The antitriangular factorization of skew-symmetric matrices $\stackrel{\text{\tiny{$\stackrel{\approx}}}}{\to}$

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Abstract

In this paper we develop algorithms for orthogonal similarity transformations of skew-symmetric matrices to simpler forms. The first algorithm is similar to the algorithm for the block antitriangular factorization of symmetric matrices, but in the case of skew-symmetric matrices, an antitriangular form is always obtained. Moreover, a simple two-sided permutation of the antitriangular form transforms the matrix into a multiarrowhead matrix. In addition, we show that the block antitriangular form of the skew-Hermitian matrices has the same structure as the block antitriangular form of the symmetric matrices.

Keywords: skew-symmetric matrices, antitriangular form, multi-arrowhead matrices, skew-Hermitian matrices

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1. Introduction

Skew-symmetric matrices are significantly less used than symmetric ones. Many algorithms designed for symmetric matrices have been transformed in the course of last two decades to work with the skew-symmetric and other structured matrices, to avoid the algorithms for the general, nonstructured, matrices.

Mastronardi and Van Dooren in [4] showed that every symmetric and indefinite matrix $A \in \mathbb{R}^{n \times n}$ can be transformed into a block antitriangular form by orthogonal similarities. More precisely, if $\operatorname{inertia}(A) = (n_-, n_0, n_+), n_1 = \min(n_-, n_+), n_2 = \max(n_-, n_+) - n_1$, there exists an orthogonal matrix $Q \in \mathbb{R}^{n \times n}$ such that

$$M = Q^{T} A Q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & Y^{T} \\ 0 & 0 & X & Z^{T} \\ 0 & Y & Z & W \end{bmatrix},$$
(1.1)

where $Y \in \mathbb{R}^{n_1 \times n_1}$ is nonsingular and lower antitriangular, $W \in \mathbb{R}^{n_1 \times n_1}$ is symmetric, $X \in \mathbb{R}^{n_2 \times n_2}$ is symmetric and definite, and $Z \in \mathbb{R}^{n_1 \times n_2}$.

Bujanović and Kressner in [1] derived a computationally effective block algorithm that computes the block antitriangular factorization (1.1). Unfortunately that algorithm sometimes fails to detect the inertia. A new algorithm for the antitriangular factorization was presented in [3].

Pestana and Wathen in [5] simplified the algorithm for the special saddle point matrices

$$A = \begin{bmatrix} H & B^T \\ B & 0 \end{bmatrix},$$

where $H \in \mathbb{R}^{k \times k}$ is symmetric, but not necessarily positive definite, and $B \in \mathbb{R}^{m \times k}$, $m \ge k$.

In this paper we show that skew-symmetric matrices have antitriangular form, while skew-Hermitian ones have a block antitriangular form similar to the block antitriangular form of real symmetric matrices.

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In the next section of the paper we constructively prove that every skew-symmetric matrix can be transformed into lower antitriangular form, and establish the connection between the number of nontrivial antidiagonals and the rank of the skew-symmetric matrix. In Section 3 a stable numerical procedure for computing the antitriangular form is derived. In Section 4 we show that the antitriangular form can be reorganized to the multi-arrowhead form. Section 5 contains the results about block antitriangular form of Hermitian, and, therefore, skew-Hermitian matrices.

2. Factorization of a skew-symmetric matrix into antitriangular form

In this section we constructively prove that every skew-symmetric matrix can be reduced to antitriangular form by orthogonal similarity transformations.

To this end we use Givens rotations, since Jacobi rotations $Q_{ij} := Q(i, j, \varphi_{ij})$ cannot annihilate the element at the position (i, j) in a skew symmetric matrix A. Suppose that A_{ij} is a skew-symmetric matrix of order 2, and Q_{ij} is a rotation. Then we have

$$Q_{ij}^T A_{ij} Q_{ij} = \begin{bmatrix} \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{bmatrix} \begin{bmatrix} 0 & a_{ij} \\ -a_{ij} & 0 \end{bmatrix} \begin{bmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{bmatrix} = \begin{bmatrix} 0 & a_{ij} \\ -a_{ij} & 0 \end{bmatrix} = A_{ij}.$$

Therefore, we use the Givens rotation Q_{ij} to annihilate the elements at positions (i, k) and (k, i), $k \neq j$, or at positions (k, j) and (j, k), $k \neq i$.

Theorem 2.1. Let $A \in \mathbb{R}^{n \times n}$ be a skew-symmetric matrix. Then A can be factored as

$$A = QMQ^T,$$

where Q is an orthogonal matrix, and M is an antitriangular matrix.

Proof. The proof is by induction over the number of already annihilated antidiagonals of a skew-symmetric matrix A.

Note that A has a zero on its position (1, 1), and this fact serves as the basis of induction.

Suppose that after k-1 annihilated antidiagonals M_{k-1} has the following form,

$$M_{k-1} := Q_{k-1}^T A Q_{k-1} = \begin{bmatrix} M_{11} & M_{12} \\ -M_{12}^T & M_{22} \end{bmatrix},$$
(2.1)

where

$$M_{11} = \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & & \ddots & m_{2,k-1} \\ \vdots & & \ddots & \vdots \\ 0 & -m_{2,k-1} & \cdots & 0 \end{bmatrix},$$
 (2.2)

while the matrices M_{12} and M_{22} are generally full. In the matrix Q_{k-1} we keep the product of the applied rotations. If n = k, we have completed the job. Otherwise, in the next step we annihilate the kth antidiagonal.

First we annihilate elements at positions (1, k) and (k, 1) by a rotation $Q_{k,k+1}$ in the plane (k, k+1) that is equal to the identity matrix except at the crossings of the kth and the (k + 1)th rows and columns, where

$$\widehat{Q}_{k,k+1} = \begin{bmatrix} \cos\varphi_{k,k+1} & -\sin\varphi_{k,k+1} \\ \sin\varphi_{k,k+1} & \cos\varphi_{k,k+1} \end{bmatrix}.$$
(2.3)

We may assume that the elements at the positions (1, k) and (k, 1) are nonzero. Otherwise, we may skip this transformation.

Since the element at the position (1, k) is transformed only from the right-hand side (and the element at the position (k, 1) only from the left-hand side), the new elements at these positions are

$$m'_{1k} = m_{1k} \cos \varphi_{k,k+1} + m_{1,k+1} \sin \varphi_{k,k+1}, m'_{k1} = -(m_{1k} \cos \varphi_{k,k+1} + m_{1,k+1} \sin \varphi_{k,k+1}) = -m'_{1k}.$$

By choosing

$$\cot \varphi_{k,k+1} = -\frac{m_{1,k+1}}{m_{1k}},\tag{2.4}$$

from the basic identity for the trigonometric functions $\sin^2 \varphi_{k,k+1} + \cos^2 \varphi_{k,k+1} = 1$, it is easy to derive that the sines and the cosines in (2.3) (which annihilate $m'_{1,k}$) are

$$\sin\varphi_{k,k+1} = \pm \frac{1}{\sqrt{1 + \cot^2\varphi_{k,k+1}}}, \qquad \cos\varphi_{k,k+1} = \sin\varphi_{k,k+1}\cot\varphi_{k,k+1},$$

where $\cot \varphi_{k,k+1}$ is defined by (2.4).

The next step is to annihilate the elements at the positions (2, k-1) and (k-1, 2) by a rotation in the plane (k-1, k). This transformation will not destroy the zero pattern, since the rows/columns k-1 and k already have zeroes as the first elements in the corresponding row/column.

In a similar way all the elements of the kth antidiagonal will be annihilated without destroying the already introduced zeroes.

After the annihilation in this step we obtain M_k , which has the same form as M_{k-1} from (2.1), but the matrix M_{11} , still antitriangular, has one row and one column more than the matrix M_{11} from (2.2). This was the step of the induction.

We proceed with the annihilation of one antidiagonal after another until k becomes n.

As one can expect, since the skew-symmetric matrices have the eigenvalues in pairs of the form $\pm \lambda i$, one 'positive' and one 'negative' on the imaginary axis, there is no submatrix X in the symmetric block antitriangular form (1.1), whose dimension corresponds to the difference between the number of positive and negative eigenvalues of the symmetric matrix.

If a skew-symmetric matrix A of order n = 2p is given by its antitriangular factor, then the determinant of A is

$$det(A) = det(QMQ^{T}) = det(M)$$

= $(-1)^{2p+1} \cdot (-1)^{2p} \cdots (-1)^{3} \cdot (-1)^{2} \cdot (-1)^{p} m_{1,2p}^{2} m_{2,2p-1}^{2} \cdots m_{p,p+1}^{2}$
= $(-1)^{2(p^{2}+2p)} m_{1,2p}^{2} m_{2,2p-1}^{2} \cdots m_{p,p+1}^{2} = m_{1,2p}^{2} m_{2,2p-1}^{2} \cdots m_{p,p+1}^{2}.$

Therefore, A (of even order) is singular if and only if at least one of the antidiagonal entries is zero. If A is of odd order, one of the zeroes of the main diagonal is on the antidiagonal, which proves the well-known fact that any skew-symmetric matrix of odd order is always singular. Now suppose that A is of even order and singular, and the antidiagonal entry at the position $(\ell, n - \ell + 1), \ell \leq n - \ell + 1$ is zero. Obviously, due to skew-symmetry, the element at the position $(n - \ell + 1, \ell)$ is also zero. If there is more than one pair of zeroes on the antidiagonal, we start from a zero with the smallest difference of its column and row indices.

Now we apply a procedure similar to the procedure of annihilation of the elements of the antidiagonal from the previous theorem, but starting with the annihilation of the element at the position $(\ell + 1, n - \ell)$ by a rotation in the plane $(n - \ell, n - \ell + 1)$. This rotation will also annihilate the element at the position $(n - \ell, \ell + 1)$. We proceed with this annihilation process until all the elements on the antidiagonal between $(\ell, n - \ell + 1)$ and $(n - \ell + 1, \ell)$ are zeroes.

If A is of even order, after the previous sequence of transformations, our matrix has a middle part of the antidiagonal equal to zero. After such a preparation, a procedure for the annihilation of the nonzero elements on the antidiagonal is similar for odd and even orders. If A is of odd order, the elements at the positions $(\lceil n/2 \rceil + 1, \lceil n/2 \rceil - 1)$ and $(\lceil n/2 \rceil - 1, \lceil n/2 \rceil + 1)$ are the first to be annihilated, by a rotation in

the plane $(\lceil n/2 \rceil - 1, \lceil n/2 \rceil)$. If A is of even order, we proceed with the annihilation of the elements at the positions $(n - \ell + 2, \ell - 1)$ and $(\ell - 1, n - \ell + 2)$ by a rotation in the plane $(\ell - 1, \ell)$. The process is finished when the elements at the positions (1, n) and (n, 1) are annihilated by a rotation in the plane (1, 2).

If all the elements on the first nontrivial antidiagonal of the final matrix are nonzero, the matrix has rank n-1. Otherwise, we continue the process until all elements of some antidiagonal are nonzero. The count of such elements is the rank of the matrix.

The process of detecting the rank is illustrated in Figures 2.1 and 2.2. The first of them is for a matrix of even order, and the second for a matrix of odd order.

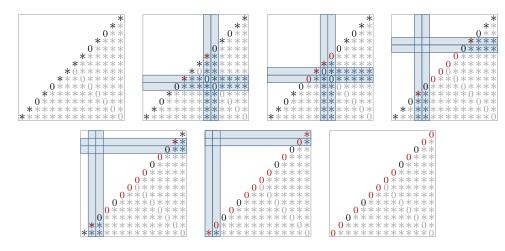


Figure 2.1: From left to right, top to bottom – annihilation of the antidiagonal of a matrix of **even** order: the first subfigure is the state before annihilation, while the last is after the completion of the process for the first antidiagonal. The horizontal and the vertical stripes show the application of the Givens rotations from the left and from the right, respectively.

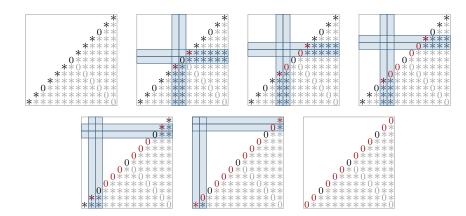


Figure 2.2: From left to right, top to bottom – annihilation of the antidiagonal of a matrix of **odd** order: the first subfigure is the state before annihilation, while the last is after the completion of the process for the first antidiagonal. The horizontal and the vertical stripes show the application of the Givens rotations from the left and from the right, respectively.

3. Numerical computation of the antitriangular form of a skew-symmetric matrix

When the QR factorization is used for the numerical rank detection, it is always computed with column pivoting. Here we derive a similar algorithm for the antitriangular factorization. For a purely practical reason we reduce an antitriangular matrix to the upper antitriangular form, which can easily be 'flipped' over the main antidiagonal to the lower antitriangular form.

In addition to the procedure described in the previous section, here we derive a reduction to the antitriangular form by applying the ordinary Householder reflectors.

Before the annihilation process in each step, a pivot column is chosen. The pivot column has maximal norm in the unreduced part of the matrix. In the first step, the unreduced part is the whole matrix. Then the whole pivot column (not only its unreduced part) is swapped with the last column in the whole matrix by a permutation P_1^T , applied from the right, while P_1 is applied from the left to swap the corresponding rows of the matrix.

An orthogonal matrix H_1 , that consists of a Householder reflector \tilde{H}_1 of order n-1 complemented with the identity matrix of order 1,

$$H_1 = \operatorname{diag}(H_1, 1),$$

is then applied to the first n-1 rows of A such that the last column is reduced to a single element at the position (1, n). Note that this element is the largest by absolute value in the matrix $H_1P_1AP_1^T$. After the completion of the left-hand-side transformation, the right-hand side transformation with the same H_1 (since $H_1^T = H_1$) is applied from to the first n-1 columns of $H_1P_1AP_1^T$.

In the second step we proceed by reducing the last-but-one row and column of $\hat{H}_1 A \hat{H}_1^T$, where $\hat{H}_1 = H_1 P_1$, while the first and the last rows and columns of the whole matrix remain intact. After the appropriate pivoting by a permutation P_2 , an orthogonal matrix H_2 ,

$$H_2 = \text{diag}(1, H_2, 1),$$

where \tilde{H}_2 is a Householder reflector, is chosen such that the submatrix $(\hat{H}_2\hat{H}_1A\hat{H}_1^T\hat{H}_2^T)(2:n-1,2:n-1)$, with $\hat{H}_2 = H_2P_2$, has its last column (and row) equal to $ce_1(-ce_1^T)$, where |c| is the norm of the unreduced part of the pivot (now, the penultimate) column.

The process is repeated in the same way until the unreduced part of the pivot column is of length 1, as shown in Algorithm 3.1, while the first steps of the reduction process are illustrated in Figure 3.1.

Note that Algorithm 3.1 could be written to work only on one triangle of the matrix, as is customary in LAPACK. In that case the skew-symmetrization in the last step of the algorithm would not be needed. Also, the generating vector v_j of the Householder reflector $\tilde{H}_j = I - v_j v_j^T$ could be stored after the *j*th step below the main diagonal in the upper antitriangular case, i.e., the *k*th element of v_j in the place A(j+k, j).

Algorithm 3.1 can be stopped earlier if the pivot column norm in the unreduced part of the matrix is (numerically) zero. Since this is the largest column norm in the unreduced part, the whole submatrix is then zero. Therefore, the right-hand-side transformation will not spoil the zeroes in this submatrix.

Suppose that the reduction process illustrated in Figure 3.1 is completed, i.e., the norms of the unreduced part of the columns are zeroes. This situation is displayed in Figure 3.2.

Of course, the situation in the real process is not as ideal as in the Figure 3.2 since the small rounding errors shift the zeroes in the shaded region to the elements with the small absolute values, such that the column norm of each shaded column is less than or equal to some *tol*. The first question is how to choose *tol*. An experience from similar factorizations shows that *tol* should include the machine epsilon, ε , the number of transformations applied to each element (of order n), and the largest element in the process. Since the norms in the first step are chosen such that the norm of the last column is the largest, this is a good candidate for the largest element in the process. Therefore, *tol* is set as

$$tol = n\varepsilon \max_{k=1,\dots,n} \|A(:,k)\|_2.$$

Note that the determination of *tol* is directly related to the determination of the rank of the matrix. After the selection of *tol*, the elements in the shaded region in Figure 3.2 should be set to zeroes.

As we have already seen in the previous section, this antitriangular form can be further reduced to an antitriangular form with nonzero antidiagonal elements. Once again, this process can be done using Householder reflectors.

First, denote by ℓ , $\ell \leq n/2$, the last column with a nonzero antidiagonal element. Then apply an orthogonal transformation

$$H'_{\ell} = \operatorname{diag}(I_{\ell}, H'_{\ell}, I_{\ell-1}),$$

Algorithm 3.1: Reduction to the upper antitriangular form.

Input: A, a skew-symmetric matrix of order n, and tol, a numerical tolerance. **Output:** A reduced to the upper antitriangular form.

begin

for step = 1 to n/2 do i1 = step; i2 = n - step + 1;for k = i1 to i2 do compute $n_k = ||A(i1:i2,k)||_2$ end for compute imax – the index of the column with the largest norm, $n_{imax} = \max_{k=i_1,\ldots,i_2} n_k$; if $imax \neq i2$ then swap A(:, imax) and A(:, i2); swap A(imax, :) and A(i2, :); end if if $n_k > tol$ then compute the Householder reflector \widetilde{H}_{step} from A(i1:i2,i2); apply \tilde{H}_{step} from the left to the columns 1 to i2; set A(i1 + 1 : i2, i2) = 0;apply H_{step} from the right to the rows 1 to i2; set A(i2, i1 + 1 : i2 - 1) = 0;skew-symmetrize matrix $A(i1:i2,i1:i2) = (A(i1:i2,i1:i2) - A^T(i1:i2,i1:i2))/2$ end if end for end

where H'_{ℓ} is the Householder reflector of order $m := n - 2\ell + 1$, from the left to reduce the part of the ℓ th column (from the $(\ell + 1)$ th row) to a single element at the position $(\ell + 1, \ell)$. Due to skew-symmetry, the application of the right-hand-side transformation is not needed. The elements of the first ℓ columns that have been transformed could be transposed, with a change of sign, and written to the elements of the first ℓ rows.

The next transformation $H'_{\ell-1}$,

$$H'_{\ell-1} = \operatorname{diag}(I_{\ell+1}, H'_{\ell-1}, I_{\ell-2}),$$

has its Householder reflector $\widetilde{H}'_{\ell-1}$ of the same order m as \widetilde{H}'_{ℓ} , but "shifted down" one place. This transformation is applied to the part of $(\ell - 1)$ th column (from row $\ell + 2$) to reduce it to a single element at the position $(\ell + 2, \ell - 1)$.

This sequence of transformations ends after transforming the first column (row). Algorithm 3.2 describes this reduction. An illustration of the reduction process for the matrix from Figure 3.2 is given in Figure 3.3.

Algorithm 3.2 was tested for the matrices with various ranks in Fortran's double precision. The test matrices were constructed by a procedure similar to the LAPACK's dlarge by setting their eigenvalues in the Murnaghan form,

$$D = \operatorname{diag} \left(\begin{bmatrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \lambda_2 \\ -\lambda_2 & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & \lambda_r \\ -\lambda_r & 0 \end{bmatrix}, 0, \dots, 0 \right).$$

To avoid the unnecessary errors, the random orthogonal matrices Q_1, \ldots, Q_n were generated in quadruple precision and then applied to D,

$$A' = Q_n \cdots Q_1 D Q_1^T \cdots Q_n^T.$$

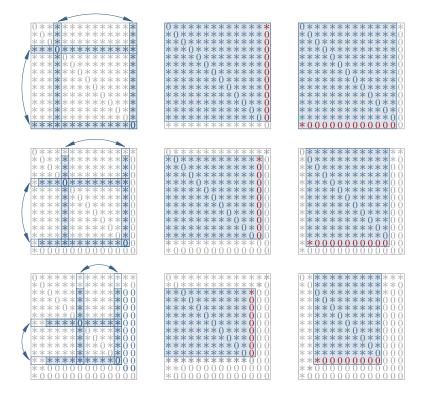


Figure 3.1: The first three steps of the reduction process. The first subfigure in each row shows the symmetric pivoting. The second shows the application of the left-hand-side, while the third shows the right-hand-side orthogonal transformation. The shaded regions in the second and the third subfigure show the part of the matrix affected by the Householder reflector \tilde{H}_{step} .

$\left[0 \right]$	*	*	*	*	*	*	*	*	*	*	*	*	*
							\ast						0
*	*	0	*	*	\ast	\ast	\ast	*	\ast	*	*	0	0
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*	\ast	\ast	0	0	0	0	0	0	0	0	0	0	0
*	\ast	\ast	0	0	0	0	0	0	0	0	0	0	0
							0					0	0
*	*	*	0	0	0	0	0	0	0	0	0	0	0
*	*	*	0	0	0	0	0	0	0	0	Ō	Ō	Ō
*	*	0	0	0	0	0	0	0	0	0	0	0	0
*	0	0	0	0	0	0	0	0	0	0	0	0	0

Figure 3.2: A possible termination of the antitriangularization process.

Final A was obtained from A' by rounding the quadruple precision result to double precision. Our test collection consists of matrices A_k of order 108 with ranks of $2, 4, 6, \ldots, 108$. A_k of rank 2r has

$ \begin{array}{c} 0 & * * * * * * * * * * * * * * * * * *$	$\begin{array}{c} 0 * * * * * * * * * * * * * * * * * * $	$ \begin{array}{c} () * * * * * * * * * * * * * * * * * * $	$ \begin{bmatrix} * * * * * * & 0 & 0 & 0 & 0 & 0 & 0 \\ * 0 & * * * * & 0 & 0 & 0 & 0 & 0 & 0 &$	$ \begin{smallmatrix} & \ast & \ast & \ast & \ast & \ast & \circ & \circ & \circ & \circ & \circ$
***0000000000 0000	* * 0 0 0 0 0 0 0 0 0 0 0 0 0	*000000000000000	000000000000000000	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0

Figure 3.3: The first subfigure shows the initial state of a matrix before the first iteration of the ℓ -loop of Algorithm 3.2. The next three subfigures show the effects of the transformations H'_3 , H'_2 , and H'_1 , in that order. The last subfigure shows the fully reduced matrix.

Algorithm 3.2: Reduction to the upper antitriangular form with a nonzero diagonal.

Input: A, a skew-symmetric matrix of order n.

Output: A reduced to the upper antitriangular form with a nontrivial antidiagonal.

begin

 $tol = n \cdot \varepsilon \cdot \max_{k=1,\dots,n} \|A(:,k)\|_2;$ for step = 1 to n/2 do i1 = step; i2 = n - step + 1;for k = i1 to i2 do compute $n_k = ||A(i1:i2,k)||_2$ end for compute imax – the index of the column with the largest norm, $n_{imax} = \max_{k=i_1,\ldots,i_2} n_k$; if $imax \neq i2$ then swap A(:, imax) and A(:, i2): swap A(imax, :) and A(i2, :); end if if $n_k > tol$ then compute the Householder reflector \widetilde{H}_{step} from A(i1:i2,i2); apply H_{step} from the left to the columns 1 to i2; set A(i1 + 1 : i2, i2) = 0;apply \tilde{H}_{step} from the right to the rows 1 to i2; set A(i2, i1 + 1 : i2 - 1) = 0;skew-symmetrize matrix $A(i1:i2,i1:i2) = (A(i1:i2,i1:i2) - A^T(i1:i2,i1:i2))/2$ else A(i1:i2,i1:i2) = 0;break; end if end for p1 = step - 1; p2 = n - step;for $\ell = step - 1$ to 1 do $p1 = p1 + 1; \quad p2 = p2 + 1;$ compute the Householder reflector \widetilde{H}'_{ℓ} from $A(p1:p2,\ell)$; apply \tilde{H}'_{ℓ} from the left to the columns 1 to ℓ ; set $A(p1 + 1 : p2, \ell) = 0;$ $A(1:\ell, p1:p2) = -A^T(p1:p2, 1:\ell);$ end for \mathbf{end}

nonzero eigenvalues $\{\pm i, \pm 2^{-1}i, \ldots, \pm 2^{r-1}i\}$. Note that for the matrices of higher ranks, eigenvalues are gradually tending to zero, and therefore are very hard to detect. The following results were obtained.

Matrix rank	2 - 96	98	100 - 108
Detected rank	correct	96	98

The test collection and a Fortran implementation of the Algorithms 3.1 and 3.2 is freely available at https://github.com/venovako/ATFact repository.

4. Multi-arrowhead form of a skew-symmetric matrix

In Section 2 we transform a full skew-symmetric matrix to antitriangular form. From the antitriangular form of a skew-symmetric matrix it is easy to obtain a new form – the multi-arrowhead form of a matrix.

Theorem 4.1. Let $M \in \mathbb{R}^{n \times n}$ be a skew-symmetric matrix in the antitriangular form. By the two-sided permutations P,

$$P = \begin{cases} [e_k, e_{k-1}, e_{k+1}, e_{k-2}, e_{k+2}, \dots, e_1, e_n], & \text{if } n = 2k - 1, \\ [e_k, e_{k+1}, e_{k-1}, e_{k+2}, e_{k-2}, \dots, e_1, e_n], & \text{if } n = 2k, \end{cases}$$

$$(4.1)$$

the matrix M can be transformed into

$$M = PSP^T,$$

where S has the following multi-arrowhead form. If n is odd, then

	0	0	s_{13}	0	s_{15}	0	•••	0	s_{1n}	
	0	0	s_{23}	0	s_{25}	0		0	s_{2n}	
	$-s_{13}$	$-s_{23}$	0	0	s_{35}	0		0	s_{3n}	
	0	0	0	0	s_{45}	0		0	s_{4n}	
S =	$-s_{15}$	$-s_{25}$	$-s_{35}$	$-s_{45}$	0	0	•••	0	s_{5n}	
	0	0	0	0	0	0	•••	0	s_{6n}	
	:	:		:	:	:	·		÷	
	0	0	0	0	0	0		0	$s_{n-1,n}$	
	$\lfloor -s_{1n} \rfloor$	$-s_{2n}$	$-s_{3n}$	$-s_{4n}$	$-s_{5n}$	$-s_{6n}$	•••	$-s_{n-1,n}$	0	

and if n is even, then

$$S = \begin{bmatrix} 0 & s_{12} & 0 & s_{14} & 0 & \cdots & 0 & s_{1n} \\ -s_{12} & 0 & 0 & s_{24} & 0 & \cdots & 0 & s_{2n} \\ 0 & 0 & 0 & s_{34} & 0 & \cdots & 0 & s_{3n} \\ -s_{14} & -s_{24} & -s_{34} & 0 & 0 & \cdots & 0 & s_{4n} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & s_{5n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & s_{n-1,n} \\ -s_{1n} & -s_{2n} & -s_{3n} & -s_{4n} & -s_{5n} & \cdots & -s_{n-1,n} & 0 \end{bmatrix}.$$

Moreover, if n is odd, the first row and the first column can become a zero row and a zero column with an additional sequence of rotations at the positions $(1, 2), (1, 4), \ldots, (1, n - 1)$.

Proof. The required result is obtained by a symmetric permutation $P^T M P$, where P is given by (4.1).

The remaining part of the proof for the skew-symmetric matrices of odd order is straightforward. By a rotation at the position (1,2) we annihilate the elements at the positions (1,3) and (3,1). Then we use a rotation at the position (1,4) and annihilate the elements at the positions (1,5) and (5,1), and so on until the rotation at the position (1, n - 1) which annihilates the elements at the positions (1, n) and (n, 1).

5. Factorization of a skew-Hermitian matrix into the block antitriangular form

Skew-Hermitian matrices are the complex generalizations of the skew-symmetric matrices, with purely imaginary eigenvalues, but now they need not be in complex-conjugate pairs. Therefore, we can have a surplus of 'positive' or 'negative' signs on the imaginary axis.

For example, if Q is any unitary matrix, then a matrix A = aiI, where $a \in \mathbb{R}$ and $a \neq 0$, cannot be transformed into antitriangular form since $Q^*AQ = aiI$.

On the other hand, H := iA is a Hermitian matrix if A is skew-Hermitian. Therefore, if H can be transformed into block antitriangular form, a relation between skew-Hermitian and Hermitian matrices is used to obtain the the block antitriangular form of A.

If we look at the proof of Theorem 2.1 from [4], that theorem is also valid for the Hermitian matrices if in the statement of the Theorem orthogonal matrices are replaced by unitary matrices and the transpose operation is replaced by the conjugate transpose. That proof relies on the properties of the nonnegative, nonpositive, neutral and null-spaces. In [2], all the required properties are derived, not only for the complex Euclidean scalar products, but for the indefinite complex scalar products. Therefore, it is easy to prove the following theorem.

Theorem 5.1. Let $A \in \mathbb{C}^{n \times n}$ be a Hermitian indefinite matrix with $inertia(A) = (n_-, n_0, n_+)$, $n_1 = \min(n_-, n_+)$, $n_2 = \max(n_-, n_+) - n_1$. Then, there exists a unitary matrix $Q \in \mathbb{C}^{n \times n}$ such that

	0	0	0	0	
M 0* 40	0	0	0	Y^*	
M = Q AQ =	0	0	X	Z^*	,
$M = Q^* A Q =$	0	Y	Z	W	

where $Y \in \mathbb{C}^{n_1 \times n_1}$ is nonsingular and lower antitriangular, $W \in \mathbb{C}^{n_1 \times n_1}$ is Hermitian, $X \in \mathbb{C}^{n_2 \times n_2}$ is Hermitian and definite, and $Z \in \mathbb{C}^{n_1 \times n_2}$.

In the next Corollary we abuse the notation for the inertia of the skew-Hermitian matrices. If the skew-Hermitian matrix A has n_{-} eigenvalues on the negative part of the imaginary axis, n_{0} zeroes as eigenvalues and n_{+} eigenvalues on the positive part of the imaginary axis, we denote this by inertia(A) = $i(n_{-}, n_{0}, n_{+})$.

Corollary 5.2. Let $A \in \mathbb{C}^{n \times n}$ be a skew-Hermitian matrix, and let $inertia(A) = i(n_-, n_0, n_+)$, such that neither $n_- = n$, nor $n_+ = n$, and $n_1 = \min(n_-, n_+)$, $n_2 = \max(n_-, n_+) - n_1$. Then, there exists a unitary matrix $Q \in \mathbb{C}^{n \times n}$ such that

$$M = Q^* A Q = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -Y^* \\ 0 & 0 & X & -Z^* \\ 0 & Y & Z & W \end{vmatrix},$$
(5.1)

where $Y \in \mathbb{C}^{n_1 \times n_1}$ is nonsingular and lower antitriangular, $W \in \mathbb{C}^{n_1 \times n_1}$ and $X \in \mathbb{C}^{n_2 \times n_2}$ are skew-Hermitian, and $Z \in \mathbb{C}^{n_1 \times n_2}$. Then, either inertia $(X) = i(n_2, 0, 0)$ or inertia $(X) = i(0, 0, n_2)$.

Proof. If the previous Theorem 5.1 is applied to H = iA, it holds

$$\widetilde{M} = Q^* H Q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \widetilde{Y}^* \\ 0 & 0 & \widetilde{X} & \widetilde{Z}^* \\ 0 & \widetilde{Y} & \widetilde{Z} & \widetilde{W} \end{bmatrix}.$$
(5.2)

If (5.2) is multiplied by -i we obtain (5.1) by setting $M = -i\widetilde{M}$, $Y := -i\widetilde{Y}$, $X := -i\widetilde{X}$, $Z := -i\widetilde{Z}$, $W := -i\widetilde{W}$. It is easy to see that $X = -X^*$ and $W = -W^*$. According to Theorem 5.1, the matrix \widetilde{X} is definite, therefore all eigenvalues of X,

$$\lambda_k(X) = \lambda_k(-i\widetilde{X}) = -i\lambda_k(\widetilde{X}),$$

are placed at the same part of the imaginary axis.

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