# Triple Roman Domination in Graphs 

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#### Abstract

The Roman domination in graphs is well-studied in graph theory. The topic is related to a defensive strategy problem in which the Roman legions are settled in some secure cities of the Roman Empire. The deployment of


the legions around the Empire is designed in such a way that a sudden attack to any undefended city could be quelled by a legion from a strong neighbour. There is an additional condition: no legion can move if doing so leaves its base city defenceless. In this manuscript we start the study of a variant of Roman domination in graphs: the triple Roman domination. We consider that any city of the Roman Empire must be able to be defended by at least three legions. These legions should be either in the attacked city or in one of its neighbours. We determine various bounds on the triple Roman domination number for general graphs, and we give exact values for some graph families. Moreover, complexity results are also obtained.
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## 1 Introduction

Only simple, undirected and non-trivial connected graphs (ntc-graphs) will be considered in this manuscript. The set of vertices of the graph $\Gamma$ is denoted by $V=V(\Gamma)$ and the edge set is $E=E(\Gamma)$. The order of a graph is the number of vertices of the graph $\Gamma$ and it is denoted by $p=p(\Gamma)$. The size of $\Gamma$ is the cardinality of the edge set and it is denoted by $q=q(\Gamma)$. An edge joining the vertices $u$ and $v$ is denoted by $e=u v$ and it is said to be incident with both end-vertices. For every vertex $v \in V$, the number of edges that are incident with $v$ is the degree of $v$ and is denoted by $d_{\Gamma}(v)$, or simply by $d(v)$ if the underlying graph is clear. Given two sets of vertices $S, T \subseteq V$, we denote by $E[S, T]$ the set of edges having one end-vertex in $S$ and the other end-vertex in $T$. The open neighbourhood $N(v)$ is the set $\{u \in V(\Gamma): u v \in E(\Gamma)\}$ and the closed neighbourhood of $v$ is the set $N[v]=N(v) \cup\{v\}$. The minimum and maximum degree of a graph $\Gamma$ are denoted by $\delta=\delta(\Gamma)$ and $\Delta=\Delta(\Gamma)$, respectively. Any vertex of degree one is called a leaf and a stem vertex is a vertex adjacent to a leaf. For a stem $v$, let $L_{v}$ be the set of leaves adjacent to $v$.

A path $P_{t+1}$ in a graph is a set of different vertices $\left\{v_{i}: i=0, \ldots, t\right\}$ and the set of edges $\left\{v_{i} v_{i+1}: i=0, \ldots, t-1\right\}$. A cycle $C_{t}$ is a path in which $v_{0}=v_{t}$. A connected graph is a graph in which any pair or vertices could be joined by a path. The distance $d_{G}(u, v)$ between two vertices $u$ and $v$ in a connected graph $G$ is the length of a shortest $u-v$ path in $G$. The diameter of $G$, denoted by diam $(G)$, is the maximum value among distances between all pair of vertices of $G$. The girth of
$G$, denoted by $g(G)$, is the minimum length of a cycle in $G$.
A tree is an acyclic connected graph. A tree $T$ is a double star if it contains exactly two vertices that are not leaves. A double star with, respectively $r$ and $s$ leaves attached at each stem is denoted $D S_{r, s}$. For a vertex $v$ in a rooted tree $T$, let $C(v)$ and $D(v)$ denote the set of children and descendants of $v$, respectively and let $D[v]=D(v) \cup\{v\}$. Also, the depth of $v$, depth $(v)$, is the largest distance from $v$ to a vertex in $D(v)$. The maximal subtree rooted at $v$, denoted by $T_{v}$, consists of $v$ and all its descendants.

A dominating set (DS) in a graph $\Gamma$ is a set of vertices $S \subseteq V(\Gamma)$ such that any vertex of $V-S$ is connecting to at least one vertex of $S$. The domination number $\gamma(\Gamma)$ equals the minimum cardinality of a DS in $\Gamma$.

Let $h: V(\Gamma) \rightarrow\{0,1,2, \ldots, k\}$ be a function, and let $\left(V_{0}, V_{1}, V_{2}, \ldots, V_{k}\right)$ be the ordered partition of $V=V(\Gamma)$ induced by $h$, where $V_{i}=\{v \in V: h(v)=i\}$ for $i \in\{0,1, \ldots, k\}$. There is a 1-1 correspondence between the functions $h: V \rightarrow$ $\{0,1,2, \ldots, k\}$ and the ordered partitions $\left(V_{0}, V_{1}, V_{2}, \ldots, V_{k}\right)$ of $V$, so we will write $h=\left(V_{0}, V_{1}, V_{2}, \ldots, V_{k}\right)$. For any subset $A \subseteq V$ we denote by $h_{\mid A}: A \rightarrow\{0,1, \ldots, k\}$ the restriction of the function $h$ to the smaller domain $A$.

A function $h: V(\Gamma) \rightarrow\{0,1,2\}$ is a Roman dominating function (RDF) on $\Gamma$ if every vertex $u \in V$ for which $h(u)=0$ is connecting to at least one vertex $v$ for which $h(v)=2$. The weight of a RDF is the value $h(V(\Gamma))=\sum_{u \in V(\Gamma)} h(u)$. The Roman domination number RD-number $\gamma_{R}(\Gamma)$ is the minimum weight of an RDF on $\Gamma$. The concept of Roman domination was introduced by Cockayne et al. in 13 and was inspired by the manuscript of ReVelle and Rosing [20], and Stewart [21] about the defensive strategy of the Roman Empire decreed by Constantine I The Great. The strategy establishes that any undefended place should be a neighbour to a 'strong' city in which there are (exactly) 2 legions deployed. In that way, one of these two legions could quell any sudden attack to the undefended city, without leaving its base site undefended. Clearly, taking into account the defensive strategy decreed by the Roman Emperor, the aim is to minimize the total cost of the legions deployed along the empire.

The topic of RD-number in graphs has been extensively studied for the last 15 years, where many manuscripts have been published. Moreover, several new variations of RD-number were introduced. For example, weak Roman domination [16], independent Roman domination [9, maximal Roman domination [3], mixed Roman domination [8], Roman $\{2\}$-domination [12] and recently total Roman $\{2\}$ domination [7], for more see [5, 6, 17].

In 2016, Beeler et al. [10] introduced the double Roman domination number DRD-number, by proposing a stronger version of the RD-number in which any city of the empire could be defended by at least two legions. In order to achieve this goal, up to three legions could be placed in a given location. This provides a defensive capacity which doubles the one of the original strategy and, simultaneously, does not increase the total cost as much as one would expect. Formally, a double Roman dominating function DRDF in a graph $\Gamma$ is a function $h: V(\Gamma) \rightarrow\{0,1,2,3\}$ that satisfies the following conditions:
a) If $h(v)=0$, then vertex $v$ must have at least two neighbours in $V_{2}$ or at least one neighbour in $V_{3}$.
b) If $h(v)=1$, then vertex $v$ must have at least one neighbour in $V_{2} \cup V_{3}$.

The minimum weight $h(V(\Gamma))=\sum_{v \in V} h(v)$ of a DRDF is the DRD-number $\gamma_{d R}(\Gamma)$ of the graph $\Gamma$. For more see [1, 2, 4, 11, 15, 18, 22, 23].

Beeler et al. pointed out in [10] that sometimes doubling the defence capacity (that is to say, defend against any attack with at least two legions), only increases the cost up to, at most, $50 \%$, as can be easily verified for the star $K_{1, p-1}$ and the complete bipartite graph $K_{2, p-2}$.

In this manuscript, we introduce a generalization of the DRD-number in which we assume that any undefended place could be defended from a sudden attack with, at least, $k$ legions without leaving any 'neighbouring strong-city' without military forces. We can think of this generalization as in an umpteenth Roman Domination or a $k$-th Roman Domination.

More precisely, let $h$ be a function that assigns labels from the set $\{0,1, \ldots, k+1\}$ to the vertices of the graph $\Gamma$. Given a vertex $v \in V(\Gamma)$, the active neighbourhood of $v$, denoted by $A N(v)$, is the set of vertices $w \in N_{\Gamma}(v)$ such that $h(w) \geq 1$. Let $A N[v]=A N(v) \cup\{v\}$.

A $[\mathrm{k}]-R D F$ is a function $h: V \rightarrow\{0,1, \ldots, k+1\}$ such that for any vertex $v \in V$ with $h(v)<k$,

$$
h(A N[v]) \geq|A N(v)|+k
$$

The weight of a $[\mathrm{k}]-\mathrm{RDF}$ is the value $h(V)=\sum_{v \in V} h(v)$, and the minimum weight of such a function $[\mathrm{k}]-R D$-number of $\Gamma$, denoted by $\gamma_{[k R]}(\Gamma)$.

Let us point out that for $k=2$ the previous definition matches that of the DRDnumber. In this manuscript we focus our attention to the triple Roman domination
number TRD-number case, so that for any vertex $v \in V$ with $h(v)<3$, it must happen that

$$
h(A N[v]) \geq|A N(v)|+3 .
$$

That is to say, we have to label the vertices of the graph, with labels from the set $\{0,1,2,3,4\}$, so that:

1. If $h(v)=0$, then $v$ must have either one neighbour in $V_{4}$, or either two neighbours in $V_{2} \cup V_{3}$ (one neighbour in $V_{2}$ and another one in $V_{3}$ ) or either three neighbours in $V_{2}$.
2. If $h(v)=1$, then $v$ must have either one neighbour in $V_{3} \cup V_{4}$ or either two neighbours in $V_{2}$.
3. If $h(v)=2$, then $v$ must have one neighbour in $V_{2} \cup V_{3} \cup V_{4}$.

Let us denote by $\gamma_{[3 R]}(\Gamma)$ the minimum weight of a triple Roman domination function (3RDF) in $\Gamma$.


Figure 1: Increasing the defence up to three times does not always increase the cost to triple.

It is worth noting that there is a significant benefit in considering the TRD-number strategy in some cases. For example, the RD-number for the bipartite graph $K_{3,4}$ equals 4 while the corresponding TRD-number would be only 8 (see Fig. [1). That is to say, we are able to increase the defensive capacity up to three times whilst the total cost is only double.

Moreover, in the case of the pentagon (see the right side in Fig. (1), we need 4 legions to defend the entire graph under the RD-number $\left(\gamma_{R}\left(C_{5}\right)=4\right)$ and we
only need 7 legions to defend any place, with at least three legions, according to the definition of the TRD-number $\left(\gamma_{[3 R]}\left(C_{5}\right)=7\right)$. This means that we may increase the defence to $300 \%$, because any sudden attack to an unsafe place could be defended with at least three legions, with only an increase in the cost of $75 \%$, since the number of legions needed goes from 4 to 7 .

In this manuscript, we initiate the study of TRD-number. We first show that the problem of computing $\gamma_{[3 R]}(\Gamma)$ is NP-complete for bipartite and chordal graphs. Moreover, we show that it is possible to compute this parameter in linear time for bounded clique-width graphs including the class of trees. Also, we establish various bounds on the TRD-number for general graphs. In particular, we show that for any ntc-graph $\Gamma$ of order $p \geq 2, \gamma_{[3 R]}(\Gamma) \leq \frac{7 p}{4}$, and we characterize the ntc-graphs attaining the upper bound. Finally the exact values of the TRD-number for some graph families are given.

## 2 Complexity results

Our goal in here is to establish the NP-complete result for the TRD-number problem in bipartite and chordal graphs.

TRIPLE ROM-DOM
Instance: Graph $\Gamma=(V, E)$, positive integer $k \leq|V|$.
Question: Does $\Gamma$ have a 3 RDF of weight at most $k$ ?

We show that this problem is NP-complete by reducing the well-known NPcomplete problem, Exact-3-Cover (X3C), to TRIPLE ROM-DOM.

EXACT 3-Cover (X3C)
Instance: A finite set $X$ with $|X|=3 q$ and a collection $C$ of 3 -member subsets of $X$.
Question: Is there a subcollection $C^{\prime}$ of $C$ for which any member of $X$ appears in perfectly one member of $C^{\prime}$ ?

Theorem 1 Problem TRIPLE ROM-DOM is NP-Complete for bipartite graphs.
Proof. Clearly TRIPLE ROM-DOM is a member of $\mathcal{N} \mathcal{P}$, since we can check in polynomial time that a function $l: V \rightarrow\{0,1,2,3,4\}$ has weight at most $k$
and is a 3RDF. Given an instance $(X, C)$ of X3C with $X=\left\{x_{1}, x_{2}, \ldots, x_{3 q}\right\}$ and $C=\left\{C_{1}, C_{2}, \ldots, C_{t}\right\}$.


Figure 2: NP-completeness of 3 RDF for bipartite.
We construct the bipartite graph $\Gamma$ as follows: for any $x_{i} \in X$, we create a path $P_{2}: x_{i} y_{i}$; for any $C_{j} \in C$ we build a star $H_{j}=K_{1,4}$ centered at $w_{j}$ with one of its leaves labeled $c_{j}$. Let $Z=\left\{c_{1}, c_{2}, \ldots, c_{t}\right\}$. To achieve the construction of $\Gamma$, we add edges $c_{j} x_{i}$ when $x_{i} \in C_{j}$. Set $k=4 t+11 q$.

Assume that the instance $X, C$ of X 3 C has a solution $C^{\prime}$. We construct a 3 RDF $l$ on $\Gamma$ of weight $k$. We label a 4 to all $w_{j}$ 's, a 3 to all $y_{i}$ 's, a 0 to all $x_{i}$ 's and a 0 to leaves of each $w_{j}$. For every $c_{j}$, label a 2 if $C_{j} \in C^{\prime}$, and a 0 if $C_{j} \notin C^{\prime}$. Note that since $C^{\prime}$ exists, its cardinality is exactly $q$, and so the number of $c_{j}$ 's with weight 2 is $q$, having disjoint neighbourhoods in $\left\{x_{1}, x_{2}, \ldots, x_{3 q}\right\}$. Note that $C^{\prime}$ is a solution for X3C, every vertex $x_{i} \in X$ satisfies $l\left(A N\left[x_{i}\right]\right) \geq\left|A N\left(x_{i}\right)\right|+3$. Thus, it is straightforward to check that $l$ is a 3 RDF with weight $l(V)=4 t+2 q+9 q=k$.

Conversely, let $\Gamma$ be a 3 RDF with weight at most $k$. Let $z=\left(V_{0}, V_{1}, V_{2}, V_{3}, V_{4}\right)$ be one that labels large values to $w_{j}$ 's and $y_{i}$ 's. By this choice of $z$, it is clear that for any $j$, vertex $w_{j}$ is labeled a 4 and its leaves are labeled a 0 . Observe that vertex $c_{j}$ may also be labeled a 0 under $z$. Hence the total weight labeled to the vertices of $H_{j}$ 's is at least $4 t$. Moreover, $z\left(x_{i}\right)+z\left(y_{i}\right) \geq 3$ for every $i$, and since $k=4 t+11 q$, we deduce that $z\left(x_{i}\right)+z\left(y_{i}\right) \leq 4$. We note that by the definition of a 3RDF and our choice of $z, z\left(y_{i}\right) \in\{3,4\}$ and $z\left(x_{i}\right)=0$ for every $i$. We also note that if $z\left(y_{i}\right)=3$ for some $i$, then $x_{i}$ needs that $z\left(c_{r}\right)=2$ for some $c_{r} \in N\left(x_{i}\right)$. Now let $\left|\left\{y_{1}, y_{2}, \ldots, y_{3 q}\right\} \cap V_{3}\right|=a$ and $\left|Z \cap V_{2}\right|=b$. Clearly, $a \leq 3 q$. Also, since every $c_{j}$ has exactly three neighbours in $X$, we have $3 b \geq a$. Now, since $z(V(G))=4 t+2 b+3 a+4(3 q-a) \leq k=4 t+11 q$, we deduce that $a-2 b \geq q$. Combining the previous three inequalities we arrive at $b=q$ and $a=3 q$. Consequently, $C^{\prime}=\left\{C_{j}: z\left(c_{j}\right)=2\right\}$ is an exact cover for $C$.

Now we can consider the same graph $\Gamma$ built for the transformation in the proof of Theorem 1 and add all edges between the $c_{j}$ 's to obtain a chordal graph $\Gamma^{\prime}$. Therefore the next result is obtained by using the same proof as before on the graph $\Gamma^{\prime}$.

Theorem 2 Problem TRIPLE ROM-DOM is NP-Complete for chordal graphs.

Finally, we will show that we can solve TRIPLE ROM-DOM in linear time for the class of graphs with bounded clique-width. Clearly, this fact implies the mentioned problem can also be solved in linear time for the class of trees.

In what follows, we use several known results related to logic structures. We refer the reader to the manuscripts by Courcelle et al. [14] and by Liedloff et al. [19] for formal details. Namely, we call a $k$-expression of $\Gamma$, on the vertices $\left\{v_{i}\right\}$, with labels $\{1,2, \ldots, k\}$ to an expression describing the graph by using the following operations:

$$
\begin{aligned}
\bullet i(x) & \text { create a new vertex } x \text { with label } i \\
\Gamma_{1} \oplus \Gamma_{2} & \text { create a new graph which is the disjoint union of the graphs } \Gamma_{i} \\
\eta_{i j}(\Gamma) & \text { add all edges in } \Gamma \text { joining vertices with label } i \text { to vertices with label } j \\
\rho_{i \rightarrow j}(\Gamma) & \text { change the label of all vertices with label } i \text { into label } j
\end{aligned}
$$

The clique-width of a graph $\Gamma$ is the minimum integer $k$ which is needed to give a $k$-expression of the graph $\Gamma$. As an example, we can describe the complete graph $K_{3}$, whose set of vertices is $\{a, b, c\}$, by means of the 2 -expression,

$$
\rho_{2 \rightarrow 1}\left(\eta_{12}\left(\rho_{2 \rightarrow 1}\left(\eta_{12}(\bullet 1(a) \oplus \bullet 2(b))\right) \oplus \bullet 2(c)\right)\right)
$$

Let us denote by $\operatorname{MSOL}\left(\tau_{1}\right)$ the monadic second order logic with quantification over subsets of vertices. We also write $\Gamma\left(\tau_{1}\right)$ for the logic structure $\langle V(\Gamma), R\rangle$, where $R$ is a binary relation such that $R(u, v)$ holds whenever $u v$ is an edge in $\Gamma$.

An optimization problem is said to be a $\operatorname{LinEMSOL}(\tau)$ optimization problem when it is possible to describe it in the following way (see [19] for more details, since this is a version of the definition given by [14] restricted to finite simple graphs),

$$
\text { Opt }\left\{\sum_{1 \leq i \leq l} a_{i}\left|X_{i}\right|:<G\left(\tau_{1}\right), X_{1}, \ldots, X_{l}>\vDash \theta\left(X_{1}, \ldots, X_{l}\right)\right\}
$$

where $\theta$ is an $\operatorname{MSOL}\left(\tau_{1}\right)$ formula that contains free set-variables $X_{1}, \ldots, X_{l}$, integers $a_{i}$ and the operator $O p t$ is either min or max.

We use the following result on LinEMSOL optimization problems.

Theorem 3 (Courcelle et all. [14]) Let $k \in \mathbb{N}$ and let $\mathcal{C}$ be a class of graphs of clique-width at most $k$. Then every LinEMSOL $\left(\tau_{1}\right)$ optimization problem on $\mathcal{C}$ can be solved in linear time if a $k$-expression of the graph is part of the input.

We extend a result proved by Liedloff et al. (see Th. 31 in [19]) regarding the complexity of the RD-number decision problem to the corresponding decision problem for the TRD-number.

Theorem 4 Problem TRIPLE ROM-DOM is a LinEMSOL $\left(\tau_{1}\right)$ optimization problem.

Proof. Let us prove that the TRIPLE ROM-DOM can be expressed as a $\operatorname{LinEMSOL}\left(\tau_{1}\right)$ optimization problem. Let $f=\left(V_{0}, V_{1}, V_{2}, V_{3}, V_{4}\right)$ be a 3RDF in $\Gamma=(V, E)$ and let us define the free set-variables $X_{i}$ such that $X_{i}(y)=1$ whenever $y \in V_{i}$ and $X_{i}(y)=0$, for the remaining vertices. For the sake of congruence with the logical system notation, we denote by $\left|X_{i}\right|=\sum_{y \in V} X_{i}(y)$, even when, clearly, is $\left|X_{i}\right|=\left|V_{i}\right|$.

Note that to solve the decision problem associated with the TRD-number problem is exactly the same to that to reach the optimum for the following expression.

$$
\min _{X_{i}}\left\{\sum_{i=1}^{4} i\left|X_{i}\right|:<G\left(\tau_{1}\right), X_{0}, X_{1}, X_{2}, X_{3}, X_{4}>\vDash \theta\left(X_{0}, X_{1}, X_{2}, X_{3}, X_{4}\right)\right\}
$$

where $\theta$ is the formula given by

$$
\begin{aligned}
\theta\left(X_{0}, \ldots, X_{4}\right)= & \left(\forall y\left(X_{3}(y) \vee X_{4}(y)\right)\right) \vee \\
& \left(X_{2}(y) \wedge \exists z\left(R(z, y) \wedge\left(X_{2}(z) \vee X_{3}(z) \vee X_{4}(z)\right)\right)\right) \vee \\
& \left(X_{1}(y) \wedge \exists z\left(R(z, y) \wedge\left(X_{3}(z) \vee X_{4}(z)\right)\right)\right) \vee \\
& \left(X_{1}(y) \wedge \exists z, t\left(R(z, y) \wedge R(t, y) \wedge X_{2}(z) \wedge X_{2}(t)\right)\right) \vee \\
& \left(X_{0}(y) \wedge \exists z\left(R(z, y) \wedge X_{4}(z)\right)\right) \vee \\
& \left(X_{0}(y) \wedge \exists z, t\left(R(z, y) \wedge R(t, y) \wedge X_{2}(z) \wedge X_{3}(t)\right)\right) \vee \\
& \left(X_{0}(y) \wedge \exists z, t, s\left(R(z, y) \wedge R(t, y) \wedge R(s, y) \wedge X_{2}(z) \wedge X_{2}(t) \wedge X_{2}(s)\right)\right)
\end{aligned}
$$

It is straightforward to verify that $\theta\left(X_{i}\right)$ is an $\operatorname{MSOL}\left(\tau_{1}\right)$ formula that corresponds to the conditions required for a labeling of the vertices of the graph to be a TRDnumber assignment. Namely, the formula consists of seven clauses and one of which,
at least, must be true. The first clause of the formula verifies whether a vertex has a label 3 or a label 4, in which case, no additional condition has to be demanded. In case of the vertex has a label 2 , the second clause checks if it has a neighbour with a label 2,3 or 4 , and so on. Hence, when the formula $\theta\left(X_{i}\right)$ is satisfied, the requirements of a 3RDF in $G$ occurs.

As a consequence, we may derive the following corollary

Corollary 5 Problem TRIPLE ROM-DOM can be solved in linear time on any graph $\Gamma$ with clique-width bounded by a constant $k$, provided that either there exists a linear-time algorithm to construct a $k$-expression of $\Gamma$, or a $k$-expression of $\Gamma$ is part of the input.

Since any graph with bounded treewidth is also a bounded clique-width graph, and it is well-known that any tree graph has treewidth equal to 1 , then we can deduce that the TRIPLE ROM-DOM can be solved in linear time for the class of trees. Besides, there are several classes of graphs $\Gamma$ with bounded clique-width $c w(\Gamma)$ like, for example, the cographs $(c w(\Gamma) \leq 2)$ and the distance hereditary graphs $(c w(\Gamma) \leq 3)$, for which it is also possible to solve TRIPLE ROM-DOM in linear time.

## 3 Bounds in terms of $p, \Delta$ and $\delta$

The purpose of this section is to provide various upper and lower bounds on the TRD-number in terms of the order, maximum and minimum degrees of a graph. Since the function that labels 2 to each vertex in an ntc-graph $\Gamma$ is a 3 RDF , the following observation is immediate.

Observation 6 For any ntc-graph the inequality $\gamma_{[3 R]}(\Gamma) \leq 2 p$ holds.

Through the following results, we prove upper bounds on the TRD-number of an ntc-graph improving, in some cases, the one given in Observation 6 .

Proposition 7 Let $\Gamma$ be an ntc-graph with $p \geq 2$ vertices and maximum degree $\Delta \geq 1$. Then $\gamma_{[3 R]}(\Gamma) \leq 3 p-3 \Delta+1$.

Proof. Consider a vertex $v \in V(\Gamma)$ of maximum degree $\Delta$ and define the function $h: V \rightarrow\{0,1,2,3,4\}$ as follows: $h(v)=4, h(w)=0$ for all $w \in N(v)$ and $h(w)=3$ for the remaining vertices. It is straightforward to see that $h$ is a 3RDF, and hence

$$
\gamma_{[3 R]}(\Gamma) \leq h(V)=4+3(p-1-\Delta)=3 p-3 \Delta+1
$$

Note that for any ntc-graph satisfying that $\Delta(\Gamma)>\frac{p+1}{3}$, the upper bound given in Proposition 7 is better than the more general upper bound pointed out in Observation 6.

Corollary 8 Let $\Gamma$ be an ntc-graph with $p \geq 2$ vertices, girth $\geq 4$, maximum degree $\Delta \leq p-2$ and minimum degree $\delta \geq 2$. Then $\gamma_{[3 R]}(\Gamma) \leq 3 p-3 \Delta$.

Proof. It is straightforward to check that the function defined in the proof of Proposition 7 can be modified such that $h(v)=3$ and, since $\delta \geq 2$ and $g(\Gamma) \geq 4$, it is still a 3RDF.

Next, we prove that the previous upper bound can be improved if the graph meets certain requirements.

Proposition 9 Let $\Gamma$ be an ntc-graph with $p \geq 2$ vertices, minimum degree $\delta(\Gamma) \geq 2$ and girth at least 5 . Then $\gamma_{[3 R]}(\Gamma) \leq 2 p-2 \Delta(\Gamma)+1$.

Proof. Let us denote by $v$ a vertex with maximum degree $d(v)=\Delta(\Gamma)$. Consider the function $l: V(\Gamma) \rightarrow\{0,1,2,3,4\}$ defined as follows: $l(v)=3, l(w)=0$ for all $w \in N(v)$ and $l(w)=2$ for the remaining vertices. Since $\delta(\Gamma) \geq 2$ and $g(\Gamma) \geq 5$, each vertex in $V(\Gamma)-\{v\}$ must have at least a neighbour in the set $V(\Gamma)-N[v]$. Therefore, any vertex labeled a 2 is connecting to, at least, another vertex with the same value. it follows that $l$ is a 3 RDF in $\Gamma$ and, in consequence,

$$
\gamma_{[3 R]}(\Gamma) \leq 3+2(p-\Delta-1)=2 p-2 \Delta+1
$$

It is worth noting that the condition required in Proposition 9 about the minimum degree is essential. As an example, we can consider the path $P_{4}$, where $\gamma_{[3 R]}\left(P_{4}\right)=7$ which implies that the upper bound given by Proposition 7 is tight. However, since $2 p-2 \Delta+1=5$, then for this particular graph the bound proved in Proposition

9 does not apply because $\delta(\Gamma)=1$. Similarly, the condition about the girth of the graph is also essential, as we may observe by the cycle $C_{4}$ for which $\gamma_{[3 R]}\left(C_{4}\right)=6>$ $2 p-2 \Delta+1=5$, or the graph depicted in Figure 3. The graph has girth 3 and, in this case, $\gamma_{[3 R]}(\Gamma)=7$ which shows that the bound of Proposition 77 is tight, while by applying Proposition 9, it would be $\gamma_{[3 R]}(\Gamma) \leq 5$.


Figure 3: The condition $g(\Gamma) \geq 5$ is necessary in Proposition 10.

Corollary 10 Let $\Gamma$ be a connected $r$-regular graph, with girth $g(\Gamma) \geq 7$ and order $p$. Then $\gamma_{[3 R]}(\Gamma) \leq 2 p-2 r^{2}+3 r-2$.

Proof. Let $v$ be any vertex of $\Gamma$ and let us consider $v$ as the root of a spanning tree in $\Gamma$. So that, $N_{0}=\{v\}$ is the root, $N_{1}=N(v)$ is the first level and $\left|N_{1}(v)\right|=r$, $N_{2}=N\left(N_{1}\right)-N_{0}$ is the second level and $\left|N_{2}(v)\right|=r(r-1)$ and so on. Define the function $h: V \rightarrow\{0,1,2,3,4\}$ as follows: $h(w)=3$ for all $w \in N_{1}, h(w)=0$ for all $w \in N_{0} \cup N_{2}$ and $h(w)=2$ for the remaining vertices. Clearly, $h$ is a 3RDF and the weight of $h$ is:

$$
h(V)=3 r+2(p-1-r-r(r-1))
$$

which is well-defined since $g(\Gamma) \geq 7$, thus $p \geq r^{3}-r^{2}+r+1$, and every vertex in $N_{3} \cap V_{2}$ must have a neighbour in $V_{2}$. Therefore,

$$
\gamma_{[3 R]}(\Gamma) \leq h(V)=3 r+2\left(p-r^{2}-1\right)=2 p-2 r^{2}+3 r-2 .
$$

Observe that $2 r^{2}-3 r+2 \geq 2 r-1$ for all $r \geq 1$ and therefore the upper bound given by Corollary 10 is better than the one given in Proposition 9 for $r$-regular graphs with girth at least 7. Besides, the upper bound given in Corollary 10 is tight for $C_{7}$, among other graphs.

Using a probabilistic method, we provide the next upper bound on $\gamma_{[3 R]}(G)$.

Proposition 11 Let $\Gamma$ be a graph of order $p$, minimum degree $\delta$ and maximum degree $\Delta$. Then

$$
\gamma_{[3 R]}(\Gamma) \leq\left\lfloor\frac{4 p}{\delta+1}\left(\ln \left(\frac{3(\delta+1)}{4}\right)+1\right)\right\rfloor
$$

Proof. Let us denote by $p^{\prime} \in(0,1)$ the probability that any vertex $v$ belongs to certain set of vertices $A \subseteq V(\Gamma)$. Let us denote by $B=N[A]^{c}=A^{c} \cap N(A)^{c}$. This means that

$$
\begin{aligned}
P[v \in B] & =P[\text { neither v nor its neighbours belong to A }] \\
& =\left(1-p^{\prime}\right)\left(1-p^{\prime}\right)^{d(v)} \\
& =\left(1-p^{\prime}\right)^{d(v)+1} \leq\left(1-p^{\prime}\right)^{\delta(\Gamma)+1}
\end{aligned}
$$

because $0<p^{\prime}<1$. Clearly, the expected size of the set $A$ is equal to $E[|A|]=p p^{\prime}$ and the corresponding expected value of $|B|$ is $E[|B|] \leq p\left(1-p^{\prime}\right)^{\delta(\Gamma)+1}$. Now, let us define the following function on $V(\Gamma)$,

$$
f(v)= \begin{cases}4 & \text { if } v \in A \\ 0 & \text { if } v \in N(A) \\ 3 & \text { if } v \in B=V(\Gamma)-N[A]\end{cases}
$$

The expected value of $f(V)$ would be

$$
E[f(V)]=4 E[|A|]+3 E[|B|] \leq 4 p p^{\prime}+3 p\left(1-p^{\prime}\right)^{\delta(\Gamma)+1}
$$

Taking into account that $\left(1-p^{\prime}\right)<\mathrm{e}^{-p^{\prime}}$ when $0<p^{\prime}<1$, it is deduced that

$$
E[f(V)] \leq 4 p p^{\prime}+3 p \mathrm{e}^{-p^{\prime}(\delta(\Gamma)+1)}
$$

The value $p^{\prime}$ for which the minimum value of the latter expression is reached satisfies $4 p-3 p(\delta(\Gamma)+1) \mathrm{e}^{-p^{\prime}(\delta(\Gamma)+1)}=0$, that implies that

$$
\mathrm{e}^{-p^{\prime}(\delta(\Gamma)+1)}=\frac{4}{3(\delta(\Gamma)+1)}
$$

This leads us to the desired value of the probability $p^{\prime}=\frac{1}{\delta(\Gamma)+1} \ln \left(\frac{3(\delta(\Gamma)+1)}{4}\right)$, and finally

$$
\gamma_{[3 R]}(\Gamma) \leq 4 p \frac{1}{\delta(\Gamma)+1} \ln \left(\frac{3(\delta(\Gamma)+1)}{4}\right)+3 p \frac{4}{3(\delta(\Gamma)+1)}
$$

which concludes the proof.
It is worth noting that the upper bound provided by Proposition 11 may be smaller or larger than the one given by Proposition 7. To see consider the cycle $C_{5}$, where by Proposition 7, we obtain that $\gamma_{[3 R]}\left(C_{5}\right) \leq 10$ which is better than the bound derived from the probabilistic procedure, $\gamma_{[3 R]}\left(C_{5}\right) \leq 12$. On the contrary, for the cycle $C_{9}$, Proposition 11 provides $\gamma_{[3 R]}\left(C_{9}\right) \leq 21$ while by Proposition 7, we have $\gamma_{[3 R]}\left(C_{9}\right) \leq 22$.

Our next result shows that any ntc-graph $\Gamma$ with order $p \geq 2, \gamma_{[3 R]}(\Gamma) \leq \frac{7 p}{4}$. Since removing an edge cannot decrease the TRD-number, it suffices to prove the result for trees. We then characterize the ntc-graphs attaining the upper bound.

Let $\mathcal{F}$ be the family of all trees that can be built from $k \geq 1$ paths $P_{4}^{i}:=$ $v_{1}^{i} v_{2}^{i} v_{3}^{i} v_{4}^{i}(1 \leq i \leq k)$ by adding $k-1$ edges incident with the $v_{2}^{i}$ 's so that they induce a connected subgraph. We write "Ind-Hyp" instead of "induction hypothesis".

Proposition 12 If $T \in \mathcal{F}$, then $\gamma_{[3 R]}(T) \geq \frac{7 p(T)}{4}$.

Proof. Suppose $T$ is a tree in $\mathcal{F}$ and let $T$ be built from $k$ copies of $P_{4}$. Note that if $H$ is an induced subgraph of $T$ for which $H \cong P_{4}$ and the leaves of $H$ are leaves of $T$, then every 3RDF of $T$ labels a weight of at least 7 to $H$. Since $T$ has $p(T) / 4$ disjoint copies of the induced subgraph $P_{4}$, where the leaves of the $P_{4}$ are leaves of $T$, we have $\gamma_{[3 R]}(T) \geq \frac{7 p(T)}{4}$.

Theorem 13 If $T$ is a tree with order $p \geq 3$, then $\gamma_{[3 R]}(T) \leq \frac{7 p}{4}$.

Proof. The proof is by induction on $p$. Since $p \geq 3$, we have $\operatorname{diam}(T) \geq 2$. If $\operatorname{diam}(T)=2$, then $T$ is a star and we have $\gamma_{[3 R]}(T)=4<\frac{7 p}{4}$. If the diameter of $T$ is 3 , then $T$ is a double star $D S_{r, s}$ for $r \geq s \geq 1$. If $s=1$, then $\gamma_{[3 R]}(T)=7 \leq \frac{7(3+r)}{4}$ with equality if and only if $r=1$, that is $T=P_{4}$. If $s \geq 2$, then $p \geq 6$ and $\gamma_{[3 R]}(T)=8<\frac{7 p}{4}$. Hence, we may suppose that $\operatorname{diam}(T) \geq 4$ and this implies that $p \geq 5$. Assume that the statement is true for each tree $T^{\prime}$ of order $p^{\prime}$ with $3 \leq p^{\prime}<p$. Let $T$ be a tree of order $p$ and let $v_{1} v_{2} \ldots v_{k}$ be a diametral path in $T$ such that $d\left(v_{2}\right)$ is as large as possible. Assume, without loss of generality, that the tree $T$ is rooted at the vertex $v_{k}$. For sake of notation, we will often write $T-T_{v_{j}}$ to denote the tree obtained from $T$ by removing $v_{j}$ and all its descendants. If $d\left(v_{2}\right) \geq 3$, then any $\gamma_{[3 R]}\left(T-T_{v_{2}}\right)$-function $h$ can be extended to a 3RDF of $T$ by keeping the
assignments given to vertices of $T-T_{v_{2}}$ under $h$ to which we also label $v_{2}$ by 4 and the leaves in $L_{v_{2}}$ by 0 . It follows from the Ind-Hyp that

$$
\gamma_{[3 R]}(T) \leq \gamma_{[3 R]}\left(T-T_{v_{2}}\right)+4 \leq \frac{7(p-3)}{4}+4<\frac{7 p}{4} .
$$

Hence assume that $d\left(v_{2}\right)=2$. By the choice of diametral path, we may suppose that any child of $v_{3}$ with depth one is of degree 2 . If $d\left(v_{3}\right)=2$ and $T-T_{v_{3}}=P_{2}$, then $T=P_{5}$ and clearly $\gamma_{[3 R]}(T)<\frac{7 p}{4}$ and if $d\left(v_{3}\right)=2$ and $p\left(T-T_{v_{3}}\right) \geq 3$, then any $\gamma_{[3 R]}\left(T-T_{v_{3}}\right)$-function can be extended to a 3 RDF of $T$ by labeling a 4 to $v_{2}$ and a 0 to $v_{1}, v_{3}$ and as above we have $\gamma_{[3 R]}(T)<\frac{7 p}{4}$. Henceforth we assume that $d\left(v_{3}\right) \geq 3$. We consider the following cases.

Case 1. $d\left(v_{3}\right) \geq 4$.
Let $t_{1}$ be the number of children of $v_{3}$ with depth one and $t_{2}=\left|L_{v_{3}}\right|, t_{1}+t_{2} \geq 3$ and $t_{1} \geq 1$. Set $t=0$ if $t_{2}=0$ and $t=1$ if $t_{2} \geq 1$. Let $T^{\prime}=T-T_{v_{3}}$. If $p\left(T^{\prime}\right)=2$, then $T$ is a spider and labeling $3+t$ to $v_{3}$ and a 3 to each leaf at distance two from $v_{3}$ provides a 3 RDF of $T$ of weight $3 t_{1}+3+t$ and this implies that $\gamma_{[3 R]}(T) \leq 3 t_{1}+3+t<\frac{7\left(2 t_{1}+1+t_{2}\right)}{4}=\frac{7 p}{4}$. Let $p\left(T^{\prime}\right) \geq 3$. Then any $\gamma_{[3 R]}\left(T-T_{v_{3}}\right)-$ function can be extended to a 3RDF of $T$ by labeling $3+t$ to $v_{3}$, a 3 to all leaves of $T_{v_{3}}$ at distance two from $v_{3}$ and 0 to other vertices and by the Ind-Hyp we have

$$
\gamma_{[3 R]}(T) \leq \gamma_{[3 R]}\left(T-T_{v_{3}}\right)+3 t_{1}+3+t \leq \frac{7\left(p-\left(2 t_{1}+1+t_{2}\right)\right)}{4}+3 t_{1}+3+t<\frac{7 p}{4}
$$

Case 2. $d\left(v_{3}\right)=3$ and $v_{3}$ is not a stem vertex.
Then $T_{v_{3}}$ is a healthy spider with two feet. The result is immediate if $T-T_{v_{3}}=P_{2}$. So assume that $p\left(T-T_{v_{2}}\right) \geq 3$. Then any $\gamma_{[3 R]}\left(T-T_{v_{3}}\right)$-function can be extended to a 3 RDF of $T$ by labeling 4 to children of $v_{3}$ and a 0 to $v_{3}$ and all leaves of $T_{v_{3}}$ and by the Ind-Hyp we have

$$
\gamma_{[3 R]}(T) \leq \gamma_{[3 R]}\left(T-T_{v_{3}}\right)+8 \leq \frac{7(p-5)}{4}+8<\frac{7 p}{4} .
$$

Case 3. $d\left(v_{3}\right)=3$ and $v_{3}$ is a stem vertex.
Let $w$ be the leaf connecting to $v_{3}$. Then $T_{v_{3}} \cong P_{4}$. If $p=6$, then clearly $\gamma_{[3 R]}(T) \leq$ $10<\frac{7 p}{4}$. Hence let $p \geq 7$. Then any $\gamma_{[3 R]}\left(T-T_{v_{3}}\right)$-function can be extended to a 3RDF of $T$ by labeling 4 to $v_{3}$, a 3 to $v_{1}$ and a 0 to $v_{2}, w$ and by the Ind-Hyp we obtain

$$
\begin{equation*}
\gamma_{[3 R]}(T) \leq \gamma_{[3 R]}\left(T-T_{v_{3}}\right)+7 \leq \frac{7(p-4)}{4}+7=\frac{7 p}{4} . \tag{1}
\end{equation*}
$$

Theorem 14 Let $T$ be a tree of order $p \geq 3$. Then $\gamma_{[3 R]}(T)=\frac{7 p}{4}$ if and only if $T \in \mathcal{F}$.

Proof. The sufficiency follows from Proposition 12 and Theorem 13, To show that every tree $T$ with $\gamma_{[3 R]}(T)=\frac{7 p}{4}$ is in $\mathcal{F}$, we demand to the proof of Theorem [13, Assume that $\gamma_{[3 R]}(T)=\frac{7 p}{4}$. The proof is by the induction on $p$. Since $P_{4} \in \mathcal{F}$, and $p$ is a multiple of 4 , we may assume that $p \geq 8$. Following the proof of Theorem [13, there is only one case, namely, Case 3 , where it is possible to achieve equality. Using the terminology from this proof, we have $\gamma_{[3 R]}\left(T-T_{v_{3}}\right)=\frac{7 p\left(T-T_{v_{3}}\right)}{4}$. It follows from the Ind-Hyp that $T-T_{v_{3}} \in \mathcal{F}$. Thus $T-T_{v_{3}}$ is built from $l=p / 4-1$ paths $P_{4}^{i}:=v_{1}^{i} v_{2}^{i} v_{3}^{i} v_{4}^{i}(1 \leq i \leq \ell)$ by adding $\ell-1$ edges incident to the vertices $v_{2}^{1}, v_{2}^{2}, \ldots, v_{2}^{\ell}$ so that they induce a subtree. Hence vertex $v_{4}$ belongs to some path $P_{4}^{i}$, that we denote, without loss of generality, by $H=P_{4}^{\ell}$, constituting the tree $T-T_{v_{3}}$. Note that any $\gamma_{[3 R]}\left(T-T_{v_{3}}\right)$-function labels a weight of 7 to each path $P_{4}^{i}(1 \leq i \leq \ell)$. We claim that $v_{4}$ is a stem of $H=P_{4}^{\ell}$. Suppose, to the contrary, that $v_{4}$ is a leaf in $H=P_{4}^{\ell}$. If $v_{4}=v_{4}^{\ell}$, then the function $z: V(T) \rightarrow\{0,1,2,3,4\}$ defined by $z\left(v_{1}\right)=3, z\left(v_{3}\right)=4, z\left(v_{2}^{i}\right)=4$ for $1 \leq i \leq \ell, z\left(v_{4}^{i}\right)=3$ for each $1 \leq i \leq \ell-1$ and $z(x)=0$ for the remaining vertices, is a 3RDF of $T$ of weight $\frac{7 p}{4}-3$ which leads to a contradiction. If $v_{4}=v_{1}^{\ell}$, then the function $z: V(T) \rightarrow\{0,1,2,3,4\}$ defined by $z\left(v_{1}\right)=3, z\left(v_{3}\right)=4, z\left(v_{3}^{\ell}\right)=4, z\left(v_{2}^{i}\right)=4$ for $1 \leq i \leq \ell-1, z\left(v_{4}^{i}\right)=3$ for each $1 \leq i \leq \ell-1$ and $z(x)=0$ for the remaining vertices, is a 3RDF of $T$ of weight $\frac{7 p}{4}-3$ which is a contradiction again. Thus $v_{4}$ is a stem vertex of $H=P_{4}^{\ell}$. If $\ell=1$, then clearly $T \in \mathcal{F}$. Assume that $\ell \geq 2$. If $v_{4}=v_{3}^{\ell}$, then the function $z: V(T) \rightarrow\{0,1,2,3,4\}$ defined by $z\left(v_{1}\right)=3, z\left(v_{3}\right)=4, z\left(v_{1}^{\ell}\right)=z\left(v_{4}^{\ell}\right)=3$, $z\left(v_{2}^{i}\right)=4$ for $1 \leq i \leq \ell-1, z\left(v_{4}^{i}\right)=3$ for $1 \leq i \leq \ell-1$ and $z(x)=0$ for the remaining vertices, is a 3 RDF of $T$ of weight $\frac{7 p}{4}-3$ which leads to a contradiction. Thus $v_{4}=v_{2}^{\ell}$ and this implies that $T \in \mathcal{F}$.

The next result is an immediate consequence of Theorem 13 ,

Corollary 15 If $\Gamma$ is an ntc-graph with order $p \geq 3$, then $\gamma_{[3 R]}(\Gamma) \leq \frac{7 p}{4}$.

Assume $\mathcal{H}$ is the family of ntc-graphs $\Gamma$ of order $p$ that can be built from $\ell=p / 4$ paths $P_{4}:=v_{1}^{i} v_{2}^{i} v_{3}^{i} v_{4}^{i}(1 \leq i \leq \ell)$ by adding some edges between the vertices $v_{2}^{1}, v_{2}^{2}, \ldots, v_{2}^{\ell}$ so that they induce a connected subgraph.

Theorem 16 The family $\mathcal{H}$ is precisely the family of ntc-graphs $\Gamma$ such that $\gamma_{[3 R]}(\Gamma)=$ $\frac{7 p}{4}$.

Proof. If $\Gamma \in \mathcal{H}$, then as we did for the family $\mathcal{F}$, we can see that $\gamma_{[3 R]}(\Gamma)=\frac{7 p}{4}$. Now assume that $\gamma_{[3 R]}(\Gamma)=\frac{7 p}{4}$. Since removing edges cannot decrease $\gamma_{[3 R]}(\Gamma)$, we deduce that any spanning tree of $\Gamma$ has TRD-number $\frac{7 p}{4}$. Hence, every spanning tree of $\Gamma$ is in $\mathcal{F}$. In fact since any spanning tree $T$ of $\Gamma$ can be built from $\ell=p / 4$ paths $P_{4}^{i}:=v_{1}^{i} v_{2}^{i} v_{3}^{i} v_{4}^{i}(1 \leq i \leq \ell)$ by adding $\ell-1$ edges incident with vertices $v_{2}^{1}, v_{2}^{2}, \ldots, v_{2}^{\ell}$ so that they induce a subtree, any edge that does not belong to $T$ has its endvertices in $\left\{v_{2}^{1}, v_{2}^{2}, \ldots, v_{2}^{\ell}\right\}$, and therefore, $\Gamma \in \mathcal{H}$.

## 4 Relationships with some Roman domination parameters

In 2004, Cockayne et al. showed that for any graph, $\gamma(\Gamma) \leq \gamma_{R}(\Gamma) \leq 2 \gamma(\Gamma)$, (see [13], Prop. 1). In 2016, Beeler et al. proved that $2 \gamma(\Gamma) \leq \gamma_{d R}(\Gamma)$, (see [10], Prop. 8). Moreover, in the same manuscript, it is proved that $\gamma_{d R}(\Gamma) \leq 2 \gamma_{R}(\Gamma)$. Next, we give relations involving $\gamma_{[3 R]}(\Gamma)$ with $\gamma(\Gamma), \gamma_{R}$ and $\gamma_{d R}(\Gamma)$ that complete known chain of inequalities.

Proposition 17 For any ntc-graph $\Gamma$, there exists a $\gamma_{[3 R]}(\Gamma)$-function that does not assign a 1 to any vertex in $\Gamma$.

Proof. Among all $\gamma_{[3 R]}(\Gamma)$-functions let $h=\left(V_{0}, V_{1}, V_{2}, V_{3}, V_{4}\right)$ be one such that $\left|V_{1}\right|$ is as small as possible. If $V_{1}=\emptyset$, then $h$ is the desired function. Hence we assume that $V_{1} \neq \emptyset$, and let $v$ be a vertex of $\Gamma$ such that $h(v)=1$. Since $h$ is a 3RDF then either there is a vertex $w \in N_{\Gamma}(v)$ with $h(w) \geq 3$ or there are two vertices $w_{1}, w_{2} \in N(v) \cap V_{2}$. In the first case, taking into account that $h$ has minimum weight, it is clear that $h(w)=3$. In the former case, the new function $l$ defined as follows, $l(u)=h(u)$ for every $u \in V(\Gamma)-\{v, w\}$ and $l(v)=0, l(w)=4$ is a $\gamma_{[3 R]}(\Gamma)$-function. In the latter case, the function $z$ defined by $z(u)=h(u)$ for every $u \in V(\Gamma)-\left\{v, w_{1}\right\}, l(v)=0, l\left(w_{1}\right)=3$ is a $\gamma_{[3 R]}(\Gamma)$-function. In either case, $l$ or $z$ labels 1 s to vertices less than $h$, a contradiction.

Proposition 18 Let $\Gamma$ be any ntc-graph. Then

$$
\gamma(\Gamma) \leq \gamma_{R}(\Gamma) \leq 2 \gamma(\Gamma) \leq \gamma_{d R}(\Gamma)<\gamma_{[3 R]}(\Gamma) \leq \min \left\{\frac{3}{2} \gamma_{d R}(\Gamma), 4 \gamma(\Gamma)\right\}
$$

Proof. As mentioned before, it is sufficient to prove that $\gamma_{d R}(\Gamma)<\gamma_{[3 R]}(G) \leq$ $\min \left\{\frac{3}{2} \gamma_{d R}(\Gamma), 4 \gamma(\Gamma)\right\}$. By Proposition (17, there is a $\gamma_{[3 R]}(\Gamma)$-function with no vertex labeled a 1 . So, let $h=\left(V_{0}, \emptyset, V_{2}, V_{3}, V_{4}\right)$ be such a 3RDF of minimum weight, and let us define the function $l$ as follows: $l(v)=3$ for all $v \in V_{3} \cup V_{4}$ and $l(v)=h(v)$ for the remaining vertices. Clearly, $l$ is a DRDF in $\Gamma$ and

$$
\gamma_{d R}(\Gamma) \leq l(V)=h(V)-\left|V_{4}\right| \leq h(V)=\gamma_{[3 R]}(\Gamma) .
$$

From the last expression we may derive that $\gamma_{d R}(\Gamma)=\gamma_{[3 R]}(\Gamma)$ implies that $V_{4}=\emptyset$, and therefore $V_{3} \neq \emptyset$. So, there exists a $\gamma_{[3 R]}(\Gamma)$-function $h=\left(V_{0}, \emptyset, V_{2}, V_{3}, \emptyset\right)$ such that $l=\left(V_{0}, \emptyset, V_{2}, V_{3}\right)$ is also a $\gamma_{d R}$-function in $\Gamma$. Let $v$ be any vertex of $V_{3}$ and let $z$ be the function defined by $z(v)=2$ and $z(w)=l(w)$ for all $w \neq v$. Taking into account that for any vertex $u \in N(v) \cap V_{0}$ it must be $l(N[u]) \geq 5$ we have that $z(N[u]) \geq 4$. Hence $z$ is a DRDF in $\Gamma$ of weight $z(V)=l(V)-1$, which is a contradiction. Thus, $\gamma_{d R}(\Gamma)<\gamma_{[3 R]}(\Gamma)$.

To prove the remaining inequality, we first consider a $\gamma(\Gamma)$-set $D$, where each of $D$ is labeled a 4 and each vertex not in $D$ is labeled a 0 . Clearly we obtain a 3 RDF of $\Gamma$ and so $\gamma_{[3 R]}(\Gamma) \leq 4 \gamma(\Gamma)$.

Now, to see that $\gamma_{[3 R]}(\Gamma) \leq \frac{3}{2} \gamma_{d R}(\Gamma)$, let us consider a $\gamma_{d R}(\Gamma)$-function with no 1 labels, $h=\left(V_{0}, V_{2}, V_{3}\right)$. Note that such a function $h$ exists (see [10]). Let $l$ be a function defined as: $l(v)=4$ for all $v \in V_{3}, l(v)=3$ for all $v \in V_{2}$ and $l(v)=0$ for the remaining vertices. Then $l$ is a 3 RDF on $V(\Gamma)$, and so

$$
\gamma_{[3 R]}(\Gamma) \leq l(V)=3\left|V_{2}\right|+4\left|V_{3}\right| \leq 3\left|V_{2}\right|+\frac{9}{2}\left|V_{3}\right|=\frac{3}{2}\left(2\left|V_{2}\right|+3\left|V_{3}\right|\right)=\frac{3}{2} \gamma_{d R}(\Gamma) .
$$

Our next result is a lower bound for the TRD-number of a graph in terms of the order, the maximum degree and the domination number of a graph $\Gamma$.

Proposition 19 Let $\Gamma$ be an ntc-graph with order $p$, maximum degree $\Delta$ and domination number $\gamma=\gamma(\Gamma)$. Then

$$
\gamma_{[3 R]}(\Gamma) \geq\left\lceil\frac{2 p+(\Delta-1) \gamma(\Gamma)}{\Delta}\right\rceil
$$

This bound is tight for any graph with a universal vertex.

Proof. Let $h=\left(V_{0}, \emptyset, V_{2}, V_{3}, V_{4}\right)$ be a $\gamma_{[3 R]}(\Gamma)$-function in $\Gamma$, where $\gamma_{[3 R]}(\Gamma)=$ $2\left|V_{2}\right|+3\left|V_{3}\right|+4\left|V_{4}\right|$. Let $V_{0}=V_{0}^{2} \cup V_{0}^{3} \cup V_{0}^{4}$ be a partition of the set $V_{0}$ such that
$V_{0}^{4}=V_{0} \cap N\left(V_{4}\right), V_{0}^{3}=\left(V_{0} \cap N\left(V_{3}\right)\right)-V_{0}^{4}$, and $V_{0}^{2}=V_{0}-\left(V_{0}^{4} \cup V_{0}^{3}\right) \subseteq N\left(V_{2}\right)$. Since the maximum degree is $\Delta$, any vertex of $V_{4}$ can be connecting to at most $\Delta$ vertices in $V_{0}$. That is to say, $\left|V_{0}^{4}\right| \leq \Delta\left|V_{4}\right|$. On the other hand, each vertex in $V_{0}^{3}$ must have at least two neighbours is $V_{2} \cup V_{3}$, so $2\left|V_{0}^{3}\right| \leq\left|E\left[V_{2} \cup V_{3}, V_{0}^{3}\right]\right| \leq\left|E\left[V_{2}, V_{0}^{3}\right]\right|+\Delta\left|V_{3}\right|$. Finally, any vertex in $V_{0}^{2}$ is connecting to at least three vertices in $V_{2}$, and thus $3\left|V_{0}^{2}\right| \leq\left|E\left[V_{2}, V_{0}^{2}\right]\right|$. Summing up the latter bounds we have that

$$
\begin{aligned}
\left|V_{0}\right| & =\left|V_{0}^{2}\right|+\left|V_{0}^{3}\right|+\left|V_{0}^{4}\right| \\
& \leq \frac{\left|E\left[V_{2}, V_{0}^{2}\right]\right|}{3}+\frac{\left|E\left[V_{2}, V_{0}^{3}\right]\right|}{2}+\frac{\Delta}{2}\left|V_{3}\right|+\Delta\left|V_{4}\right| \\
& \leq \frac{\left|E\left[V_{2}, V_{0}^{2}\right]\right|+\left|E\left[V_{2}, V_{0}^{3}\right]\right|}{2}+\frac{\Delta}{2}\left|V_{3}\right|+\Delta\left|V_{4}\right| \\
& \leq \frac{\Delta-1}{2}\left|V_{2}\right|+\frac{\Delta}{2}\left|V_{3}\right|+\Delta\left|V_{4}\right|
\end{aligned}
$$

or, equivalently,

$$
\frac{2}{\Delta}\left|V_{0}\right| \leq \frac{\Delta-1}{\Delta}\left|V_{2}\right|+\left|V_{3}\right|+2\left|V_{4}\right|
$$

Now, taking into account that $V_{2} \cup V_{3} \cup V_{4}$ is a dominating set in $V(\Gamma)$, it is possible to estimate the value of the parameter $\gamma_{[3 R]}(\Gamma)$ as follows,

$$
\begin{aligned}
\gamma_{[3 R]}(\Gamma) & =2\left|V_{2}\right|+3\left|V_{3}\right|+4\left|V_{4}\right| \\
& \geq\left|V_{2}\right|+\left|V_{3}\right|+\left|V_{4}\right|+\frac{\Delta-1}{\Delta}\left|V_{2}\right|+\left|V_{3}\right|+2\left|V_{4}\right|+\frac{1}{\Delta}\left|V_{2}\right|+\left|V_{3}\right|+\left|V_{4}\right| \\
& \geq\left|V_{2}\right|+\left|V_{3}\right|+\left|V_{4}\right|+\frac{2\left|V_{0}\right|}{\Delta}+\frac{1}{\Delta}\left|V_{2}\right|+\left|V_{3}\right|+\left|V_{4}\right| \\
& =\left|V_{2}\right|+\left|V_{3}\right|+\left|V_{4}\right|+\frac{2 p-2\left(\left|V_{2}\right|+\left|V_{3}\right|+\left|V_{4}\right|\right)}{\Delta}+\frac{1}{\Delta}\left|V_{2}\right|+\left|V_{3}\right|+\left|V_{4}\right| \\
& =\frac{2 p+(\Delta-1)\left(\left|V_{2}\right|+\left|V_{3}\right|+\left|V_{4}\right|\right)}{\Delta}+\frac{\Delta-1}{\Delta}\left(\left|V_{3}\right|+\left|V_{4}\right|\right) \\
& \geq \frac{2 p+(\Delta-1) \gamma(\Gamma)}{\Delta}
\end{aligned}
$$

and the result holds.

## 5 TRD-number in some families of graphs.

In this section, we determine exact values for the TRD-number in some particular families of graphs. To do that, we introduce the following notation: given a positive
integer $p \geq 2$ let us denote by $M_{p}$ the value:

$$
M_{p}= \begin{cases}4\left\lfloor\frac{p}{3}\right\rfloor, & \text { if } p \equiv 0(\bmod 3) \\ 4\left\lfloor\frac{p}{3}\right\rfloor+3, & \text { if } p \equiv 1(\bmod 3) \\ 4\left\lfloor\frac{p}{3}\right\rfloor+4, & \text { if } p \equiv 2(\bmod 3)\end{cases}
$$

Proposition 20 Let $p \geq 2$ be a positive integer. Then $\gamma_{[3 R]}\left(P_{p}\right)=M_{p}$.
Proof. Let $P_{p}$ be a path on $p$ vertices with vertex set $V\left(P_{p}\right)=\left\{v_{11}, v_{12}, v_{13}, v_{21}, v_{22}\right.$, $\left.v_{23}, \ldots, v_{m 1}, v_{m 2}, v_{m 3}\right\} \cup V_{t}^{\prime}$ where $p=3 m+t$, with $0 \leq t \leq 2$, and

$$
V_{t}^{\prime}= \begin{cases}\emptyset & \text { if } t=0 \\ \left\{w_{1}\right\} & \text { if } t=1 \\ \left\{w_{1}, w_{2}\right\} & \text { if } t=2\end{cases}
$$

the set of (ordered) vertices of the path. First of all, let us define a family of functions $h_{t}$ in $V\left(P_{p}\right)$ in the following terms,

$$
\begin{gathered}
h_{t}\left(v_{i j}\right)=\left\{\begin{array}{ll}
4 & \text { if } j=2 \\
0 & \text { if } j \neq 2
\end{array} \text {, for all } i \in\{1,2, \ldots, m\}, t \in\{0,1,2\}\right. \\
\text { and } h_{1}\left(w_{1}\right)=3, h_{2}\left(w_{1}\right)=h_{2}\left(w_{2}\right)=2
\end{gathered}
$$

Clearly, $h_{t}$ are 3 RDF in $P_{p}$ for all $t=p-3\left\lfloor\frac{p}{3}\right\rfloor \in\{0,1,2\}$ and the weight of $h_{t}$ is exactly $M_{p}$. Therefore, $\gamma_{[3 R]}\left(P_{p}\right) \leq M_{p}$.

On the other hand, let $h$ be a $\gamma_{[3 R]}\left(P_{p}\right)$-function such that no vertex is labeled a 1 under $h$. Let us denote by $t=p-3\left\lfloor\frac{p}{3}\right\rfloor \in\{0,1,2\}$ and proceed by induction in the number of vertices $p$. Clearly, $\gamma_{[3 R]}\left(P_{2}\right)=4 \geq M_{2}, \gamma_{[3 R]}\left(P_{3}\right)=4 \geq M_{3}, \gamma_{[3 R]}\left(P_{4}\right)=$ $7 \geq M_{4}$ and $\gamma_{[3 R]}\left(P_{5}\right)=8 \geq M_{5}$. Let $p \geq 6$, and assume that $\gamma_{[3 R]}\left(P_{p^{\prime}}\right) \geq M_{p^{\prime}}$ for all $2 \leq p^{\prime}<p$. Let $V\left(P_{p}\right)=\left\{y_{1}, y_{2}, \ldots, y_{p}\right\}$ be the set of (ordered) vertices of the path $P_{p}$. Consider the following two cases.

Case 1. $h\left(y_{p-3}\right) \geq 3$ or $h\left(y_{p-3}\right) \leq 2$ and $h_{\mid P_{p-3}}$ is a 3 RDF.
Then $h_{\mid P_{p-3}}$ is a 3 RDF of $P_{p-3}$ and $h\left(y_{p-2}\right)+h\left(y_{p-1}\right)+h\left(y_{p}\right) \geq 4$. Hence

$$
\gamma_{[3 R]}\left(P_{p}\right)=h\left(V\left(P_{p}\right)\right) \geq h_{\mid P_{p-3}}\left(V\left(P_{p-3}\right)\right)+4 \geq M_{p-3}+4=M_{p}
$$

Case 2. $h\left(y_{p-3}\right) \leq 2$ and $h_{\mid P_{p-3}}$ is not a 3RDF. Clearly if $h\left(y_{p-3}\right)=h\left(y_{p-2}\right)=0$, then $h_{\mid P_{p-3}}$ would be a 3 RDF, which is a contradiction. Thus either $h\left(y_{p-3}\right) \neq 0$ or $h\left(y_{p-2}\right) \neq 0$. Consider the following situations.
(a) $h\left(y_{p-3}\right)=0$ and $h\left(y_{p-2}\right)=4$. Then it is easy to check that it must be $h\left(y_{p}\right)+$ $h\left(y_{p-1}\right) \geq 3$. We define the function $l$ in $V\left(P_{p}\right)$ as follows $l\left(y_{p}\right)=l\left(y_{p-2}\right)=0$, $l\left(y_{p-1}\right)=4, l\left(y_{p-3}\right)=3$ and $l(w)=h(w)$ for the remaining vertices.
(b) $h\left(y_{p-3}\right)=0$ and $2 \leq h\left(y_{p-2}\right) \leq 3$. Then by considering again that $h\left(y_{p}\right)+$ $h\left(y_{p-1}\right) \geq 3$, we may deduce that

$$
\sum_{i=p-4}^{p} h\left(y_{i}\right) \geq 8
$$

Define the function $l$ in $V\left(P_{p}\right)$ as follows $l\left(y_{p}\right)=l\left(y_{p-2}\right)=l\left(y_{p-3}\right)=0, l\left(y_{p-1}\right)=$ $l\left(y_{p-4}\right)=4$ and $l(w)=h(w)$ for the remaining vertices.
(c) $h\left(y_{p-3}\right)=2$. Since $h_{\mid P_{p-3}}$ is not a 3 RDF , we have that $h\left(y_{n-2}\right) \geq 2$. Therefore, $\sum_{i=p-3}^{p} h\left(y_{i}\right) \geq 7$. We can define the function $l$ in $V\left(P_{p}\right)$ as follows $l\left(y_{p}\right)=l\left(y_{p-2}\right)=$ $0, l\left(y_{p-1}\right)=4, l\left(y_{p-3}\right)=3$ and $l(w)=h(w)$ for the remaining vertices.

In either case, $l$ is a 3 RDF in $P_{p}$ such that $l_{\mid P_{p-3}}$ is also a 3 RDF and hence

$$
\gamma_{[3 R]}\left(P_{p}\right)=h\left(V\left(P_{p}\right)\right) \geq l\left(V\left(P_{p}\right)\right) \geq l_{\mid P_{p-3}}\left(V\left(P_{p-3}\right)\right)+4 \geq M_{p-3}+4=M_{p}
$$

Proposition 21 Let $\Gamma$ be a connected graph with maximum degree $\Delta=2$ and let $h$ be a $\gamma_{[3 R]}(\Gamma)$-function. If $P: y_{1} y_{2} \ldots y_{t}$ is a path in $\Gamma$ such that $h\left(y_{j}\right)=2$ for all $j=1, \ldots, t$, then $t \leq 3$. Moreover, there exists a $\gamma_{[3 R]}(\Gamma)$-function such that $h(N(y))=\{0,2\}$, for all vertex $y \in V_{2}$ with degree 2 .

Proof. let $h$ be a $\gamma_{[3 R]}(\Gamma)$-function. Let $P: y_{1} y_{2} \ldots y_{t}$ be a path in $\Gamma$ such that $h\left(y_{j}\right)=2$ for all $j=1, \ldots, t$. If $t \geq 4$ then the function $l$ defined as follows: $l\left(y_{1}\right)=3, l\left(y_{2}\right)=0$ and $l\left(y^{\prime}\right)=h\left(y^{\prime}\right)$ for each $y^{\prime} \in V(\Gamma)-\left\{y_{1}, y_{2}\right\}$ would be a 3RDF in $\Gamma$ with $l(V(\Gamma))=h(V(\Gamma))-1$, which is impossible. So, $t \leq 3$.

On the other hand, let $l$ be a $\gamma_{[3 R]}(\Gamma)$-function that minimizes the size of the set $V_{2}$ of vertices assigned a 2 under $h$. Let $y \in V_{2}$ be a vertex of degree two and let $N(y)=\{v, w\}$. If $N(y) \subseteq V_{2}$, then the function $l$ defined in this way: $l(y)=0$, $l(v)=l(w)=3$ and $l(z)=h(z)$ for each vertex $z \in V(\Gamma)-\{y, v, w\}$ would be
a $\gamma_{[3 R]}(\Gamma)$-function with fewer vertices assigned a 2 than $h$. Hence we can assume, without loss of generality, that $v \in V_{0}$ and $w \in V_{3} \cup V_{4}$. By the definition of a 3RDF, $v$ must have either one neighbour in $V_{4}$, or either one neighbour in $V_{2}$ and another one in $V_{3}$ or either three neighbours in $V_{2}$. If $N(v) \cap V_{4} \neq \emptyset$, then reassigning $y$ the value 1 instead of 2 provides a 3 RDF with weight $h(\Gamma)-1$, a contradiction. If $v$ has one neighbour in $V_{2}$ and another one $z$ in $V_{3}$, then reassigning vertices $y, v, w, z$ the values $0,0,4$, 4, respectively provides a $\gamma_{[3 R]}(\Gamma)$-function with fewer vertices assigned a 2 than $h$. Finally, assume that $v$ has three neighbours in $V_{2}$. Then reassigning vertices $y, v, w$ the values $0,1,4$, respectively provides a $\gamma_{[3 R]}(\Gamma)$-function with fewer vertices assigned a 2 than $h$.

Proposition 22 For any integer $p \geq 3$,

$$
\gamma_{[3 R]}\left(C_{p}\right)= \begin{cases}\left\lceil\frac{4 p}{3}\right\rceil \quad & \text { if either } p=4,5,7,10 \text { or } p \equiv 0(\bmod 3) \\ \left\lceil\frac{4 p}{3}\right\rceil+1 & \text { if } p \neq 4,5,7,10 \text { and } p \equiv 1,2(\bmod 3)\end{cases}
$$

Proof. First of all, observe that $M_{p}=\left\lceil\frac{4 p}{3}\right\rceil+t$ where $t=0$, if $p \equiv 0(\bmod 3)$; and $t=1$, if $p>0(\bmod 3)$.

Let us begin with the cycles $C_{4}, C_{5}, C_{7}$ and $C_{10}$ by considering the 3RDF depicted in Figure 4.


Figure 4: 3 RDFs for $C_{4}, C_{5}, C_{7}, C_{10}$
Hence, we deduce that $\gamma_{[3 R]}\left(C_{4}\right) \leq 6, \gamma_{[3 R]}\left(C_{5}\right) \leq 7, \gamma_{[3 R]}\left(C_{7}\right) \leq 10$ and $\gamma_{[3 R]}\left(C_{10}\right) \leq$ 14. Now, let $C_{p}: y_{1} y_{2} \ldots y_{p} y_{1}$ be a cycle, with $p \neq 4,5,7,10$. Since each $\gamma_{[3 R]^{-}}$ function in a path induces a 3 RDF in the corresponding cycle, then $\gamma_{[3 R]}\left(C_{p}\right) \leq$ $\gamma_{[3 R]}\left(C_{p}-y_{1} y_{2}\right)=M_{p}$, by applying the Proposition 20.

On the other hand, let $h=\left(V_{0}, \emptyset, V_{2}, V_{3}, V_{4}\right)$ be a $\gamma_{[3 R]}\left(C_{p}\right)$-function. If there exists a vertex $y_{i}$ such that either $h\left(y_{i}\right)=h\left(y_{i+1}\right)=0$ or either $h\left(y_{i+1}\right)=0, h\left(y_{i+2}\right)=4$ or either $h\left(y_{i}\right), h\left(y_{i+1}\right) \geq 3$ (where the subscripts are considered modulus $p$ ), then $h_{\mid C_{p}-y_{i} y_{i+1}}$ is a 3 RDF in the path $C_{p}-y_{i} y_{i+1}$. Therefore,

$$
\begin{equation*}
\gamma_{[3 R]}\left(C_{p}\right)=h\left(V\left(C_{p}\right)\right)=w\left(h_{\mid C_{p}-y_{i} y_{i+1}}\right) \geq \gamma_{[3 R]}\left(C_{p}-y_{i} y_{i+1}\right)=M_{p} . \tag{2}
\end{equation*}
$$

So, assume that $h=\left(V_{0}, \emptyset, V_{2}, V_{3}, \emptyset\right)$ is a $\gamma_{[3 R]}\left(C_{p}\right)$-function such that $E\left[V_{0}, V_{0}\right]=$ $E\left[V_{3}, V_{3}\right]=\emptyset$, that is sets $V_{0}$ and $V_{3}$ are independent. Let $r$ (resp. $s, t$ ) be the number of vertices in $C_{p}$ labeled with the label 3 (resp. 2,0) under $h$. Clearly, $r+s+t=p, 3 r+2 s=h\left(V\left(C_{p}\right)\right)$ and, since each 0 must be connecting to at least a 3 , we have that $t \leq 2 r$.

Besides, as $h\left(V\left(C_{p}\right)\right)$ is minimum, if $h(y)=3$, then either $h(N(y))=\{0\}$ or either $h(N(y))=\{0,2\}$. Next, we shall show that it is possible to have $E\left[V_{2}, V_{3}\right]=$ $\emptyset$. Assume that there are two adjacent vertices $y_{i}, y_{i+1}$ such that $h\left(y_{i}\right)=2$ and $h\left(y_{i+1}\right)=3$. In this situation, it must be $h\left(y_{i+2}\right)=0$, (where all the subscripts are considered to be modulus $p$ ). It is straightforward to see that $h\left(y_{i-1}\right)=0$, because for the remaining vertices $h\left(y_{i-1}\right)+h\left(y_{i}\right)+h\left(y_{i+1}\right) \geq 7$ and the function $l\left(y_{i-1}\right)=3$, $l\left(y_{i}\right)=0$ and $l\left(y^{\prime}\right)=h\left(y^{\prime}\right)$ for all $y^{\prime} \neq y_{i-1}, y_{i}$ would be a 3 RDF with weight less than $h\left(V\left(C_{p}\right)\right)$, which is impossible. Since $h\left(y_{i-1}\right)=0$ then it must be $h\left(y_{i-2}\right)=3$ because $h\left(y_{i}\right)=2$, which implies that $h\left(y_{i-3}\right) \leq 2$ due to $E\left[V_{3}, V_{3}\right]=\emptyset$. Consider the two possible situations.

If $h\left(y_{i-3}\right)=2$ then, reasoning analogously to when we showed that it must necessarily be $h\left(y_{i-1}\right)=0$, we have that $h\left(y_{i-4}\right)=0$. Hence, the function $l\left(y_{i-3}\right)=4$, $l\left(y_{i-2}\right)=0, l\left(y_{i-1}\right)=3, l\left(y_{i}\right)=0$ and $l\left(y^{\prime}\right)=h\left(y^{\prime}\right)$ for the remaining vertices is a 3RDF with $h\left(V\left(C_{p}\right)\right)=l\left(V\left(C_{p}\right)\right)$. Besides, since $l\left(y_{i-3}\right)=4$ and $l\left(y_{i-2}\right)=0$, by applying (2), we have that $h\left(V\left(C_{p}\right)\right)=l\left(V\left(C_{p}\right)\right) \geq M_{p}$.

If $h\left(y_{i-3}\right)=0$ then $h\left(y_{i-4}\right) \in\{2,3\}$. In case that $h\left(y_{i-4}\right)=3$ we consider the function $l\left(y_{i-4}\right)=4, l\left(y_{i-3}\right)=0, l\left(y_{i-2}\right)=0, l\left(y_{i-1}\right)=4, l\left(y_{i}\right)=0$ and $l\left(y^{\prime}\right)=h\left(y^{\prime}\right)$ for the remaining vertices. Clearly, $l$ is a 3RDF with $h\left(V\left(C_{p}\right)\right)=l\left(V\left(C_{p}\right)\right)$ and since $l\left(y_{i-2}\right)=0, l\left(y_{i-1}\right)=4$ we may apply (21) to deduce that $h\left(V\left(C_{p}\right)\right)=l\left(V\left(C_{p}\right)\right) \geq M_{p}$. Hence let $h\left(l_{i-4}\right)=2$. Then either $h\left(y_{i-5}\right)=2$ and $h\left(y_{i-6}\right)=0$ or either $h\left(y_{i-5}\right)=3$ and $h\left(y_{i-6}\right)=0$. In the former situation, let us consider the function $l\left(y_{i-5}\right)=3$, $l\left(y_{i-4}\right)=0, l\left(y_{i-3}\right)=3, l\left(y_{i-2}\right)=0, l\left(y_{i-1}\right)=3, l\left(y_{i}\right)=0$ and $l\left(y^{\prime}\right)=h\left(y^{\prime}\right)$ for the remaining vertices. Again it is easy to deduce that $h\left(V\left(C_{p}\right)\right)=l\left(V\left(C_{p}\right)\right) \geq M_{p}$. In the latter situation, we define the function $l$ as follows: $l\left(y_{i-4}\right)=0, l\left(y_{i-3}\right)=3$, $l\left(y_{i-2}\right)=0, l\left(y_{i-1}\right)=4, l\left(y_{i}\right)=0$ and $l\left(y^{\prime}\right)=h\left(y^{\prime}\right)$ for the remaining vertices. So, $l$ is a 3RDF with $h\left(V\left(C_{p}\right)\right)=l\left(V\left(C_{p}\right)\right)$ and the joining of a vertex with label 2 with
a vertex labeled with a 3 is avoided.
Summing up, without loss of generality, we may suppose that $E(G)=E\left[V_{0}, V_{2}\right] \cup$ $E\left[V_{2}, V_{2}\right] \cup E\left[V_{0}, V_{3}\right]$. Moreover, since each vertex labeled with a 3 is connecting to two vertices labeled with 0 , then $\left|E\left[V_{0}, V_{3}\right]\right|=2 r$. According to Proposition 21, it is clear that $h(N(y))=\{0,2\}$ for all vertex $y \in V_{2}$. Therefore we derive that $E\left[V_{2}, V_{2}\right]=\left|V_{2}\right| / 2=s / 2$, and $\left|E\left[V_{0}, V_{2}\right]\right|=\left|V_{2}\right|=s$. The latter lead us to the following relations between $r, s, t$

$$
\left\{\begin{aligned}
r+s+t & =p \\
4 r+3 s & =2 p \\
3 r+2 s & =w(f)
\end{aligned}\right.
$$

from which we deduce that: $r=3 h\left(V\left(C_{p}\right)\right)-4 p, s=6 p-4 h\left(V\left(C_{p}\right)\right), t=h\left(V\left(C_{p}\right)\right)-$ $p$. Let us note that in this situation $r=0$ implies $t=0$, because $t \leq 2 r$ and hence, by Proposition [21, we may assume that $r>0$. If $p=3 m$, then we have that $\gamma_{[3 R]}\left(C_{p}\right)=w(f)>4 p / 3=M_{p}$, a contradiction. If $p=3 m+1$ then it must be $h\left(V\left(C_{p}\right)\right)>\frac{4}{3}(3 m+1)=4 m+4 / 3$ and therefore $h\left(V\left(C_{p}\right)\right) \geq 4 m+2$. But, if $h\left(V\left(C_{p}\right)\right)=4 m+2$ then $r=2, s=2 m-2$ and $t=m+1 \leq 2 r=4$ which implies that $1 \leq m \leq 3$. For $m=1(m=2,3$ respectively) we obtain that $\gamma_{[3 R]}\left(C_{4}\right)=6,\left(\gamma_{[3 R]}\left(C_{7}\right)=10, \gamma_{[3 R]}\left(C_{10}\right)=14\right.$, respectively) as we needed to show. Therefore, $\gamma_{[3 R]}\left(C_{p}\right)=h\left(V\left(C_{p}\right)\right)=4 m+3=\left\lceil\frac{4 p}{3}\right\rceil+1=M_{p}$, for all $p=3 m+1$ with $p \neq 4,7,10$.

If $p=3 m+2$ then it must be $h\left(V\left(C_{p}\right)\right)>\frac{4}{3}(3 m+2)=4 m+8 / 3$ and therefore $w(h) \geq 4 m+3$. Now, if $h\left(V\left(C_{p}\right)\right)=4 m+3$ then $r=1, s=2 m$ and $t=m+1 \leq$ $2 r=2$ which implies that $m=1$, and hence, we obtain that $\gamma_{[3 R]}\left(C_{5}\right)=7$. Besides, we deduce that $\gamma_{[3 R]}\left(C_{p}\right)=h\left(V\left(C_{p}\right)\right)=4 m+4=\left\lceil\frac{4 p}{3}\right\rceil+1=M_{p}$, for all $p=3 m+2$ with $p \neq 5$.

Finally, we give two upper bounds of $\gamma_{[3 R]}(\Gamma)$ in terms of the diameter and the girth of the graphs that are immediate consequences of Propositions 20 and 22.

Proposition 23 Let $\Gamma$ be an ntc-graph with $p$ vertices and diameter diam $(\Gamma)$. Then

$$
\gamma_{[3 R]}(\Gamma) \leq 3 p-\frac{5 \operatorname{diam}(\Gamma)}{3}+\frac{7}{3}
$$

Proof. Let us denote by $d=\operatorname{diam}(\Gamma)$ and let $P_{d+1}$ be a diametral path with $d+1$ vertices. By Proposition 20, it is easy to check that $\gamma_{[3 R]}\left(P_{d+1}\right) \leq 4\left\lfloor\frac{d+1}{3}\right\rfloor+4$.

Any $\gamma_{[3 R]}\left(P_{d+1}\right)$-function can be extended to a 3DRF in $\Gamma$ by assigning a 3 to every vertex in $V(\Gamma) \backslash V\left(P_{d+1}\right)$. Hence

$$
\begin{aligned}
\gamma_{[3 R]}(\Gamma) & \leq \gamma_{[3 R]}\left(P_{d+1}\right)+3(p-d-1) \\
& \leq 4\left\lfloor\frac{d+1}{3}\right\rfloor+4+3 p-3 d-3 \\
& \leq \frac{4}{3} d+\frac{4}{3}+4+3 p-3 d-3=3 p-\frac{5 d}{3}+\frac{7}{3}
\end{aligned}
$$

Proposition 24 Let $\Gamma$ be an ntc-graph of order $p$ and with girth $g(\Gamma)$. Then

$$
\gamma_{[3 R]}(\Gamma) \leq 3 p+2-\frac{5 g(\Gamma)}{3} .
$$

Proof. Let us denote by $g=\operatorname{girth}(\Gamma)$ and let $C_{g}$ be a cycle in $\Gamma$ with $g$ vertices. By Proposition [22, we have that $\gamma_{[3 R]}\left(C_{g}\right) \leq\left\lceil\frac{4 g}{3}\right\rceil+1$. Any $\gamma_{[3 R]}\left(C_{g}\right)$-function can be extended to a 3 DRF in $\Gamma$ by assigning a 3 to every vertex in $V(\Gamma) \backslash V\left(C_{g}\right)$. Hence

$$
\begin{aligned}
\gamma_{[3 R]}(\Gamma) & \leq \gamma_{[3 R]}\left(C_{g}\right)+3(p-g) \leq\left\lceil\frac{4 g}{3}\right\rceil+1+3 p-3 g \\
& \leq \frac{4 g}{3}+2+3 p-3 g=3 p-\frac{5 g}{3}+2
\end{aligned}
$$

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