_R - 1) \beta S R + (-a_1 b_E + a_2 c_E + a_3 p_E) E \\
&+ (S^* \beta + a_1 c_R - a_2 b_R + a_3 p_R + a_4 \sigma q_E \beta V^*) R + (a_1 \alpha_E - a_3 b_T) T + (c_V - a_4 b_V) V \\
&+ q_E \beta \sigma (a_1 - a_4) V R - p_S \Lambda \frac{S^*}{S} - a_3 p_E T^* \frac{E}{T} - a_2 c_E R^* \frac{E}{R} - a_3 p_R T^* \frac{R}{T} - a_1 \alpha_E E^* \frac{T}{E} \\
&- a_1 c_R E^* \frac{R}{E} - a_1 \beta q_E E^* \frac{SR}{E} - a_1 \sigma q_E \beta E^* \frac{VR}{E} - a_4 p_V \Lambda \frac{V^*}{V} - c_V V \frac{S^*}{S}.
\end{aligned}$$

where $C = \Lambda + \mu S^* + a_1 b_E E^* + a_2 b_R R^* + a_3 b_T T^* + a_4 b_V V^*$. For simplicity denote $w = \frac{S}{S^*}$, $x = \frac{E}{E^*}$, $y = \frac{R}{R^*}$, $z = \frac{T}{T^*}$, and $v = \frac{V}{V^*}$. Then

$$\begin{aligned}W' =& C - (\mu + a_2\beta q_R R^*)S^* w + (a_1 q_E + a_2 q_R - 1)\beta S^* R^* wy + (-a_1 b_E + a_2 c_E + a_3 p_E)E^* x\\ &+ (S^*\beta + a_1 c_R - a_2 b_R + a_3 p_R + a_4 \sigma q_E \beta V^*)R^* y + (a_1\alpha_E - a_3 b_T)T^* z + (c_V - a_4 b_V)V^* v\\ &+ q_E\beta\sigma(a_1 - a_4)V^* R^* vy - p_S\Lambda\frac{1}{w} - a_3 p_E E^*\frac{x}{z} - a_2 c_E E^*\frac{x}{y} - a_3 p_R R^*\frac{y}{z} - a_1\alpha_E T^*\frac{z}{x}\\ &- a_1 c_R R^*\frac{y}{x} - a_1\beta q_E S^* R^*\frac{wy}{x} - a_1\sigma q_E\beta V^* R^*\frac{vy}{x} - a_4 p_V\Lambda\frac{1}{v} - c_V V^*\frac{v}{w}.\end{aligned}$$

Following the method used in [12] and [11] we introduce the following set

$$\mathcal{D} = \left\{ v, w, x, y, z, vy, wy, \frac{1}{v}, \frac{1}{w}, \frac{v}{w}, \frac{x}{z}, \frac{x}{y}, \frac{y}{z}, \frac{z}{x}, \frac{y}{x}, \frac{vy}{x}, \frac{wy}{x} \right\}.$$

We now list all the subsets of $\mathcal{D}$ for which the product of all functions within each subset is equal to one

$$\begin{aligned}&\left\{v, \frac{1}{v}\right\}, \left\{w, \frac{1}{w}\right\}, \left\{\frac{x}{y}, \frac{y}{x}\right\}, \left\{\frac{x}{z}, \frac{z}{x}\right\},\\ &\left\{\frac{1}{v}, \frac{vy}{x}, \frac{x}{y}\right\}, \left\{\frac{1}{w}, \frac{wy}{x}, \frac{x}{y}\right\}, \left\{\frac{z}{x}, \frac{y}{z}, \frac{x}{y}\right\}, \left\{w, \frac{1}{v}, \frac{v}{w}\right\}, \left\{\frac{1}{v}, \frac{v}{w}, \frac{x}{y}, \frac{wy}{x}\right\}.\end{aligned} \tag{5.5}$$

We associate the following terms to the subsets in equation (5.5):

$$\begin{aligned}&\left(2 - v - \frac{1}{v}\right), \left(2 - w - \frac{1}{w}\right), \left(2 - \frac{x}{y} - \frac{y}{x}\right), \left(2 - \frac{x}{z} - \frac{z}{x}\right),\\ &\left(3 - \frac{1}{v} - \frac{vy}{x} - \frac{x}{y}\right), \left(3 - \frac{1}{w} - \frac{wy}{x} - \frac{x}{y}\right), \left(3 - \frac{z}{x} - \frac{y}{z} - \frac{x}{y}\right), \left(3 - w - \frac{1}{v} - \frac{v}{w}\right),\\ &\left(4 - \frac{1}{v} - \frac{v}{w} - \frac{x}{y} - \frac{wy}{x}\right).\end{aligned}$$

As in [12], we define a Lyapunov function using the terms above:

$$\begin{aligned}H(v, w, x, y, z) = & b_1\left(2 - v - \frac{1}{v}\right) + b_2\left(2 - w - \frac{1}{w}\right) + b_3\left(2 - \frac{x}{y} - \frac{y}{x}\right)\\ &+ b_4\left(2 - \frac{x}{z} - \frac{z}{x}\right) + b_5\left(3 - \frac{1}{v} - \frac{vy}{x} - \frac{x}{y}\right)\\ &+ b_6\left(3 - \frac{1}{w} - \frac{wy}{x} - \frac{x}{y}\right) + b_7\left(3 - \frac{z}{x} - \frac{y}{z} - \frac{x}{y}\right)\\ &+ b_8\left(3 - w - \frac{1}{v} - \frac{v}{w}\right), + b_9\left(4 - \frac{1}{v} - \frac{v}{w} - \frac{x}{y} - \frac{wy}{x}\right)\end{aligned} \tag{5.6}$$

where $b_1, \dots, b_9$ are unknown constants. We look for solutions of the equation $G(v, x, y, z) = H(v, w, x, y, z)$ with $a_i > 0$ $(i = 1, \dots, 4)$ and $b_k \geq 0$ $(k = 1, \dots, 9)$

Setting like terms in G and H equal yields:

$$
\begin{aligned}
vy: &\quad q_E\beta\sigma(a_1 - a_4) = 0\\
wy: &\quad a_1q_E + a_2q_R - 1 = 0\\
x: &\quad -a_1b_E + a_2c_E + a_3p_E = 0\\
y: &\quad S^*\beta + a_1c_R - a_2b_R + a_3p_R + a_4\sigma q_E\beta V^* = 0\\
z: &\quad a_1\alpha_E - a_3b_T = 0\\
&\\
v: &\quad b_1 = -(c_V - a_4b_V)V^*\\
yx^{-1}: &\quad b_3 = a_1c_RR^*\\
xz^{-1}: &\quad b_4 = a_3p_EE^*\\
vyx^{-1}: &\quad b_5 = a_1\sigma q_E\beta V^*R^*\\
yz^{-1}: &\quad b_7 = a_3p_RR^*\\
zx^{-1}: &\quad b_4 + b_7 = a_1\alpha_ET^*\\
&\\
v^{-1}: &\quad b_1 + b_5 + b_8 + b_9 = a_4p_V\Lambda\\
xy^{-1}: &\quad b_3 + b_5 + b_6 + b_7 + b_9 = a_2c_EE^*\\
&\\
v^0: &\quad 2(b_1 + b_2 + b_3 + b_4) + 3(b_5 + b_6 + b_7 + b_8) + 4b_9 = C\\
w: &\quad b_2 + b_8 = (\mu + a_2\beta q_RR^*)S^*\\
w^{-1}: &\quad b_2 + b_6 = p_S\Lambda\\
cw^{-1}: &\quad b_8 + b_9 = c_VV^*\\
wyx^{-1}: &\quad b_6 + b_9 = a_1\beta q_ES^*R^*
\end{aligned}
\tag{5.7}
$$

The first five lines of (5.7) form an overdetermined consistent system of five equations in the variables $a_1, \dots, a_4$, with solution

$$
\begin{aligned}
a_1 = a_4 &= \frac{c_E}{c_Eq_E + b_Eq_R - \frac{p_E}{b_T}\alpha_Eq_R} > 0\\
a_2 &= \frac{1}{q_R} - \frac{\frac{q_E}{q_R}c_E}{c_Eq_E + b_Eq_R - \frac{p_E}{b_T}\alpha_Eq_R} > 0\\
a_3 &= \frac{\frac{c_E\alpha_E}{b_T}}{c_Eq_E + b_Eq_R - \frac{p_E}{b_T}\alpha_Eq_R} > 0.
\end{aligned}
$$

The following six lines of (5.7) form an overdetermined consistent system of six equations in the variables b_1, b_3, b_4, b_5, b_7, with a unique solution where $b_1, b_3, b_4, b_5, b_7 > 0$. The next two lines can be easily seen to be consistent with the other equations, but because they are dependent from other equations they can be discarded.

The last group of equations in (5.7) is a system of five equations in the variables b_2, b_6, b_8 and b_9. It remains to prove that this system has a solution with $b_2, b_6, b_8, b_9 > 0$. The augmented matrix of the system given by the last five equations of (5.7) is

$$\left[\begin{array}{cccc|c} 2 & 3 & 3 & 4 & d_1 \\ 1 & 0 & 1 & 0 & d_2 \\ 1 & 1 & 0 & 0 & d_3 \\ 0 & 0 & 1 & 1 & d_4 \\ 0 & 1 & 0 & 1 & d_5 \end{array}\right]$$

where $d_1 = C - 2(b_1 + b_3 + b_4) - 3(b_5 + b_7)$, $d_2 = (\mu + a_2\beta q_R R^*)S^*$, $d_3 = p_S\Lambda$, $d_4 = c_V V^*$, and $d_5 = a_1\beta q_E S^* R^*$. Performing row operations we can reduce the augmented matrix to row echelon form

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & d_2 \\ 0 & 1 & -1 & 0 & d_3 - d_2 \\ 0 & 0 & 1 & 1 & d_4 \\ 0 & 0 & 0 & 0 & \frac{1}{4}(d_1 + d_2 - 3d_3) - d_4 \\ 0 & 0 & 0 & 0 & d_5 - d_3 + d_2 - d_4 \end{array}\right]$$

A computations involving the last two rows of the augmented matrix above shows that the system is consistent, but since only the first three rows are independent it follows that the solution is not unique. Solutions of the system are given by

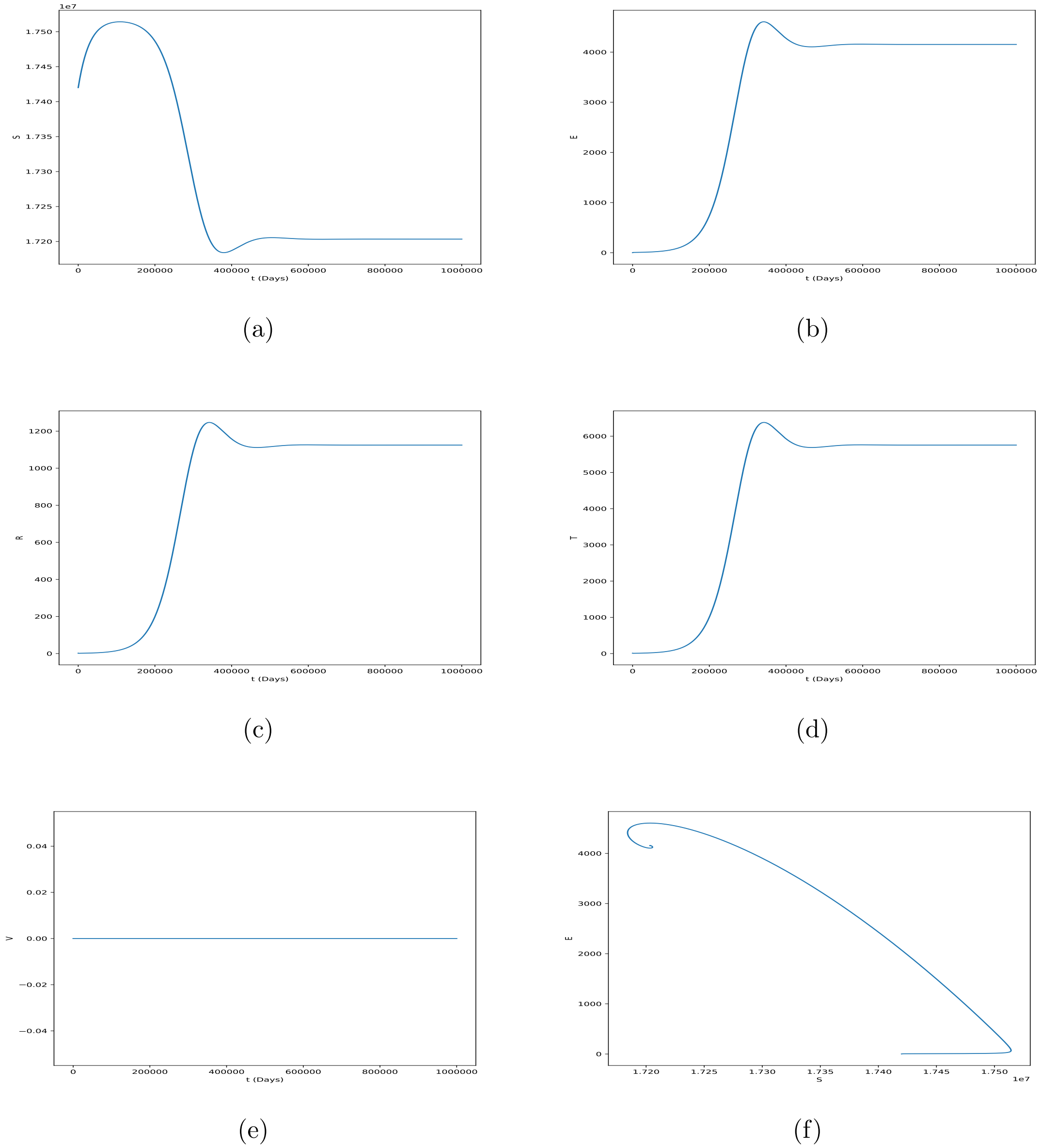
$$\begin{aligned} b_2 &= d_2 - t \\ b_6 &= (d_3 - d_2) + t \\ b_9 &= d_4 - t \end{aligned}$$

with $b_8 = t$. For the solutions to be positive we must have $\max(0, d_2 - d_3) < t < \min(d_2, d_4)$. Hence, for this system of equation to always have a positive solution we must have that $d_2 - d_3 < d_2$, $d_2 - d_3 < d_4$ and $d_2, d_4 > 0$. The first inequality is always satisfied since $d_3 > 0$. The second inequality holds since $d_5 - d_3 + d_2 - d_4 = 0$, so that $-d_3 + d_2 - d_4 = -d_5 < 0$. Clearly, $d_4 > 0$. The fact that $d_2 > 0$ follows from the fact that $a_2 > 0$, which completes the proof. □

6. Numerical Simulations

We now use numerical simulations of system (2.1) to illustrate and support the results of our mathematical analysis.

We assume that the life expectancy is 80 years, which implies that death rate is $\mu = 0.000034247\,(\text{days})^{-1}$ [9]. We take $\Lambda = 600\,(\text{days})^{-1}$, which corresponds to a population size of about 17.5 million. The remaining parameters are taken to be $\beta = 0.0000005\,(\text{days})^{-1}$, $d_E = 0.00083\,(\text{days})^{-1}$, $d_R = 0.00083\,(\text{days})^{-1}$, $p_E = 0.00175$, $p_R = 0.0019\,(\text{days})^{-1}$, $c_E = 0.0006\,(\text{days})^{-1}$, $c_R = 0.0008\,(\text{days})^{-1}$, and $\delta = 0.0016\,(\text{days})^{-1}$, $q_E = 0.86$, $q_R = 0.14$ $\sigma = 0.2$



(a) (b) (c) (d) (e) (f)

FIGURE 1. Time history and phase portraits of system (2.1) for $\beta = 0.0000005$, $q_E = 0.86$, $q_R = 0.14$, $d_E = 0.00083$, $d_R = 0.000083$, $p_E = 0.00175$, $p_R = 0.0019$, $p_S = 1$, $p_V = 0$, $\sigma = 0.8$, $c_E = 0.0006$, $c_R = 0.0008$, $k = 0.66$, $\delta = 0.0016$, $\mu = 0.000034247$, $\Lambda = 600$

and $k = 0.66$. We also assume that, initially, there is no prevention program by choosing $p_S = 1$, $p_V = 0$.

In this case we find that $\mathcal{R}_0 = 1.0183912670368627$, and hence, x^* is globally asymptotically stable in $\mathbb{R}^5_{>0}$ by Theorem 5.1.

Figures 1 (a)-(d) show that the number of individuals in the compartments S, E, R, and T, approach a constant value. Figure 1 (e) shows that the number of individuals in the vaccination compartment is, in this case zero. Figure (f), instead, is a phase portrait for the system. These figures confirm that the solutions approach a globally asymptotically stable equilibrium point. This case illustrates the unwanted scenario where terrorists and recruiters become endemic to the population.

If we raise p_V to $p_V = 0.1\,(\text{days})^{-1}$ without changing the other parameters, then $\mathcal{R}_0 = 1.0126247813740998$ and the endemic equilibrium x^* is still globally asymptotically stable. S, E, R, and T still approach a constant value as $t \to \infty$, however, the number of individuals in the compartments E and R is much less than before, see Figure 2 (a)-(f).

If we increase p_V to $p_V = 0.9$, keeping the other parameters constant, then $\mathcal{R}_0 = 0.9664928960719988$, so that x^0 is globally asymptotically stable. Figures 3 (a)–(d) show that $S, E, R, V \to 0$, as $t \to \infty$, supporting our result on the asymptotic stability of x_0. This means that if a large enough number of susceptible individuals is targeted by effective prevention programs it is possible to change the stability of the equilibria, ensuring that extremists and recruiters eventually disappear from the population.

7. Conclusions

This paper considers a model of radicalization in which the population is divided into five compartments: Susceptible, Vaccinated, Extremists, Recruiters and Treated. The model incorporates as part of the analysis two key strategy of CVE, namely, prevention and de-radicalization. This work builds upon the papers of McCluskey and Santoprete [14], and Santoprete and Xu [22].

The model considered in this paper has a threshold dynamics governed by the basic reproduction number $\mathcal{R}_0$. If $\mathcal{R}_0 \leq 1$, then there is only one equilibrium, free from extremists and recruiters, which is globally asymptotically stable provided $Ap_V > 4$. In this case the extremist ideology will be eradicated. If $\mathcal{R}_0 > 1$, the equilibrium mentioned above becomes unstable while an additional equilibrium, which we call "endemic", appears. This second equilibrium is globally asymptotically stable for $\mathcal{R}_0 > 1$. In this case the ideology will become endemic, that is, recruiters and extremists will establish themselves in the population.

The analysis in [14] showed that an increase in police and military action, that is, increasing the parameters d_E and d_R, decreased $\mathcal{R}_0$. The same conclusion applies in the model studied in this paper.

In [22] we established that an increase in the success rate of the de-radicalization programs, or an increase in the rates p_R and p_E at which individuals in the R and E compartments enter the Treated class caused $\mathcal{R}_0$ to decrease. Similar conclusions apply in the present model, proving that de-radicalization programs can be an integral part of a successful effort to combat violent extremism. If it is not possible to change k, p_E or p_R it is often possible to increase $\frac{1}{\delta}$, the average prison sentence, which in turn decreases $\mathcal{R}_0$. As observed in

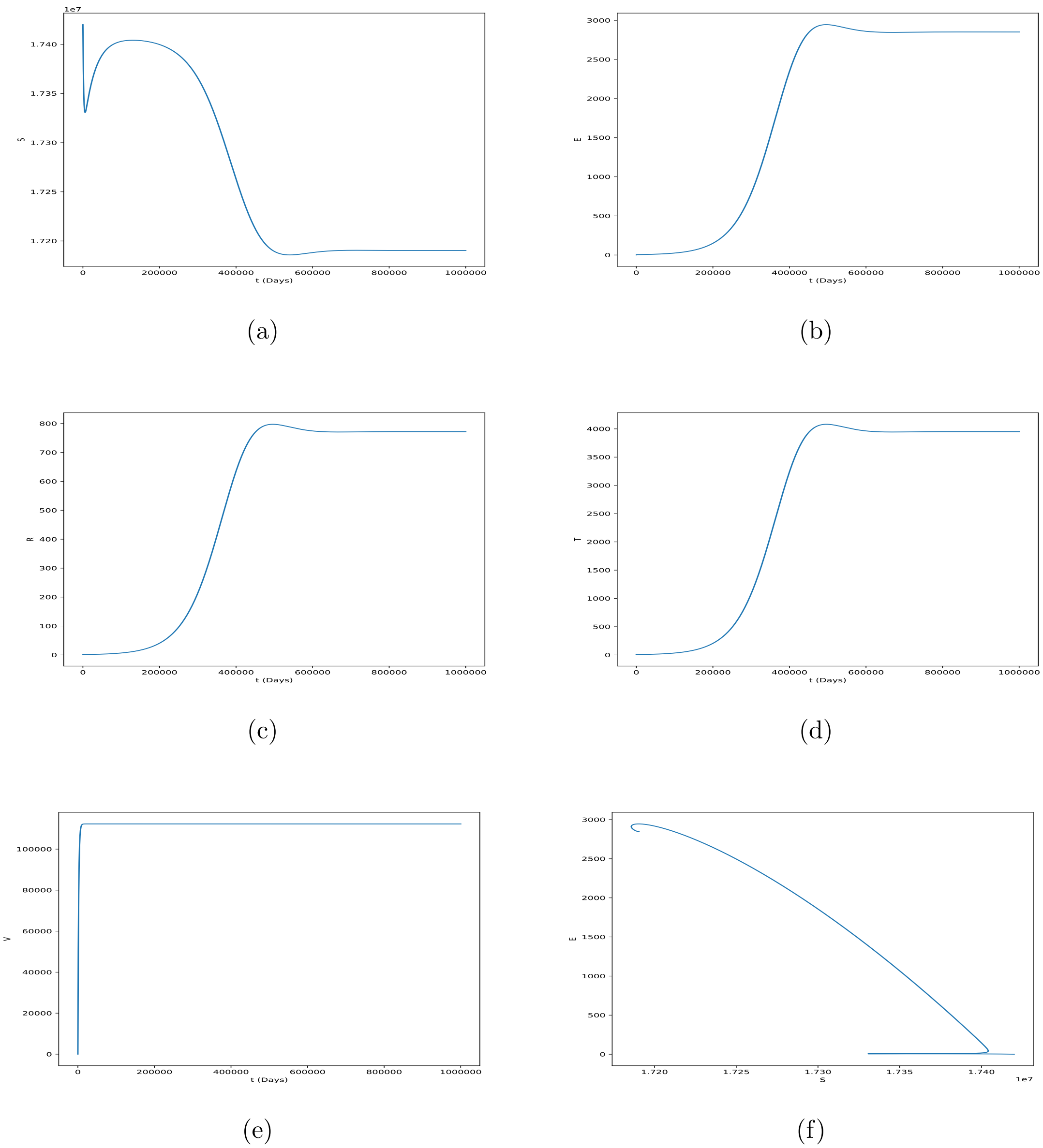
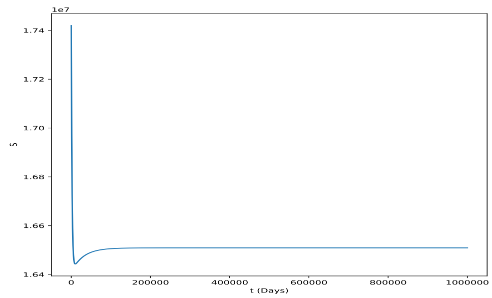
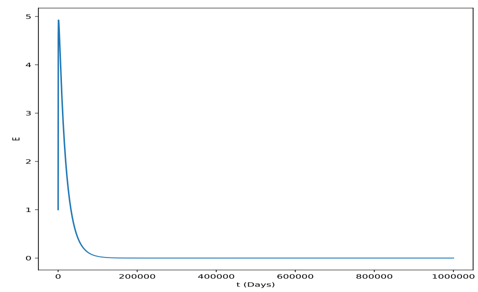


FIGURE 2. Time history and phase portraits of system (2.1) for $\beta = 0.0000005,\ q_E = 0.86,\ q_R = 0.14,\ d_E = 0.00083,\ d_R = 0.000083,\ p_E = 0.00175,\ p_R = 0.0019,\ p_S = 0.9,\ p_V = 0.1,\ \sigma = 0.8,\ c_E = 0.0006,\ c_R = 0.0008,\ k = 0.66,\ \delta = 0.0016,\ \mu = 0.000034247,\ \Lambda = 600$

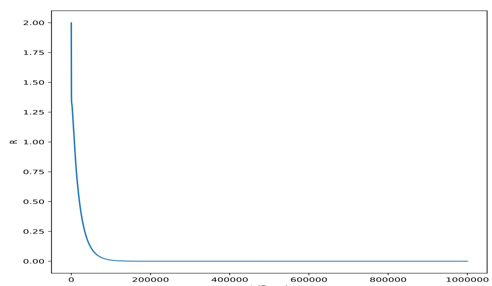
[22], this illustrates that incrementing prison sentences length could be used to remedy the shortcoming of poorly designed de-radicalization programs.



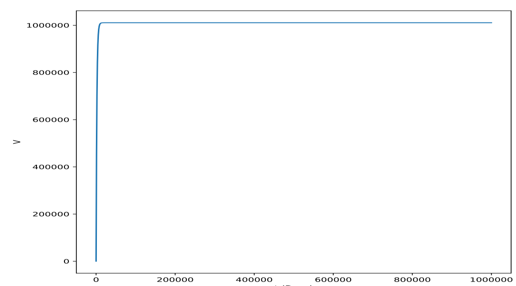
(a)



(b)



(c)



(d)

FIGURE 3. Time history of system (2.1) for $\beta = 0.0000005,\ q_E = 0.86,\ q_R = 0.14,\ d_E = 0.00083,\ d_R = 0.000083,\ p_E = 0.00175,\ p_R = 0.0019,\ p_S = 0.1,\ p_V = 0.9,\ \sigma = 0.8,\ c_E = 0.0006,\ c_R = 0.0008,\ k = 0.66,\ \delta = 0.0016,\ \mu = 0.000034247,\ \Lambda = 600$

Moreover, an increase in the fraction p_V of individuals entering the Vaccinated class produces a reduction in $\mathcal{R}_0$. Similarly, a decrease in σ, which accounts for the decreased rate at which vaccinated individuals enter the Extremist class, corresponds to a decrease in $\mathcal{R}_0$. This demonstrates that prevention strategies can be effective in reducing extremism.

As we have seen, since the basic reproduction number is expressed in terms of the model parameters, it is easy to evaluate various strategies to be used in combating violent extremism. However, there are two shortcoming in our analysis. The first one is that it is difficult to obtain realistic estimates of the parameters. This is, at least in part, due to a lack of empirical data regarding CVE programs [13]. The second is that we made many simplifying assumptions in the model. Some of these are listed in [22]. These simplifying assumptions open up a number of research paths that we intend to address in future work.

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