

Traceability of Connected Domination Critical Graphs

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Abstract

A dominating set in a graph G is a set S of vertices of G such that every vertex outside S is adjacent to a vertex in S . A connected dominating set in G is a dominating set S such that the subgraph $G[S]$ induced by S is connected. The connected domination number of G , $\gamma_c(G)$, is the minimum cardinality of a connected dominating set of G . A graph G is said to be k - γ_c -critical if the connected domination number $\gamma_c(G)$ is equal to k and $\gamma_c(G + uv) < k$ for every pair of non-adjacent vertices u and v of G . Let ζ be the number of cut-vertices of G . It is known that if G is a k - γ_c -critical graph, then G has at most $k - 2$ cut-vertices, that is $\zeta \leq k - 2$. In this paper, for $k \geq 4$ and $0 \leq \zeta \leq k - 2$, we show that every k - γ_c -critical graph with ζ cut-vertices has a hamiltonian path if and only if $k - 3 \leq \zeta \leq k - 2$.

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1 Introduction

A *dominating set* in a graph G is a set S of vertices of G such that every vertex in $V(G) \setminus S$ is adjacent to at least one vertex in S . The *domination number* of G , denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G . A graph G is said to be k - γ -critical if $\gamma(G) = k$ and $\gamma(G + uv) < k$ for every pair of non-adjacent vertices u and v of G . Such a graph G is called a *domination critical graph*. If S is a dominating set of G , we write $S \succ G$, and if $X = \{v\}$, we also write $v \succ G$ rather than $\{v\} \succ G$. The concept of domination and its variations have been widely studied in the literature; a rough estimate says that it occurs in more than 6,000 papers to date. A thorough treatment of the fundamentals of domination theory in graphs can be found in the books [15, 16].

A *connected dominating set*, abbreviated a CD-set, of a connected graph G is a dominating set S of G such that the subgraph $G[S]$ induced by S is connected. The *connected domination number* of G , denoted by $\gamma_c(G)$, is the minimum cardinality of a CD-set of G . A CD-set of G of cardinality $\gamma_c(G)$ is called a γ_c -set of G . A graph G is said to be k - γ_c -critical if $\gamma_c(G) = k$ and $\gamma_c(G + uv) < k$ for every pair of non-adjacent vertices u and v of G . Such a graph G is called a *connected domination critical graph*. If S is a CD-set of G , we write $S \succ_c G$, and if $X = \{v\}$, we also write $v \succ_c G$ rather than $\{v\} \succ_c G$. The concept of connected domination was studied at least in the early 1970s, although it was first formally defined by Sampathkumar and Walikar in their 1979 paper [27]. Subsequently over the past forty years, the connected domination number has been extensively studied in the literature; a rough estimate says that it occurs in more than 400 papers to date. For a small sample of papers on the connected domination we refer the reader to [4, 9, 10, 25, 26, 28].

We remark that the concept of connected domination in graphs is application driven, as evidenced by the earlier papers on the concept. For example, Wu and Li [32] show that connected dominating sets are useful in the computation of routing for mobile ad hoc networks. In this application, a minimum connected dominating set is used as a backbone for communications, and vertices that are not in this set communicate by passing messages through neighbors that are in the set.

We also remark that finding connected dominating sets and Steiner trees in a graph are closely related [7, 8]. Moreover, determining the connected domination number of a connected graph G is equivalent to finding the largest possible number of leaves among all spanning trees of G . A *maximum leaf spanning tree* of G is a spanning tree that has the largest possible number of leaves among all spanning trees of G , and the *max leaf number*, denoted $\ell_{\max}(G)$, of G is the number of leaves in a maximum leaf spanning tree of G . Since $n(G) = \ell_{\max}(G) + \gamma_c(G)$, the problems of a connected dominating set and a maximum leaf spanning tree are closely connected. The maximum leaf spanning tree problem is MAX-SNP hard, implying that no polynomial time approximation scheme is likely [14]. We remark, however, that both the minimum connected dominating set problem and the maximum leaf spanning tree problem are fixed-parameter tractable [3]. The connected dominating set problem is polynomially solvable for distance-hereditary graphs [8].

1.1 Terminology and Notation

For notation and graph theory terminology, we in general follow [17]. Specifically, let $G = (V, E)$ be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$, and let v be a vertex in V . A *neighbor* of a vertex is a vertex adjacent to it. The *open neighborhood* of v is the set $N_G(v)$ of all neighbors of v , and so $N_G(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood of v* is $N_G[v] = \{v\} \cup N_G(v)$. A vertex v is said to *dominate* a vertex u in G if $u = v$ or if u is a neighbor of v . The *degree* of a vertex v is $|N_G(v)|$ and is denoted by $d_G(v)$. An *end vertex* is a vertex of degree 1 and a *support vertex* is a vertex adjacent to an end vertex. For a set S of vertices in G , the subgraph induced by S in G is denoted by $G[S]$. If G is a graph, the *complement* of G , denoted by \overline{G} , is formed by taking the vertex set of G and joining two vertices by an edge whenever they are not joined in G . If the graph G is clear from the context, we omit it in the above expressions. For example, we write $N(v)$ and $N[v]$ rather than $N_G(v)$ and $N_G[v]$, respectively. We use the standard notation $[k] = \{1, \dots, k\}$.

Two vertices u and v in a graph G are *connected* if there exists a (u, v) -path in G . A graph G is *connected* if every two vertices in G are connected. We denote the number of components in a graph G by $\omega(G)$. The *distance* $d_G(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest (u, v) -path in G . A *hamiltonian cycle* (respectively, *hamiltonian path*) of a graph is a cycle (path) passing through all vertices of the graph. A graph G is *traceable* if it contains a hamiltonian path. Moreover, a graph G is *hamiltonian* if it contains a hamiltonian cycle. For any subgraph F of G and distinct vertices a and b of G , aP_Fb denotes an (a, b) -path in G all of whose internal vertices are in $V(F)$. We note that a and b need not be in $V(F)$. If P is an (a, b) -path in G , we sometimes write the path P by aPb to indicate the start and end vertices of the path P .

We denote the *path*, *cycle*, and *complete graph* on n vertices by P_n , C_n , and K_n , respectively, and we denote the *complete bipartite graph* with partite sets of cardinality n and m by $K_{n,m}$. A *star* is the graph $K_{1,k}$, where $k \geq 1$. The graph $K_{1,3}$ is called a *claw*. A graph G is *claw-free* if it does not contain a claw as an induced subgraph. A *tree* is a connected graph with no cycle.

For vertex subsets $X, Y \subseteq V(G)$, we let $N_Y(X)$ be the set of all vertices in Y that have a neighbor that belongs to X in G , that is, $N_Y(X) = \{y \in Y \mid y \in N_G(x) \text{ for some } x \in X\}$. For a subgraph H of G , we use $N_Y(H)$ instead of $N_Y(V(H))$ and we use $N_H(X)$ instead of $N_{V(H)}(X)$. If $X = \{x\}$, we use $N_Y(x)$ instead of $N_Y(\{x\})$. The *open neighborhood* of a set S of vertices in G is the set $N_G(S) = \bigcup_{v \in S} N_G(v)$ and its *closed neighborhood* is the set $N_G[S] = N_G(S) \cup S$.

A subset $S \subseteq V(G)$ is a *vertex cut set* of G if the number of components of $G - S$ is more than the number of components of G ; that is, of $\omega(G - S) > \omega(G)$. In particular, if $S = \{v\}$, then v is called a *cut-vertex* of G . We let $\zeta(G)$ be the number of cut-vertices of G . When no ambiguity can occur, we write ζ instead of $\zeta(G)$. A *block* of a graph G is a maximal connected subgraph of G has no cut-vertex of its own. Thus, a block is a maximal 2-connected subgraph of G . Any two blocks of a graph have at most one vertex in common, namely a cut-vertex. A block of G containing exactly one cut-vertex of G is called an *end block*. If a connected graph contains a single block, we call the graph itself a *block*.

For $\ell \geq 2$ and a finite sequence G_1, \dots, G_ℓ of vertex disjoint graphs, we let the *join* $G_1 \vee \dots \vee G_\ell$ be the graph obtained from the disjoint union of G_1, \dots, G_ℓ by joining each vertex in G_i to all vertices in G_{i+1} for $i \in [\ell - 1]$. If $V(G_i) = \{x\}$, then we write $G_1 \vee \dots \vee G_{i-1} \vee x \vee G_{i+1} \vee \dots \vee G_\ell$. Moreover, for vertex disjoint graphs G_1 and G_2 and for a subgraph H of G_2 , the *join* $G_1 \vee_H G_2$ is the graph obtained from the disjoint union of G_1 and G_2 by joining each vertex in G_1 to each vertex in H .

1.2 Domination Critical Graphs

A study of properties of domination critical graphs was initiated by Sumner and Blich in their classical 1983 paper [29]. Among other results, they showed that every connected 3- γ -critical graph of even order contains a perfect matching. Wojcicka [31] subsequently studied hamiltonian properties of domination critical graphs and showed every connected 3- γ -critical graph on at least seven vertices is traceable. Favaron et al. [11], Flandrin et al. [13] and Tian et al. [30] proved further that all connected 3- γ -critical graphs with minimum degree at least 2 are hamiltonian. Motivated in part by these results, Sumner and Wojcicka (Chapter 16 in [15]) conjectured in 1998 that all $(k-1)$ -connected k - γ -critical graphs are hamiltonian for all $k \geq 4$. However, their conjecture was disproved seven years later by Yuansheng et al. [33] who constructed a 3-connected 4- γ -critical non-hamiltonian graph containing 13 vertices. On the positive side, Kaemawichanurat and Caccetta [22] proved the Sumner-Wojcicka Conjecture is true if $k = 4$ and the graphs are claw-free.

1.3 Connected Domination Critical Graphs

Kaemawichanurat [18] initiated a study of connected domination critical graphs. Hamiltonian properties of connected domination critical graphs were subsequently studied by Kaemawichanurat, Caccetta and Ananchuen [23] who showed that every 2-connected k - γ_c -critical graph is hamiltonian for all $k \in [3]$. Further, they constructed k - γ_c -critical graphs that are non-hamiltonian for all $k \geq 4$. Recently, Kaemawichanurat and Caccetta [22] proved that every 2-connected 4- γ_c -critical claw-free graph is hamiltonian, and they constructed 2-connected k - γ_c -critical claw-free graphs that are non-hamiltonian for all $k \geq 5$. For $5 \leq k \leq 6$, they proved that every 3-connected k - γ_c -critical claw-free graph is hamiltonian. Recall that $\zeta(G)$ denotes the number of cut-vertices of G , and that if the graph G is clear from the context, we simply write ζ instead of $\zeta(G)$. Kaemawichanurat and Ananchuen [21] showed that a connected domination critical graph cannot have too many cut-vertices.

Theorem 1 ([21]) *For $k \geq 2$, every k - γ_c -critical graph has at most $k - 2$ cut-vertices, that is, $\zeta \leq k - 2$.*

2 Main Result

Our aim in this paper is to determine a connection between the traceability of a k - γ_c -critical graph and the number of cut-vertices in the graph. More precisely, we shall prove the following result.

Theorem 2 *For $k \geq 4$ and $0 \leq \zeta \leq k - 2$, every k - γ_c -critical graph with ζ cut-vertices has a hamiltonian path if and only if $k - 3 \leq \zeta \leq k - 2$.*

3 Preliminary Results

In this section, we present some preliminary results that we will need to prove our main theorem, namely Theorem 2. The following result is a simple exercise in most graph theory textbooks.

Observation 1 *Let G be a graph and let S be a nonempty proper subset of $V(G)$. If G is traceable, then $\omega(G - S) \leq |S| + 1$.*

By Observation 1, if S is a vertex cut set of a graph G satisfying $|S| + 1 < \omega(G - S)$, then G is non-traceable. Kaemawichanurat, Caccetta and Ananchuen [23] showed that connected domination critical graphs with small connected domination number are hamiltonian.

Theorem 3 ([23]) *Every k - γ_c -critical graph is hamiltonian for all $k \in [3]$.*

Chen, Sun, and Ma [5] characterized all k - γ_c -critical graphs for $k \in [2]$.

Theorem 4 ([5]) *A graph G is 1- γ_c -critical if and only if G is a complete graph. Moreover, a graph G is 2- γ_c -critical if and only if $\overline{G} = \cup_{i=1}^k K_{1,n_i}$ where $k \geq 2$ and $n_i \geq 1$ for all $i \in [k]$.*

Chen et al. [5] also established fundamental properties of k - γ_c -critical graphs for $k \geq 2$.

Lemma 1 ([5]) *Let G be a k - γ_c -critical graph, and let x and y be a pair of non-adjacent vertices of G . If D_{xy} is a γ_c -set of $G + xy$, then the following holds.*

- (a) $k - 2 \leq |D_{xy}| \leq k - 1$.
- (b) $D_{xy} \cap \{x, y\} \neq \emptyset$.
- (c) If $\{x\} = \{x, y\} \cap D_{xy}$, then $N_G(y) \cap D_{xy} = \emptyset$.

Ananchuen [1] established the following properties and structural results of k - γ_c -critical graphs that possess cut-vertices.

Lemma 2 ([1]) *For $k \geq 3$, if G is a k - γ_c -critical graph with a cut-vertex c and if D is a CD -set of G , then the following holds.*

- (a) $G - c$ contains exactly two components.
- (b) If C_1 and C_2 are the components of $G - c$, then $G[N_{C_1}(c)]$ and $G[N_{C_2}(c)]$ are complete.
- (c) $c \in D$.

As remarked earlier, Kaemawichanurat and Ananchuen [21] showed in Theorem 5 that for $k \geq 2$, every k - γ_c -critical graph has at most $k - 2$ cut-vertices, that is, $\zeta \leq k - 2$. Further, they also characterized the k - γ_c -critical graph with exactly $k - 2$ cut-vertices. To state their results, let \mathcal{S} be a set of stars $G_1, G_2, \dots, G_{|\mathcal{S}|}$ where $|\mathcal{S}| \geq 2$, $G_i \cong K_{1, n_i}$ and $V(G_i) = \{s_0^i, s_1^i, \dots, s_{n_i}^i\}$ where s_0^i is the center of the star G_i for $i \in [|\mathcal{S}|]$. Let

$$S = \bigcup_{i=1}^{|\mathcal{S}|} \{s_0^i\} \quad \text{and} \quad S' = \bigcup_{i=1}^{|\mathcal{S}|} \{s_1^i, s_2^i, \dots, s_{n_i}^i\}.$$

Moreover, let S'' be a (possibly empty) set of isolated vertices. We note that $|S| = |\mathcal{S}| \geq 2$. Let T be the vertex disjoint union of these stars $G_1, G_2, \dots, G_{|\mathcal{S}|}$. Thus, the complement \overline{T} of T is a complete graph obtained by removing the edges from the stars in \mathcal{S} . We are now in a position to describe the following classes of graphs.

The class \mathcal{B}_1 . A graph G in the class \mathcal{B}_1 is constructed from the complement \overline{T} of T by adding a new vertex b and joining it to every vertex of S' . The vertex b of G is called the *head* of G . A graph in the class \mathcal{B}_1 is illustrated in Figure 1.

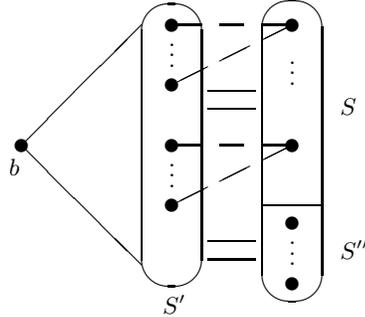


Figure 1: A graph G in the class \mathcal{B}_1

The class $\mathcal{U}(k)$. Let B be a graph in the class \mathcal{B}_1 defined earlier. A graph G in the class $\mathcal{U}(k)$ is constructed from the graph B and a path $P_{k-2}: c_0 c_1 \dots c_{k-3}$ of order $k - 2$ by joining c_{k-3} to b . A graph G in the class $\mathcal{U}(k)$ is illustrated by Figure 2.

We are now in a position to state the characterization of k - γ_c -critical graphs with $k - 2$ cut-vertices.

Theorem 5 ([21]) *For $k \geq 2$, if G is a k - γ_c -critical graph, then $\zeta \leq k - 2$. Moreover, $\zeta = k - 2$ if and only if $G \in \mathcal{U}(k)$.*

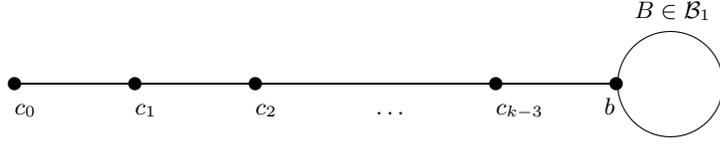


Figure 2: A graph G in the class $\mathcal{U}(k)$

In order to present the characterization due to Kaemawichanurat [19] of k - γ_c -critical graphs with $\zeta = k - 3$ cut-vertices, we describe next some additional classes of graphs. Let $\mathbf{i} = (i_1, i_2, \dots, i_{k-3})$ be a $(k - 3)$ -tuple such that $i_1, i_2, \dots, i_{k-3} \in \{0, 1\}$ and $\sum_{j=1}^{k-3} i_j = 1$. Thus, there is exactly one $\ell \in [k - 3]$ such that $i_\ell = 1$ and $i_{\ell'} = 0$ for all $\ell' \in [k - 3] \setminus \{\ell\}$.

The class $\mathcal{G}_1(i_1, i_2, \dots, i_{k-3})$. For a $(k - 3)$ -tuple $\mathbf{i} = (0, 0, \dots, i_\ell, \dots, 0)$ where $i_\ell = 1$ and $i_{\ell'} = 0$ for $1 \leq \ell \leq k - 4$ and $1 \leq \ell' \leq k - 3$ where $\ell \neq \ell'$, a graph G in the class $\mathcal{G}_1 \mathbf{i}$ can be constructed from the vertex disjoint paths $c_0 c_1 \dots c_{\ell-1}$ and $c_\ell c_{\ell+1} \dots c_{k-4}$, a copy of a complete graph K_{n_ℓ} and a block $B \in \mathcal{B}_1$ by adding edges according the join operations

$$c_{\ell-1} \vee K_{n_\ell} \vee c_\ell \quad \text{and} \quad c_{k-4} \vee b$$

where b is the head of B . Thus, the vertices $c_{\ell-1}$ and c_ℓ are joined to every vertex in the complete graph K_{n_ℓ} , and the vertices c_{k-4} and b are joined. Two examples of graphs in this case when $1 \leq \ell \leq k - 4$ are illustrated by Figure 3 and Figure 4.

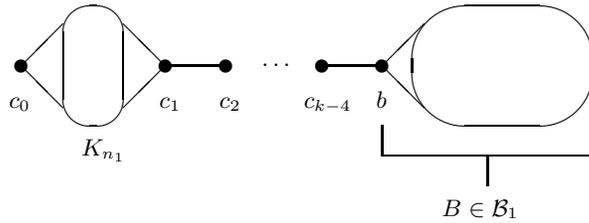


Figure 3: A graph G in the class $\mathcal{G}_1(i_1 = 1, 0, 0, \dots, 0)$

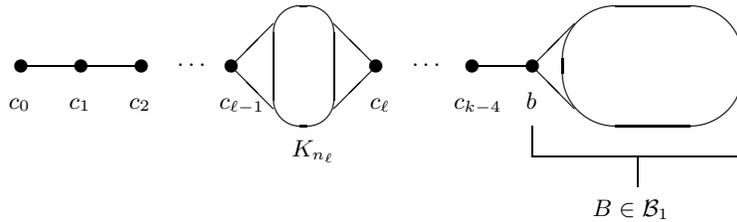


Figure 4: A graph G in the class $\mathcal{G}_1(0, 0, \dots, i_\ell = 1, 0, \dots, 0)$

Further, for a $(k - 3)$ -tuple $\mathbf{i} = (0, 0, \dots, 1)$ where $i_{k-3} = 1$ and $i_{\ell'} = 0$ for $\ell' \in [k - 4]$, a

graph G in the class $\mathcal{G}_1 \mathbf{i}$ can be constructed from a path $c_0 c_1 \dots c_{k-4}$, a copy of a complete graph $K_{n_{k-3}}$ and a block $B \in \mathcal{B}_1$ by adding edges according the join operation $c_{k-4} \vee K_{n_{k-3}} \vee b$, where b is the head of B . Thus, the vertices c_{k-4} and b are joined to every vertex in the complete graph $K_{n_{k-3}}$. An example of a graph in this case is illustrated in Figure 5.

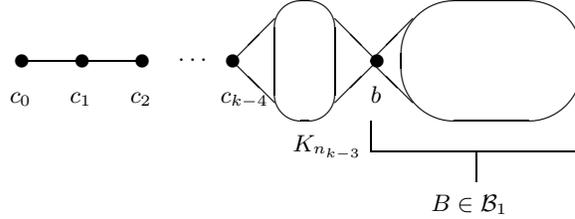


Figure 5: A graph G in the class $\mathcal{G}_1(0, 0, \dots, 1)$

We proceed further by defining a special class of end blocks.

The class \mathcal{B}_2 . Let H be a block graph, and so H is a connected graph that contains a single block. The block H belongs to the family \mathcal{B}_2 if $\gamma_c(H) = 3$ and H has the following properties.

- (a) The block H contains a vertex b such that $N_H(b)$ is a complete graph.
- (b) Every vertex v of H different from b belongs to some γ_c -set of H of size 3.
- (c) For every pair of non-adjacent vertices x and y in $H - b$, there exists a γ_c -set of $H + xy$ of size 2 that contains a neighbor of b in H and contains at least one of x and y .

The vertex b is called the *head* of the block $H \in \mathcal{B}_2$. We note that in property (b) defined above, the γ_c -set of H that contains the vertex $v \in V(H) \setminus \{b\}$ must contain a neighbor of b in H in order to dominate the vertex b .

The class $\mathcal{G}_2(k)$ for $k \geq 5$. A graph G belongs to the class $\mathcal{G}_2(k)$ for $k \geq 5$ if it can be constructed from the vertex disjoint union of a path $c_0 c_1 \dots c_{k-4}$ and a block graph $H \in \mathcal{G}_2$ with head b by adding the edge bc_{k-4} .

We are now in a position to state the characterization of k - γ_c -critical graphs with $\zeta = k - 3$ cut-vertices due to Kaemawichanurat [19].

Theorem 6 ([19]) *For $k \geq 4$, if G is a k - γ_c -critical graph with $k - 3$ cut-vertices, then $G \in \mathcal{G}_1(i_1, i_2, \dots, i_{k-3}) \cup \mathcal{G}_2(k)$.*

4 Traceability of k - γ_c -Critical Graphs

In this section, we show that, for $k \geq 4$ and $k - 3 \leq \zeta \leq k - 2$, every k - γ_c -critical graph with ζ cut-vertices contains a hamiltonian path. We first prove basic properties of k - γ_c -critical graphs.

In what follows, let B be a graph in the class \mathcal{B}_1 of order n_0 and let the vertex b be the head of B . For notational convenience, we sometimes rename the vertex b as the vertex c_{k-2} .

We show first that there exists a hamiltonian path in B that contains the vertex b as one of its ends.

Lemma 3 *If $B \in \mathcal{B}_1$ with the vertex b as its head, then there exists a hamiltonian path P_B of B having b as one of its ends.*

Proof. By the construction of the graph $B \in \mathcal{B}_1$, we have $S' = N_B(b)$ and $S \cup S'' = V(B) \setminus N_B[b]$. Further, we note that $B[S' \cup \{b\}]$ and $B[S \cup S'']$ are complete subgraphs. Since $|S| \geq 2$, every vertex in S' has at least one neighbor in S . Let uv be an arbitrary edge in G where $u \in S'$ and $v \in S$. Further, let u' be an arbitrary vertex in S' different from u . Let P_u be a hamiltonian path in $G[S']$ that starts at the vertex u' and ends at the vertex u . Let P_v be a hamiltonian path in $G[S \cup S'']$ that starts at the vertex v . Let P_B be the hamiltonian path of B that starts at the vertex b , proceeds along the edge bu' to u' , follows the hamiltonian path P_u from u' to u , proceeds along the edge uv to v , and then follows the hamiltonian path P_v starting at the vertex v . By construction, the hamiltonian path P_B of B has the vertex b as one of its ends. \square

By Lemma 3, there exists a hamiltonian path P_B of B having b as one of its ends. Let b' be the other end of the path P_B . As a consequence of Theorem 5 and Lemma 3, we obtain the following lemma.

Lemma 4 *If G is a k - γ_c -critical graph with $k - 2$ cut-vertices, then G is traceable.*

Proof. Let G be a k - γ_c -critical graph with $k - 2$ cut-vertices. By Theorem 5, the graph $G \in \mathcal{U}(k)$. Therefore, G is constructed from a graph $B \in \mathcal{B}_1$ with head b and a path $P_{k-2}: c_0c_1 \dots c_{k-3}$ by joining c_{k-3} to b . The path $c_0c_1 \dots c_{k-3}$ can therefore be extended to a hamiltonian path of G by proceeding along the edge $c_{k-3}b$ from c_{k-3} to b , and then following the hamiltonian path P_B from b to b' to yield the hamiltonian path $c_0c_1 \dots c_{k-3}bP_Bb'$ of G . \square

We show next that every graph in the class $\mathcal{G}_1(i_1, i_2, \dots, i_{k-3})$ has a hamiltonian path.

Lemma 5 *If $G \in \mathcal{G}_1(i_1, i_2, \dots, i_{k-3})$, then G is traceable.*

Proof. Suppose that $G \in \mathcal{G}_1(i_1, i_2, \dots, i_{k-3})$, where $i_\ell = 1$ and $i_{\ell'} = 0$ for $\ell, \ell' \in [k - 4]$ and $\ell \neq \ell'$. Let Q be a hamiltonian path in the copy of K_{n_ℓ} used in the construction of G , and let w_1 and w_{n_ℓ} be the start and final vertex of the path Q .

We first consider the case when $\ell = 1$, and so $G \in \mathcal{G}_1(1, 0, 0, \dots, 0)$. The path that starts at the vertex c_0 , proceeds along the edge c_0w_1 to w_1 , follows the hamiltonian path Q from w_1 to w_{n_1} , proceeds along the edge $w_{n_1}c_1$ to c_1 , follows the path $c_1c_2 \dots c_{k-4}$, proceeds along the edge $c_{k-4}b$ to b , and then follows the hamiltonian path P_B from b to b' yield the hamiltonian path $c_0P'c_1 \dots c_{k-4}bP_Bb'$ of G .

Secondly we consider the case when $2 \leq \ell \leq k - 3$. We note that $c_{\ell-1} \succ K_{n_\ell}$ and $c_\ell \succ K_{n_\ell}$. Starting with the path $c_0c_1 \dots c_{\ell-1}$ from c_0 to $c_{\ell-1}$, we proceed along the edge $c_{\ell-1}w_1$ from

$c_{\ell-1}$ to w_1 , follow the hamiltonian path Q from w_1 to w_{n_ℓ} , proceed along the edge $w_{n_\ell}c_\ell$ from w_{n_ℓ} to c_ℓ , follow the path $c_\ell \dots c_{k-4}b$ from c_ℓ to b , and follow the hamiltonian path P_B from b to b' to yield the hamiltonian path $c_0c_1 \dots c_{\ell-1}Qc_\ell \dots bP_Bb'$ of G .

Thirdly we consider the case when $\ell = k - 3$, and so $G \in \mathcal{G}_1(0, 0, \dots, 1)$. In this case, $c_{k-4} \succ K_{n_{k-3}}$ and $b \succ K_{n_{k-3}}$. Starting with the path $c_0c_1 \dots c_{k-4}$ from c_0 to c_{k-4} , we proceed along the edge $c_{k-4}w_1$ from c_{k-4} to w_1 , follow the hamiltonian path Q from w_1 to $w_{n_{k-3}}$, proceed along the edge $w_{n_{k-3}}b$ from $w_{n_{k-3}}$ to b , and follow the hamiltonian path P_B from b to b' to yield the hamiltonian path $c_0c_1 \dots c_{k-4}QbP_Bb'$ of G . This completes the proof of Lemma 5. \square

We are now in a position to prove that all k - γ_c -critical graphs with ζ cut-vertices are traceable when $\zeta \in \{k - 3, k - 2\}$.

Theorem 7 *For $k \geq 4$ and $\zeta \in \{k - 3, k - 2\}$, if G is a k - γ_c -critical graph with ζ cut-vertices, then G is traceable.*

Proof. For $k \geq 4$ and $\zeta \in \{k - 3, k - 2\}$, let G be a k - γ_c -critical graph with ζ cut-vertices. If $\zeta = k - 2$, then by Lemma 4, the graph G is traceable. Hence we may assume that $\zeta = k - 3$, for otherwise the desired result follows. By Theorem 6, $G \in \mathcal{G}_1(i_1, i_2, \dots, i_{k-3}) \cup \mathcal{G}_2(k)$. If $G \in \mathcal{G}_1(i_1, i_2, \dots, i_{k-3})$, then, by Lemma 5, the graph G is traceable. Hence we may assume that $G \in \mathcal{G}_2(k)$, for otherwise the desired result follows. Thus, $k \geq 5$ and G can be constructed from the vertex disjoint union of a path $P: c_0c_1 \dots c_{k-4}$ and a block graph $H \in \mathcal{G}_2$ with head b by adding the edge bc_{k-4} . Let

$$A = N_H(b) \quad \text{and} \quad \overline{A} = V(H) \setminus N_H[b].$$

By construction of the graph $H \in \mathcal{G}_2$, we note that $G[A]$ is a complete subgraph. We now consider $G[\overline{A}]$. Let $P^1: x_1^1x_2^1 \dots x_{n_1}^1$ be a longest path in $G[\overline{A}]$. We note that P^1 is a subgraph of $G[\overline{A}]$ and thus, x_j^1 and $x_{j'}^1$ may be adjacent for $1 \leq j \leq j' + 2 \leq n_1$. If $\overline{A}_1 = V(P^1)$ and $\overline{A} \setminus \overline{A}_1 \neq \emptyset$, then we let $P^2: x_1^2x_2^2 \dots x_{n_2}^2$ be a longest path in $G[\overline{A} \setminus \overline{A}_1]$. Continuing in this way, for $i \geq 1$ if the paths P^1, P^2, \dots, P^i are defined and $\overline{A} \setminus \overline{A}_i \neq \emptyset$ where

$$\overline{A}_i = \bigcup_{j=1}^i V(P^j),$$

then we let $P^{i+1}: x_1^{i+1}x_2^{i+1} \dots x_{n_{i+1}}^{i+1}$ be a longest path in $G[\overline{A} \setminus \overline{A}_i]$. Continuing in this way, let $z \geq 1$ be the smallest integer such that $\overline{A} \setminus \overline{A}_z = \emptyset$. Thus either $z = 1$, in which case $\overline{A} = V(P^1)$, or $z \geq 2$, in which case $(V(P^1), V(P^2), \dots, V(P^z))$ is a partition of \overline{A} where each set $V(P^i)$ is nonempty for all $i \in [z]$. By definition of the paths P^i for $i \in [z]$, we note that

$$|V(P^1)| \geq |V(P^2)| \geq \dots \geq |V(P^z)|.$$

The structure of $G[\overline{A}]$ is illustrated in Figure 6.

We proceed further with the following series of claims.

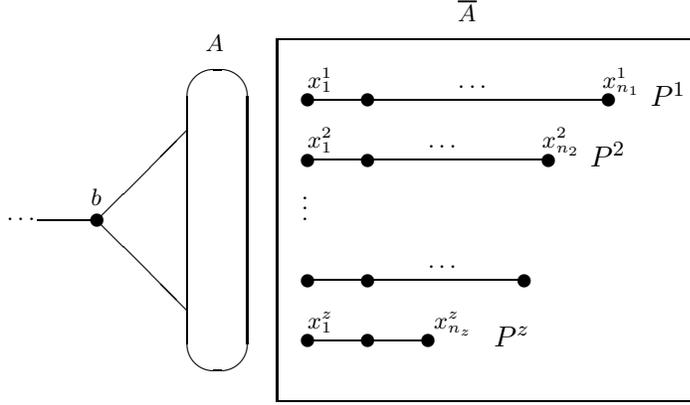


Figure 6: The structure of $G[\overline{A}]$ in the proof of Theorem 7

Claim 1 *The set $\{x_{j_1}^1, x_{j_2}^2, \dots, x_{j_z}^z\}$ is an independent set for all $j_i \in \{1, n_i\}$ and $i \in [z]$.*

Proof. Suppose, to the contrary, that $x_{j_i}^i x_{j_{i'}}^{i'} \in E(G)$ for some i and i' where $1 \leq i < i' \leq z$ where $j_i \in \{1, n_i\}$ and $j_{i'} \in \{1, n_{i'}\}$. Renaming the vertices on the path P^i and $P^{i'}$ if necessary, we may assume without loss of generality that $j_i = n_i$ and $j_{i'} = 1$. We now consider the path P^* obtained from P^i by proceeding along the edge $x_{n_i}^i x_1^{i'}$ from $x_{n_i}^i$ to $x_1^{i'}$, and then following the path $P^{i'}$ from $x_1^{i'}$ to $x_{n_{i'}}^{i'}$. If $i = 1$, then P^* is a longer path in $G[\overline{A}]$ than P^1 , contradicting the maximality of P^1 . If $i \geq 2$, then P^* is a longer path in $G[\overline{A} \setminus \overline{A}_{i-1}]$ than P^i , contradicting the maximality of P^i . \square

In what follows, we adopt the following notation. If x and y are two non-adjacent vertices of G , then we let D_{xy} denote a γ_c -set of $G + xy$.

Claim 2 *If x and y are two non-adjacent vertices of $G[\overline{A}]$, then $|D_{xy} \cap (A \cup \overline{A})| = 2$, implying that $|D_{xy} \cap \{x, y\}| = 1$ and $|D_{xy} \cap A| = 1$.*

Proof. Let x, y and D_{xy} be as defined in the statement of the claim. We now consider the graph $G + xy$. Since G is a k - γ_c -critical graph, Lemma 1(a) implies that $|D_{xy}| \leq k - 1$. Further, Lemma 1(b) implies that $D_{xy} \cap \{x, y\} = 1$. Renaming x and y if necessary, we may assume that $x \in D_{xy}$. If $c_0 \notin D_{xy}$, then $c_1 \in D_{xy}$ to dominate c_0 . If $c_0 \in D_{xy}$, then, since the subgraph, $(G + xy)[D_{xy}]$, of $G + xy$ induced by the set D_{xy} is connected and since c_1 is the only neighbor of c_0 in $G + xy$, we must have $c_1 \in D_{xy}$. Hence, in both cases, $c_1 \in D_{xy}$. Recall that $x \in D_{xy} \cap \overline{A}$. Since $(G + xy)[D_{xy}]$ is a connected graph that contains both c_1 and x , the structure of the graph G implies that D_{xy} contains all vertices of the path $P: c_0 c_1 \dots c_{k-4}$ except possibly for the vertex c_0 , the vertex b , at least one neighbor of b in A , and at least one vertex in \overline{A} , namely the vertex x . Thus, D_{xy} contains at least $(|V(P)| - 1) + 3 = (k - 4) + 3 = k - 1$ vertices, and so $|D_{xy}| \geq k - 1$. As observed earlier, by Lemma 1(a) we have $|D_{xy}| \leq k - 1$. Consequently, $|D_{xy}| = k - 1$, implying that $D_{xy} = (V(P) \setminus \{c_0\}) \cup \{b, u, x\}$, where $u \in A$ and $ux \in E(G)$. In particular, we note

that $|D_{xy} \cap V(H - b)| = |D_{xy} \cap (A \cup \overline{A})| = |\{u, x\}| = 2$, and so $|D_{xy} \cap \{x, y\}| = 1$ and $|D_{xy} \cap A| = 1$. \square

In what follows, for notational convenience we let $A_0 = V(P) \cup \{b\} \cup A$, and so $A_0 = V(G) \setminus \overline{A}$.

Claim 3 *If R is a proper subset of vertices of A , where possibly $R = \emptyset$, and v is an arbitrary vertex in $A \setminus R$, then there exists a path $P_{R,v}$ from c_0 to v containing every vertex in $A_0 \setminus R$.*

Proof. Recall that $A = N_B(H)$ and $G[A]$ is a complete graph. Since $R \subset A$, we note therefore that $G[A - R]$ is a complete subgraph. Let P_v be a hamiltonian path in $G[A - R]$ that ends at the vertex v , and let v' be the start vertex of P_v (possibly, $v = v'$). The path $P_{R,v}$ that starts at the vertex c_0 , follows the path P to c_{k-4} , proceeds along the edge $c_{k-4}b$ from c_{k-4} to b , along the edge bv' from b to v' , and then follows the path P_v is a path from c_0 to v containing every vertex in $A_0 \setminus R$. \square

Claim 4 *If $z = 1$, then G is traceable.*

Proof. Suppose that $z = 1$, and so $\overline{A} = V(P^1)$. Suppose that x_1^1 or $x_{n_1}^1$ is adjacent to some vertex y of A . Renaming vertices if necessary, we may assume that y is adjacent to x_1^1 . By Claim 3 with $R = \emptyset$, there exists a path $P_{R,y}$ from c_0 to y containing every vertex in A_0 . The path $P_{R,y}$ can be extended to a hamiltonian path of G by proceeding along the edge yx_1^1 from y to x_1^1 , and then following the path P^1 from x_1^1 to $x_{n_1}^1$. Thus, we may assume that neither x_1^1 nor $x_{n_1}^1$ is adjacent to any vertex of A , for otherwise G is traceable as desired. Since $H \in \mathcal{B}_2$ is a connected graph, this implies that $|V(P^1)| \geq 3$.

We show next that $x_1^1 x_{n_1}^1 \in E(G)$. Suppose, to the contrary, that $x_1^1 x_{n_1}^1 \notin E(G)$. In this case, we consider $G + x_1^1 x_{n_1}^1$. For notational simplicity, let $D^* = D_{x_1^1 x_{n_1}^1}$. By Claim 2, we have $|D^* \cap \overline{A}| = |D^* \cap A| = 1$. Further, $|D^* \cap \{x_1^1, x_{n_1}^1\}| = 1$. Renaming x_1^1 and $x_{n_1}^1$ if necessary, we may assume that $D^* \cap \{x_1^1, x_{n_1}^1\} = \{x_1^1\}$. Let $D^* \cap A = \{u\}$. By the connectedness of $(G + x_1^1 x_{n_1}^1)[D^*]$, this implies that $x_1^1 u \in E(G)$, contradicting our earlier assumption that x_1^1 is not adjacent to any vertex in A . Hence, $x_1^1 x_{n_1}^1 \in E(G)$.

Since $x_1^1 x_{n_1}^1 \in E(G)$, we note that $C: P^1 + x_1^1 x_{n_1}^1$ is a hamiltonian cycle of $G[\overline{A}]$. Since G is a connected graph, there exists a vertex v in A which is adjacent to a vertex of P^1 , say to x_j^1 for some j where $1 < j < n_1$. By Claim 3 with $R = \emptyset$, there exists a path $P_{R,v}$ from c_0 to v containing every vertex in A_0 . The path $P_{R,v}$ can be extended to a hamiltonian path of G by proceeding along the edge vx_j^1 from v to x_j^1 , and then following a hamiltonian path in the cycle C starting at the vertex x_j^1 . Thus, G is traceable. \square

Claim 5 *If $z = 2$, then G is traceable.*

Proof. Suppose that $z = 2$, and so $\overline{A} = V(P^1) \cup V(P^2)$. Recall that $|V(P^1)| \geq |V(P^2)|$. By Claim 1, the vertex x_1^1 (respectively, $x_{n_1}^1$) is adjacent to neither x_1^2 nor $x_{n_2}^2$. In particular, $x_1^1 x_1^2 \notin E(G)$. We now consider the graph $G + x_1^1 x_1^2$. For notational simplicity, let $D_{1,2} = D_{x_1^1 x_1^2}$. By Claim 2, we have $|D_{1,2} \cap \overline{A}| = |D_{1,2} \cap A| = 1$. Further, $|D_{1,2} \cap \{x_1^1, x_1^2\}| = 1$. Let $D_{1,2} \cap \overline{A} = \{y\}$. We consider the cases $x_1^1 \in D_{1,2}$ and $x_1^2 \in D_{1,2}$ separately.

Claim 5.1 *If $x_1^1 \in D_{1,2}$, then G is traceable.*

Proof. Suppose that $x_1^1 \in D_{1,2}$. Thus in this case, $D_{1,2} \cap (A \cup \overline{A}) = \{x_1^1, y\}$. Since $(G + x_1^1 x_1^2)[D_{1,2}]$ is a connected graph, $x_1^1 y \in E(G)$. Moreover by Lemma 1(c), $yx_1^2 \notin E(G)$.

Claim 5.1.1 *If $|V(P^1)| \leq 2$, then G is traceable.*

Proof. Suppose that $|V(P^1)| \leq 2$. Suppose that $|V(P^1)| = 1$. Since $|V(P^1)| \geq |V(P^2)|$, we therefore have $|V(P^2)| = 1$, and so P^1 and P^2 consists of the single vertices x_1^1 and x_2^2 , respectively. But then $\{y_1, y_2\}$ is a CD-set of H , where y_i is an arbitrary neighbor of x_i^i that belongs to A for $i \in [2]$, and so $\gamma_c(H) \leq 2$, contradicting the fact that $\gamma_c(H) = 3$. Hence, $|V(P^1)| = 2$, and so $n_1 = 2$ and P^1 is the path $x_1^1 x_2^1$. As observed earlier, $|V(P^2)| \leq 2$.

Suppose firstly that $|V(P^2)| = 1$. Thus, P^2 consists of the single vertex x_2^2 , and $\overline{A} = \{x_1^1, x_2^1, x_1^2\}$. By Claim 1, the vertex x_2^2 is adjacent to neither x_1^1 nor x_2^1 . Let u be an arbitrary neighbor of x_2^2 in the connected graph G . We note that $u \in A$. If $u = y$, then $\{y, x_1^1\} \succ_c H$, implying that $\gamma_c(H) \leq 2$, a contradiction. Thus, $u \neq y$. Since H is a 2-connected graph, the vertex x_2^2 has a neighbor, w say, different from x_1^1 . We note that $w \in A$. If $w \in \{u, y\}$, then $\{u, y\} \succ_c H$, a contradiction. Hence, the vertices u, w and y are distinct vertices in A . By Claim 3 with $R = \{u, w\}$, there exists a path $P_{R,y}$ from c_0 to y containing every vertex in $A_0 \setminus \{u, w\}$. The path $P_{R,y}$ can be extended to a hamiltonian path of G by proceeding along the edge yx_1^1 from y to x_1^1 , and then following the path $x_1^1 x_2^1 w u x_2^2$ from x_1^1 to x_2^2 ; that is, the path

$$c_0 P_{R,y} y, x_1^1 x_2^1 w u x_2^2$$

is a hamiltonian path in G . Hence we may assume that $|V(P^2)| = 2$, for otherwise G is traceable, as desired. Thus, $\overline{A} = \{x_1^1, x_2^1, x_1^2, x_2^2\}$. Recall that $D_{1,2} \cap (A \cup \overline{A}) = \{x_1^1, y\}$ and that the vertex x_1^1 is adjacent to neither x_1^2 nor x_2^2 . Further, $yx_1^2 \notin E(G)$. These observations imply that $yx_2^2 \in E(G)$ in order for $D_{1,2}$ to dominate the vertex x_2^2 in $G + x_1^1 x_1^2$. By Claim 1, the vertex x_2^2 is adjacent to neither x_1^1 nor x_2^1 . Since H is a 2-connected graph, the vertex x_2^2 has a neighbor, u say, different from x_1^1 . We note that $u \in A$. If $u = y$, then $\{y, x_2^2\} \succ_c H$, a contradiction. Hence, $u \neq y$. By Claim 3 with $R = \{y\}$, there exists a path $P_{R,u}$ from c_0 to u containing every vertex in $A_0 \setminus \{y\}$. The path $P_{R,u}$ can be extended to a hamiltonian path of G by proceeding along the edge ux_2^2 from u to x_2^2 , and then following the path $x_2^2 x_1^1 y x_2^1 x_1^2$ from x_2^2 to x_1^2 ; that is, the path

$$c_0 P_{R,u} u, x_2^2 x_1^1 y x_2^1 x_1^2$$

is a hamiltonian path in G . This completes the proof of Claim 5.1.1. \square

By Claim 5.1.1, we may assume that $|V(P^1)| \geq 3$, for otherwise G is traceable and the desired result holds.

Claim 5.1.2 *If $x_1^1 x_{n_1}^1 \notin E(G)$, then G is traceable.*

Proof. Suppose that $x_1^1 x_{n_1}^1 \notin E(G)$. This implies that $|V(P^1)| \geq 3$. In order to dominate the vertex $x_{n_1}^1$ in $G + x_1^1 x_{n_1}^1$, we must have that $yx_{n_1}^1 \in E(G)$. We now consider the graph $G + x_1^1 x_{n_1}^1$. For notational simplicity, let $D_{1,n_1} = D_{x_1^1 x_{n_1}^1}$. By Claim 2, we have $|D_{1,n_1} \cap \overline{A}| = |D_{1,n_1} \cap A| = 1$. Further, $|D_{1,n_1} \cap \{x_1^1, x_{n_1}^1\}| = 1$. Let $\{u\} = D_{1,n_1} \cap \overline{A}$. As observed earlier, the vertex x_1^1 is adjacent to neither x_1^1 nor $x_{n_1}^1$. In order to dominate the vertex x_1^1 in $G + x_1^1 x_{n_1}^1$, we must have that $ux_1^1 \in E(G)$.

By Claim 3 with $R = \{u\}$, there exists a path $P_{R,y}$ from c_0 to y containing every vertex in $A_0 \setminus \{u\}$. By the connectedness of $(G + x_1^1 x_{n_1}^1)[D_{1,n_1}]$, the vertex u is adjacent to the vertex in $D_{1,n_1} \cap \{x_1^1, x_{n_1}^1\}$. Renaming the vertices x_1^1 and $x_{n_1}^1$ if necessary, we may assume without loss of generality that $x_{n_1}^1 \in D_{1,n_1}$. With this assumption, $ux_{n_1}^1 \in E(G)$. The path $P_{R,y}$ can be extended to a hamiltonian path of G by proceeding along the edge yx_1^1 from y to x_1^1 , following the path P^1 from x_1^1 to $x_{n_1}^1$, proceeding along the edge $x_{n_1}^1 u$ from $x_{n_1}^1$ to u , proceeding along the edge ux_1^1 from u to x_1^1 , and then following the path P^2 from x_1^1 to $x_{n_2}^2$; that is, the path

$$c_0 P_{R,y} y, x_1^1 P^1 x_{n_1}^1, u, x_1^1 P^2 x_{n_2}^2$$

is a hamiltonian path in G . \square

By Claim 5.1.2, we may assume that $x_1^1 x_{n_1}^1 \in E(G)$, for otherwise G is traceable and the desired result holds. Since $x_1^1 x_{n_1}^1 \in E(G)$, we note that $C: P^1 + x_1^1 x_{n_1}^1$ is a cycle in $G[\overline{A}]$.

Claim 5.1.3 *If $|V(P^2)| = 1$, then G is traceable.*

Proof. Suppose that $|V(P^2)| = 1$. Thus, P^2 consists of the single vertex x_1^2 . By Claim 1, the vertex x_1^2 is adjacent to neither x_1^1 nor $x_{n_1}^1$. If x_1^2 is adjacent to x_j^1 for some $1 < j < n_1$, then the (x_{j+1}^1, x_j^1) -path on C that does not contain the edge $x_j^1 x_{j+1}^1$ can be extended to a longer path in $G[\overline{A}]$ by adding to it the vertex x_1^2 and the edge $x_j^1 x_1^2$, contradicting the maximality of the path P^1 . Hence, the vertex x_1^2 is adjacent to no vertex of P^1 . By the connectivity of G and the maximality of the path P^2 , the vertex x_1^2 is adjacent to a vertex, u say, in A .

If $y \succ P^1$, then $\{y, u\} \succ_c H$, implying that $\gamma_c(H) \leq 2$, a contradiction. Thus, y does not dominate P^1 . Let j be the smallest integer so that yx_j^1 is not an edge. Since $x_1^1 y \in E(G)$, we note that $j \in [n_1] \setminus \{1\}$. By the choice of j , we note that $yx_\ell^1 \in E(G)$ for all $\ell \in [j-1]$.

We now consider the graph $G + x_j^1 x_1^2$. For notational simplicity, let $D^* = D_{x_j^1 x_1^2}$. By Claim 2, we have $|D^* \cap \overline{A}| = |D^* \cap A| = 1$. Further, $|D^* \cap \{x_j^1, x_1^2\}| = 1$. Let $\{w\} = D^* \cap A$. Since y is adjacent to neither x_j^1 nor x_1^2 , we note that $w \neq y$. If $x_1^2 \in D^*$, then since x_1^2 is not

adjacent to any vertex of P^1 , we note that $w \succ P^1 - x_j^1$. implying that $\{w, x_{j-1}^1\} \succ_c H$, and so $\gamma_c(H) \leq 2$, a contradiction. Hence, $x_1^2 \notin D^*$, implying that $x_j^1 \in D^*$; that is, $D^* \cap \bar{A} = \{x_j^1\}$. By Lemma 1(c), we note that $w x_1^2 \notin E(G)$. Since $(G + x_j^1 x_1^2)[D^*]$ is connected, we therefore have $w x_j^1 \in E(G)$. Since G is connected, the vertex x_1^2 is adjacent to a vertex, say v , that belongs to A . Since the vertex x_1^2 is adjacent neither w nor y , the vertices v , w and y are distinct.

By Claim 3 with $R = \{v, w\}$, there exists a path $P_{R,y}$ from c_0 to y containing every vertex in $A_0 \setminus \{v, w\}$. The path $P_{R,y}$ can be extended to a hamiltonian path of G by proceeding along the edge $y x_{j-1}^1$ from y to x_{j-1}^1 , and then following the (x_{j-1}^1, x_j^1) -path, say P^* , on C that does not contain the edge $x_{j-1}^1 x_j^1$ (and contains all vertices of P^1) from x_{j-1}^1 to x_j^1 , and then following the path $x_j^1 w v x_1^2$ from x_j^1 to x_1^2 ; that is, the path

$$c_0 P_{R,y} y, x_{j-1}^1 P^* x_j^1, w v x_1^2$$

is a hamiltonian path in G . This completes the proof of Claim 5.1.3. (\square)

By Claim 5.1.3, we may assume that $|V(P^2)| \geq 2$, for otherwise G is traceable and the desired result holds. Recall that $D_{1,2} \cap \bar{A} = \{y\}$ and $D_{1,2} \cap \bar{A} = \{x_1^1\}$. Further, the vertex x_1^1 is adjacent to neither x_1^2 nor $x_{n_2}^2$. In particular, $x_1^1 x_{n_2}^2 \notin E(G)$. In order for the set $D_{1,2}$ to dominate the vertex $x_{n_2}^2$, we note that $y x_{n_2}^2 \in E(G)$.

We now consider the graph $G + x_1^1 x_{n_2}^2$. For notational simplicity, let $D^* = D_{x_1^1 x_{n_2}^2}$. By Claim 2, we have $|D^* \cap \bar{A}| = |D^* \cap A| = 1$. Further, $|D^* \cap \{x_1^1, x_{n_2}^2\}| = 1$. Let $D^* \cap A = \{u\}$. By Lemma 1(c), the vertex u is adjacent to exactly one of x_1^1 and $x_{n_2}^2$. Therefore since y is adjacent to both x_1^1 and $x_{n_2}^2$, we note that $u \neq y$.

Suppose firstly that $x_1^1 \in D^*$. In this case, u is adjacent to x_1^1 but not to $x_{n_2}^2$. Since $x_1^1 x_{n_2}^2 \notin E(G)$, in order for the set D^* to dominate the vertex $x_{n_2}^2$, we note that $u x_1^1 \in E(G)$. By Claim 3 with $R = \{u\}$, there exists a path $P_{R,y}$ from c_0 to y containing every vertex in $A_0 \setminus \{u\}$. The path $P_{R,y}$ can be extended to a hamiltonian path of G by proceeding along the edge $y x_{n_2}^2$ from y to $x_{n_2}^2$, following the path P^2 in reverse direction from $x_{n_2}^2$ to x_1^2 , proceeding along the path $x_1^2 u x_1^1$ from x_1^2 to x_1^1 , and then following the path P^1 from x_1^1 to $x_{n_1}^1$; that is, the path

$$c_0 P_{R,y} y, x_{n_2}^2 P^2 x_1^2, u, x_1^1 P^1 x_{n_1}^1$$

is a hamiltonian path in G . Suppose next that $x_{n_2}^2 \in D^*$. In this case, u is adjacent to $x_{n_2}^2$ but not to x_1^1 . Since $x_{n_1}^1 x_{n_2}^2 \notin E(G)$, in order for the set D^* to dominate the vertex $x_{n_1}^1$, we note that $u x_{n_1}^1 \in E(G)$. By Claim 3 with $R = \{u\}$, there exists a path $P_{R,y}$ from c_0 to y containing every vertex in $A_0 \setminus \{u\}$. The path $P_{R,y}$ can be extended to a hamiltonian path of G by proceeding along the edge $y x_1^1$ from y to x_1^1 , following the path P^1 from x_1^1 to $x_{n_1}^1$, proceeding along the path $x_{n_1}^1 u x_{n_2}^2$ from $x_{n_1}^1$ to $x_{n_2}^2$, and following the path P^2 in reverse direction from $x_{n_2}^2$ to x_1^2 ; that is, the path

$$c_0 P_{R,y} y, x_1^1 P^1 x_{n_1}^1, u, x_{n_2}^2 P^2 x_1^2$$

is a hamiltonian path in G . This completes the proof of Claim 5.1. (\square)

By Claim 5.1, we may assume that $x_1^2 \in D_{1,2}$, for otherwise G is traceable and the desired result follows. Thus in this case, $D_{1,2} \cap (A \cup \bar{A}) = \{x_1^2, y\}$. Since $(G + x_1^1 x_1^2)[D_{1,2}]$ is a connected graph, $x_1^2 y \in E(G)$. Moreover by Lemma 1(c), $yx_1^1 \notin E(G)$. Since $x_{n_1}^1 x_1^2 \notin E(G)$, in order for the set $D_{1,2}$ to dominate the vertex $x_{n_1}^1$, we note that $yx_{n_1}^1 \in E(G)$.

Claim 5.2 *If $|V(P^2)| = 1$, then G is traceable.*

Proof. Suppose that $|V(P^2)| = 1$. Thus, P^2 consists of the single vertex x_1^2 . We show firstly that $x_1^1 x_{n_1}^1 \notin E(G)$. Suppose, to the contrary, that $x_1^1 x_{n_1}^1 \in E(G)$. We show that x_1^2 is not adjacent to any vertex of P^1 . By our earlier observations, the vertex x_1^2 is adjacent to neither x_1^1 nor $x_{n_1}^1$. Suppose that x_1^2 is adjacent to x_j^1 for some j where $1 < j < n_1$. In this case, $|V(P^1)| \geq 3$, and so $G[\bar{A}]$ has a cycle $C: P^1 + x_1^1 x_{n_1}^1$ as a subgraph. The (x_{j+1}^1, x_j^1) -path on C that does not contain the edge $x_j^1 x_{j+1}^1$ can be extended to a longer path in $G[\bar{A}]$ by adding to it the vertex x_1^2 and the edge $x_j^1 x_1^2$, contradicting the maximality of the path P^1 . Hence, the vertex x_1^2 is adjacent to no vertex of P^1 . Thus since $\{x_1^2, y\}$ dominates all vertices of P^1 different from x_1^1 , this implies that $y \succ P^1 - x_1^1$. Therefore, $\{y, x_{n_1}^1\} \succ_c H$, and so $\gamma_c(H) \leq 2$, a contradiction. Hence, $x_1^1 x_{n_1}^1 \notin E(G)$.

We now consider the graph $G + x_{n_1}^1 x_1^2$. For notational simplicity, let $D^* = D_{x_{n_1}^1 x_1^2}$. By Claim 2, we have $|D^* \cap \bar{A}| = |D^* \cap A| = 1$. Further, $|D^* \cap \{x_{n_1}^1, x_1^2\}| = 1$. Let $D^* \cap A = \{u\}$. By Lemma 1(c), the vertex u is adjacent to exactly one of $x_{n_1}^1$ and x_1^2 . Therefore since y is adjacent to both $x_{n_1}^1$ and x_1^2 , we note that $u \neq y$. If $x_1^2 \in D^*$, then since $x_1^2 x_1^1 \notin E(G)$, we have $ux_1^1 \in E(G)$. If $x_{n_1}^1 \in D^*$, then since $x_1^1 x_{n_1}^1 \notin E(G)$, we must have $ux_1^1 \in E(G)$. In both cases, $ux_1^1 \in E(G)$.

By Claim 3 with $R = \{y\}$, there exists a path $P_{R,u}$ from c_0 to u containing every vertex in $A_0 \setminus \{y\}$. The path $P_{R,u}$ can be extended to a hamiltonian path of G by proceeding along the edge ux_1^1 from u to x_1^1 , following the path P^1 from x_1^1 to $x_{n_1}^1$, and then proceeding along the path $x_{n_1}^1 y x_1^2$ from $x_{n_1}^1$ to x_1^2 ; that is, the path

$$c_0 P_{R,u} u, x_1^1 P^1 x_{n_1}^1, y, x_1^2$$

is a hamiltonian path in G . This completes the proof of Claim 5.2. \square

By Claim 5.2, we may assume that $|V(P^2)| \geq 2$, for otherwise G is traceable and the desired result holds. We now consider the graph $G + x_{n_1}^1 x_1^2$. For notational simplicity, let $D^* = D_{x_{n_1}^1 x_1^2}$. By Claim 2, we have $|D^* \cap \bar{A}| = |D^* \cap A| = 1$. Further, $|D^* \cap \{x_{n_1}^1, x_1^2\}| = 1$. Let $D^* \cap A = \{u\}$. By Lemma 1(c), the vertex u is adjacent to exactly one of $x_{n_1}^1$ and x_1^2 . Therefore since y is adjacent to both $x_{n_1}^1$ and x_1^2 , we note that $u \neq y$.

Suppose firstly that $x_1^2 \in D^*$. In this case, u is adjacent to x_1^2 but not to $x_{n_1}^1$. Since $x_1^1 x_1^2 \notin E(G)$, in order for the set D^* to dominate the vertex x_1^1 , we note that $ux_1^1 \in E(G)$. By Claim 3 with $R = \{u\}$, there exists a path $P_{R,y}$ from c_0 to y containing every vertex in $A_0 \setminus \{u\}$. The path $P_{R,y}$ can be extended to a hamiltonian path of G by proceeding along the edge $yx_{n_1}^1$ from y to $x_{n_1}^1$, following the path P^1 in the reverse direction from $x_{n_1}^1$ to x_1^1 ,

proceeding along the path $x_1^1 u x_1^2$ from x_1^1 to x_1^2 , and following the path P^2 from x_1^2 to $x_{n_2}^2$; that is, the path

$$c_0 P_{R,y} y, x_{n_1}^1 P^1 x_1^1, u, x_1^2 P^2 x_{n_2}^2$$

is a hamiltonian path in G . Suppose next that $x_{n_1}^1 \in D^*$. In this case, u is adjacent to $x_{n_1}^1$ but not to x_1^2 . Since $x_{n_1}^1 x_{n_2}^1 \notin E(G)$ and $x_1^2 \neq x_{n_2}^2$, in order for the set D^* to dominate the vertex $x_{n_2}^2$, we note that $u x_{n_2}^2 \in E(G)$.

By Claim 3 with $R = \{y\}$, there exists a path $P_{R,u}$ from c_0 to u containing every vertex in $A_0 \setminus \{y\}$. The path $P_{R,u}$ can be extended to a hamiltonian path of G by proceeding along the edge $u x_{n_2}^2$ from u to $x_{n_2}^2$, following the path P^2 in the reverse direction from $x_{n_2}^2$ to x_1^2 , proceeding along the path $x_1^2 y x_{n_1}^1$ from x_1^2 to $x_{n_1}^1$, and following the path P^1 in the reverse direction from $x_{n_1}^1$ to x_1^1 ; that is, the path

$$c_0 P_{R,u} u, x_{n_2}^2 P^2 x_1^2, y, x_{n_1}^1 P^1 x_1^1$$

is a hamiltonian path in G . This completes the proof of Claim 5. \square

By Claims 4 and 5, we may assume that $z \geq 3$, for otherwise G is traceable and the desired result holds. The following claim uses similar ideas to those presented in [20]. However for completeness, we provide a proof of this claim.

Claim 6 *If I is an independent set of \overline{A} where $|I| = t \geq 3$, then all the vertices of I can be ordered u_1, u_2, \dots, u_t in such a way that there exist $t - 1$ different vertices v_1, v_2, \dots, v_{t-1} of A satisfying $\{u_i, v_i\} \succ_c H - u_{i+1}$ for all $i \in [t - 1]$.*

Proof. We will construct a tournament T (a digraph which any two vertices are joined by an arc) with vertex set $V(T) = I$ and where the arcs of T are defined as follow. For every two distinct vertices u and v in I , we choose a fixed γ_c -set, say D_{uv} , of $G + uv$. By Claim 2, $|D_{uv} \cap (A \cup \overline{A})| = 2$, implying that $|D_{uv} \cap \{u, v\}| = 1$ and $|D_{uv} \cap A| = 1$. Let $D_{uv} \cap A = \{x\}$. If $u \in D_{uv}$, then since A is a complete subgraph, it follows that $\{u, x\} \succ_c H - v$. In this case, we orient the arc from u to v . If $v \in D_{uv}$, then $\{v, x\} \succ_c H - u$, and we orient the arc from v to u . We do this for every two distinct vertices u and v in I . This defines the arcs of the resulting tournament T . Since every tournament has a directed hamiltonian path, we let $u_1 u_2 \dots u_t$ be a directed hamiltonian path in T . This implies that there exists a vertex $v_i \in A$ such that $\{u_i, v_i\} \succ_c H - u_{i+1}$ for every $i \in [t - 1]$. Since $I = \{u_1, u_2, \dots, u_t\}$ is an independent set, it follows that the vertex v_i is adjacent to every vertex in I except for the vertex u_{i+1} for all $i \in [t - 1]$. This implies that the vertices v_1, v_2, \dots, v_t are all distinct. \square

We now return to the proof of Theorem 7. Since $\{x_1^1, x_1^2, \dots, x_1^z\}$ is an independent set of size $z \geq 3$, by Claim 6 there exists an ordering u_1, u_2, \dots, u_z of the vertices of $\{x_1^1, x_1^2, \dots, x_1^z\}$ such that there exist vertices v_1, v_2, \dots, v_{z-1} of A satisfying

$$\{u_i, v_i\} \succ_c H - u_{i+1}$$

for all $i \in [z - 1]$. Let $R = \{v_1, v_2, \dots, v_{z-1}\}$. For notational convenience, if $u_j = x_1^i$ for some $i, j \in [z]$, then we relabel the path P^i as the path T^j . Further we let $u'_j = x_{n_i}^i$. We note that

$v_i \succ \{u_1, u_2, \dots, u_z\} \setminus \{u_{i+1}\}$ for all $i \in [z-1]$. Further, we note that the collection of paths P^1, P^2, \dots, P^z is therefore precisely the collection of paths T^1, T^2, \dots, T^z . If $|V(T^i)| = 1$ for all $i \in [z]$, then $\{v_1, v_2\} \succ_c H$, and so $\gamma_c(H) \leq 2$, a contradiction. Hence, $|V(T^i)| > 1$ for some $i \in [z]$. For $i \in [z]$, we let

$$\ell = \max\{i : |V(T^i)| > 1\}.$$

Therefore if $\ell < z$, then $|V(T^j)| = 1$ for all j where $\ell < j \leq z$, implying that $u_j = u'_j$ for such values of j . We remark that it is possible that $|V(T^j)| = 1$ for some $j < \ell$. The structure of $G[\bar{A}]$ is now illustrated by Figure 7.

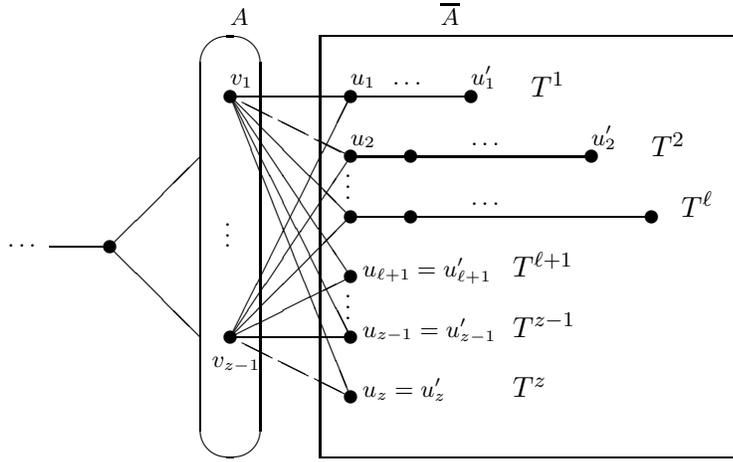


Figure 7: The structure of $G[\bar{A}]$ after rearranging the paths

Claim 7 *If $\ell > 1$, then G is traceable.*

Proof. Suppose that $\ell > 1$. Therefore, $u'_\ell \neq u'_1$ and $u'_\ell \neq u_\ell$. We now consider the graph $G + u_1 u'_\ell$. For notational simplicity, let $D_{1,\ell} = D_{u_1 u'_\ell}$. By Claim 2, we have $|D_{1,\ell} \cap \bar{A}| = |D_{1,\ell} \cap A| = 1$. Further, $|D_{1,\ell} \cap \{u_1, u'_\ell\}| = 1$. Let $D_{1,\ell} \cap A = \{w\}$. By Lemma 1(c), the vertex w is adjacent to exactly one of u_1 and u'_ℓ . By Claim 6, $\{u_i, v_i\} \succ_c H - u_{i+1}$ for all $i \in [z-1]$. Therefore since $u_i u'_{i-1} \notin E(G)$, this implies that $v_i u'_{i-1} \in E(G)$ for all $i \in [z] \setminus \{1\}$.

Suppose firstly that $u_1 \in D_{1,\ell}$. Thus, w is adjacent to u_1 but not to u'_ℓ . If $w = v_j$ for some $j \in [z-1]$, then $\{u_1, w\}$ does not dominate the vertex u_{j+1} , contradicting the fact that $D_{1,\ell}$ is a (connected) dominating set of $G + u_1 u'_\ell$. Hence, $w \notin R$. Since $u_1 u_z \notin E(G)$, we note that $w u_z \in E(G)$. Further since $u_1 u'_z \notin E(G)$ and $\{u_1, v_1\} \succ_c H - u_2$ and $z \geq 3$, it follows that $v_1 u'_z \in E(G)$. (Possibly, $u_z = u'_z$.) As observed earlier, $v_i u'_{i-1} \in E(G)$ for all $i \in [z] \setminus \{1\}$. By Claim 3, there exists a path $P_{R,w}$ from c_0 to w containing every vertex in $A_0 \setminus R$. The path $P_{R,w}$ can be extended to a hamiltonian path of G by proceeding along the edge $w u_z$ from w to u_z , following the path T^z from u_z to u'_z , proceeding along the path $u'_z v_1 u_1$ from u'_z to u_1 , following the path T^1 from u_1 to u'_1 , proceeding along the path $u'_1 v_2 u_2$ from u'_1 to u_2 ,

following the path T^2 from u_2 to u'_2 , proceeding along the path $u'_2v_3u_3$ from u_3 to u'_3 , and, continuing in this way, finally proceeding along the path $u'_{z-2}v_{z-1}u_{z-1}$ from u'_{z-2} to u_{z-1} , and then following the path T^{z-1} from u_{z-1} to u'_{z-1} ; that is, the path

$$c_0P_{R,w}u_z, u_zT^zu'_z, v_1, u_1T^1u'_1, v_2, u_2T^2u'_2, \dots, u_{z-2}T^{z-2}u'_{z-2}, v_{z-1}, u_{z-1}T^{z-1}u'_{z-1}$$

is a hamiltonian path in G , as desired. Suppose next that $u'_\ell \in D_{1,\ell}$. Thus, w is adjacent to u'_ℓ but not to u_1 . If $\ell \leq z-2$, then u'_ℓ is adjacent to neither u'_{z-1} nor u'_z , implying that w is adjacent to both u'_{z-1} and u'_z . If $\ell = z-1$, then w is adjacent to u'_{z-1} . Further since u'_{z-1} is not adjacent to u'_z , the vertex w is adjacent to u'_z in this case. If $\ell = z$, then w is adjacent to u'_z . Further since $z \geq 3$, the vertex $u'_{z-1} \neq u_1$. Thus since the vertex u'_z is not adjacent to u'_{z-1} , the vertex w is therefore adjacent to u'_{z-1} in this case. Thus in all cases, we note that the vertex w is adjacent to both u'_{z-1} and u'_z .

Let $R_1 = (R \setminus \{v_1\}) \cup \{w\}$. By Claim 3, there exists a path P_{R_1} from c_0 to v_1 containing every vertex in $A_0 \setminus R_1$. The path P_{R_1, v_1} can be extended to a hamiltonian path of G by proceeding along the edge v_1u_1 from v_1 to u_1 , following the path T^1 from u_1 to u'_1 , proceeding along the path $u'_1v_2u_2$ from u'_1 to u_2 , following the path T^2 from u_2 to u'_2 , proceeding along the path $u'_2v_3u_3$ from u_3 to u'_3 , and, continuing in this way, finally proceeding along the path $u'_{z-2}v_{z-1}u_{z-1}$ from u'_{z-2} to u_{z-1} , and then following the path T^{z-1} from u_{z-1} to u'_{z-1} , proceeding along the path $u'_{z-1}wu'_z$ from u'_{z-1} to u'_z , following the path T^z in the reverse direction from u'_z to u_z ; that is, the path

$$c_0P_{R_1, v_1}v_1, u_1T^1u'_1, v_2, u_2T^2u'_2, \dots, u_{z-1}T^{z-1}u'_{z-1}, w, u'_zT^zu_z$$

is a hamiltonian path in G , as desired. (\square)

By Claim 7, we may assume that $\ell = 1$, for otherwise G is traceable and the desired result follows. Thus, $P^1 = T^1$ and $|V(P^1)| = n_1 \geq 2$, and so $u_1 \neq u'_1$. Moreover, $u_i = u'_i$ for all $i \in [z] \setminus \{1\}$.

Suppose firstly that $u_1u'_1 \in E(G)$, and so $G[\overline{A}]$ contains a cycle $C: T^1 + u_1u'_1$ as a subgraph. If there exist integers j and r where $j \in [n_1 - 1] \setminus \{1\}$ and $r \in [z] \setminus \{1\}$ such that $u_r x_j^1 \in E(G)$, then the (x_{j+1}^1, x_j^1) -path on C that does not contain the edge $x_j^1 x_{j+1}^1$ can be extended to a longer path in $G[\overline{A}]$ by adding to it the vertex u_r and the edge $u_r x_j^1$, contradicting the maximality of the path P^1 . Hence, no vertex of T^1 is adjacent to any vertex from the set $\{u_2, u_3, \dots, u_z\}$, implying that the vertex u_i is an isolated vertex in $G[\overline{A}]$ for all $i \in [z] \setminus \{1\}$. Since $\{u_2, v_2\} \succ_c H - u_3$ and u_2 is isolated in $G[\overline{A}]$, this implies that $v_2 \succ H - u_3$. Therefore, $\{v_1, v_2\} \succ_c H$, and so $\gamma_c(H) \leq 2$, a contradiction. Hence, $u_1u'_1 \notin E(G)$.

We now consider the graph $G + u_1u'_1$. For notational simplicity, let $D_{1,1} = D_{u_1u'_1}$. By Claim 2, we have $|D_{1,1} \cap \overline{A}| = |D_{1,1} \cap A| = 1$. Further, $|D_{1,1} \cap \{u_1, u'_1\}| = 1$. Let $D_{1,1} \cap A = \{w\}$. By Lemma 1(c), the vertex w is adjacent to exactly one of u_1 and u'_1 . If $w = v_i$ for some $i \in [z-1]$, then $D_{1,1}$ does not dominate u_{i+1} , a contradiction. Hence, $w \in A \setminus R$. Since neither u_1 nor u'_1 is adjacent to u_z or u_{z-1} , the vertex w is necessarily adjacent to both u_z and u_{z-1} . Recall that $v_i u_{i-1} \in E(G)$ for all $i \in [z-1] \setminus \{1\}$, and recall that $v_2 u'_1 \in E(G)$. Let $R_1 = (R \setminus \{v_1\}) \cup \{w\}$. By Claim 3, there exists a path P_{R_1} from c_0 to v_1 containing

every vertex in $A_0 \setminus R_1$. The path P_{R_1, v_1} can be extended to a hamiltonian path of G by proceeding along the edge $v_1 u_1$ from v_1 to u_1 , following the path T^1 from u_1 to u'_1 , and then proceeding along the path $u'_1 v_2 u_2 v_3 \dots v_{z-1} u_{z-1} w u_z$; that is, the path

$$c_0 P_{R_1, v_1} v_1, u_1 T^1 u'_1 v_2 u_2 v_3 \dots v_{z-1} u_{z-1} w u_z$$

is a hamiltonian path in G . This completes the proof Theorem 7. \square

5 k - γ_c -Critical Graphs which are Non-Traceable

In this section, we establish the realizability result that for $k \geq 4$, there exist k - γ_c -critical graphs which is non-traceable containing ζ vertices for all $0 \leq \zeta \leq k-4$. For this purpose, for $k \geq 3$ we introduce a class $\mathcal{P}(k)$ of k - γ_c -critical graphs such that, for every graph $G \in \mathcal{P}(k)$ and every integer $\ell \geq 1$, there exists a $(k+\ell)$ - γ_c -critical graph that contains G as an induced subgraph. Further, we construct a class $\mathcal{N}(s)$ of graphs for all $s \geq 6$.

The class $\mathcal{P}(k)$ for $k \geq 3$. A k - γ_c -critical graph G is in the class $\mathcal{P}(k)$ if there exists a maximal complete subgraph H of G of order at least 2 satisfies the following two properties.

- (a) Every vertex of G belongs to some γ_c -set of G that contains a vertex of H .
- (b) For every pair of non-adjacent vertices x and y in G , there exists a CD-set D'_{xy} of $G+xy$ such that $D'_{xy} \cap V(H) \neq \emptyset$ and $|D'_{xy}| < k$ (we remark that D'_{xy} need not necessarily be a γ_c -set of $G+xy$).

The class $\mathcal{N}(s)$ for $s \geq 6$. For a set $S = [s]$ where $s \geq 6$, we let $B_1 = \{a_i : i \in [s]\}$ and $B_2 = \{b_i : i \in [s]\}$ be two disjoint sets of vertices, and let

$$B_3 = \left\{ z_{i,j} : \{i, j\} \in \binom{S}{2} \right\}$$

where $\binom{S}{2}$ is a set of all pairs (regardless of order) of the members in S , and so $|B_3| = \binom{s}{2}$.

A graph G in the class $\mathcal{N}(s)$ can be constructed from the disjoint sets B_1 , B_2 and B_3 by adding a new vertex x and adding edges as follows:

- Add edges so that B_1 and B_2 form two complete subgraphs.
- Add all edges between B_1 and B_2 except for the edges $a_i b_i$ for $i \in [s]$.
- Join x to every vertex of B_3 .
- Join b_i to $z_{j,\ell}$ for $1 \leq i \neq j \neq \ell \leq s$.
- Join a_i to $z_{i,j}$ for $1 \leq i \neq j \leq s$.

We note that for $i \in [s]$, $N_{B_3}(a_i) = \{z_{i,1}, z_{i,2}, \dots, z_{i,i-1}, z_{i,i+1}, z_{i,i+2}, \dots, z_{i,s}\}$ and

$$N_{B_3}(b_i) = \left\{ z_{j,\ell} : \{j, \ell\} \in \binom{S \setminus \{i\}}{2} \right\}.$$

A graph in the class $\mathcal{N}(s)$ is illustrated by Figure 8.

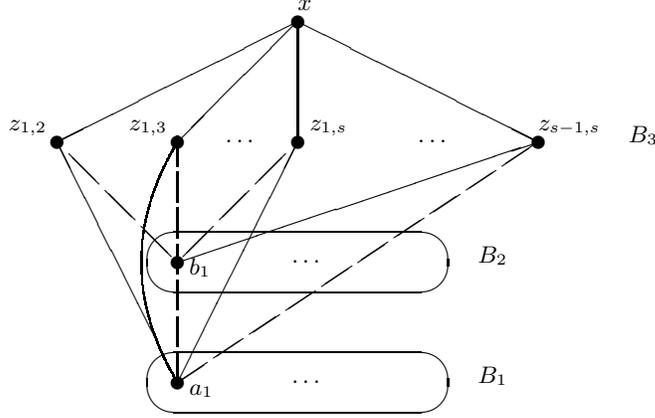


Figure 8: A graph G in the class $\mathcal{N}(s)$

Let $\mathcal{F}(k, \zeta)$ be the class of k - γ_c -critical graphs with ζ cut-vertices which are non-traceable. In view of Theorem 3, $\mathcal{F}(k, \zeta) = \emptyset$ for all $k \in [3]$. We show next that $\mathcal{N}(s) \subseteq \mathcal{P}(4)$ and $\mathcal{N}(s) \subseteq \mathcal{F}(4, 0)$. In particular, this implies that the class $\mathcal{P}(4)$ is not empty when $k = 4$.

Lemma 6 *For all $s \geq 6$, $\mathcal{N}(s) \subseteq \mathcal{F}(4, 0)$. Moreover, $\mathcal{N}(s) \subseteq \mathcal{P}(4)$ where in the construction of $\mathcal{P}(4)$ here we take H as the maximal complete subgraph $G[B_2]$.*

Proof. Let $G \in \mathcal{N}(s)$. We show that G is a 4 - γ_c -critical non-traceable graph. Let H be the maximal complete subgraph $G[B_2]$ of G . We show firstly that $\gamma_c(G) \geq 4$. Suppose, to the contrary, that there exists a CD-set D of G of size 3. Suppose that $x \in D$. If $D = \{x, z_{i,j}, z_{i',j'}\}$, then D does not dominate a_ℓ where $\ell \in S \setminus \{i, j, i', j'\}$. If $D = \{x, z_{i,j}, a_i\}$, then D does not dominate b_i . If $D = \{x, z_{i,j}, b_\ell\}$, then D does not dominate a_ℓ . In all three cases we produce a contradiction. Hence, $x \notin D$. In order to dominate the vertex x , we have $z_{i,j} \in D$ for some i and j where $1 \leq i \neq j \leq s$. If $D = \{z_{i,j}, a_i, a_j\}$, then D does not dominate $z_{i',j'}$ where $\{i', j'\} \cap \{i, j\} = \emptyset$. If $D = \{z_{i,j}, a_i, a_\ell\}$ where $\ell \notin \{i, j\}$, then D does not dominate $z_{i',j'}$ where $\{i', j'\} \cap \{i, j, \ell\} = \emptyset$. If $D = \{z_{i,j}, a_i, b_\ell\}$ where $\ell \notin \{i, j\}$, then D does not dominate $z_{j,\ell}$. If $D = \{z_{i,j}, a_i, z_{i',j'}\}$ or $D = \{z_{i,j}, b_\ell, z_{i',j'}\}$, then D does not dominate $z_{j,\ell}$ where $\ell \notin \{i, i', j, j'\}$. If $D = \{z_{i,j}, b_\ell, b_{\ell'}\}$ where $\ell, \ell' \notin \{i, j\}$, then D does not dominate $z_{\ell,\ell'}$. In all cases, we have a contradiction. We deduce, therefore, that not such CD-set D of size 3 exists. Hence, $\gamma_c(G) \geq 4$.

We show next that property (a) holds in the construction of $\mathcal{P}(4)$, and, simultaneously, we show that $\gamma_c(G) = 4$. For all i and j where $1 \leq i \neq j \leq s$, we note that $xz_{i,j}a_ib_j$ is an induced path in G and the set $D_{i,j} = \{x, z_{i,j}, a_i, b_j\}$ is a CD-set of G . This implies that $\gamma_c(G) \leq 4$ and every vertex of G belongs to some CD-set of G of size 4 that contains a vertex of H , where recall that $V(H) = B_2$. As observed earlier, $\gamma_c(G) \geq 4$. Consequently, $\gamma_c(G) = 4$ and every vertex of G belongs to some γ_c -set of G that contains a vertex of H . This establishes property (a) in the construction of $\mathcal{P}(4)$.

We show next that property (b) in the construction of $\mathcal{P}(4)$ holds. Let u and v be an arbitrary pair of non-adjacent vertices of G . We show that there exists a CD-set D_{uv} of $G + uv$ such that $|D_{uv}| = 3$ and the set D_{uv} contains at least one vertex in B_2 .

Suppose that $x \in \{u, v\}$. Renaming vertices if necessary, we may assume that $x = u$. Thus, $v \in B_1 \cup B_2$. If $v = a_i$, then let $D_{uv} = \{x, a_i, b_j\}$ where $i \neq j$. If $v = b_i$, then let $D_{uv} = \{x, a_i, b_j\}$ where $i \neq j$. In both cases, $|D_{uv}| = 3$, the set D_{uv} contains a vertex of B_2 and $D_{uv} \succ_c G + uv$, as desired. Hence, we may assume that $x \notin \{u, v\}$, for otherwise the desired result holds.

Suppose next that $z_{i,j} \in \{u, v\}$ for some i and j where $1 \leq i \neq j \leq s$. Renaming vertices if necessary, we may assume that $z_{i,j} = u$. If $v = a_\ell$, then necessarily $\ell \neq i$ and we let $D_{uv} = \{a_\ell, b_\ell, z_{i,j}\}$. If $v = b_i$ or $v = b_j$, then we let $D_{uv} = \{z_{i,j}, b_i, b_j\}$. If $v = z_{i',j'}$ where $\{i, j\} \neq \{i', j'\}$, then $z_{i,j}$ is adjacent to at least one of the vertices $b_{i'}$ or $b_{j'}$ and we let $D_{uv} = \{z_{i,j}, b_{i'}, b_{j'}\}$. In all cases, $|D_{uv}| = 3$, the set D_{uv} contains a vertex of B_2 and $D_{uv} \succ_c G + uv$, as desired. Hence, we may assume that $\{u, v\} \subseteq B_1 \cup B_2$, for otherwise the desired result holds. Thus, $\{u, v\} = \{a_i, b_i\}$ for some $i \in [s]$. We now let $D_{uv} = \{a_i, b_i, z_{i,j}\}$ where $i \neq j$. Once again in this case, $|D_{uv}| = 3$, the set D_{uv} contains a vertex of B_2 and $D_{uv} \succ_c G + uv$, as desired.

Thus, for an arbitrary pair x and y of non-adjacent vertices of G , there exists a CD-set D_{xy} of $G + xy$ such that $|D_{xy}| = 3$ and the set D_{xy} contains at least one vertex of H , where $V(H) = B_2$. As observed earlier, $\gamma_c(G) = 4$. Therefore, property (b) in the construction of $\mathcal{P}(4)$ holds. In particular, we note G is a $4\text{-}\gamma_c$ -critical graph. Since G is an arbitrary graph in $\mathcal{N}(s)$, we have $\mathcal{N}(s) \subseteq \mathcal{P}(4)$.

By construction, we note that G has no cut-vertex. Finally, to show that $\mathcal{N}(s) \subseteq \mathcal{F}(4, 0)$, it remains to show that G is non-traceable. Let $S = B_1 \cup B_2 \cup \{x\}$ and consider the graph $G - S$. We note that $|S| = 2s + 1$ and $G - S$ consists of $\binom{s}{2}$ isolated vertices, namely the vertices $z_{i,j}$ where $1 \leq i \neq j \leq s$. Since $s \geq 6$, we note that $2s + 2 < \binom{s}{2} = \frac{s(s-1)}{2}$. Hence,

$$|S| = 2s + 1 < 2s + 2 < \frac{s(s+1)}{2} = |B_3| = \omega(G - S).$$

Therefore, by Observation 1 the graph G is non-traceable. Thus, $\mathcal{N}(s) \subseteq \mathcal{F}(4, 0)$. This completes the proof of Lemma 6. \square

For $k \geq 4$ and for $\ell \geq 1$, we next give a construction of a $(k + \ell)\text{-}\gamma_c$ -critical graph that contains a graph G in the class $\mathcal{P}(k)$ as an induced subgraph. Let G be a graph in the class $\mathcal{P}(k)$, and let H be a maximal complete subgraph of G having properties (a) and (b) in the construction of the class $\mathcal{P}(k)$. For $\ell \geq 1$, let G_1, \dots, G_ℓ be ℓ vertex disjoint complete graphs where $G_i = K_{n_i}$ and $n_i \geq 1$ for $i \in [\ell]$. Let $G(n_1, n_2, \dots, n_\ell)$ be the graph constructed from an isolated vertex x_0 , vertex disjoint copies of the complete graphs G_1, \dots, G_ℓ and $G \in \mathcal{P}(k)$ by adding edges according to the join operations

$$x_0 \vee G_1 \vee G_2 \vee \dots \vee G_\ell \vee HG.$$

Let $\mathcal{P}(k, \ell)$ be the class of all such graphs $G(n_1, n_2, \dots, n_\ell)$. A graph in the class $\mathcal{P}(k, \ell)$ is illustrated in Figure 9.

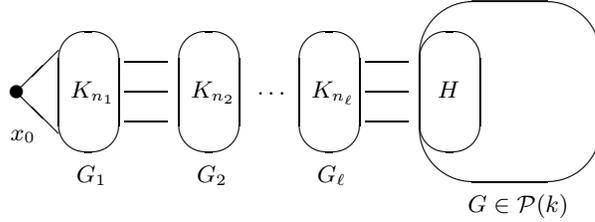


Figure 9: The graph in the class $\mathcal{P}(k, \ell)$

Theorem 8 For $k \geq 4$ and $\ell \geq 1$, every graph in the class $\mathcal{P}(k, \ell)$ is a $(k + \ell)$ - γ_c -critical graph.

Proof. For $k \geq 4$ and $\ell \geq 1$, let $G(n_1, n_2, \dots, n_\ell)$ be a graph in the class $\mathcal{P}(k, \ell)$ that is constructed from a graph $G \in \mathcal{P}(k)$ with H as the maximal complete subgraph of G having properties (a) and (b) in the construction of G . For notation convenience, we write the graph $G(n_1, n_2, \dots, n_\ell)$ simply as $G_{k, \ell}$. Let G_1, \dots, G_ℓ be the ℓ vertex disjoint complete graphs used to construct $G_{k, \ell}$, where $G_i = K_{n_i}$ for $i \in [\ell]$. Let x_i be an arbitrary vertex of G_i for $i \in [\ell]$, and let $X = \{x_1, \dots, x_\ell\}$. Let D be a γ_c -set of G that contains a vertex of H . Thus, $|D| = \gamma_c(G) = k$ and $D \cap V(H) \neq \emptyset$.

We show firstly that $\gamma_c(G_{k, \ell}) = k + \ell$. Since $D \cup X \succ_c G$, we note that $\gamma_c(G_{k, \ell}) \leq |D| + |X| = k + \ell$. To show that $\gamma_c(G_{k, \ell}) \geq k + \ell$, let D' be an arbitrary γ_c -set of $G_{k, \ell}$. If D' contains the vertex x_0 , then since D' is a CD-set of $G_{k, \ell}$ it also contains a vertex of G_1 . If D' does not contain the vertex x_0 , then in order to dominate the vertex x_0 , we note that D' contains a vertex of G_1 . Hence in both cases, $D' \cap V(G_1) \neq \emptyset$. Since $G - H$ is not the empty graph, the set D' contain at least one vertex of G . By the connectedness of $G_{k, \ell}[D']$, the set D' therefore contains at least one vertex from each of the sets G_i for $i \in [\ell]$ and $D' \cap V(H) \neq \emptyset$. Therefore, $|D' \cap (\cup_{i=1}^{\ell} V(G_i))| \geq \ell$, that is, $|D' \cap (V(G_{k, \ell}) \setminus V(G))| \geq \ell$. Since H is a complete subgraph of G and $D' \cap V(H) \neq \emptyset$, we note that the set $D' \cap V(G)$ is a CD-set of G , implying that $|D' \cap V(G)| \geq \gamma_c(G) = k$. Hence, $\gamma_c(G_{k, \ell}) = |D'| = |D' \cap V(G)| + |D' \cap (V(G_{k, \ell}) \setminus V(G))| \geq k + \ell$. Consequently, $\gamma_c(G_{k, \ell}) = k + \ell$.

We establish next the criticality of $G_{k, \ell}$. Let u and v be an arbitrary pair of non-adjacent vertices of $G_{k, \ell}$. We show that there exists a CD-set D_{uv} of $G_{k, \ell} + uv$ such that $|D_{uv}| < k + \ell$. We first consider the case when $\{u, v\} \subseteq V(G_{k, \ell}) \setminus V(G)$. Renaming vertices if necessary, we may assume that $u = x_j$ and $v = x_{j'}$ where $0 \leq j < j' \leq \ell$. Since u and v are non-adjacent vertices of $G_{k, \ell}$, we note that $j + 2 \leq j'$. If $j > 0$, then let $D_{uv} = D \cup (X \setminus \{x_{j+1}\})$. If $j = 0$, then let $D_{uv} = D \cup (X \setminus \{x_1\})$. In both cases, $|D_{uv}| = k + \ell - 1$ and $D_{uv} \succ_c G_{k, \ell} + uv$. Hence, we may assume that at least one of u and v belongs to G , for otherwise the desired result follows. Renaming vertices if necessary, we may assume that $u \in V(G)$. By property (a) in the construction of the graph $G \in \mathcal{P}(4)$, there exists a γ_c -set D_u of G that contains the vertex u and contains a vertex of H .

Suppose next that $v \notin V(G)$, and so $v = x_0$ or $v \in V(G_i)$ for some $i \in [\ell]$. Renaming

vertices if necessary, we may assume that $v = x_i$ for some $i \in [\ell] \cup \{0\}$. Suppose that $u \in V(H)$, implying that $v \neq x_\ell$. If $v = x_0$, then let $D_{uv} = D_u \cup (X \setminus \{x_1\})$. If $v \neq x_0$, then let $D_{uv} = D_u \cup (X \setminus \{x_\ell\})$. In both cases, $|D_{uv}| = k + \ell - 1$ and $D_{uv} \succ_c G_{k,\ell} + uv$. Hence, we may assume that $u \in V(G) \setminus V(H)$. Once again if $v \neq x_0$, then let $D_{uv} = D_u \cup (X \setminus \{x_\ell\})$, and if $v = x_i$ for some $i \in [\ell]$, then let $D_{uv} = D_u \cup (X \setminus \{x_\ell\})$. In both cases, $|D_{uv}| = k + \ell - 1$ and $D_{uv} \succ_c G_{k,\ell} + uv$. Hence, we may assume that $v \in V(G)$. By property (b) in the construction of the graph $G \in \mathcal{P}(4)$, there exists a γ_c -set D'_{uv} of G that contains a vertex of H and such that $|D'_{uv}| \leq k - 1$. In this case, we let $D_{uv} = D'_{uv} \cup X$ and note that $|D_{uv}| \leq |D'_{uv}| + |X| \leq k + \ell - 1$ and $D_{uv} \succ_c G_{k,\ell} + uv$. These observations imply that $G_{k,\ell}$ is a $(k + \ell)$ - γ_c -critical graph. This completes the proof of Theorem 8. \square

We are now ready to establish the realisability of k - γ_c -critical non-traceable graphs containing ζ cut-vertices for all $k \geq 4$ and $0 \leq \zeta \leq k - 4$. Recall that $\mathcal{F}(k, \zeta)$ is the class of k - γ_c -critical graphs with ζ cut-vertices which are non-traceable.

Theorem 9 *For integers $k \geq 1$ and $\zeta \geq 0$, $\mathcal{F}(k, \zeta) \neq \emptyset$ if and only if $k \geq 4$ and $0 \leq \zeta \leq k - 4$.*

Proof. We first show that if $k \geq 4$ and $0 \leq \zeta \leq k - 4$, then $\mathcal{F}(k, \zeta) \neq \emptyset$. In view of Lemma 6, $\mathcal{F}(4, 0) \neq \emptyset$. Thus if $k = 4$ and $\zeta = k - 4 = 0$, then $\mathcal{F}(k, \zeta) \neq \emptyset$. Hence, we may assume that $k \geq 5$, for otherwise the desired result follows. Let $G \in \mathcal{N}(s)$ for some integer $s \geq 6$. Adopting our earlier notation, Lemma 6 yields also that $G \in \mathcal{P}(4)$ where here we take H as the maximal complete subgraph $G[B_2]$ having properties (a) and (b) in the construction of G . For a given $\zeta \in [k - 4] \cup \{0\}$, let $G^* = G(n_1, n_2, \dots, n_{k-4})$ be a graph in the class $\mathcal{P}(4, k - 4)$ that is constructed from the graph G by taking $n_i \geq 2$ for $i \in [k - 4]$ in the case when $\zeta = 0$ and taking $n_i = 1$ for $i \in [\zeta]$ and $n_i \geq 2$ for $i \in [k - 4] \setminus [\zeta]$ in the case when $\zeta \in [k - 4]$ for the $k - 4$ complete graphs $K_{n_1}, K_{n_2}, \dots, K_{n_{k-4}}$ used in the construction of G^* . We note that if $\zeta = 0$, then G^* has no cut-vertex, while if $\zeta \in [k - 4]$, then G^* has exactly ζ cut-vertices, namely the singleton vertices of the ζ complete graphs $K_{n_1}, \dots, K_{n_\zeta}$. In both cases, G^* has exactly ζ cut-vertices. Moreover, by Theorem 8 the graph G^* is a k - γ_c -critical.

We show next that G^* is non-traceable. Adopting our earlier notation used in the construction of the graph $G \in \mathcal{P}(4)$, let $S = B_1 \cup B_2 \cup \{x\}$ and consider the graph $G^* - S$. We note that $|S| = 2s + 1$ and $G^* - S$ consists of $\binom{s}{2}$ isolated vertices, namely the vertices $z_{i,j}$ where $1 \leq i \neq j \leq s$, together with an additional component containing the vertex x_0 and the $k - 4$ complete graphs $K_{n_1}, K_{n_2}, \dots, K_{n_{k-4}}$. Since $s \geq 6$, we note that $2s + 2 < \binom{s}{2} = \frac{s(s-1)}{2}$. Hence,

$$|S| = 2s + 1 < 2s + 2 < \frac{s(s+1)}{2} + 1 = |B_3| + 1 = \omega(G^* - S).$$

Therefore, by Observation 1 the graph G^* is non-traceable, implying that $G^* \in \mathcal{F}(k, \zeta)$. Hence, if $k \geq 4$ and $0 \leq \zeta \leq k - 4$, then $\mathcal{F}(k, \zeta) \neq \emptyset$.

Conversely, suppose that $\mathcal{F}(k, \zeta) \neq \emptyset$. Since every hamiltonian graph is traceable, by Theorem 3, we must have that $k \geq 4$. Theorem 5 implies that $\mathcal{F}(k, \zeta) = \emptyset$ when $\zeta \geq k - 1$. Thus, $\zeta \leq k - 2$. By Theorem 7 also implies that $\mathcal{F}(k, \zeta) = \emptyset$ when $\zeta \in \{k - 3, k - 2\}$. These results imply that $0 \leq \zeta \leq k - 4$ and $k \geq 4$. This completes the proof of Theorem 9. \square

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