# Algorithmic Counting of Zero-Dimensional Finite Topological Spaces With Respect to the Covering Dimension

Rudolf Berghammer<sup>a,\*</sup>, Steffen Börm<sup>b</sup>, Michael Winter<sup>c,1</sup>

<sup>a</sup>Institut für Informatik, Universität Kiel, 24098 Kiel, Germany <sup>b</sup>Mathematisches Seminar, Universität Kiel, 24098 Kiel, Germany <sup>c</sup>Department of Computer Science, Brock University, St. Catharines, ON, Canada

# Abstract

Taking the covering dimension dim as notion for the dimension of a topological space, we first specify the number  $z\dim_{T_0}(n)$  of zero-dimensional  $T_0$ -spaces on  $\{1, ..., n\}$  and the number  $z\dim(n)$  of zero-dimensional arbitrary topological spaces on  $\{1, ..., n\}$  by means of two mappings po and P that yield the number po(n) of partial orders on  $\{1, ..., n\}$  and the set P(n) of partitions of  $\{1, ..., n\}$ , respectively. Algorithms for both mappings exist. Assuming one for po to be at hand, we use our specification of  $z\dim_{T_0}(n)$  and modify one for P in such a way that it computes  $z\dim_{T_0}(n)$  instead of P(n). The specification of  $z\dim(n)$  then allows to compute this number from  $z\dim_{T_0}(1)$  to  $z\dim_{T_0}(n)$  and the Stirling numbers of the second kind S(n, 1) to S(n, n). The resulting algorithms have been implemented in C and we also present results of practical experiments with them. To considerably reduce the running times for computing  $z\dim_{T_0}(n)$ , we also describe a backtracking approach and its parallel implementation in C using the OpenMP library.

*Keywords:* Finite topological space, covering dimension, specialisation pre-order, partial order, partition, backtracking algorithm, parallelisation, GMP, OpenMP

### 1. Introduction

Motivated by application in image processing, in [19] the small inductive dimension ind of Alexandroff  $T_0$ -spaces is investigated. As main result it is shown that the dimension ind(X, T) of an Alexandroff  $T_0$ -space (X, T) equals the height of the specialisation order of  $\mathcal{T}$ . This work is continued in [2, 10, 11, 12, 13, 14, 15], thereby generalising it to Alexandroff spaces which are not  $T_0$ -spaces and to the other two important notions of a dimension in topology. The latter are the large inductive dimension Ind and the covering dimension dim. Especially [2] contains a comprehensive investigation of the dimensions ind, Ind and dim for Alexandroff spaces and finite topological spaces. These dimensions are specified via the specialisation order of the topology of the Kolmogoroff quotient. This leads to algorithms for their computation which are, except that for the dimension Ind, of polynomial order in the size of the carrier set if the specialisation pre-order is taken as input. It also allows to clarify how the dimensions ind, Ind and dim are related. For finite topological spaces further consequences are sharp upper bounds for all three dimensions, characterisations of the maximal-dimensional spaces, how many such spaces exist in case of ind and dim and, if the number of points is odd, in case of Ind and whether they are homeomorphic and/or  $T_0$ -spaces. The same problems are solved for zerodimensional finite spaces, too, except the number of zero-dimensional finite spaces with respect to the dimension dim. Notice, that dim $(X, \mathcal{T}) = 0$  iff  $Ind(X, \mathcal{T}) = 0$ , for all topological spaces  $(X, \mathcal{T})$  Using order- and graph-theoretic means, meanwhile an efficient algorithm for the computation of the dimension Ind has been developed; see [3]. This result solves the first one of the three open problems mentioned above. The present paper treats the third one, the counting of the zero-dimensional finite topological spaces with respect to the dimension dim.

Section 2 presents the mathematical preliminaries. In Section 3 we specify the number  $z\dim_{T_0}(n)$  of zerodimensional (with respect to the dimension dim)  $T_0$ -spaces on the set  $\{1, \ldots, n\}$  by means of two mappings po and

<sup>\*</sup>Corresponding author: Email rub@informatik.uni-kiel.de

<sup>&</sup>lt;sup>1</sup>The author gratefully acknowledges support from the Natural Sciences and Engineering Research Council of Canada (283267).

P that yield the number po(n) of partial orders on  $\{1, ..., n\}$  and the set P(n) of partitions of  $\{1, ..., n\}$ , respectively. In combination with the Stirling numbers of the second kind S(n, 1) to S(n, n), the numbers  $zdim_{T_0}(1)$  to  $zdim_{T_0}(n)$  allow a simple specification of the number zdim(n) of zero-dimensional (with respect to the dimension dim) arbitrary spaces on  $\{1, ..., n\}$ . As Stirling numbers easily can be computed, an algorithm for  $zdim_{T_0}(n)$  immediately leads to an algorithm for zdim(n). The main part of the paper, Sections 4 to 7, is devoted to algorithms for  $zdim_{T_0}(n)$ , their implementation in the programming language C and results of practical experiments.

First, we use our specification of  $zdim_{T_0}(n)$  and modify an algorithm of B. Djokić et al. (published in [4]) in such a way that it computes this number instead of the set P(n). This approach assumes the numbers po(1) to po(n-1)to be at hand. How to count partial orders on  $\{1, \ldots, n\}$  is shown in [8, 9]. There also the values of po(1) to po(18)are given. To our knowledge up to now no number po(n) seems to be computed and published for n > 18. Using the data of [8, 9], we have implemented the modified algorithm in the programming language C. We present results of practical experiments with it for  $n \le 18$ . As for n = 19 we expect a running time of almost two weeks, we have not tried to compute  $zdim_{T_0}(19)$  this way. To considerably reduce the running time of algorithms for  $zdim_{T_0}(n)$  that are based on our specification of  $zdim_{T_0}(n)$  via the numbers po(1) to po(n-1) and the mapping P, the use of a parallel algorithm for P(n) seems to be promising. Such an algorithm is presented in [5], again by B. Djokić et al. In view of easy realisability in C we take another approach. First, we develop a simple recursive backtracking algorithm for P(n). It is very similar to the algorithm published by M.C. Er in [7]. A modification, similar to that of the algorithm for P(n) of [4] to compute  $zdim_{T_0}(n)$ , then leads to a recursive backtracking algorithm for  $zdim_{T_0}(n)$ . Next, we optimise this algorithm by an incremental computation of the auxiliary array introduced in the second step. The algorithm obtained this way easily can be implemented and parallelised in C by means of the OpenMP library. As we will demonstrate, with this C-program  $zdim_{T_0}(n)$  can be obtained significantly faster than with the sequential C-program that implements the above mentioned modification of the algorithm of [4].

# 2. Mathematical Preliminaries

There are some equivalent ways to define topologies; see e.g., [16] for more details. If they are defined by means of open sets, then a subset  $\mathcal{T}$  of the powerset  $2^X$  is a *topology* on X iff  $\emptyset \in \mathcal{T}, X \in \mathcal{T}$ , any union  $\bigcup \mathcal{A}$  of an arbitrary subset  $\mathcal{A}$  of  $\mathcal{T}$  is in  $\mathcal{T}$  and any intersection  $A \cap B$  of sets  $A, B \in \mathcal{T}$  is in  $\mathcal{T}$ . The sets of  $\mathcal{T}$  are defined as *open* and  $(X, \mathcal{T})$  is a *topological space* with carrier set X. Usually carrier sets are assumed to be non-empty and their elements are called *points*. Dimension theory, however, also allows the empty space  $(\emptyset, \{\emptyset\})$ .

In topology certain separation axioms are used to distinguish topological spaces in view of the separation of sets and points. In this paper only the  $T_0$ -axiom is of interest. Suppose  $(X, \mathcal{T})$  to be a topological space. Then the points  $x, y \in X$  are topologically distinguishable iff there exists a set  $A \in \mathcal{T}$  such that  $x \in A$  and  $y \notin A$  or  $x \notin A$  and  $y \in A$ . The topology  $\mathcal{T}$  satisfies the  $T_0$  axiom iff any two distinct points are topologically distinguishable. In this case  $\mathcal{T}$  is a  $T_0$ -topology and  $(X, \mathcal{T})$  is a  $T_0$ -space. To be topologically indistinguishable defines an equivalence relation  $\equiv$  on Xand the quotient space  $(X_{\equiv}, \mathcal{T}_{\equiv})$  is the *Kolmogorov quotient* of  $(X, \mathcal{T})$ . Here  $X_{\equiv}$  is the set of equivalence classes of  $\equiv$ and  $\mathcal{T}_{\equiv} := \{B \in 2^{X_{\equiv}} | \pi^{-1}[B] \in \mathcal{T}\}$  is the quotient topology of  $\mathcal{T}$  with respect to  $\equiv$ , with  $\pi^{-1}[B]$  as the inverse image of the set B under the canonical epimorphism  $\pi : X \to X_{\equiv}, \pi(x) = [x]$ . Kolmogorov quotients are  $T_0$ -spaces.

Given a topological space  $(X, \mathcal{T})$ , another relation on X we will use in this paper is the *specialisation pre-order*  $\leq_{\mathcal{T}}$  of the topology  $\mathcal{T}$ , defined by  $x \leq_{\mathcal{T}} y$  iff for all  $A \in \mathcal{T}$  from  $y \in A$  it follows  $x \in A$ , for all  $x, y \in X$ . This relation is especially of interest in case of *Alexandroff spaces*, that is, in case of topological spaces  $(X, \mathcal{T})$  where not only  $A \cap B \in \mathcal{T}$ , for all  $A, B \in \mathcal{T}$ , but even  $\bigcap \mathcal{R} \in \mathcal{T}$ , for all non-empty subsets  $\mathcal{R}$  of  $\mathcal{T}$ . (Such a topology is an *Alexandroff topology* on X.) The importance of the specialisation pre-order with regard to Alexandroff spaces is mainly due to the following property, shown in [1]: For all sets X the mapping  $\mathcal{T} \mapsto \leq_{\mathcal{T}}$ , that maps an Alexandroff topology  $\mathcal{T}$ on X to its specialisation pre-order  $\leq_{\mathcal{T}}$ , is a bijection between the set of Alexandroff topologies on X and the set of pre-orders on X and the Alexandroff space  $(X, \mathcal{T})$  is a  $T_0$ -space iff  $(X, \leq_{\mathcal{T}})$  is a partially ordered set. In [2] it is shown that, if  $(X, \mathcal{T})$  is an Alexandroff space,  $(X_{\equiv}, \mathcal{T}_{\equiv})$  is its Kolmogorov quotient,  $\leq_{\mathcal{T}}$  is the specialisation pre-order of  $\mathcal{T}$ and  $\leq_{\mathcal{T}_{\equiv}}$  is the specialisation order of  $\mathcal{T}_{\equiv}$ , then  $x \leq_{\mathcal{T}} y$  iff  $[x] \leq_{\mathcal{T}_{\equiv}} [y]$ , for all  $x, y \in X$ . I.e., the partial order  $\leq_{\mathcal{T}_{\equiv}}$  is the quotient order of the pre-order  $\leq_{\mathcal{T}}$  with respect to the equivalence relation  $\equiv$  of topological indistinguishability. Finite topological spaces are Alexandroff spaces and for computational problems on finite spaces usually the specialisation pre-order is taken as input. See e.g., [2, 12, 14, 15] for such problems with regard to dimension theory. Let  $(X, \mathcal{T})$  be a topological space. A *finite open covering* of  $(X, \mathcal{T})$  is a finite subset  $\mathcal{A}$  of  $\mathcal{T}$  with  $\bigcup \mathcal{A} = X$ . The finite open covering  $\mathcal{A}$  of  $(X, \mathcal{T})$  is *finer* than the finite open covering  $\mathcal{B}$  of  $(X, \mathcal{T})$  iff for all  $A \in \mathcal{A}$  there exists  $B \in \mathcal{B}$  such that  $A \subseteq B$ . The *order*  $\operatorname{ord}(\mathcal{A})$  of a finite open covering  $\mathcal{A}$  of  $(X, \mathcal{T})$  is defined as the largest integer  $n \in \mathbb{N} \cup \{-1\}$  such that  $\mathcal{A}$  contains n + 1 distinct sets with a non-empty intersection. Hence,  $\operatorname{ord}(\mathcal{A}) = -1$  iff  $\mathcal{A} = \{\emptyset\}$ , such that  $(X, \mathcal{T})$  is the empty space in this case. For  $n \in \mathbb{N}$  it holds  $\operatorname{ord}(\mathcal{A}) = n$  iff there exist distinct sets  $A_1, \ldots, A_{n+1} \in \mathcal{A}$  with  $\bigcap_{i=1}^{n+1} A_i \neq \emptyset$  and  $\bigcap_{i=1}^{n+2} B_i = \emptyset$ , for all distinct sets  $B_1, \ldots, B_{n+2} \in \mathcal{A}$ . Hence, in this case the space  $(X, \mathcal{T})$  is non-empty.

Using finite open coverings, to every topological space  $(X, \mathcal{T})$  the *covering dimension* dim $(X, \mathcal{T}) \in \mathbb{N} \cup \{-1, \infty\}$  is assigned by the following rules:

- a) dim $(X, \mathcal{T}) \leq n$ , where  $n \in \mathbb{N} \cup \{-1\}$ , iff for all finite open coverings  $\mathcal{A}$  of  $(X, \mathcal{T})$  there exists a finite open covering  $\mathcal{B}$  of  $(X, \mathcal{T})$  such that  $\mathcal{B}$  is finer than  $\mathcal{A}$  and  $\operatorname{ord}(\mathcal{B}) \leq n$ .
- b)  $\dim(X, \mathcal{T}) = n$ , where  $n \in \mathbb{N}$ , iff  $\dim(X, \mathcal{T}) \le n$  and not  $\dim(X, \mathcal{T}) \le n 1$ .
- c) dim(*X*,  $\mathcal{T}$ ) =  $\infty$  iff there exists no  $n \in \mathbb{N} \cup \{-1\}$  such that dim(*X*,  $\mathcal{T}$ )  $\leq n$

In this definition, as in [6], implicitly the equivalence of  $\dim(X, \mathcal{T}) \leq -1$  and  $\dim(X, \mathcal{T}) = -1$  is assumed such that  $\dim(X, \mathcal{T}) = -1$  iff  $(X, \mathcal{T})$  is the empty space.

#### 3. Specifying the Number of Zero-Dimensional Finite Topological Spaces with Respect to Dimension dim

Given  $n \in \mathbb{N}_{>0}$ , where  $\mathbb{N}_{>0} := \{k \in \mathbb{N} \mid k > 0\}$ , we denote by  $\operatorname{zdim}_{T_0}(n)$  the number of  $T_0$ -topologies  $\mathcal{T}$  on the set  $\{1, \ldots, n\}$  such that  $\dim(\{1, \ldots, n\}, \mathcal{T}) = 0$  and by  $\operatorname{zdim}(n)$  the number of arbitrary topologies  $\mathcal{T}$  on  $\{1, \ldots, n\}$  such that  $\dim(\{1, \ldots, n\}, \mathcal{T}) = 0$ . Our ultimative goal is the computation of the numbers  $\operatorname{zdim}_{T_0}(n)$  and  $\operatorname{zdim}(n)$ . Corresponding algorithms are presented in Sections 4 and 6. Sections 5 and 7 discuss their implementations in C and present experimental results. In this section we develop specifications of  $\operatorname{zdim}_{T_0}(n)$  and  $\operatorname{zdim}(n)$  on which the algorithms are based upon. Decisive for our approach is the following characterisation of zero-dimensionality with respect to the dimension dim, which is proved in [2].

**Theorem 3.1.** Assume  $(X, \mathcal{T})$  to be a finite and non-empty topological space and  $\leq_{\mathcal{T}_{\equiv}}$  to be the specialisation order of the topology  $\mathcal{T}_{\equiv}$  of the Kolmogorov quotient  $(X_{\equiv}, \mathcal{T}_{\equiv})$ . Then dim $(X, \mathcal{T}) = 0$  iff  $(X_{\equiv}, \leq_{\mathcal{T}_{\equiv}})$  is the disjoint union of partially ordered sets with greatest elements.

In other words, dim $(X, \mathcal{T}) = 0$  iff there exist a partition  $\{A_1, \ldots, A_k\}$  of the set  $X_{\equiv}$  and partial orders  $\leq_i$  on the sets  $A_i$ , for all  $i \in \{1, \ldots, k\}$ , such that each partially ordered set  $(A_i, \leq_i)$  has a greatest element and  $\leq_{\mathcal{T}_{\equiv}} = \bigcup_{i=1}^k \leq_i$ . If  $(X, \mathcal{T})$  is a  $T_0$ -space, then dim $(X, \mathcal{T}) = 0$  iff there exist a partition  $\{A_1, \ldots, A_k\}$  of the set X and partial orders  $\leq_i$  on the sets  $A_i$ , for all  $i \in \{1, \ldots, k\}$ , such that each partially ordered set  $(A_i, \leq_i)$  has a greatest element and  $\leq_{\mathcal{T}} = \bigcup_{i=1}^k \leq_i$ .

Because of Theorem 3.1 partitions of  $\{1, ..., n\}$  and partially ordered sets with greatest elements play a decisive role with regard to our problems. To this end, for a given  $n \in \mathbb{N}_{>0}$  we denote by P(n) the set of partitions of the set  $\{1, ..., n\}$  and by  $po^*(n)$  the number of partial orders  $\leq$  on  $\{1, ..., n\}$  such that  $(\{1, ..., n\}, \leq)$  has a greatest element. In [8] the counting of finite partial orders on the set  $\{1, ..., n\}$  is investigated and, with  $P_n$  as number of partial orders on  $\{1, ..., n\}$ , the numbers  $P_1$  to  $P_{14}$  are given (see also Section 4). We use the notation po(n) instead of  $P_n$ . The following lemma shows how the values of the mapping  $po^*$  can be obtained from the values of the mapping po. In the proof we use a specific operation on partial orders  $\leq$ , viz.  $\leq_{a,b}$ , where *a* and *b* are elements of the carrier set of  $\leq$ . The partial order  $\leq_{a,b}$  is obtained from the partial order  $\leq$  by interchanging *a* and *b*, i.e., by a (simultaneous) replacement of each pair (a, x) by (b, x), of each pair (x, a) by (x, b), of each pair (b, x) by (a, x) and of each pair (x, b) by (x, a). Furthermore, we apply the restriction of a partial order  $\leq$  to a subset *A* of its carrier set, with the usual notation  $\leq_{|A|}$ .

**Lemma 3.1.** We have  $po^*(n + 1) = (n + 1)po(n)$ , for all  $n \in \mathbb{N}_{>0}$ .

**Proof:** To prove estimate " $\geq$ ", consider an arbitrary partial order  $\leq$  on  $\{1, ..., n\}$ . Then  $\leq^* := \leq \cup (\{1, ..., n+1\} \times \{n+1\})$  defines a partial order on  $\{1, ..., n+1\}$ . It is obtained from  $\leq$  by adding n + 1 as greatest element. Furthermore, for all  $x \in \{1, ..., n\}$  by  $\leq_x := (\leq^*)_{x,n+1}$  we get partial orders on  $\{1, ..., n+1\}$  with greatest elements x and it holds  $\leq_x \neq \leq_y$ , for all  $x, y \in \{1, ..., n\}$  with  $x \neq y$ . Hence, from the partial order  $\leq$  on  $\{1, ..., n\}$  we obtain the n + 1 different partial orders  $\leq^*, \leq_1, ..., \leq_n$  on  $\{1, ..., n+1\}$  such that in each case  $\{1, ..., n+1\}$  has a greatest element. It is easy to verify

that for two different partial orders  $\leq$  and  $\sqsubseteq$  on  $\{1, \ldots, n\}$  the partial orders  $\leq^*$  and  $\sqsubseteq^*$  are different. As a consequence, for all  $x \in \{1, \ldots, n\}$  the partial orders  $\leq_x$  and  $\sqsubseteq_x$  are different, too. So, the po(*n*) partial orders on  $\{1, \ldots, n\}$  lead to (n + 1)po(*n*) partial orders on  $\{1, \ldots, n + 1\}$  with the additional property that  $\{1, \ldots, n + 1\}$  has a greatest element. This implies po\* $(n + 1) \geq (n + 1)$ po(*n*).

To show " $\leq$ " we use contradiction. Assume  $po^*(n + 1) > (n + 1)po(n)$ . Because of the proof of " $\geq$ ", there exists a partial order  $\sqsubseteq$  on  $\{1, \ldots, n + 1\}$  such that  $\{1, \ldots, n + 1\}$  has a greatest element with respect to  $\sqsubseteq$  but  $\sqsubseteq$  is not obtained from a partial order  $\leq$  on  $\{1, \ldots, n\}$  by means of  $\sqsubseteq = \leq^*$  or  $\sqsubseteq = \leq_x$ , where  $x \in \{1, \ldots, n\}$ . Then n + 1 is not the greatest element of  $\{1, \ldots, n + 1\}$  with respect to  $\sqsubseteq$  since, otherwise,  $\leq := \bigsqcup_{i=1,\ldots,n}$  would lead to a partial order  $\leq$  on  $\{1, \ldots, n\}$  with  $\sqsubseteq = \leq^*$ , i.e., to a contradiction. Let  $a \in \{1, \ldots, n\}$  be the greatest element of  $\{1, \ldots, n + 1\}$  with respect to  $\sqsubseteq$ . Then  $\leq := (\bigsqcup_{n+1,a})_{i=1,\ldots,n}$  leads to a partial order  $\leq$  on  $\{1, \ldots, n\}$  with  $\sqsubseteq = (\leq^*)_{a,n+1} = \leq_a$ . This is again a contradiction.

Using this lemma, we are able to specify for all  $n \in \mathbb{N}_{>0}$  the number of  $T_0$ -topologies  $\mathcal{T}$  on the set  $\{1, \ldots, n\}$  with the property dim $(\{1, \ldots, n\}, \mathcal{T}) = 0$  by means of the two mappings P and po as follows, where the sets  $\mathcal{P}_{>1}$  are defined as  $\mathcal{P}_{>1} := \{A \in \mathcal{P} \mid |A| > 1\}$ , for all  $\mathcal{P} \in \mathsf{P}(n)$ .

**Theorem 3.2.** We have  $\operatorname{zdim}_{T_0}(n) = \sum_{\mathcal{P} \in \mathsf{P}(n)} \prod_{A \in \mathcal{P}_{>1}} |A| \operatorname{po}(|A| - 1)$ , for all  $n \in \mathbb{N}_{>0}$ .

**Proof:** We denote the specialisation order of a  $T_0$ -topology  $\mathcal{T}$  again as  $\leq_{\mathcal{T}}$ . From Section 2 we then know that  $\mathcal{T} \mapsto \leq_{\mathcal{T}}$  establishes a 1-1-correspondence between the  $T_0$ -topologies on  $\{1, \ldots, n\}$  and the partial orders on  $\{1, \ldots, n\}$ . Because of this and Theorem 3.1, the number  $\operatorname{zdim}_{T_0}(n)$  equals the number N(n) of sets of pairs  $\{(A_1, \leq_1), \ldots, (A_k, \leq_k)\}$  with the following two properties:

- a) The set  $\{A_1, \ldots, A_k\}$  is a partition of the set  $\{1, \ldots, n\}$ .
- b) Each pair  $(A_i, \leq_i)$ , where  $i \in \{1, \dots, k\}$ , constitutes a partially ordered set with a greatest element.

We have  $N(n) = \sum_{\mathcal{P} \in \mathsf{P}(n)} \prod_{A \in \mathcal{P}} \mathsf{po}^*(|A|)$ , since for all partitions  $\{A_1, \ldots, A_k\}$  of the set  $\{1, \ldots, n\}$  there are precisely  $\prod_{i=1}^k \mathsf{po}^*(|A_i|)$  possibilities to select a set of partial orders  $\{\leq_1, \ldots, \leq_k\}$  such that each pair  $(A_i, \leq_i)$ , where  $i \in \{1, \ldots, k\}$ , constitutes a partially ordered set with a greatest element. This yields:

$$\mathsf{zdim}_{T_0}(n) = N(n) = \sum_{\mathcal{P} \in \mathsf{P}(n)} \prod_{A \in \mathcal{P}} \mathsf{po}^*(|A|) \tag{1}$$

Using  $po^*(1) = 1$  and Lemma 3.1, for an arbitrary  $\mathcal{P} \in P(n)$  we get:

$$\prod_{A \in \mathcal{P}} \mathsf{po}^*(|A|) = \prod_{A \in \mathcal{P}_{>1}} \mathsf{po}^*(|A|) = \prod_{A \in \mathcal{P}_{>1}} |A| \operatorname{po}(|A| - 1)$$

In combination with equation (1) this yields the desired result.

As next result we show how the two mappings P and  $z\dim_{T_0}$  can be used to specify for all  $n \in \mathbb{N}_{>0}$  the number  $z\dim(n)$ , that is, the number of arbitrary topologies  $\mathcal{T}$  on the set  $\{1, \ldots, n\}$  such that  $\dim(\{1, \ldots, n\}, \mathcal{T}) = 0$ .

**Theorem 3.3.** We have  $\operatorname{zdim}(n) = \sum_{\mathcal{P} \in \mathsf{P}(n)} \operatorname{zdim}_{T_0}(|\mathcal{P}|)$ , for all  $n \in \mathbb{N}_{>0}$ .

**Proof:** Because of the characterisation of zero-dimensionality with respect to the dimension dim given in Theorem 3.1, the number zdim(n) equals the number of partially ordered sets  $(\{1, ..., n\}/\equiv, \leq)$  such that the following two properties hold:

- a) Each relation  $\equiv$  is an equivalence relation on the set  $\{1, ..., n\}$  (the classes of which consist of the topologically indistinguishable points).
- b) Each partially ordered set  $(\{1, ..., n\} / \equiv, \leq)$  is the disjoint union of partially ordered sets with greatest elements.

If we use partitions instead of equivalence relations, then we get zdim(n) as the number of partially ordered sets  $(\mathcal{P}, \leq)$  such that  $\mathcal{P} \in \mathsf{P}(n)$  and each  $(\mathcal{P}, \leq)$  is the disjoint union of partially ordered sets with greatest elements. This yields:

$$z\dim(n) = \sum_{\mathcal{P} \in \mathsf{P}(n)} N(|\mathcal{P}|)$$
<sup>(2)</sup>

As in the proof of Theorem 3.2 in (2) by  $N(|\mathcal{P}|)$  we denote the number of sets of pairs  $\{(A_1, \leq_1), \ldots, (A_k, \leq_k)\}$  such that the following two properties hold:

- c) The set  $\{A_1, \ldots, A_k\}$  is a partition of the set  $\{1, \ldots, |\mathcal{P}|\}$ .
- d) Each pair  $(A_i, \leq_i)$ , where  $i \in \{1, \ldots, k\}$ , constitutes a partially ordered set with a greatest element.

Similar to the proof of Theorem 3.2 we can show that  $\mathsf{zdim}_{T_0}(|\mathcal{P}|) = N(|\mathcal{P}|)$ , for all  $\mathcal{P} \in \mathsf{P}(n)$ . Together with equation (2) this yields the desired result.

For all  $n \in \mathbb{N}_{>0}$  and  $i \in \{1, ..., n\}$  we define  $\mathsf{P}_i(n) := \{\mathcal{P} \in \mathsf{P}(n) \mid |\mathcal{P}| = i\}$  and  $S(n, i) := |\mathsf{P}_i(n)|$ . The numbers S(n, i) are called the *Stirling numbers of the second kind*. With these notions we obtain from Theorem 3.3 the following specification of the numbers  $\mathsf{zdim}(n)$ , which avoids the use of the sets  $\mathsf{P}(n)$ .

**Corollary 3.1.** We have  $\operatorname{zdim}(n) = \sum_{i=1}^{n} S(n, i) \operatorname{zdim}_{T_0}(i)$ , for all  $n \in \mathbb{N}_{>0}$ .

**Proof:** Via Theorem 3.3 and the definitions of the sets  $P_i(n)$  and the Stirling numbers of the second kind S(n, i) we obtain the claim as follows.

$$\operatorname{zdim}(n) = \sum_{\mathcal{P} \in \mathsf{P}(n)} \operatorname{zdim}_{T_0}(|\mathcal{P}|) = \sum_{i=1}^n \sum_{\mathcal{P} \in \mathsf{P}_i(n)} \operatorname{zdim}_{T_0}(|\mathcal{P}|) = \sum_{i=1}^n S(n,i) \operatorname{zdim}_{T_0}(i) \qquad \Box$$

#### 4. Iterative Algorithms for zdim(n) and $zdim_{T_0}(n)$

In this section, we mainly concentrate on the development of an (iterative, sequential) algorithm  $zdim_{T_0}(n)$  for computing the number  $zdim_{T_0}(n)$ . Having such an algorithm at hand and also an integer array *S* such that *S*[*i*] contains the Stirling number *S*(*n*, *i*), for all  $i \in \{1, ..., n\}$ , from Corollary 3.1 we then immediately obtain the following simple algorithm for computing the number zdim(n).

$$zdim(n)$$
  
 $z := 0;$   
**for**  $i = 1$  **to**  $n$  **do**  
 $z := z + S[i] zdim_{T_0}(i)$  **od**;  
**return**  $z;$   
**end**;

The computation of the array *S* used in algorithm zdim(n) is rather easy. This is due to the fact that the Stirling numbers of the second kind obey the recursive equation S(n, i) = iS(n - 1, i) + S(n - 1, i - 1), for all  $n \in \mathbb{N}_{>1}$  and  $i \in \{2, ..., n\}$ , with the initial values S(n, 1) = 1 and S(n, n) = 1, for all  $n \in \mathbb{N}_{>0}$ . As a consequence, they can be computed very efficiently by means of a triangle array. This algorithm is quite similar to the computation of the binomial coefficients using the well-known Pascal triangle. The array *S* used in zdim(n) consists of the *n*-th row of the triangle array.

For the development of the algorithm  $zdim_{T_0}(n)$  we assume an algorithm for the mapping po of Section 3 to be at hand, that returns for the input  $n \in \mathbb{N}_{>0}$  the number of partial orders on the set  $\{1, ..., n\}$ . In the concrete realisation of our algorithms in C (which we will discuss in Sections 5 and 7) we have used the table of Figure 1, where the data po(n) for  $1 \le n \le 14$  are taken from [8] and for  $15 \le n \le 18$  they are taken from [9].

Next, we consider the generation of all partitions of the set  $\{1, ..., n\}$ , where  $n \in \mathbb{N}_{>0}$ . During the last years some set-partition generation algorithms have been presented, see, e.g., [4, 7, 17, 18]. Such algorithms usually do not compute the set P(n) directly, but instead of that the set of the corresponding *codewords*, also known as *restricted growth strings*. The codeword corresponding to a partition  $\{A_1, ..., A_k\}$  of the set  $\{1, ..., n\}$  is a vector  $c := (c_1, ..., c_n) \in \{1, ..., n\}^n$  such that  $c_i = j$  iff  $i \in A_j$ , for all  $i \in \{1, ..., n\}$  and  $j \in \{1, ..., k\}$ .

The table of Figure 2 shows in the second column all 15 partitions of the set {1, 2, 3, 4} and in the third column the corresponding 15 codewords. Besides these data, we additionally show in the last column for each codeword *c* a corresponding vector  $d = (d_1, d_2, d_3, d_4)$  such that  $d_i$  equals the number of occurrences of *i* in the codeword *c*, for all  $i \in \{1, 2, 3, 4\}$ . Generalising the example to arbitrary  $n \in \mathbb{N}_{n>0}$ , we will use the auxiliary vectors  $d \in \{0, \dots, n\}^n$  later to get from the codewords  $c \in \{1, \dots, n\}^n$  of the partitions from P(n) the number  $\operatorname{zdim}_{T_0}(n)$ . Notice that the codewords *c* and the corresponding vectors *d* depend on the order of the sets of the partitions. If, for instance, the second partition

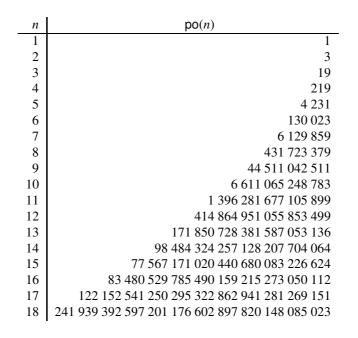


Figure 1: Number of partial orders on the set  $\{1, \ldots, n\}$ .

of the table of Figure 2 is written as  $\{\{4\}, \{1, 2, 3\}\}$ , then its codeword changes to c = (2, 2, 2, 1) and the corresponding vector to d = (1, 3, 0, 0).

In [4] the following iterative algorithm *setpart1(n)* for the generation of the codewords of the partitions of the set  $\{1, ..., n\}$  is presented, where *c* and *g* are two arrays with 0, ..., n as indices and integers as values, *calc* is a Boolean variable and *r* and *j* are two integer variables. For n > 1 each run through the **repeat**-loop generates a codeword as part (*c*[1],...,*c*[*n*]) of the array *c* and prints it. The algorithm of [4] contains two flaws and, strictly speaking, the algorithm *setpart1(n)* given below is not identical to the algorithm *setpart1(n)* of [4] but a (slightly) refined version of it. The first flaw of the original algorithm is that it works only for inputs n > 1 (and does not terminate for n = 1). In our version this is corrected via a conditional. The second flaw of the original algorithm is that it prints the first generated codeword (1, 1, ..., 1) (for the partition  $\{\{1, 2, ..., n\}\}$ ) twice. In our version this is corrected by the use of the Boolean variable *calc*.

```
setpart1(n)
 calc := false;
   if n = 1 then print (1)
           else r := 0; c[0] := 0; g[0] := 0;
                 repeat while r < n - 1 do
                           r := r + 1; c[r] := 1; g[r] := g[r - 1] od;
                         for i = 1 to g[n-1] + 1 do
                           c[n] := j;
                           if calc then print (c[1], ..., c[n]) fi;
                           calc := true od;
                         while c[r] > g[r-1] do
                           r := r - 1 od;
                         c[r] := c[r] + 1;
                         if c[r] > g[r] then g[r] := c[r] fi;
                 until r = 1 fi;
   end:
```

chu,

Using the specification of  $zdim_{T_0}(n)$  given in Theorem 3.2, it is easy to modify the above algorithm setpart1(n)

	Partition $\mathcal{P}$	Codeword c	Vector d	
1	{ {1,2,3,4} }	(1,1,1,1)	(4,0,0,0)	
2	{ {1,2,3},{4} }	(1,1,1,2)	(3,1,0,0)	
3	{ {1,2,4},{3} }	(1,1,2,1)	(3,1,0,0)	
4	{ {1,2},{3,4} }	(1,1,2,2)	(2,2,0,0)	
5	$\{ \{1,2\}, \{3\}, \{4\} \}$	(1,1,2,3)	(2,1,1,0)	
6	{ {1,3,4},{2} }	(1,2,1,1)	(3,1,0,0)	
7	{ {1,3},{2,4} }	(1,2,1,2)	(2,2,0,0)	
8	$\{ \{1,3\},\{2\},\{4\} \}$	(1,2,1,3)	(2,1,1,0)	
9	{ {1,4},{2,3} }	(1,2,2,1)	(2,2,0,0)	
10	{ {1},{2,3,4} }	(1,2,2,2)	(1,3,0,0)	
11	{ {1},{2,3},{4} }	(1,2,2,3)	(1,2,1,0)	
12	$\{ \{1,4\},\{2\},\{3\} \}$	(1,2,3,1)	(2,1,1,0)	
13	{ {1},{2,4},{3} }	(1,2,3,2)	(1,2,1,0)	
14	{ {1},{2},{3,4} }	(1,2,3,3)	(1,1,2,0)	
15	$\{ \{1\}, \{2\}, \{3\}, \{4\} \}$	(1,2,3,4)	(1,1,1,1)	

Figure 2: Partitions of the set {1, 2, 3, 4} with codewords and auxiliary vectors.

in such a way that, instead of printing the codewords corresponding to the elements of P(n) one after the other, the number  $zdim_{T_0}(n)$  is returned. This is the place where the auxiliary vectors *d* come into the play, such that, when these are implemented via an auxiliary array *d* with 1,..., *n* as indices and integers as values, the formula

$$\forall i \in \{1, \dots, n\} : d[i] = |\{k \in \{1, \dots, n\} \mid c[k] = i\}|$$
(3)

is an invariant of the **repeat**-loop. This means that during the execution of the **repeat**-loop the component d[i] equals the number of occurrences of *i* in the part (c[1], ..., c[n]) of *c*, for all  $i \in \{1, ..., n\}$ . Here is the complete algorithm.

$$zdim_{T_0}(n)$$

$$calc := false;$$
if  $n = 1$  then return 1  
else  $z := 0; r := 0; c[0] := 0; g[0] := 0;$   
repeat while  $r < n - 1$  do  
 $r := r + 1; c[r] := 1; g[r] := g[r - 1]$  od;  
for  $j = 1$  to  $g[n - 1] + 1$  do  
 $c[n] := j;$   
if calc then for  $i = 1$  to  $n$  do  
 $d[i] := 0$  od;  
for  $i = 1$  to  $n$  do  
 $d[c[i]] := d[c[i]] + 1$  od;  
 $z := z + \prod_{d[i]>1} d[i] \operatorname{po}(d[i] - 1)$  fi;  
 $calc := true$  od;  
while  $c[r] > g[r - 1]$  do  
 $r := r - 1$  od;  
 $c[r] := c[r] + 1;$   
if  $c[r] > g[r]$  then  $g[r] := c[r]$  fi;  
until  $r = 1;$   
return  $z$  fi;

end;

In this algorithm the computation of the auxiliary array d from the array c is done in lines 9 to 12. If at this point of the algorithm the part  $(c[1], \ldots, c[n])$  stores the codeword  $c = (c_1, \ldots, c_n)$  for the partition  $\{A_1, \ldots, A_k\}$  of  $\{1, \ldots, n\}$ ,

n	$zdim_{T_0}(n)$	Time (ulli)	Time (GMP)
1	1	0.000	0.000
2	3	0.000	0.000
3	16	0.000	0.000
4	137	0.000	0.000
5	1 826	0.000	0.000
6	37 777	0.000	0.000
7	1 214 256	0.000	0.000
8	60 075 185	0.000	0.001
9	4 484 316 358	0.003	0.007
10	493 489 876 721	0.016	0.023
11	78 456 654 767 756	0.054	0.102
12	17 735 173 202 222 665	0.263	0.564
13	5 630 684 018 989 523 274	1.872	3.748
14	2 486 496 790 249 207 894 159		27.303
15	1 515 191 575 312 017 424 784 521		233.218
16	1 265 630 395 473 933 567 972 009 297		1916.205
17	1 440 898 175 760 773 111 084 979 329 715		14 228.470
18	2 224 880 834 303 273 680 055 277 143 713 603		122918.117

Figure 3: Number of zero-dimensional  $T_0$  spaces on the set  $\{1, \ldots, n\}$ .

then via the loops of lines 9 to 12 the invariant (3) is maintained and  $(d[1], \ldots, d[n])$  stores the auxiliary vector  $d = (d_1, \ldots, d_n)$ . Hence,  $d[i] = |A_i|$ , for all  $i \in \{1, \ldots, k\}$ , and d[i] = 0, for all  $i \in \{k + 1, \ldots, n\}$ . A consequence of this and Lemma 3.1 is

$$\prod_{A \in \mathcal{P}_{>1}} \mathsf{po}^*(|A|) = \prod_{d[i]>1} \mathsf{po}^*(d[i]) = \prod_{d[i]>1} d[i] \,\mathsf{po}(d[i]-1),$$

where in the second and third product the notation means that the variable *i* only ranges over the elements of the set  $\{1, ..., n\}$  with a *d*-value greater than 1. So, to get the desired result  $z\dim_{T_0}(n)$ , only all products  $\prod_{d[i]>1} d[i] \operatorname{po}(d[i]-1)$  computed during the run through the **repeat**-loop have to be added. In the algorithm  $z\dim_{T_0}(n)$  above this is realised via an integer variable *z*, its initialisation by 0 in line 4 and its update in line 13.

#### 5. Sequential Implementation and Experimental Results

As already mentioned, we have realised our algorithms in C. Both,  $zdim_{T_0}(n)$  and zdim(n), have been implemented in two versions. The first C-implementations use the pre-defined data type unsigned long long int of C. Since normally this data type ranges from 0 to 18 446 744 073 709 551 615, with these C-programs only inputs up to n = 13are possible. For larger inputs we have implemented the algorithms by means of GMP, the *GNU Multiple Precision Arithmetic library*, and its data type mpz\_t for integers (cf. [20] for details on GMP). In the remainder of the section we present results of practical experiments with these C-programs. All tests have been performed on a computer with two CPUs of type Intel<sup>®</sup> Xeon<sup>®</sup> Gold 6242, each with 2.80 GHz base frequency and 16 cores, 1.5 TByte RAM and running Ubuntu 18.04.3 LTS. The C-programs described in this section are available via [22].

In the table of Figure 3 the results of our experiments with the two C-implementations of the algorithm  $zdim_{T_0}(n)$  are presented up to n = 18. The second column of the table shows the numbers  $zdim_{T_0}(n)$ , where  $1 \le n \le 18$ , the third one shows the corresponding running times (in seconds) for the C-implementation using the data type unsigned long long int and the fourth one shows the corresponding running times (again in seconds) for the C-implementation using the GMP library.

For gaining efficiency, in both C-implementations of the algorithm  $zdim_{T_0}(n)$  we work with the mapping po<sup>\*</sup> instead of the mapping po and represent po<sup>\*</sup> by a global array *P*. Concretely, before calling the C-function for  $zdim_{T_0}(n)$  in the C-function main the numbers po<sup>\*</sup>(1) to po<sup>\*</sup>(13) (respectively po<sup>\*</sup>(1) to po<sup>\*</sup>(18) in case of the GMP-version) are stored in the integer array *P*, where a small auxiliary C-program has been used to obtain the numbers

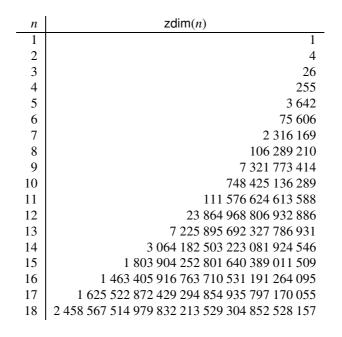


Figure 4: Number of zero-dimensional spaces on the set  $\{1, \ldots, n\}$ .

 $po^*(i)$  from the data of Figure 1. To avoid an index transformation, we start with array index 1 and store the value of  $po^*(i)$  in P[i], for all array indices i > 0. Thereby the expression  $\prod_{d[i]>1} d[i] po(d[i] - 1)$  of the algorithm  $zdim_{T_0}(n)$  reduces to  $\prod_{d[i]>1} P[d[i]]$ . In both C-programs the latter expression is then computed via a simple loop from 1 to n and a conditional assignment as its body.

We also have experimented with a loop without a conditional by considering the expression  $\prod_{i=1}^{n} P[d[i]]$ , where additionally to the above initialisation the array component P[0] is initialised as 1. Due to a lot of (unnecessary) multiplications with 1 this, however, had led to larger running times, in case of the GMP-version up to 20% larger than the times of the table of Figure 3.

Of course, the use of the GMP library makes the C-programs considerably slower. This can be seen by comparing the third and fourth column of the table of Figure 3. From these columns it also can be seen that the running time of the GMP-version for input n + 1 is about 8 times the running time for input n. As a consequence, for computing the number  $zdim_{T_0}(19)$  by means of the above mentioned computer and the C-program using the GMP library, we expect a running time of 12 to 13 days.

The table of Figure 4 shows in the second column the numbers zdim(n) up to n = 18. (In [2] already zdim(1) to zdim(7) are given, computed via a relation-algebraic approach and by means of the Kiel ReLVIEW tool.) For gaining efficiency, in both C-implementations of the algorithm zdim(n) we use the same technique as in case of the C-implementations of the algorithm  $zdim_{T_0}(n)$  and store the results of  $zdim_{T_0}(1)$  to  $zdim_{T_0}(13)$  (respectively  $zdim_{T_0}(1)$  to  $zdim_{T_0}(18)$  in case of the GMP-version) in an array such that its *i*-th component directly yields the corresponding number of the table of Figure 3. With this array-implementation we have been able to perform the C-program for zdim(n) up to n = 13 (respectively up to n = 18 in case of the GMP-version) in 0.0001 seconds. But, of course, the entire running time for getting the number zdim(n) consists of the sum of the times of the third/fourth column of the table of Figure 3 up to n, plus 0.0001. For example, in case of n = 14 this leads to 69.341 seconds as total running time for getting zdim(n).

# 6. A Recursive Algorithm for $zdim_{T_0}(n)$

Let C(n) denote the set of codewords for the partitions of the set P(n), for all  $n \in \mathbb{N}_{>0}$ . Based on recursive specifications of the sets P(n) and C(n), in this section we first develop a simple recursive backtracking algorithm for the computation of the set C(n). With modifications quite similar to those of the algorithm *setpart1(n)* of Section 4

to obtain the algorithm  $zdim_{T_0}(n)$ , this leads to a recursive backtracking algorithm for the computation of the number  $zdim_{T_0}(n)$ . Finally, we optimise this algorithm by the incremental computation of the array, where the auxiliary vectors d of the codewords c are stored.

The theoretical background of the approach is Theorem 6.1 below. In it, for all  $n \in \mathbb{N}_{>1}$ , partitions  $\mathcal{P} :=$  $\{A_1,\ldots,A_k\} \in \mathsf{P}(n-1) \text{ and } i \in \{1,\ldots,k\}$  we denote the replacement of the set  $A_i$  in  $\mathcal{P}$  by  $A_i \cup \{n\}$  as  $\mathcal{P} \oplus_i n$  and the insertion of the singleton set  $\{n\}$  into  $\mathcal{P}$  as  $\mathcal{P} \oplus \{n\}$ . For instance, in case n = 5 and  $\mathcal{P} := \{\{1, 4\}, \{2\}, \{3\}\} \in \mathsf{P}(4)$  we get  $\mathcal{P} \oplus_3 5 = \{\{1, 4\}, \{2\}, \{3, 5\}\}$  and  $\mathcal{P} \oplus \{5\} = \{\{1, 4\}, \{2\}, \{3\}, \{5\}\}$ . Obviously we have  $\mathcal{P} \oplus_3 5 \in \mathsf{P}(5)$  and  $\mathcal{P} \oplus \{5\} \in \mathsf{P}(5)$ . It can easily be verified that  $\mathcal{P} \oplus_i n$  as well as  $\mathcal{P} \oplus \{n\}$  are partitions of  $\{1, \ldots, n\}$ , for all  $n \in \mathbb{N}_{>1}$ ,  $\mathcal{P} \in \mathsf{P}(n-1)$  and  $i \in \{1, \ldots, |\mathcal{P}|\}.$ 

**Theorem 6.1.** We have  $\mathsf{P}(n) = \{\mathcal{P} \oplus_i n \mid \mathcal{P} \in \mathsf{P}(n-1) \land 1 \le i \le |\mathcal{P}|\} \cup \{\mathcal{P} \oplus \{n\} \mid \mathcal{P} \in \mathsf{P}(n-1)\}$ , for all  $n \in \mathbb{N}_{>1}$ , and  $P(1) = \{\{1\}\}.$ 

**Proof:** Equation  $P(1) = \{\{1\}\}\$  is obvious. To prove the remaining claim, let an arbitrary  $n \in \mathbb{N}_{>1}$  be given.

For a proof of inclusion " $\subseteq$ " suppose an arbitrary  $\mathcal{P} := \{A_1, \ldots, A_k\} \in \mathsf{P}(n)$ . If  $\{n\} \notin \mathcal{P}$ , then there exists  $i \in$  $\{1,\ldots,k\} = \{1,\ldots,|\mathcal{P}|\}$  such that  $n \in A_i$ . We define  $\mathcal{P}' := \{A_1,\ldots,A_{i-1},A_i \setminus \{n\},A_{i+1},\ldots,A_k\}$  and get  $\mathcal{P}' \in \mathsf{P}(n-1)$ and  $\mathcal{P} = \mathcal{P}' \oplus_i n$  by simple calculations. Hence,  $\mathcal{P}$  is contained in the right-hand side of the equation we want to prove. If  $\{n\} \in \mathcal{P}$ , then we define  $\mathcal{P}' := \mathcal{P} \setminus \{\{n\}\}$  and get  $\mathcal{P}' \in \mathsf{P}(n-1)$  and  $\mathcal{P} = \mathcal{P}' \oplus \{n\}$ , which again shows that  $\mathcal{P}$  is contained in the right-hand side of this equation.

To verify inclusion " $\supseteq$ " we take an arbitrary  $\mathcal{P} \in \mathsf{P}(n-1)$ . We have already mentioned that then  $\mathcal{P} \oplus_i n \in \mathsf{P}(n)$ , for all  $i \in \{1, ..., |\mathcal{P}|\}$ , and  $\mathcal{P} \oplus \{n\} \in \mathsf{P}(n)$ . This concludes the proof.  $\square$ 

In the following we use c: i to denote that the number i is appended to the codeword c from the right. For example, we have (1, 2, 3, 1): 4 = (1, 2, 3, 1, 4). Then the recursive specification of P(n) given in Theorem 6.1 immediately leads to the recursive specification of C(n) given in Corollary 6.1 below. This is due to the fact that, assuming an arbitrary partition  $\mathcal{P} := \{A_1, \ldots, A_k\} \in \mathsf{P}(n-1)$  with corresponding codeword  $(c_1, \ldots, c_{n-1}) \in \mathsf{C}(n-1)$ , the codeword corresponding to  $\mathcal{P} \oplus_i n$  is  $(c_1, \ldots, c_{n-1}, i) \in C(n)$ , for all  $i \in \{1, \ldots, k\}$ , and the codeword corresponding to  $\mathcal{P} \oplus \{n\}$  is  $(c_1, \ldots, c_{n-1}, max(c) + 1) \in C(n)$ , with max(c) as the maximal element of c.

and  $C(1) = \{(1)\}.$ 

By means of the well-known backtracking technique the recursive specification of Corollary 6.1 immediately can be translated into a recursive algorithm for computing the set C(n) from the input  $n \in \mathbb{N}_{>0}$ . The result looks as given below. In the recursive procedure generate the inputs n, m and i are natural numbers and the input c is an array with  $1, \ldots, n$  as indices and integers as values. Whereas n, m and i are read-only parameters, the array c is changed during the execution of generate(n, m, c, i).

generate(n, m, c, i)  
**if** 
$$i = n$$
 **then** print (c[1], ..., c[n])  
**else for**  $j = 1$  **to** m **do**  
 $c[i + 1] := j$ ; generate(n, m, c, i + 1) **od**;  
 $c[i + 1] := m + 1$ ;  
generate(n, m + 1, c, i + 1) **fi**;  
**end**:

enu;

*setpart(n)* c[1] := 1; generate(n, 1, c, 1);end;

An invariant of the recursion is that the part  $(c[1], \ldots, c[i])$  of the array c is a codeword from the set C(i) and equation  $m = max(c[1], \dots, c[i])$  holds. As a consequence we get: if the codeword  $(c[1], \dots, c[i])$  corresponds to the partition  $\mathcal{P} \in \mathsf{P}(i)$ , then  $m = |\mathcal{P}|$ .

The structure of the above algorithm is rather similar to that of the recursive set partition algorithm published by M.C. Er in [7]. In [7] the recursive procedure SP (corresponding to the procedure *generate* above) is locally declared within the main procedure *SetPartitions* (corresponding to the procedure *setpart* above) and the array c and the input n are global variables within SP. Also the initialsation and update of c are done completely within SP, but in a slightly different way to those of *setpart*(n).

To compute the number  $\operatorname{zdim}_{T_0}(n)$  instead of the set C(n), first, as in case of the algorithm  $\operatorname{zdim}_{T_0}(n)$  of Section 4, the output of the array *c* in the algorithm  $\operatorname{generate}(n, m, c, i)$  (see line 2) has to be replaced by the computation of the auxiliary array *d*, such that formula (3) holds, followed by the computation of the expression  $\prod_{d[j]>1} d[j] \operatorname{po}(d[j]-1)$ . Finally, all the numbers produced this way during the recursive execution have to be added. The latter can be obtained by an additional integer parameter *z* of the recursive procedure *generate*, a variable parameter (or pointer to integers), which is initialised by 0 before the recursion starts and then is changed to  $z + \prod_{d[j]>1} d[j] \operatorname{po}(d[j] - 1)$  after the computation of the auxiliary array *d*. Renaming, finally, the main procedure *setpart* into  $\operatorname{zdim}_{T_0}$ , we get the following recursive backtracking algorithm for computing  $\operatorname{zdim}_{T_0}(n)$ .

generate(z, n, m, c, i)  
if 
$$i = n$$
 then for  $j = 1$  to n do  
 $d[j] := 0$  od;  
for  $j = 1$  to n do  
 $d[c[j]] := d[c[j]] + 1$  od;  
 $z := z + \prod_{d[j]>1} d[j] \operatorname{po}(d[j] - 1)$   
else for  $j = 1$  to m do  
 $c[i + 1] := j$ ; generate(z, n, m, c, i + 1) od;  
 $c[i + 1] := m + 1$ ;  
generate(z, n, m + 1, c, i + 1) fi;  
end;

 $zdim_{T_0}(n)$  c[1] := 1; z := 0; generate(z, n, 1, c, 1); return z;end;

To optimise this algorithm, we now incrementally compute the auxiliary array *d* during its execution. To this end, we introduce *d* as an additional parameter of the recursive procedure *generate*. Furthermore, we define and initialise an array *d* in the algorithm  $zdim_{T_0}(n)$  and update *d* in the algorithm generate(z, n, m, c, d, i) in such a way, that the initialisation establishes the formula

$$\forall l \in \{1, \dots, i\} : d[l] = |\{k \in \{1, \dots, i\} \mid c[k] = l\}|$$
(4)

and the update maintains the validity of (4) during the entire execution. Due to this invariant property, the computation of the array *d* in the **then**-case of the algorithm *generate*(*z*, *n*, *m*, *c*, *d*, *i*) via the two **for**-loops then can be removed and this case reduces to the assignment  $z := z + \prod_{d[j]>1} d[j] \operatorname{po}(d[j] - 1)$ . In the C-implementation (which we will describe in Section 7) the assignment is again realised via a simple loop from 1 to *n* and a conditional assignment as its body.

It is obvious how to initialise the array d in  $zdim_{T_0}(n)$ . Because of the call generate(z, n, 1, c, d, 1) in  $zdim_{T_0}(n)$ , we have to initialise the component d[1] as 1 and the remaining components d[2] to d[n] as 0 before this call. Then formula (4) is true for the call's fourth, fifth and sixth argument.

The update of the array *d* in generate(*z*, *n*, *m*, *c*, *d*, *i*) consists of two cases. First, we have to consider the assignment c[i+1] := j and the subsequent recursive call generate(*z*, *n*, *m*, *c*, *d*, *i*+1) (see line 8 of the above procedure generate). If before the execution of c[i+1] := j formula (4) is true, its validity is maintained if after the assignment d[j] is incremented by 1. Of course, after termination of the call generate(*z*, *n*, *m*, *c*, *d*, *i*+1) the update of *d* has to be canceled and the algorithm has to continue with the original array. Summing up, we have to insert d[j] := d[j] + 1 in front of the call generate(*z*, *n*, *m*, *c*, *d*, *i*+1) and d[j] := d[j] - 1 after this call. As second case we have to consider the assignment c[i+1] := m + 1 and the subsequent recursive call generate(*z*, *n*, *m*+1, *c*, *d*, *i*+1) (see lines 9 and 10 of the above procedure generate). Here we get in a similar way that d[m + 1] := 1 has to be inserted in front of the call

generate(z, n, m + 1, c, d, i + 1) and d[m + 1] := 0 has to be inserted after this call. Altogether, we obtain the following refined algorithm for  $zdim_{T_0}(n)$ .

generate(z, n, m, c, d, i)  
**if** 
$$i = n$$
 **then**  $z := z + \prod_{d[j]>1} d[j] \operatorname{po}(d[j] - 1)$   
**else for**  $j = 1$  **to**  $m$  **do**  
 $c[i+1] := j; d[j] := d[j] + 1;$   
 $generate(z, n, m, c, d, i + 1); d[j] := d[j] - 1$  **od**;  
 $c[i+1] := m + 1; d[m+1] := 1;$   
 $generate(z, n, m + 1, c, d, i + 1); d[m + 1] := 0$  **fi**;  
**end**;

*zdim*<sub>T<sub>0</sub></sub>(*n*) *c*[1] := 1; **for** *j* = 1 **to** *n* **do if** *j* = 1 **then** *d*[*j*] := 1 **else** *d*[*j*] := 0 **fi od**; *z* := 0; generate(*z*, *n*, 1, *c*, *d*, 1); *return z*; **end**;

Having a closer look to this algorithm we see that now the array *c* is no longer necessary for computing the result *z*. Thus, as a further optimisation step we remove all assignments referring to *c* from both procedures, i.e., c[i+1] := j and c[i+1] := m+1 from *generate* and c[1] := 1 from  $zdim_{T_0}$ , and also *c* from the parameter list of *generate*. This leads to the following result.

```
generate(z, n, m, d, i)
if i = n then z := z + \prod_{d[j]>1} d[j] \operatorname{po}(d[j] - 1)
else for j = 1 to m do

d[j] := d[j] + 1;

generate(z, n, m, d, i + 1); d[j] := d[j] - 1 od;

d[m + 1] := 1;

generate(z, n, m + 1, d, i + 1); d[m + 1] := 0 fi;

end;

zdim_{T_0}(n)
```

```
for j = 1 to n do
    if j = 1 then d[j] := 1 else d[j] := 0 fi od;
    z := 0; generate(z, n, 1, d, 1);
return z;
end;
```

# 7. Parallel Implementation and Experimental Results

The final algorithm we have obtained in Section 6 after removing *c* easily can be implemented in C. In this section we describe how this C-program can be parallelised using the OpenMP library (cf. [21]) for parallel programming in the symmetric multiprocessing (SMP) model. We also present results of practical experiments. They demonstrate that the parallel C-program allows to compute the number  $zdim_{T_0}(n)$  much faster than the sequential C-programs described in Section 5. For reasons of space we only present and describe the decisive parts of the C-code. The complete C-program is again available via [22].

OpenMP is an API that supports the shared memory parallel programming in C (and some other programming languages) on many platforms. It follows the multi-thread model of parallel programming, where identical processors are connected to a single, shared main memory. The compiler can be directed to create a *parallel region* that will be

executed by a *team* of concurrent threads. Within this region, *work-sharing directives* can be used to distribute computations to the different threads. The two pre-defined functions omp\_get\_num\_threads and omp\_get\_thread\_num can be used to obtain the total number of threads in the current team and the index of the current thread within this team, respectively.

For recursive algorithms like ours, *OpenMP tasks* are particularly useful. Such a task consists of a section of code and corresponding variables and stack that can be assigned to threads in the current team for execution. The number of tasks can be significantly larger than the number of threads, and it is up to the OpenMP run-time system to ensure that the tasks are executed as efficiently as possible, minimizing the number of idle threads.

A naïve implementation of the recursive algorithm generate(z, n, m, d, i) obtained at the end of Section 6 could be to create a task for each call of the procedure generate. But because of the very large number of partitions of the set  $\{1, ..., n\}$  (even in case of a small *n*) and the relatively long time needed to set up a task, this would be fairly inefficient. Therefore we follow a different approach. In it, the calls of the first few levels of the activation tree of the recursive procedure generate are executed by just one thread and do not benefit from parallelisation. Once a certain recursion depth has been reached, tasks are created for all calls of generate.

When implementing the final algorithm of Section 6 as a parallel C-program by means of the OpenMP library, for the sake of brevity and in order to improve efficiency we collect the parameters n, m and d of generate(z, n, m, d, i) into a single object of the following data type:

#### typedef struct { int n; int m; int \*d; } partition

Via partition the recursive algorithm generate(z, n, m, d, i) can be implemented as given below. In the C-function generate the first parameter is a dynamic integer array. Recall that GMP uses mpz\_t as data type for integers. The function generate is based on a global integer array P such that  $P[i] = po^*(i)$ , for all array indices i > 0, and uses the GMP-functions mpz\_init\_set\_str for the initialisation of integer variables, mpz\_mul for integer multiplication, mpz\_add for integer addition and mpz\_clear for the deallocation of storage.

```
void generate(mpz_t *z, partition *p, int i) {
  int *d = p ->d;
  int n = p - n, m = p - m;
  mpz_t oN;
  int j, tn;
  if (i >= n) {
    mpz_init_set_str(oN, "1", 10);
    for (j=0; j<m; j++)</pre>
      if (d[j] > 1) mpz_mul(oN, oN, P[d[j]]);
    tn = omp_get_thread_num();
    mpz_add(z[tn], z[tn], oN);
    mpz_clear(oN); }
  else {
    for (j=0; j<m; j++) {</pre>
      d[j]++; generate(z, p, i+1); d[j]--; }
    p \rightarrow m = m+1; d[m] = 1;
    generate(z, p, i+1);
    d[m] = 0; p \rightarrow m = m; 
}
```

If  $i \ge n$  holds in line 6 of this C-function, a partition of  $\{1, ..., n\}$  has been constructed and from the array d we can compute its contribution  $\prod_{d[i]>1} \text{po}^*(d[i]) = \prod_{d[i]>1} P[d[i]]$  to the final result. The lines 7 to 11 evaluate  $\prod_{d[i]>1} P[d[i]]$  and add the result to the array component z[tn], where tn is the current thread number. This approach avoids race conditions that would arise if we added all contributions to the same integer variable. If  $i \ge n$  does not hold, then the construction of a partition of  $\{1, ..., n\}$  is not yet completed. In this case we proceed by recursion exactly as in algorithm generate(z, n, m, d, i).

For the first few levels of the recursion, we do not use tasks, but execute the algorithm in a single thread. The following C-function generate\_omp implementing this approach is very similar to the above function generate, with some adjustments to ensure a correct parallel execution.

```
void generate_omp(mpz_t *z, partition *p, int i, int depth) {
  partition **q;
  int *d = p - >d;
  int n = p - n, m = p - m;
  mpz_t oN;
  int j, tn;
  if (depth < 1) {
    #pragma omp task firstprivate(p,i)
    generate(z, p, i); }
  else if (i >= n) {
    mpz_init_set_str(oN, "1", 10);
    for (j=0; j<m; j++)
      if (d[j] > 1) mpz_mul(oN, oN, P[d[j]]);
    tn = omp_get_thread_num();
    mpz_add(z[tn], z[tn], oN);
    mpz_clear(oN); }
  else {
    q = (partition **) malloc(sizeof(partition *)*(m+1));
    for (j=0; j<m; j++) {</pre>
      q[j] = clone_partition(p); q[j]->d[j]++;
      generate_omp(z, q[j], i+1, depth-1); }
    q[m] = clone_partition(p); q[m] \rightarrow m = m+1; q[m] \rightarrow d[m] = 1;
    generate_omp(z, q[m], i+1, depth-1); }
}
```

Line 7 tests whether the given parallelisation depth depth is reached. If we have, lines 8 and 9 generate a new task that creates partitions by adding the missing elements and updates the corresponding components of the array z. If we have not yet reached the parallelisation depth, we essentially follow the structure of the above C-function generate. But instead of modifying the partitions in place, we use an auxiliary C-function clone\_partition to create copies of the partition that can be modified by parallel threads without interference. For performance reasons, the allocated storage is not freed.

Having the C-functions generate and generate\_omp at hand, now we only have to initialise the array P with the numbers  $po^*(i)$ , to initialise a partition variable with the empty partition, to start the function generate\_omp in a parallel region and to collect the results. If we use an auxiliary function initP for the first task and an auxiliary function new\_partition for the second one, the decisive part of the C-function main looks as follows.

```
initP();
p = new_partition(n);
#pragma omp parallel
#pragma omp single
{ z = malloc(sizeof(mpz_t) * omp_get_num_threads());
    for (i=0; i<omp_get_num_threads(); i++)
        mpz_init_set_str(z[i], "0", 10);
    generate_omp(z, p, 0, depth);
    #pragma omp taskwait
    mpz_init_set_str(zdimT0, "0", 10);
    for (i=0; i<omp_get_num_threads(); i++)
        mpz_add(zdimT0, zdimT0, z[i]); }
gmp_printf("%Zd\n", zdimT0);
```

n	13	14	15	16	17	18	19
Avg. time	0.551	3.207	25.253	202.377	1714.375	13279.527	117 374.934
Min. time	0.502	2.944	20.583	165.730	1 410.881	11202.906	96 249.940
Max. time	0.618	3.749	28.101	225.410	1 797.865	14 182.732	124 956.047
Difference	123%	127%	136%	136%	127%	126%	129%

Figure 5: Running times of the parallel algorithms with depth 9.

Line 3 creates a parallel region, line 4 ensures that the following block is only executed by one thread, and lines 5 to 7 allocate variables for the partial sums corresponding to all threads. In line 8, we call generate\_omp, wait then for all created tasks to complete in line 9, and collect the partial sums in zdimT0 in lines 10 to 12.

Since tasks are only created at a prescribed parallelisation depth, we can control the number of tasks and therefore balance the organisational overhead of task creation versus the potential for parallelisation. If the number of tasks is too small to utilise all available threads, we should increase depth. If the number of tasks is too large, i.e., if the administrative overhead of task management outweighs the actual computation, we should decrease depth. The proper choice of the parallelisation depth significantly impacts the run-time of the algorithm, and we have determined suitable choices for depth in a series of experiments.

With the parallel C-program the number  $zdim_{T_0}(19)$  can be computed much faster than estimated in the introduction and in Section 4 for the sequential C-program using GMP. On the computer with two Intel<sup>®</sup> Xeon<sup>®</sup> Gold 6242 CPUs mentioned before, we have been able to compute

 $zdim_{T_0}(19) = 4\ 639\ 372\ 746\ 385\ 389\ 556\ 519\ 264\ 489\ 422\ 075\ 597$ 

within 26.73 hours. Then again it took only 0.0001 seconds to get

## zdim(19) = 5 038 667 231 667 979 478 308 745 583 967 234 599

from the values  $z\dim_{T_0}(1)$  to  $z\dim_{T_0}(19)$ . For the computation of the latter values via the parallel C-program we have worked with 9 as depth. The experiments mentioned above have shown that depth 9 is most appropriate on our computer. The running times (in seconds) for inputs  $13 \le n \le 19$  can be found in the table of Figure 5. Since multiple executions with identical inputs led to different running times, we have performed several experiments for each input — starting with 70 experiments in case of n = 13 and decreasing this number step-by-step by 10 to 20 experiments for n = 18 and 10 experiments for n = 19. The running times of the second row of the table of Figure 5 are the arithmetical means of the actual running times. To exemplify the range of the running times of the experiments, in the third row of the table of Figure 5 the running times of the fastest executions are shown and in the fourth row the running times of the slowest execution is needed by the slowest execution. We think that the partly considerable differences of the running times are a consequence of the fact that the assignment of the tasks to the cores of the CPUs differ from case to case and the executions of the tasks may require different times. Hence, unpropitious assignments increase the total running time.

To give an impression how the running time of the parallel C-program depends on the depth, we present some further results of our experiments. The (average) time required for  $z\dim_{T_0}(13)$  increases from 0.551 seconds with depth 9 to 0.889 seconds if the depth is reduced to 6, to 1.808 seconds if it is reduced to 3 and to 2.880 seconds if it is reduced to 0, i.e., the computation is purely sequential. In case of  $z\dim_{T_0}(14)$  the time 3.207 seconds for depth 9 increases to 5.181 seconds for depth 6, to 12.781 seconds for depth 3 and to 19.921 seconds for depth 0. And for  $z\dim_{T_0}(15)$  the time 25.253 seconds for depth 9 increases to 39.010 seconds for depth 6, to 102.368 seconds for depth 3 and to 148.353 seconds for depth 0.

So, the purely sequential computation of the numbers  $zdim_{T_0}(n)$  via our parallel C-program by taking depth 0 seems to be much faster than the computation of  $zdim_{T_0}(n)$  by means of the sequential C-program resulting from the algorithm  $zdim_{T_0}(n)$  of Section 4. This matches results given in [4] with regard to a comparison of the efficiency of the algorithms of M.C. Er and B. Djokić et al. for the generation of C(n). From Table 1 of [4] it follows that on a VAX 8800 computer the algorithm of [4] is slightly faster than the algorithm of [7]. It only needs about 94% of the running

time of the latter. But, as Table 2 of [4] shows, on a Sun4/280 computer it is much slower. Here it needs about 222% of the running time of the algorithm of [7]. As reason it is stated that on the VAX recursive calls and returns consume much more time that arithmetic operations, while the RISC architecture of the Sun4 enables very fast recursive calls and returns. The latter also applies to the computer we have used for our experiments and this explains the above running times.

Finally it should be mentioned that it was important to find the optimal depth by experiments with small inputs n. Namely, if the optimal depth is exceeded then the administrative overhead of the task management grows very fast and this leads to much longer executions. For example, in case of n = 13 the running time increases to 3.502 seconds for depth 10 and to 23.208 seconds for depth 11.

## 8. Conclusion

In this paper we, first, have specified the number  $z\dim_{T_0}(n)$  of zero-dimensional  $T_0$ -spaces on  $\{1, ..., n\}$  and the number  $z\dim(n)$  of zero-dimensional arbitrary topological spaces on  $\{1, ..., n\}$ , both with respect to the dimension dim. Based on these results, we have presented algorithms for computing these numbers. We also have described their implementation in C and reported on results of practical experiments. They show that a parallel C-implementation of the recursive backtracking algorithm for  $z\dim_{T_0}(n)$  is much faster than that of a sequential iterative one which is based on an algorithm for generating the partitions of  $\{1, ..., n\}$ .

A comparison of the fourth column of the table of Figure 3 with Figure 5 shows that the parallel C-program for computing  $z\dim_{T_0}(n)$  is about 9 times faster than the sequential C-program (GMP-version). This small factor is a consequence of the low degree of parallelisation enabled by our computer. We have also experimented with a computer with two Intel<sup>®</sup> Xeon<sup>®</sup> E5-2698V4 CPUs, each with 2.20 GHz base frequecy and 20 cores, 512 GByte RAM, and running Arch Linux 5.2.0. In case of purely sequential computations it is much slower than the computer with the two Intel<sup>®</sup> Xeon<sup>®</sup> Gold 6242 CPUs. E.g., for n = 18 on it the sequential C-program needs 258474.391 seconds (or 71.79 hours) to compute  $z\dim_{T_0}(n)$ . This is 2.1 times the time 122918.117 seconds (or 34.14 hours) given in Figure 3. Because of the larger number of (physical and logical) cores, however, it allows a higher degree of parallelisation. Again for n = 18 the parallel computation of  $z\dim_{T_0}(n)$  (again with 9 as optimal depth) takes 13 365.281 seconds (or 3.71 hours). Hence, in this case on the computer with 40 (physical) cores the parallel C-program is almost 20 times faster than the sequential one. Experiments have shown that this also holds for n < 18.

Acknowledgement. We thank Mitja Kulczynski and Matthias Westphal for their support with regard to the realisation of the time-consuming practical experiments.

# References

- [1] P. Alexandroff: Diskrete Räume. Matematicheskij Sbornik NS 2, pp. 501-518, 1937.
- [2] R. Berghammer, M. Winter: Order- and graph-theoretic investigation of dimensions of finite topological spaces. Monatshefte für Mathematik 190(1), pp. 33-78, 2019.
- [3] R. Berghammer, H. Schnoor, M. Winter: Efficient computation of the large inductive dimension using order- and graph-theoretic means. Submitted for publication, 2019.
- [4] B. Djokić, M. Miyakawa, S. Sekiguch, I. Sembra, I. Stojmenovic: A fast iterative algorithm for generating set partitions. The Computer Journal 32(3), pp. 281-282, 1989.
- [5] B. Djokić, M. Miyakawa, S. Sekiguch, I. Sembra, I. Stojmenovic: Parallel algorithms for generating subsets and set partitions. In: T. Asano, T. Ibarki, T. Nishizeki (eds.), Algorithms, SIGAL 1990, Lecture Notes in Computer Science, vol. 450, pp. 76-85, 1990.
- [6] R. Engelking: Dimension theory. North-Holland Mathematical Library, vol. 19, North-Holland, 1978.
- [7] M.C. Er: A fast algorithm for generating set partitions. The Computer Journal 31(3), pp.283-284, 1988.
- [8] M. Erné, K. Stege: Counting finite posets and topologies, Order 8, pp. 247-265, 1991.
- [9] M. Erné: Ordnungskombinatorik. Lecture notes, Universität Hannover, 2001.
- [10] A.V. Evaco, R. Kopperman, Y.V. Mukhin: Dimensional properties of graphs and digital spaces. Journal of Mathematical Imaging and Vision 6, pp. 109-119, 1996.
- [11] D.N. Georgiou, A.C. Megaritis: Covering dimension and finite spaces. Applied Mathematics and Computation 218, pp. 3122-3130, 2011.
- [12] D.N. Georgiou, A.C. Megaritis: An algorithm of polynomial order for computing the covering dimension of a finite space. Applied Mathematics and Computation 231, pp. 276-283, 2014.
- [13] D.N. Georgiou, A.C. Megaritis, S.P. Moshokoa: Small inductive dimension and Alexandroff topological spaces. Topology and its Applications 168, pp. 103-119, 2014.
- [14] D.N. Georgiou, A.C. Megaritis, S.P. Moshokoa: A computing procedure for the small inductive dimension of a finite *T*<sub>0</sub>-space. Computational and Applied Mathematics 34, pp. 401-415, 2015.

- [15] D.N. Georgion, A.C. Megaritis, S.P. Moshokoa: Finite spaces: a reduction algorithm for the computation of the small inductive dimension. of a finite  $T_0$ -space. Computational and Applied Mathematics 36, pp. 791-803, 2017.
- [16] J.L. Kelley: General topology, Springer, 1975.
- [17] D.E. Knuth: The Art of Computer Programming, Vol. 4A (Combinatorial algorithms, Part 1). Addison Wesley, 2011.
- [18] I. Sembra: An efficient algorithm for generating all partitions of the set  $\{1, 2, ..., n\}$ . Journal of Information Processing 7, pp. 41-42, 1984. [19] P. Wiederhold, R.G. Wilson: Dimension for Alexandrov spaces. In: R.A. Melter, A.Y. Wu (eds.): Vision geometry. Proceedings of Society
- Phot-Optical Instrumentation Engineers, vol. 1832, pp. 13-22, 1992. [20] GMP homepage: https://www.gmplib.org.
- [21] OpenMP homepage: https://www.openmp.org.
- [22] Source code download: https://www.math.uni-kiel.de/scicom/de/veroffentlichungen/code/zerodim/