# Optimization based model order reduction for stochastic systems 

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#### Abstract

In this paper, we bring together the worlds of model order reduction for stochastic linear systems and $\mathcal{H}_{2}$-optimal model order reduction for deterministic systems. In particular, we supplement and complete the theory of error bounds for model order reduction of stochastic differential equations. With these error bounds, we establish a link between the output error for stochastic systems (with additive and multiplicative noise) and modified versions of the $\mathcal{H}_{2}$-norm for both linear and bilinear deterministic systems. When deriving the respective optimality conditions for minimizing the error bounds, we see that model order reduction techniques related to iterative rational Krylov algorithms (IRKA) are very natural and effective methods for reducing the dimension of large-scale stochastic systems with additive and/or multiplicative noise. We apply modified versions of (linear and bilinear) IRKA to stochastic linear systems and show their efficiency in numerical experiments.


Keywords: model order reduction • stochastic systems • optimality conditions • Sylvester equations • Lévy process
MSC classification: 93A15 - 93B40 • 65C30 • 93E03

## 1 Introduction

We consider the following linear stochastic systems

$$
\begin{align*}
d x(t) & =\left[A x(t)+B_{1} u(t)\right] d t+f(x(t), d M(t)), \quad x(0)=x_{0}  \tag{1a}\\
y(t) & =C x(t), \quad t \geq 0, \tag{1b}
\end{align*}
$$

and the function $f$ represents either additive or multiplicative noise, i.e.,

$$
f(x(t), d M(t))= \begin{cases}B_{2} d M(t), & \text { additive case } \\ \sum_{i=1}^{m_{2}} N_{i} x(t-) d M_{i}(t), & \text { multiplicative case }\end{cases}
$$

[^0]where $x(t-):=\lim _{s \uparrow t} x(s)$. Above, we assume that $A, N_{i} \in \mathbb{R}^{n \times n}, B_{1} \in \mathbb{R}^{n \times m_{1}}, B_{2} \in$ $\mathbb{R}^{n \times m_{2}}$, and $C \in \mathbb{R}^{p \times n}$ are constant matrices. The vectors $x(t) \in \mathbb{R}^{n}, u(t) \in \mathbb{R}^{m_{1}}$ and $y(t) \in \mathbb{R}^{p}$ are called state, control input and output vector, respectively. Moreover, let $M=\left(M_{1}, \ldots, M_{m_{2}}\right)^{T}$ be an $\mathbb{R}^{m_{2}}$-valued square integrable Lévy process with mean zero and covariance matrix $K=\left(k_{i j}\right)_{i, j=1, \ldots, m_{2}}$, i.e., $\mathbb{E}\left[M(t) M^{T}(t)\right]=K t$ for $t \geq 0$. Such a matrix exists, see, e.g., [23].
$M$ and all stochastic process appearing in this paper are defined on a filtered probability space $\left(\Omega, \mathcal{F},\left(\mathcal{F}_{t}\right)_{t \geq 0}, \mathbb{P}\right)^{1}$. In addition, $M$ is $\left(\mathcal{F}_{t}\right)_{t \geq 0}$-adapted and its increments $M(t+$ $h)-M(t)$ are independent of $\mathcal{F}_{t}$ for $t, h \geq 0$. Throughout this paper, we assume that $u$ is an $\left(\mathcal{F}_{t}\right)_{t \geq 0}$-adapted control that is square integrable meaning that
$$
\|u\|_{L_{T}^{2}}^{2}:=\mathbb{E} \int_{0}^{T}\|u(s)\|^{2} d s<\infty
$$
for all $T>0$.
In recent years, model order reduction (MOR) techniques such as balanced truncation (BT) and singular perturbation approximation (SPA), methods well-known and wellunderstood for deterministic systems [2, 20, 22] have been extended to stochastic systems of the form (1), see, for example [7, 8, 27, 28]. In this paper we discuss optimization based model order reduction techniques for stochastic systems, which will lead naturally to iterative rational Krylov algorithms (IRKA), methods well-known for deterministic systems.
IRKA was introduced in [12] (for systems (1) with $f \equiv 0$ ) and relies on finding a suitable bound ( $\mathcal{H}_{2}$-error) for the output error of two systems with the same structure but, as in the context of MOR, one is usually large-scale and the other one is of small order. Subsequently, first order optimality conditions for this $\mathcal{H}_{2}$-bound with respect to the reduced order model (ROM) coefficients were derived. These optimality conditions can be based on system Gramians [15, 29] or they can be equivalently formulated as interpolatory conditions [12, 19] associated to transfer functions of the systems. It was shown in [12] that IRKA fits these conditions. This $\mathcal{H}_{2}$-optimal scheme was extended in the linear deterministic setting to minimizing systems errors in weighted norms [1, 9, 13]. An extension of IRKA to bilinear systems, which relies on the Gramian based optimality conditions shown in [30] was given in [6]. The respective interpolatory optimality conditions in the bilinear case were proved in [11. However, in contrast to the linear case, bilinear IRKA in [6] was developed without knowing about the link between the bilinear $\mathcal{H}_{2}$-distance and the output error of two bilinear systems. Later this gap was closed in [25] showing that the bilinear $\mathcal{H}_{2}$-error bounds the output error if one involves the exponential of the control energy.
In order to establish IRKA for stochastic systems (1) as an alternative to balancing related MOR, we develop a theory as follows. We prove an output error between two stochastic systems with multiplicative noise and derive the respective first order conditions for optimality in Section 2. The bound in the stochastic case (12) covers the $\mathcal{H}_{2}$-error of two bilinear system as special cases, the same is true for the optimality

[^1]conditions which generalize the ones in [30]. However, in the stochastic case, in contrast to the bilinear case, the bound does not include the additional factor of the exponential of the control energy. Hence the bound is expected to be much tighter for stochastic system. Based on the optimality conditions for (12) we propose a modified version of bilinear IRKA. Based on the structure of the bound in (12) modified bilinear IRKA appears to be an even more natural method to reduce stochastic systems rather than bilinear systems.
For the case of additive noise, which we consider in Section 3, the first order optimality conditions are merely a special case of the ones for multiplicative noise. As an additional feature we introduce a splitting approach for systems with additive noise in this section, where we split the linear system into two subsystems; one which includes the deterministic part and one the additive noise. We reduce each subsystem independently, which allows for additional flexibility, in case that one of the systems is easier to reduce. Moreover, we consider a one step approach which reduced the deterministic and the noisy part simultaneously. Again, error bounds are provided which naturally lead to (weighted) versions of linear IRKA for each subsystem for the reduction process.
The final Section 4 contains numerical experiments for systems with both multiplicative and additive noise in order to support our theoretical results.

## 2 Systems with multiplicative noise

We study the multiplicative case first, in which (11) becomes

$$
\begin{align*}
d x(t) & =\left[A x(t)+B_{1} u(t)\right] d t+\sum_{i=1}^{m_{2}} N_{i} x(t-) d M_{i}(t), \quad x(0)=x_{0}  \tag{2a}\\
y(t) & =C x(t), \quad t \geq 0 . \tag{2b}
\end{align*}
$$

Now, the goal is to find a measure for the distance between (2) and a second system having the same structure but potentially a much smaller dimension. It is given by

$$
\begin{align*}
d \hat{x}(t) & =\left[\hat{A} \hat{x}(t)+\hat{B}_{1} u(t)\right] d t+\sum_{i=1}^{m_{2}} \hat{N}_{i} \hat{x}(t-) d M_{i}(t), \quad \hat{x}(0)=\hat{x}_{0},  \tag{3a}\\
\hat{y}(t) & =\hat{C} \hat{x}(t), \quad t \geq 0, \tag{3b}
\end{align*}
$$

where $\hat{x}(t) \in \mathbb{R}^{r}$, with $r \ll n$ and $\hat{A}, \hat{B}_{1}, \hat{C}, \hat{N}_{i}, i=1, \ldots, m_{2}$ of appropriate dimension. In order to find a distance between the above systems, a stability assumption and the fundamental solution to both systems are needed. Some results (in particular the one on the optimality for the error between two systems) obtained for systems with multiplicative noise can be transferred to the case with additive noise as we will see later.

### 2.1 Fundamental solutions and stability

We introduce the fundamental solution $\Phi$ to (2a). It is defined as the $\mathbb{R}^{n \times n}$-valued solution to

$$
\begin{equation*}
\Phi(t, s)=I+\int_{s}^{t} A \Phi(\tau, s) d \tau+\sum_{i=1}^{m_{2}} \int_{s}^{t} N_{i} \Phi(\tau-, s) d M_{i}(\tau), \quad t \geq s \tag{4}
\end{equation*}
$$

This operator maps the initial condition $x_{0}$ to the solution of the homogeneous state equation with initial time $s \geq 0$. We additionally define $\Phi(t):=\Phi(t, 0)$. Note, that $\Phi$ also includes the fundamental solution of the additive noise scenario which we will use later. It is obtained by setting $N_{1}=\ldots=N_{m_{2}}=0$ in (4), so that $\Phi(t, s)=\mathrm{e}^{A(t-s)}$. We make a stability assumption on the fundamental solution which we need to produce well-defined system norms (and distances).

Assumption 1. The fundamental solution $\Phi$ is mean square asymptotically stable, i.e., there is a constant $c>0$ such that

$$
\begin{equation*}
\mathbb{E}\|\Phi(t)\|^{2} \lesssim \mathrm{e}^{-c t} \Leftrightarrow \lambda\left(I \otimes A+A \otimes I+\sum_{i, j=1}^{m_{2}} N_{i} \otimes N_{j} k_{i j}\right) \subset \mathbb{C}_{-} \tag{5}
\end{equation*}
$$

where $\lambda(\cdot)$ denotes the spectrum of a matrix. We refer to [24] for the equivalence in (5), or to [17] for the same result in case of standard Wiener noise.

Note that, with additive noise $\left(N_{i}=0, \forall i\right.$ in (44) $)$, condition (5) simplifies to

$$
\begin{equation*}
\left\|\mathrm{e}^{A t}\right\|^{2} \lesssim \mathrm{e}^{-c t} \Leftrightarrow \lambda(A) \subset \mathbb{C}_{-} \tag{6}
\end{equation*}
$$

The fundamental solution is a vital tool to compute error bounds between two different stochastic systems. The key result to establish these bounds is the following lemma which is a generalization of [8, Proposition 4.4].

Lemma 2.1. Let $\Phi$ be the fundamental solution of the stochastic differential equation with coefficients $A, N_{i} \in \mathbb{R}^{n \times n}$ defined in (4) and let $\hat{\Phi}$ be the one of the same system, where $A, N_{i}$ are replaced by $\hat{A}, \hat{N}_{i} \in \mathbb{R}^{r \times r}$. Moreover, suppose that $L$ and $\hat{L}$ are matrices of suitable dimension. Then, the $\mathbb{R}^{n \times r}$-valued function $\mathbb{E}\left[\Phi(t, s) L \hat{L}^{T} \hat{\Phi}^{T}(t, s)\right]$, $t \geq s$, satisfies

$$
\begin{equation*}
\dot{X}(t)=X(t) \hat{A}^{T}+A X(t)+\sum_{i, j=1}^{m_{2}} N_{i} X(t) \hat{N}_{j}^{T} k_{i j}, \quad X(s)=L \hat{L}^{T} \tag{7}
\end{equation*}
$$

Proof. See Appendix B
Lemma 2.1 yields

$$
\begin{equation*}
\mathbb{E}\left[\Phi(t, s) L \hat{L}^{T} \hat{\Phi}^{T}(t, s)\right]=\mathbb{E}\left[\Phi(t-s) L \hat{L}^{T} \hat{\Phi}^{T}(t-s)\right] \tag{8}
\end{equation*}
$$

for all $t \geq s \geq 0$, since both expressions solve (7).

### 2.2 The stochastic analogue to $\mathcal{H}_{2}$-norms

For deterministic linear systems (system (1) without noise), a transfer function $G$ can be interpreted as an input-output map in the frequency domain, i.e., $\tilde{y}=G \tilde{u}$, where $\tilde{u}$ and $\tilde{y}$ are the Laplace transforms of the input and the output, receptively. A norm associated with this transfer function can subsequently be defined that provides a bound for the norm of the output. This can, e.g., be the $\mathscr{H}_{2}$-norm of $G$ which in the linear deterministic case is given by

$$
\|G\|_{\mathscr{H}_{2}}^{2}:=\frac{1}{2 \pi} \int_{-\infty}^{\infty}\|G(\mathrm{i} w)\|_{F}^{2} d w
$$

where $\|\cdot\|_{F}$ denotes the Frobenius norm and ithe imaginary unit. However, there are no transfer functions in the stochastic case but we can still define a norm that is analogue to the $\mathcal{H}_{2}$-norm. To do so, we use a connection known in the linear deterministic case. There, the $\mathcal{H}_{2}$-norm of a transfer function coincides with the $L^{2}$-norm of the impulse response of the system, that is,

$$
\|G\|_{\mathcal{H}_{2}}^{2}=\int_{0}^{\infty}\left\|C e^{A s} B_{1}\right\|_{F}^{2} d s
$$

see [2]. A generalized impulse response exists in the stochastic setting (2) and is given by $H(t):=C \Phi(t) B_{1}$, where $\Phi$ is the fundamental solution defined in (4). We introduce a space $\mathcal{L}^{2}(\mathcal{W})$ of matrix-valued and $\left(\mathcal{F}_{t}\right)_{t \geq 0}$-adapted stochastic processes $Y$ of appropriate dimension with

$$
\|Y\|_{\mathcal{L}^{2}(\mathcal{W})}^{2}:=\mathbb{E} \int_{0}^{\infty}\|Y(s) \mathcal{W}\|_{F}^{2} d s<\infty
$$

and $\mathcal{W}$ is a regular $m_{1} \times m_{1}$ matrix that can be seen as a weight (in the simplest case the identity matrix) and will be specified later. Now, the stochastic analogue to the (weighted) $\mathcal{H}_{2}$-norm for system (2) is

$$
\|H\|_{\mathcal{L}^{2}(\mathcal{W})}=\left(\mathbb{E} \int_{0}^{\infty}\left\|C \Phi(s) B_{1} \mathcal{W}\right\|_{F}^{2} d s\right)^{\frac{1}{2}}
$$

which is finite due to Assumption Let $\hat{\Phi}$ denote the fundamental solution to the reduced system (3) and $\hat{H}(t):=\hat{C} \hat{\Phi}(t) \hat{B}_{1}$ its impulse response. Then, we can specify the distance between $H$ and $\hat{H}$. A Gramian based representation is stated in the next theorem.

Theorem 2.2. Let $H$ and $\hat{H}$ be the impulse responses of systems (2) and (3), respectively. Moreover, suppose that Assumption प holds for (2) and (3). Then, we have

$$
\|H-\hat{H}\|_{\mathcal{L}^{2}(\mathcal{W})}^{2}=\left\|C \Phi B_{1}-\hat{C} \hat{\Phi} \hat{B}_{1}\right\|_{\mathcal{L}^{2}(\mathcal{W})}^{2}=\operatorname{tr}\left(C P C^{T}\right)+\operatorname{tr}\left(\hat{C} \hat{P} \hat{C}^{T}\right)-2 \operatorname{tr}\left(C P_{2} \hat{C}^{T}\right),
$$

where the matrices $P, \hat{P}$ and $P_{2}$ are the solutions to

$$
\begin{align*}
& A P+P A^{T}+\sum_{i, j=1}^{m_{2}} N_{i} P N_{j}^{T} k_{i j}=-B_{1} \mathcal{W}\left(B_{1} \mathcal{W}\right)^{T}  \tag{9}\\
& \hat{A} \hat{P}+\hat{P} \hat{A}^{T}+\sum_{i, j=1}^{m_{2}} \hat{N}_{i} \hat{P} \hat{N}_{j}^{T} k_{i j}=-\hat{B}_{1} \mathcal{W}\left(\hat{B}_{1} \mathcal{W}\right)^{T}  \tag{10}\\
& A P_{2}+P_{2} \hat{A}^{T}+\sum_{i, j=1}^{m_{2}} N_{i} P_{2} \hat{N}_{j}^{T} k_{i j}=-B_{1} \mathcal{W}\left(\hat{B}_{1} \mathcal{W}\right)^{T} \tag{11}
\end{align*}
$$

Proof. We have that

$$
\begin{aligned}
\|H-\hat{H}\|_{\mathcal{L}^{2}(\mathcal{W})}^{2}= & \mathbb{E} \int_{0}^{\infty}\left\|\left(C \Phi(s) B_{1}-\hat{C} \hat{\Phi}(s) \hat{B}_{1}\right) \mathcal{W}\right\|_{F}^{2} d s \\
= & \mathbb{E} \int_{0}^{\infty} \operatorname{tr}\left(C \Phi(s)\left(B_{1} \mathcal{W}\right)\left(B_{1} \mathcal{W}\right)^{T} \Phi^{T}(s) C^{T}\right) d s \\
& +\mathbb{E} \int_{0}^{\infty} \operatorname{tr}\left(\hat{C} \hat{\Phi}(s)\left(\hat{B}_{1} \mathcal{W}\right)\left(\hat{B}_{1} \mathcal{W}\right)^{T} \hat{\Phi}^{T}(s) \hat{C}^{T}\right) d s \\
& -2 \mathbb{E} \int_{0}^{\infty} \operatorname{tr}\left(C \Phi(s)\left(B_{1} \mathcal{W}\right)\left(\hat{B}_{1} \mathcal{W}\right)^{T} \hat{\Phi}^{T}(s) \hat{C}^{T}\right) d s
\end{aligned}
$$

using the properties of the Frobenius norm. Due to the linearity of the trace and the integral, we find

$$
\|H-\hat{H}\|_{\mathcal{L}^{2}(\mathcal{W})}^{2}=\operatorname{tr}\left(C P C^{T}\right)+\operatorname{tr}\left(\hat{C} \hat{P} \hat{C}^{T}\right)-2 \operatorname{tr}\left(C P_{2} \hat{C}^{T}\right)
$$

where we set

$$
\begin{gathered}
P:=\int_{0}^{\infty} \mathbb{E}\left[\Phi(s)\left(B_{1} \mathcal{W}\right)\left(B_{1} \mathcal{W}\right)^{T} \Phi^{T}(s)\right] d s, \quad \hat{P}:=\int_{0}^{\infty} \mathbb{E}\left[\hat{\Phi}(s)\left(\hat{B}_{1} \mathcal{W}\right)\left(\hat{B}_{1} \mathcal{W}\right)^{T} \hat{\Phi}^{T}(s)\right] d s \\
P_{2}:=\int_{0}^{\infty} \mathbb{E}\left[\Phi(s)\left(B_{1} \mathcal{W}\right)\left(\hat{B}_{1} \mathcal{W}\right)^{T} \hat{\Phi}^{T}(s)\right] d s
\end{gathered}
$$

Notice that $P, \hat{P}$ and $P_{2}$ exist due to Assumption 1 for both systems. Now, the function $t \mapsto X(t):=\mathbb{E}\left[\Phi(t)\left(B_{1} \mathcal{W}\right)\left(\hat{B}_{1} \mathcal{W}\right)^{T} \hat{\Phi}^{T}(t)\right]$ solves (7) in Lemma 2.1 with $s=0$ and $\hat{L}=\hat{B}_{1} \mathcal{W}, L=B_{1} \mathcal{W}$. Integrating (7) over $[0, v]$ yields

$$
X(v)-\left(B_{1} \mathcal{W}\right)\left(\hat{B}_{1} \mathcal{W}\right)^{T}=\int_{0}^{v} X(t) d t \hat{A}^{T}+A \int_{0}^{v} X(t) d t+\sum_{i, j=1}^{m_{2}} N_{i} \int_{0}^{v} X(t) d t \hat{N}_{j}^{T} k_{i j}
$$

Taking the limit of $v \rightarrow \infty$ above and taking into account that $\lim _{v \rightarrow \infty} X(v)=0$ by assumption, we obtain that $P_{2}$ solves (11). With the same arguments setting $\Phi=\hat{\Phi}$ and $B_{1}=\hat{B}_{1}$ and vice versa, we see that $P$ and $\hat{P}$ satisfy (9) and (10), respectively.

If $m_{1}=m_{2}$ and if we replace the noise in (2a) and (3a) by the components $u_{i}$ of the control vector $u$, i.e., $d M_{i}(t)$ is replaced by $u_{i}(t) d t$, then these systems are socalled (deterministic) bilinear systems. Let us denote these resulting bilinear systems by $\Sigma_{b i l}$ and $\hat{\Sigma}_{b i l}$ for the full and reduced system, respectively. Then, an $\mathcal{H}_{2}$-norm can be introduced for the bilinear case, too. We refer to [30 for more details. In [30] a Gramian based representation for $\left\|\Sigma_{b i l}-\hat{\Sigma}_{b i l}\right\|_{\mathcal{H}_{2}}$ is given. Interestingly, this distance coincides with the one in Theorem [2.2 if the noise and the input dimension coincide $\left(m_{1}=m_{2}\right)$, and the covariance and weight matrices are the identity $\left(K=\left(k_{i j}\right)=I\right.$ and $\mathcal{W}=I)$. Consequently, a special case of $\|H-\hat{H}\|_{\mathcal{L}^{2}(\mathcal{W})}$ yields a stochastic time-domain representation for the metric induced by the $\mathcal{H}_{2}$-norm for bilinear systems, e.g., if $M$ is an $m_{1}$-dimensional standard Wiener process. We refer to [25] for further connections between bilinear and stochastic linear systems. The next proposition deals with the distance between the outputs $y$ and $\hat{y}$, defined in (2b) and (3b), and the above stochastic $\mathcal{H}_{2}$-distance $\|H-\hat{H}\|_{\mathcal{L}^{2}(\mathcal{W})}$ when $\mathcal{W}=I$.

Proposition 2.3. Let $y$ and $\hat{y}$ be the outputs of systems (2) and (3) with $x_{0}=0, \hat{x}_{0}=0$ and let $H=C \Phi(\cdot) B_{1}, \hat{H}=\hat{C} \hat{\Phi}(\cdot) \hat{B}_{1}$ the impulse responses of these systems. Then, for $T>0$, we have

$$
\begin{equation*}
\sup _{t \in[0, T]} \mathbb{E}\|y(t)-\hat{y}(t)\| \leq\|H-\hat{H}\|_{\mathcal{L}^{2}(I)}\|u\|_{L_{T}^{2}} . \tag{12}
\end{equation*}
$$

Proof. We find a solution representation for (2a) by

$$
x(t)=\Phi(t) x_{0}+\int_{0}^{t} \Phi(t, s) B_{1} u(s) d s
$$

where $\Phi(t, s)=\Phi(t) \Phi^{-1}(s)$ satisfies (44). This is obtained by applying Ito's formula in (27) to $\Phi(t) g(t)$ with $g(t):=x_{0}+\int_{0}^{t} \Phi^{-1}(s) B_{1} u(s) d s$ using that $\Phi$ and $g$ are semimartingales, see Appendix A. Since $g$ is continuous and has a martingale part zero, (27) simply becomes the standard product rule due to (28) and one can show that $\Phi(t) g(t)$ is the solution to (2a). We proceed with the arguments used in [8]. The representations for the reduced and the full state with zero initial states yield

$$
\begin{aligned}
\mathbb{E}\|y(t)-\hat{y}(t)\| & \leq \mathbb{E} \int_{0}^{t}\left\|\left(C \Phi(t, s) B_{1}-\hat{C} \hat{\Phi}(t, s) \hat{B}_{1}\right) u(s)\right\| d s \\
& \leq \mathbb{E} \int_{0}^{t}\left\|C \Phi(t, s) B_{1}-\hat{C} \hat{\Phi}(t, s) \hat{B}_{1}\right\|_{F}\|u(s)\| d s .
\end{aligned}
$$

We apply the Cauchy-Schwarz inequality and obtain

$$
\mathbb{E}\|y(t)-\hat{y}(t)\| \leq\left(\mathbb{E} \int_{0}^{t}\left\|C \Phi(t, s) B_{1}-\hat{C} \hat{\Phi}(t, s) \hat{B}_{1}\right\|_{F}^{2} d s\right)^{\frac{1}{2}}\|u\|_{L_{t}^{2}} .
$$

Now, $\mathbb{E}\left[\Phi(t, s) B_{1} \hat{B}_{1} \hat{\Phi}(t, s)\right]=\mathbb{E}\left[\Phi(t-s) B_{1} \hat{B}_{1} \hat{\Phi}(t-s)\right]$ due to (8). Since the same property holds when considering $B_{1}=\hat{B}_{1}$ and $\Phi=\hat{\Phi}$, we find

$$
\begin{aligned}
\mathbb{E}\|y(t)-\hat{y}(t)\| & \leq\left(\mathbb{E} \int_{0}^{t}\left\|C \Phi(t-s) B_{1}-\hat{C} \hat{\Phi}(t-s) \hat{B}_{1}\right\|_{F}^{2} d s\right)^{\frac{1}{2}}\|u\|_{L_{t}^{2}} \\
& =\left(\mathbb{E} \int_{0}^{t}\left\|C \Phi(s) B_{1}-\hat{C} \hat{\Phi}(s) \hat{B}_{1}\right\|_{F}^{2} d s\right)^{\frac{1}{2}}\|u\|_{L_{t}^{2}}
\end{aligned}
$$

Taking the supremum on both sides and the upper bound of the integral to infinity implies the result.

If $\mathbb{E} \int_{0}^{\infty}\|u(s)\|^{2} d s<\infty$, then we can replace $[0, T]$ by $[0, \infty)$ in Proposition 2.3. The above result shows that one can expect a good approximation of (2) by (3) if matrices $\hat{A}, \hat{N}_{i}, \hat{B}_{1}, \hat{C}$ are chosen such that $\|H-\hat{H}\|_{\mathcal{L}^{2}(I)}$ is minimal. As mentioned above, $\|H-\hat{H}\|_{\mathcal{L}^{2}(I)}$ coincides with the $\mathcal{H}_{2}$-distance of the corresponding bilinear systems in special cases. In the bilinear case, MOR techniques have been considered that minimize the $\mathcal{H}_{2}$-error of two systems, e.g., bilinear IRKA [6]. Below, we construct related $\left(\mathcal{L}^{2}(\mathcal{W})\right.$-optimal) algorithms which are very natural for stochastic systems due to Proposition 2.3, where $\mathcal{W}=I$. Note that for bilinear systems, the above proposition can only be shown if the right hand side is additionally multiplied by $\exp \left\{0.5\|u\|_{L_{T}^{2}}^{2}\right\}$. This is a very recent result proved in [25]. Therefore, considering IRKA type methods for stochastic systems seems even more natural than in the bilinear case in terms of the expected output error.

Remark 1. The error bound in Proposition 2.3 was shown for the special case of uncorrelated noise processes $M_{i}$ in [8] and used to determine an error bound for balanced truncation. The bound in [8] was also the basis for the proof of an error bound for another balancing related method in [27].

### 2.3 Conditions for $\mathcal{L}^{2}(\mathcal{W})$-optimality and model order reduction in the multiplicative case

Motivated by Proposition [2.3, we locally minimize $\|H-\hat{H}\|_{\mathcal{L}^{2}(\mathcal{W})}$ with respect to the coefficients of the reduced systems (3) using the representation in Theorem 2.2. In particular, we find necessary conditions for optimality for the expression

$$
\mathcal{E}\left(\hat{A}, \hat{N}_{i}, \hat{B}_{1}, \hat{C}\right):=\operatorname{tr}\left(\hat{C} \hat{P} \hat{C}^{T}\right)-2 \operatorname{tr}\left(C P_{2} \hat{C}^{T}\right)
$$

with $\hat{P}$ and $P_{2}$ solving (10) and (11), respectively.
Theorem 2.4. Consider the system (2) with impulse response $H$. Let the reduced system (3) with impulse response $\hat{H}$ be optimal with respect to $\mathcal{L}^{2}(\mathcal{W})$, i.e., the matrices
$\hat{A}, \hat{N}_{i}, \hat{B}_{1}, \hat{C}$ locally minimize the error $\|H-\hat{H}\|_{\mathcal{L}^{2}(\mathcal{W})}$. Then, it holds that
(a) $\hat{C} \hat{P}=C P_{2}, \quad$ (b) $\quad \hat{Q} \hat{P}=Q_{2} P_{2}$,
(c) $\hat{Q}\left(\sum_{j=1}^{m_{2}} \hat{N}_{j} k_{i j}\right) \hat{P}=Q_{2}\left(\sum_{j=1}^{m_{2}} N_{j} k_{i j}\right) P_{2}, \quad i=1, \ldots, m_{2}$,
(d) $\hat{Q} \hat{B}_{1} \mathcal{W W}^{T}=Q_{2} B_{1} \mathcal{W W}^{T}$,
where $\hat{P}, P_{2}$ are the solutions to (10), (11) and $\hat{Q}, Q_{2}$ satisfy

$$
\begin{align*}
\hat{A}^{T} \hat{Q}+\hat{Q} \hat{A}+\sum_{i, j=1}^{m_{2}} \hat{N}_{i}^{T} \hat{Q} \hat{N}_{j} k_{i j} & =-\hat{C}^{T} \hat{C}  \tag{14}\\
\hat{A}^{T} Q_{2}+Q_{2} A+\sum_{i, j=1}^{m_{2}} \hat{N}_{i}^{T} Q_{2} N_{j} k_{i j} & =-\hat{C}^{T} C \tag{15}
\end{align*}
$$

Proof. If system (3) is optimal with respect to the $\mathcal{L}^{2}(\mathcal{W})$-norm, then we have

$$
\begin{equation*}
\partial_{z} \mathcal{\varepsilon}=0 \Longleftrightarrow \partial_{z} \operatorname{tr}\left(\hat{C} \hat{P} \hat{C}^{T}\right)=2 \partial_{z} \operatorname{tr}\left(C P_{2} \hat{C}^{T}\right) \tag{16}
\end{equation*}
$$

where $z \in\left\{\hat{a}_{k m}, \hat{n}_{k m}^{(l)}, \hat{b}_{k q}, \hat{c}_{\ell k}\right\}$, where $\hat{A}=\left(\hat{a}_{k m}\right), \hat{N}_{l}=\left(\hat{n}_{k m}^{(l)}\right), \hat{B}_{1}=\left(\hat{b}_{k q}\right)$ and $\hat{C}=\left(\hat{c}_{\ell k}\right)$. Below, $e_{k}$ denotes the $k$ th unit vector of suitable dimension. For $z=\hat{c}_{\ell k}$, (16) becomes

$$
\operatorname{tr}\left(e_{\ell} e_{k}^{T} \hat{P} \hat{C}^{T}+\hat{C} \hat{P} e_{k} e_{\ell}^{T}\right)=2 \operatorname{tr}\left(C P_{2} e_{k} e_{\ell}^{T}\right)
$$

Using the properties of the trace and $\hat{P}^{T}=\hat{P}$, this is equivalent to

$$
e_{\ell}^{T} \hat{C} \hat{P} e_{k}=e_{\ell}^{T} C P_{2} e_{k}
$$

for all $\ell=1, \ldots p$ and $k=1, \ldots, r$. This results in equality (a): $\hat{C} \hat{P}=C P_{2}$. For $z=\hat{a}_{k m}, \hat{n}_{k m}^{(l)}, \hat{b}_{k q}$, (16) becomes

$$
\operatorname{tr}\left(\left(\partial_{z} \hat{P}\right) \hat{C}^{T} \hat{C}\right)=2 \operatorname{tr}\left(\left(\partial_{z} P_{2}\right) \hat{C}^{T} C\right)
$$

which is equivalent to

$$
\begin{aligned}
& \operatorname{tr}\left(\left(\partial_{z} \hat{P}\right)\left(\hat{A}^{T} \hat{Q}+\hat{Q} \hat{A}+\sum_{i, j=1}^{m_{2}} \hat{N}_{i}^{T} \hat{Q} \hat{N}_{j} k_{i j}\right)\right) \\
& =2 \operatorname{tr}\left(\left(\partial_{z} P_{2}\right)\left(\hat{A}^{T} Q_{2}+Q_{2} A+\sum_{i, j=1}^{m_{2}} \hat{N}_{i}^{T} Q_{2} N_{j} k_{i j}\right)\right)
\end{aligned}
$$

using the equations for $\hat{Q}$ and $Q_{2}$. Again, by properties of the trace, the above can be reformulated as

$$
\begin{align*}
& \operatorname{tr}\left(\left(\hat{A}\left(\partial_{z} \hat{P}\right)+\left(\partial_{z} \hat{P}\right) \hat{A}^{T}+\sum_{i, j=1}^{m_{2}} \hat{N}_{i}\left(\partial_{z} \hat{P}\right) \hat{N}_{j}^{T} k_{i j}\right) \hat{Q}\right) \\
& =2 \operatorname{tr}\left(\left(A\left(\partial_{z} P_{2}\right)+\left(\partial_{z} P_{2}\right) \hat{A}^{T}+\sum_{i, j=1}^{m_{2}} N_{i}\left(\partial_{z} P_{2}\right) \hat{N}_{j}^{T} k_{i j}\right) Q_{2}\right), \tag{17}
\end{align*}
$$

taking into account that the covariance matrix $K=\left(k_{i j}\right)$ is symmetric. We now derive equations for $\partial_{z} \hat{P}$ and $\partial_{z} P_{2}$ for each case. Applying $\partial_{\hat{a}_{k m}}$ to (10) and (11), we obtain

$$
\begin{array}{r}
e_{k} e_{m}^{T} \hat{P}+\hat{A}\left(\partial_{\hat{a}_{k m}} \hat{P}\right)+\left(\partial_{\hat{a}_{k m}} \hat{P}\right) \hat{A}^{T}+\hat{P} e_{m} e_{k}^{T}+\sum_{i, j=1}^{m_{2}} \hat{N}_{i}\left(\partial_{\hat{a}_{k m}} \hat{P}\right) \hat{N}_{j}^{T} k_{i j}=0, \\
A\left(\partial_{\hat{a}_{k m}} P_{2}\right)+\left(\partial_{\hat{a}_{k m}} P_{2}\right) \hat{A}^{T}+P_{2} e_{m} e_{k}^{T}+\sum_{i, j=1}^{m_{2}} N_{i}\left(\partial_{\hat{a}_{k m}} P_{2}\right) \hat{N}_{j}^{T} k_{i j}=0 .
\end{array}
$$

Inserting this into (17), and using symmetry of $\hat{Q}$ and $\hat{P}$ leads to

$$
\operatorname{tr}\left(\left(e_{k} e_{m}^{T} \hat{P}+\hat{P} e_{m} e_{k}^{T}\right) \hat{Q}\right)=2 \operatorname{tr}\left(\left(P_{2} e_{m} e_{k}^{T}\right) Q_{2}\right) \Leftrightarrow e_{k}^{T} \hat{Q} \hat{P} e_{m}=e_{k}^{T} Q_{2} P_{2} e_{m}
$$

for all $k, m=1, \ldots, r$ which yields $\hat{Q} \hat{P}=Q_{2} P_{2}$, i.e. equality (b). We now define $\hat{\Psi}_{i}^{T}:=\sum_{j=1}^{m_{2}} \hat{N}_{j}^{T} k_{i j}$ and observe that $\partial_{\hat{n}_{k m}^{(l)}} \hat{\Psi}_{i}^{T}=e_{m} e_{k}^{T} k_{i l}$. Consequently, we have

$$
\begin{aligned}
\partial_{\hat{n}_{k m}^{(l)}} \sum_{i, j=1}^{m_{2}} \hat{N}_{i} \hat{P} \hat{N}_{j}^{T} k_{i j} & =\partial_{\hat{n}_{k m}^{(l)}} \sum_{i=1}^{m_{2}} \hat{N}_{i} \hat{P} \hat{\Psi}_{i}^{T} \\
& =\sum_{i=1}^{m_{2}}\left(\left(\partial_{\hat{n}_{k m}^{(l)}} \hat{N}_{i}\right) \hat{P} \hat{\Psi}_{i}^{T}+\hat{N}_{i}\left(\partial_{\hat{n}_{k m}^{(l)}} \hat{P}\right) \hat{\Psi}_{i}^{T}+\hat{N}_{i} \hat{P}\left(\partial_{\hat{n}_{k m}^{(l)}} \hat{\Psi}_{i}^{T}\right)\right) \\
& =e_{k} e_{m}^{T} \hat{P} \hat{\Psi}_{l}^{T}+\sum_{i=1}^{m_{2}} \hat{N}_{i}\left(\partial_{\hat{n}_{k m}^{l(l)}} \hat{P}\right) \hat{\Psi}_{i}^{T}+\sum_{i=1}^{m_{2}} \hat{N} \hat{i} \hat{P} e_{m} e_{k}^{T} k_{i l} \\
& =e_{k} e_{m}^{T} \hat{P} \hat{\Psi}_{l}^{T}+\hat{\Psi}_{l} \hat{P} e_{m} e_{k}^{T}+\sum_{i, j=1}^{m_{2}} \hat{N}_{i}\left(\partial_{\hat{n}_{k m}^{(l)}} \hat{P}\right) \hat{N}_{j}^{T} k_{i j},
\end{aligned}
$$

since $k_{i l}=k_{l i}$. Analogue to the above steps, we obtain

$$
\begin{aligned}
\partial_{\hat{n}_{k m}^{(l)}} \sum_{i, j=1}^{m_{2}} N_{i} P_{2} \hat{N}_{j}^{T} k_{i j} & =\partial_{\hat{n}_{k m}^{(l)}} \sum_{i=1}^{m_{2}} N_{i} P_{2} \hat{\Psi}_{i}^{T} \\
& =\sum_{i=1}^{m_{2}}\left(N_{i}\left(\partial_{\hat{n}_{k m}^{(l)}} P_{2}\right) \hat{\Psi}_{i}^{T}+N_{i} P_{2}\left(\partial_{\hat{n}_{k m}^{(l)}} \hat{\Psi}_{i}^{T}\right)\right) \\
& =\Psi_{l} P_{2} e_{m} e_{k}^{T}+\sum_{i, j=1}^{m_{2}} N_{i}\left(\partial_{\hat{n}_{k m}^{(l)}} P_{2}\right) \hat{N}_{j}^{T} k_{i j},
\end{aligned}
$$

where $\Psi_{l}:=\sum_{j=1}^{m_{2}} N_{j} k_{l j}$. Now, using these two results when applying $\partial_{\hat{n}_{k m}^{(l)}}$ to (10) and (11), we find

$$
\begin{array}{r}
\hat{A}\left(\partial_{\hat{n}_{k m}^{(l)}} \hat{P}\right)+\left(\partial_{\hat{n}_{k m}^{(l)}} \hat{P}\right) \hat{A}^{T}+e_{k} e_{m}^{T} \hat{P} \hat{\Psi}_{l}^{T}+\hat{\Psi}_{l} \hat{P} e_{m} e_{k}^{T}+\sum_{i, j=1}^{m_{2}} \hat{N}_{i}\left(\partial_{\hat{n}_{k m}^{(l)}} \hat{P}\right) \hat{N}_{j}^{T} k_{i j}=0 \\
A\left(\partial_{\hat{n}_{k m}^{(l)}} P_{2}\right)+\left(\partial_{\hat{n}_{k m}^{(l)}} P_{2}\right) \hat{A}^{T}+\Psi_{l} P_{2} e_{m} e_{k}^{T}+\sum_{i, j=1}^{m_{2}} N_{i}\left(\partial_{\hat{n}_{k m}^{(l)}} P_{2}\right) \hat{N}_{j}^{T} k_{i j}=0
\end{array}
$$

We plug these into (17) resulting in

$$
\operatorname{tr}\left(\left(e_{k} e_{m}^{T} \hat{P} \hat{\Psi}_{l}^{T}+\hat{\Psi}_{l} \hat{P} e_{m} e_{k}^{T}\right) \hat{Q}\right)=2 \operatorname{tr}\left(\Psi_{l} P_{2} e_{m} e_{k}^{T} Q_{2}\right)
$$

for all $k, m=1, \ldots, r$ and $l=1, \ldots, m_{2}$. With the above arguments, and symmetry of $\hat{P}$ and $\hat{Q}$, this is equivalent to $\hat{Q} \hat{\Psi}_{l} \hat{P}=Q_{2} \Psi_{l} P_{2}$ (equality (c)). It remains to apply $\partial_{\hat{b}_{k q}}$ to (10) and (11) providing

$$
\begin{aligned}
& \hat{A}\left(\partial_{\hat{b}_{k q}} \hat{P}\right)+\left(\partial_{\hat{b}_{k q}} \hat{P}\right) \hat{A}^{T}+\sum_{i, j=1}^{m_{2}} \hat{N}_{i}\left(\partial_{\hat{b}_{k q}} \hat{P}\right) \hat{N}_{j}^{T} k_{i j}=-\left(e_{k} e_{q}^{T} \mathcal{W} \mathcal{W}^{T} \hat{B}_{1}^{T}+\hat{B}_{1} \mathcal{W} \mathcal{W}^{T} e_{q} e_{k}^{T}\right), \\
& A\left(\partial_{\hat{b}_{k q}} P_{2}\right)+\left(\partial_{\hat{b}_{k q}} P_{2}\right) \hat{A}^{T}+\sum_{i, j=1}^{m_{2}} N_{i}\left(\partial_{\hat{b}_{k q}} P_{2}\right) \hat{N}_{j}^{T} k_{i j}=-B_{1} \mathcal{W} \mathcal{W}^{T} e_{q} e_{k}^{T} .
\end{aligned}
$$

Using this for (17), we have

$$
\operatorname{tr}\left(\left(e_{k} e_{q}^{T} \mathcal{W W}^{T} \hat{B}_{1}^{T}+\hat{B}_{1} \mathcal{W}^{T} e_{q} e_{k}^{T}\right) \hat{Q}\right)=2 \operatorname{tr}\left(B_{1} \mathcal{W}^{T} e_{q} e_{k}^{T} Q_{2}\right)
$$

for all $k=1, \ldots, r$ and $q=1, \ldots, m_{1}$. This results in the last property (d), i.e. $\hat{Q} \hat{B}_{1} \mathcal{W} \mathcal{W}^{T}=Q_{2} B_{1} \mathcal{W} \mathcal{W}^{T}$ and concludes the proof.

Remark 2. With the observations made in Section [2.2, it is not surprising that setting $m_{1}=m_{2}, K=\left(k_{i j}\right)=I$ and $\mathcal{W}=I$ in (13) leads to the necessary first-order optimality conditions in the deterministic bilinear case [30]. If $N_{i}=0$ for all $i=1, \ldots, m_{2}$, then (13) represents a special case of weighted $\mathcal{H}_{2}$-optimality conditions in the deterministic linear setting [13], where the weight is a constant matrix. If further $\mathcal{W}=I$, then one obtains the Wilson conditions [29].

With the link between the $\mathcal{L}^{2}(\mathcal{W})$-norm and the bilinear $\mathcal{H}_{2}$-norm given by Theorems 2.2 and 2.4, it is now intuitively clear how to construct a reduced order system (3) that satisfies (13). The approach is oriented on bilinear IRKA [6] which fulfills the necessary conditions for local optimality given in [30]. Its modified version designed to satisfy (13) is provided in Algorithm 1. This algorithm requires that the matrix $\hat{A}$ is diagonalizable which we assume throughout this paper. In many applications this can be guaranteed. The following theorem proves that Algorithm 1 provides reduced matrices that satisfy (13). We shall later apply Algorithm 1 with $\mathcal{W}=I$ to (2) in order to obtain a small output error for the resulting reduced system (3) using Proposition 2.3.

```
Algorithm 1 Modified Bilinear IRKA
    Input: The system matrices: \(A, B_{1}, C, N_{i}\). Covariance and weight matrices: \(K, \mathcal{W}\).
    Output: The reduced matrices: \(\hat{A}, \hat{B}_{1}, \hat{C}, \hat{N}_{i}\).
    Make an initial guess for the reduced matrices \(\hat{A}, \hat{B}_{1}, \hat{C}, \hat{N}_{i}\).
    while not converged do
        Perform the spectral decomposition of \(\hat{A}\) and define:
            \(D=S \hat{A} S^{-1}, \tilde{B}_{1}=S \hat{B}_{1}, \tilde{C}=\hat{C} S^{-1}, \tilde{N}_{i}=S \hat{N}_{i} S^{-1}\).
        Solve for \(V\) and \(W\) :
            \(V D+A V+\sum_{i, j=1}^{m_{2}} N_{i} V \tilde{N}_{j}^{T} k_{i j}=-\left(B_{1} \mathcal{W}\right)\left(\tilde{B}_{1} \mathcal{W}\right)^{T}\),
            \(W D+A^{T} W+\sum_{i, j=1}^{m_{2}} N_{i}^{T} W \tilde{N}_{j} k_{i j}=-C^{T} \tilde{C}\).
        \(V=\operatorname{orth}(V)\) and \(W=\operatorname{orth}(W)\), where orth \((\cdot)\) returns an orthonormal basis for
    the range of a matrix.
        Determine the reduced matrices:
            \(\hat{A}=\left(W^{T} V\right)^{-1} W^{T} A V, \quad \hat{B}_{1}=\left(W^{T} V\right)^{-1} W^{T} B_{1}, \quad \hat{C}=C V, \quad \hat{N}_{i}=\)
    \(\left(W^{T} V\right)^{-1} W^{T} N_{i} V\).
    end while
```

Theorem 2.5. Let $\hat{A}, \hat{N}_{i}, \hat{B}$ and $\hat{C}$ be the reduced-order matrices computed by Al-
 conditions (13) for local $\mathcal{L}^{2}(\mathcal{W})$-optimality.

Proof. The techniques used to prove the result are similar to the ones used in 6. We provide the proof in Appendix $\mathbb{C}$ but we restrict ourselves to the conditions related to $\hat{N}_{i}$ because these differ significantly from the respective bilinear conditions.

Finally, we note that, so far, many balancing related techniques for stochastic systems (2) with multiplicative noise have been studied [4, 7, 8, 10, 24, 27]. Algorithm (1) is the first alternative to such techniques for stochastic systems.

## 3 Systems with additive noise

We now focus on stochastic systems with additive noise. These are of the form

$$
\begin{align*}
d x(t) & =\left[A x(t)+B_{1} u(t)\right] d t+B_{2} d M(t), \quad x(0)=x_{0},  \tag{18a}\\
y(t) & =C x(t), \quad t \geq 0, \tag{18b}
\end{align*}
$$

for which (6) is assumed. We find a ROM based on the minimization of an error bound for this case. Fortunately, we do not have to repeat the entire theory again since it can be derived from the above results for systems with multiplicative noise. However, we will provide two different approaches to reduce system (18). The first one relies on splitting and reducing subsystems of (18) individually and subsequently obtain the reduced system as the sum of the reduced subsystems. In the second approach, (18) will be reduced directly. We assume that (6) holds for all reduced systems below.

### 3.1 Two step reduced order model

We can write the state in (18a) as $x=x_{1}+x_{2}$, where $x_{1}$ and $x_{2}$ are solutions to subsystems with corresponding outputs $y_{1}=C x_{1}$ and $y_{2}=C x_{2}$. Hence, we can rewrite (18) as follows:

$$
\begin{align*}
d x_{1}(t) & =\left[A x_{1}(t)+B_{1} u(t)\right] d t, \quad x_{1}(0)=x_{0}  \tag{19a}\\
d x_{2}(t) & =A x_{2}(t) d t+B_{2} d M(t), \quad x_{2}(0)=0  \tag{19b}\\
y(t) & =y_{1}(t)+y_{2}(t)=C x_{1}(t)+C x_{2}(t), \quad t \geq 0 \tag{19c}
\end{align*}
$$

Now, the idea is to reduce (19a) and (19b) with their associated outputs separately resulting in the following reduced system

$$
\begin{align*}
d \hat{x}_{1}(t) & =\left[\hat{A}_{1} \hat{x}_{1}(t)+\hat{B}_{1} u(t)\right] d t, \quad \hat{x}_{1}(0)=\hat{x}_{0},  \tag{20a}\\
d \hat{x}_{2}(t) & =\hat{A}_{2} \hat{x}_{2}(t) d t+\hat{B}_{2} d M(t), \quad \hat{x}_{2}(0)=0,  \tag{20b}\\
\hat{y}(t) & =\hat{y}_{1}(t)+\hat{y}_{2}(t)=\hat{C}_{1} \hat{x}_{1}(t)+\hat{C}_{2} \hat{x}_{2}(t), \quad t \geq 0, \tag{20c}
\end{align*}
$$

where $\hat{x}_{i}(t) \in \mathbb{R}^{r_{i}}(i=1,2)$ etc. with $r_{i} \ll n$. Note that there are several MOR techniques like balancing related schemes [20, 22] or $\mathcal{H}_{2}$-optimal methods [12] to reduce the deterministic subsystem (19a). Moreover, balancing related methods are available for (19b), see [14, 28]. We derive an optimization based scheme for (19b) below and combine it with [12] for (19a) leading to a new type of method to reduce (18).

The reduced system (20) provides a higher flexibility since we are not forced to choose $\hat{A}_{1}=\hat{A}_{2}=\hat{A}$ and $\hat{C}_{1}=\hat{C}_{2}=\hat{C}$ as it would be the case if we apply an algorithm to (18)) directly. Moreover, we are free in the choice of the dimension in each reduced subsystem. This is very beneficial since one subsystem might be easier to reduce than the other (note that one system is entirely deterministic, whereas the other one is stochastic). This additional flexibility is expected to give a better reduced system than by a reduction of (18) in one step. However, it is more expensive to run a model reduction procedure twice.

We now explain how to derive the reduced matrices above, assuming $x_{0}=0$ and $\hat{x}_{0}=0$. Using the inequality of Cauchy-Schwarz, we obtain

$$
\begin{aligned}
\mathbb{E}\|y(t)-\hat{y}(t)\| & \leq \mathbb{E}\left\|y_{1}(t)-\hat{y}_{1}(t)\right\|+\mathbb{E}\left\|y_{2}(t)-\hat{y}_{2}(t)\right\| \\
& \leq \mathbb{E}\left\|y_{1}(t)-\hat{y}_{1}(t)\right\|+\left(\mathbb{E}\left\|y_{2}(t)-\hat{y}_{2}(t)\right\|^{2}\right)^{\frac{1}{2}}
\end{aligned}
$$

We insert the solution representations for both $x_{1}$ and $x_{2}$ as well as for their reduced
systems into the above relation and find

$$
\begin{aligned}
\mathbb{E}\|y(t)-\hat{y}(t)\| \leq & \mathbb{E} \int_{0}^{t}\left\|\left(C \mathrm{e}^{A(t-s)} B_{1}-\hat{C}_{1} \mathrm{e}^{\hat{A}_{1}(t-s)} \hat{B}_{1}\right) u(s)\right\| d s \\
& +\left(\mathbb{E}\left\|\int_{0}^{t}\left(C \mathrm{e}^{A(t-s)} B_{2}-\hat{C}_{2} \mathrm{e}^{\hat{A}_{2}(t-s)} \hat{B}_{2}\right) d M(s)\right\|^{2}\right)^{\frac{1}{2}} \\
\leq & \left(\int_{0}^{t}\left\|C \mathrm{e}^{A(t-s)} B_{1}-\hat{C}_{1} \mathrm{e}^{\hat{A}_{1}(t-s)} \hat{B}_{1}\right\|_{F}^{2} d s\right)^{\frac{1}{2}}\|u\|_{L_{t}^{2}} \\
& +\left(\int_{0}^{t}\left\|\left(C \mathrm{e}^{A(t-s)} B_{2}-\hat{C}_{2} \mathrm{e}^{\hat{A}_{2}(t-s)} \hat{B}_{2}\right) K^{\frac{1}{2}}\right\|_{F}^{2} d s\right)^{\frac{1}{2}} \\
\leq & \left(\int_{0}^{\infty}\left\|C \mathrm{e}^{A s} B_{1}-\hat{C}_{1} \mathrm{e}^{\hat{A}_{1} s} \hat{B}_{1}\right\|_{F}^{2} d s\right)^{\frac{1}{2}}\|u\|_{L_{t}^{2}} \\
& +\left(\int_{0}^{\infty}\left\|\left(C \mathrm{e}^{A s} B_{2}-\hat{C}_{2} \mathrm{e}^{\hat{A}_{2} s} \hat{B}_{2}\right) K^{\frac{1}{2}}\right\|_{F}^{2} d s\right)^{\frac{1}{2}},
\end{aligned}
$$

where we have applied the Cauchy-Schwarz inequality to the first error term and the Ito isometry to the second one (see, e.g., [23]) and substituted $t-s \mapsto s$. Now, applying the supremum on $[0, T]$ to the above inequality yields

$$
\begin{equation*}
\sup _{t \in[0, T]} \mathbb{E}\|y(t)-\hat{y}(t)\| \leq \underbrace{\left\|C \mathrm{e}^{A \cdot} B_{1}-\hat{C}_{1} \mathrm{e}^{\hat{A}_{1} \cdot} \cdot \hat{B}_{1}\right\|_{\mathcal{L}^{2}(I)}}_{=: \varepsilon_{1}}\|u\|_{L_{T}^{2}}+\underbrace{\left\|C \mathrm{e}^{A \cdot} B_{2}-\hat{C}_{2} \mathrm{e}^{\hat{A}_{2} \cdot} \hat{B}_{2}\right\|_{\mathcal{L}^{2}\left(K^{\left.\frac{1}{2}\right)}\right.}}_{=: \varepsilon_{2}} . \tag{21}
\end{equation*}
$$

In order for the right hand side to be small, the reduced order matrices need to be chosen such that $\varepsilon_{1}$ and $\varepsilon_{2}$ are locally minimal. We observe that $\varepsilon_{1}$ and $\varepsilon_{2}$ are special cases of the $\mathcal{L}^{2}(\mathcal{W})$-distance of the impulse responses in the multiplicative noise scenario, see Section [2.2, since $\Phi(t)=\mathrm{e}^{A t}$ if $N_{i}=0$ for all $i=1, \ldots, m_{2}$. Here, we have $N_{i}=0$, $\mathcal{W}=I$ for $\mathcal{E}_{1}$ and $N_{i}=0, \mathcal{W}=K^{\frac{1}{2}}$ with $B_{1}$ replaced by $B_{2}$ for $\mathcal{E}_{2}$. Consequently, Algorithm $\mathbb{1}$ with $N_{i}=0$ and the respective choice for the weight matrices satisfies the necessary conditions for local optimality for $\mathcal{E}_{1}$ and $\varepsilon_{2}$. Therefore, $\left(\hat{A}_{i}, \hat{B}_{i}, \hat{C}_{i}\right)(i=1,2)$ can be computed from Algorithm [2 (a special version of Algorithm(1) which is a modified version of linear IRKA [12].

### 3.2 One step reduced order model

The second approach for additive noise reduces (18) directly without dividing it into subsystems. To do so, we set

$$
\begin{equation*}
\hat{A}_{1}=\hat{A}_{2}=\hat{A}, \quad \hat{C}_{1}=\hat{C}_{2}=\hat{C} \tag{22}
\end{equation*}
$$

```
Algorithm 2 Modified Two Step Linear IRKA ( \(i=1,2\) )
Input: The system matrices: \(A, B_{i}, C\). Weight: \(\mathcal{W}_{i}= \begin{cases}I, & i=1, \\ K^{\frac{1}{2}}, & i=2 .\end{cases}\)
```

Output: The reduced matrices: $\hat{A}_{i}, \hat{B}_{i}, \hat{C}_{i}$.
Make an initial guess for the reduced matrices $\hat{A}_{i}, \hat{B}_{i}, \hat{C}_{i}$.
while not converged do
Perform the spectral decomposition of $\hat{A}_{i}$ and define:

$$
D_{i}=S \hat{A}_{i} S^{-1}, \tilde{B}_{i}=S \hat{B}_{i}, \quad \tilde{C}_{i}=\hat{C}_{i} S^{-1}
$$

Solve for $V$ and $W$ :

$$
\begin{aligned}
& V D_{i}+A V=-\left(B_{i} \mathcal{W}_{i}\right)\left(\tilde{B}_{i} \mathcal{W}_{i}\right)^{T} \\
& W D_{i}+A^{T} W=-C^{T} \tilde{C}_{i} .
\end{aligned}
$$

$$
V=\operatorname{orth}(V) \text { and } W=\operatorname{orth}(W) .
$$

Determine the reduced matrices:

$$
\hat{A}_{i}=\left(W^{T} V\right)^{-1} W^{T} A V, \quad \hat{B}_{i}=\left(W^{T} V\right)^{-1} W^{T} B_{i}, \quad \hat{C}_{i}=C V .
$$

end while
in (20). This results in the following reduced system

$$
\begin{align*}
d \hat{x}(t) & =\left[\hat{A} \hat{x}(t)+\hat{B}_{1} u(t)\right] d t+\hat{B}_{2} d M(t), \quad \hat{x}(0)=\hat{x}_{0},  \tag{23a}\\
\hat{y}(t) & =\hat{C} \hat{x}(t), \quad t \geq 0, \tag{23b}
\end{align*}
$$

where $\hat{x}(t) \in \mathbb{R}^{r}$ etc. with $r \ll n$. Again, we assume that $x_{0}=0$ and $\hat{x}_{0}=0$. We insert (22) into (21) and obtain

$$
\begin{align*}
& \sup _{t \in[0, T]} \mathbb{E}\|y(t)-\hat{y}(t)\| \leq\left\|C \mathrm{e}^{A \cdot} B_{1}-\hat{C} \mathrm{e}^{\hat{A} \cdot} \hat{B}_{1}\right\|_{\mathcal{L}^{2}(I)}\|u\|_{L_{T}^{2}}+\left\|C \mathrm{e}^{A \cdot} B_{2}-\hat{C} \mathrm{e}^{\hat{A} \cdot} \hat{B}_{2}\right\|_{\mathcal{L}^{2}\left(K^{\frac{1}{2}}\right)} \\
& \leq\left(\left\|C \mathrm{e}^{A \cdot} B_{1}-\hat{C} \mathrm{e}^{\hat{A} \cdot} \hat{B}_{1}\right\|_{\mathcal{L}^{2}(I)}+\left\|C \mathrm{e}^{A \cdot} B_{2}-\hat{C} \mathrm{e}^{\hat{A} \cdot} \hat{B}_{2}\right\|_{\mathcal{L}^{2}\left(K^{\frac{1}{2}}\right)}\right) \max \left\{1,\|u\|_{L_{T}^{2}}\right\}  \tag{24}\\
& \leq \sqrt{2}\left(\left\|C \mathrm{e}^{A \cdot} B_{1}-\hat{C} \mathrm{e}^{\hat{A} \cdot} \hat{B}_{1}\right\|_{\mathcal{L}^{2}(I)}^{2}+\left\|C \mathrm{e}^{A \cdot} B_{2}-\hat{C} \mathrm{e}^{\hat{A} \cdot} \hat{B}_{2}\right\|_{\mathcal{L}^{2}\left(K^{\left.\frac{1}{2}\right)}\right.}^{2}\right)^{\frac{1}{2}} \max \left\{1,\|u\|_{L_{T}^{2}}\right\} .
\end{align*}
$$

Now, we exploit that $\left\|L_{1}\right\|_{F}^{2}+\left\|L_{2}\right\|_{F}^{2}=\left\|\left[\begin{array}{ll}L_{1} & L_{2}\end{array}\right]\right\|_{F}^{2}$ for matrices $L_{1}, L_{2}$ of suitable dimension. Hence, we have

$$
\begin{aligned}
& \left\|C \mathrm{e}^{A s} B_{1}-\hat{C} \mathrm{e}^{\hat{A} s} \hat{B}_{1}\right\|_{F}^{2}+\left\|C \mathrm{e}^{A s} B_{2} K^{\frac{1}{2}}-\hat{C} \mathrm{e}^{\hat{A} s} \hat{B}_{2} K^{\frac{1}{2}}\right\|_{F}^{2} \\
& =\left\|C \mathrm{e}^{A s}\left[\begin{array}{ll}
B_{1} & B_{2} K^{\frac{1}{2}}
\end{array}\right]-\hat{C} \mathrm{e}^{\hat{A s} s}\left[\begin{array}{ll}
\hat{B}_{1} & \hat{B}_{2} K^{\frac{1}{2}}
\end{array}\right]\right\|_{F}^{2} \\
& =\left\|\left(C \mathrm{e}^{A s}\left[\begin{array}{ll}
B_{1} & B_{2}
\end{array}\right]-\hat{C} \mathrm{e}^{\hat{A} s}\left[\begin{array}{ll}
\hat{B}_{1} & \hat{B}_{2}
\end{array}\right]\right)\left[\begin{array}{cc}
I & 0 \\
0 & K^{\frac{1}{2}}
\end{array}\right]\right\|_{F}^{2} .
\end{aligned}
$$

```
Algorithm 3 Modified One Step Linear IRKA
    Input: The system matrices: \(A, B=\left[\begin{array}{ll}B_{1} & B_{2}\end{array}\right], C\). Weight: \(\mathcal{W}=\left[\begin{array}{cc}I & 0 \\ 0 & K^{\frac{1}{2}}\end{array}\right]\).
    Output: The reduced matrices: \(\hat{A}, \hat{B}=\left[\begin{array}{ll}\hat{B}_{1} & \hat{B}_{2}\end{array}\right], \hat{C}\).
    Make an initial guess for the reduced matrices \(\hat{A}, \hat{B}=\left[\begin{array}{ll}\hat{B}_{1} & \hat{B}_{2}\end{array}\right], \hat{C}\).
    while not converged do
        Perform the spectral decomposition of \(\hat{A}_{i}\) and define:
            \(D=S \hat{A} S^{-1}, \tilde{B}=S \hat{B}, \tilde{C}=\hat{C} S^{-1}\).
        Solve for \(V\) and \(W\) :
            \(V D+A V=-(B \mathcal{W})(\tilde{B} \mathcal{W})^{T}=-B\left[\begin{array}{ll}I & 0 \\ 0 & K\end{array}\right] \tilde{B}^{T}\),
\(W D+A^{T} W=-C^{T} \tilde{C}\)
            \(W D+A^{T} W=-C^{T} \tilde{C}\).
        \(V=\operatorname{orth}(V)\) and \(W=\operatorname{orth}(W)\).
        Determine the reduced matrices:
            \(\hat{A}=\left(W^{T} V\right)^{-1} W^{T} A V, \quad\left[\begin{array}{ll}\hat{B}_{1} & \hat{B}_{2}\end{array}\right]=\hat{B}=\left(W^{T} V\right)^{-1} W^{T} B, \quad \hat{C}=C V\).
    end while
```

Plugging this into (24) yields

$$
\sup _{t \in[0, T]} \mathbb{E}\|y(t)-\hat{y}(t)\| \leq \sqrt{2} \underbrace{\left\|C \mathrm{e}^{A \cdot}\left[\begin{array}{ll}
B_{1} & B_{2}
\end{array}\right]-\hat{C} \mathrm{e}^{\hat{A} \cdot}\left[\begin{array}{ll}
\hat{B}_{1} & \hat{B}_{2} \tag{25}
\end{array}\right]\right\|_{\mathcal{L}^{2}(\mathcal{W})}}_{=: \varepsilon_{3}} \max \left\{1,\|u\|_{L_{T}^{2}}\right\},
$$

where $\mathcal{W}=\left[\begin{array}{cc}I & 0 \\ 0 & K^{\frac{1}{2}}\end{array}\right]$. We now want to find a ROM such that $\mathcal{E}_{3}$ is small leading to a small output error. Again, $\mathcal{E}_{3}$ is a special case of the impulse response error of a stochastic system with multiplicative noise, where $N_{i}=0, B_{1}$ is replaced by $B=\left[\begin{array}{ll}B_{1} & B_{2}\end{array}\right]$ and $\mathcal{W}=\left[\begin{array}{cc}I & 0 \\ 0 & K^{\frac{1}{2}}\end{array}\right]$, cf. Section [2.2. Taking this into account in Algorithm $\mathbb{1}$ we obtain a method that satisfies the necessary optimality conditions for $\mathcal{E}_{3}$. This method is given in Algorithm 3 and again represents a modified version of linear IRKA. We can therefore apply Algorithm 3 in order to compute the reduced matrices $\left(\hat{A}, \hat{B}_{1}, \hat{B}_{2}, \hat{C}\right)$ in (23). This scheme is computationally cheaper than Algorithm 2but cannot be expected to perform in the same way. The reduced system (23) is less flexible than (201) in terms of the choice of the reduced order dimensions and coefficients and it furthermore relies on the minimization of a more conservative bound in (25) in comparison to (21). Note that with Algorithm 3, an alternative method to applying balanced truncation to (18) (see, for example [5) has been found.

## 4 Numerical experiments

We now apply Algorithm 1 to a large-scale stochastic differential equation with multiplicative noise as well as Algorithms 2 and 3 to a high dimensional equation with additive noise. In both scenarios, the examples are spatially discretized versions of
controlled stochastic damped wave equations. These equations are extensions of the numerical examples considered in [26, 28]. In particular, we study

$$
\frac{\partial^{2}}{\partial t^{2}} X(t, z)+\alpha \frac{\partial}{\partial t} X(t, z)=\frac{\partial^{2}}{\partial z^{2}} X(t, z)+f_{1}(z) u(t)+\left\{\begin{array}{l}
\sum_{i=1}^{2} f_{2, i}(z) \frac{\partial}{\partial t} w_{i}(t) \\
\sum_{i=1}^{2} g_{i}(z) X(t, z) \frac{\partial}{\partial t} w_{i}(t)
\end{array}\right.
$$

for $t \in[0, T]$ and $z \in[0, \pi]$, and $w_{1}$ and $w_{2}$ are standard Wiener processes that are correlated. Boundary and initial conditions are given by

$$
X(0, t)=0=X(\pi, t) \quad \text { and } \quad X(0, z),\left.\frac{\partial}{\partial t} X(t, z)\right|_{t=0} \equiv 0
$$

We assume that the quantity of interest is either the position of the midpoint

$$
Y(t)=\frac{1}{2 \epsilon} \int_{\frac{\pi}{2}-\epsilon}^{\frac{\pi}{2}+\epsilon} X(t, z) d z,
$$

or the velocity of the midpoint

$$
Y(t)=\frac{1}{2 \epsilon} \int_{\frac{\pi}{2}-\epsilon}^{\frac{\pi}{2}+\epsilon} \frac{\partial}{\partial t} X(t, z) d z
$$

where $\epsilon>0$ is small. We can transform the above wave equation into a first order system and discretize it using a spectral Galerkin method as in [26, 28]. This leads to the following stochastic differential equations

$$
\begin{align*}
d x(t) & =\left[A x(t)+B_{1} u(t)\right] d t+\left\{\begin{array}{l}
B_{2}\left[d w_{1}(t) d w_{2}(t)\right]^{T}, \\
\sum_{i=1}^{2} N_{i} x(t) d w_{i}(t),
\end{array}\right.  \tag{26a}\\
y(t) & =C x(t), \quad x(0)=0, \quad t \in[0, T], \tag{26b}
\end{align*}
$$

where $y \approx Y$ if the dimension $n$ of $x$ is sufficiently large. Let $n$ be even. Then, for $\ell=1, \ldots, \frac{n}{2}$, the associated matrices are

- $A=\operatorname{diag}\left(E_{1}, \ldots, E_{\frac{n}{2}}\right)$ with $E_{\ell}=\left(\begin{array}{cc}0 & \ell \\ -\ell & -\alpha\end{array}\right)$,
- $B_{1}=\left(b_{k}^{(1)}\right)_{k=1, \ldots, n}$ with

$$
b_{2 \ell-1}^{(1)}=0, \quad b_{2 \ell}^{(1)}=\sqrt{\frac{2}{\pi}}\left\langle f_{1}, \sin (\ell \cdot)\right\rangle_{L^{2}([0, \pi])},
$$

- $B_{2}=\left[\begin{array}{ll}B_{2}^{(1)} & B_{2}^{(2)}\end{array}\right]$, where $B_{2}^{(i)}=\left(b_{k}^{(2, i)}\right)_{k=1, \ldots, n}$ with

$$
b_{2 \ell-1}^{(2, i)}=0, \quad b_{2 \ell}^{(2, i)}=\sqrt{\frac{2}{\pi}}\left\langle f_{2, i}, \sin (\ell \cdot)\right\rangle_{L^{2}([0, \pi])},
$$

- $N_{i}=\left(n_{k j}^{(i)}\right)_{k, j=1, \ldots, n}$ with

$$
n_{(2 \ell-1) j}^{(i)}=0, \quad n_{(2 \ell) j}^{(i)}= \begin{cases}0, & \text { if } j=2 v, \\ \frac{2}{\pi v}\left\langle\sin (\ell \cdot), g_{i} \sin (v \cdot)\right\rangle_{L^{2}([0, \pi]),}, & \text { if } j=2 v-1,\end{cases}
$$

for $i=1,2, j=1, \ldots, n$ and $v=1, \ldots, \frac{n}{2}$,

- $C^{T}=\left(c_{k}\right)_{k=1, \ldots, n}$ with

$$
c_{2 \ell}=0 \quad c_{2 \ell-1}=\frac{1}{\sqrt{2 \pi} \ell^{2} \epsilon}\left[\cos \left(\ell\left(\frac{\pi}{2}-\epsilon\right)\right)-\cos \left(\ell\left(\frac{\pi}{2}+\epsilon\right)\right)\right],
$$

if we are interested in the position as output and

$$
c_{2 \ell-1}=0 \quad c_{2 \ell}=\frac{1}{\sqrt{2 \pi} \ell \epsilon}\left[\cos \left(\ell\left(\frac{\pi}{2}-\epsilon\right)\right)-\cos \left(\ell\left(\frac{\pi}{2}+\epsilon\right)\right)\right],
$$

if we are interested in the velocity as output.
For the following examples we choose $n=1000, T=1$ and the correlation $\mathbb{E}\left[w_{1}(t) w_{2}(t)\right]=$ $0.5 t$ meaning that $K=\left[\begin{array}{ll}1 & 0.5 \\ 0.5 & 1\end{array}\right]$.

Multiplicative noise We start with the multiplicative case in (26) and compute the ROM (3) by Algorithm [1. We use $\alpha=2, f_{1}(z)=\sin (3 z), g_{1}(z)=\mathrm{e}^{-\left(z-\frac{\pi}{2}\right)^{2}}$ and $g_{2}(z)=\mathrm{e}^{-\frac{1}{2}\left(z-\frac{\pi}{2}\right)^{2}}$. As input function we take $u(t)=e^{-0.1 t}$. We compute a ROM of dimension $r=6$ using the modified bilinear IRKA algorithm. Figure 1 shows the


Figure 1: Trajectory of position and velocity in the middle of the string.


Figure 2: Trajectory errors between full model and ROM of size $r=6$.
output trajectory, i.e., the position and the velocity in the middle of the string for one particular sample over the time window $[0,1]$. The trajectory of the velocity is as rough as the noise process, whereas the trajectory for the position is smooth as it is the integral of the velocity. Figure 2 plots the respective point-wise errors between the full model


Figure 3: Mean errors for position and velocity for $r=6$.


Figure 4: Worst case mean error for several dimensions of the ROM.
of dimension $n=1000$ and ROM of dimension $r=6$ for fixed trajectories. In Figure [3 the expected value of the output error of the full and the reduced model is plotted. Both in Figure 2 for the sample error and in Figure 3 for the mean error we observe that the output error is smaller for the position than for the velocity as this is a smoother function.
Finally, we use Algorithm $\mathbb{\square}$ in order to compute several ROMs of dimensions $r=$ $2, \ldots, 18$ and the corresponding worst case error $\sup _{t \in[0, T]} \mathbb{E}\|y(t)-\hat{y}(t)\|$, which we plotted in Figure 4 We observe that the error decreases as the size of the ROM increases, as one would expect. We also see that the output error in the position is consistently about one magnitude smaller than the output error in the velocity.

Additive noise For the additive case in (26) we use $\alpha=0.1, f_{1}(z)=\cos (2 z), f_{2,1}(z)=$ $\sin (z)$ and $f_{2,2}(z)=\sin (z) \exp \left(-(z-\pi / 2)^{2}\right)$. As input function we take $u \equiv 1$, such that $\|u\|_{L_{T}^{2}}=1$. For systems with additive noise (18a) we compare the two approaches considered in this paper for computing a ROM. In this example we only consider the position for the output. Qualitatively we obtain the same results for the velocity, the error is typically larger by one magnitude, as we have seen for the case of multiplicative noise. First, we use Theorems 2.2 and 2.4 (a) in order to compute the stochastic $\mathcal{H}_{2^{-}}$ distance $\|H-\hat{H}\|_{\mathcal{L}^{2}(\mathcal{W})}^{2}=\operatorname{tr}\left(C P C^{T}\right)-\operatorname{tr}\left(\hat{C} \hat{P} \hat{C}^{T}\right)$, between a full order model and a ROM for several dimensions of the reduced system, after computing an optimal reduced system of dimensions $r=1, \ldots, 16$. This allows us to compute $\varepsilon_{1}, \varepsilon_{2}$ and $\varepsilon_{3}$ from (21) and (25), which are plotted in Figure [5. Clearly, all errors decrease with increasing size of the ROM. However, we observe that the error $\mathcal{E}_{2}$ decreases much more rapidly than the errors $\varepsilon_{1}$ and $\varepsilon_{3}$, which behave similarly. Hence, if we would like to produce a ROM which has an error of at most $2 e-02$, we can deduce from Figure 5 that we need $r=16$ for the first reduced model in (20), $r=4$ in the second reduced model in (20) and $r=16$ for the reduced model (23). One can see that there are potential savings by using a smaller reduced dimension and still obtain sufficiently small errors. For this particular


Figure 5: Bounds $\mathcal{E}_{1}, \mathcal{E}_{2}$ and $\mathcal{E}_{3}$ from (21) and (25) for different dimensions of the ROM.


Figure 6: Mean errors for the output when using the one step or two step modified linear IRKA.
case we computed the ROMs both for Algorithm 3 with $r=16$ and Algorithm 2 with $r=16$ for the first reduced model and $r=4$ for the second one. The results are shown in Figure 6. We observe that the worst case mean errors of one step and two step modified linear IRKA are of size $2.99 e-04$ and $8.20 e-04$, respectively. The errors are of the same order, despite reducing the subsystem corresponding to the stochastic part to a smaller size.

## 5 Conclusions

We have derived optimization based model order reduction methods for stochastic systems. In particular we explained the link between the output error for stochastic systems, both with additive and multiplicative noise, and modified versions of the $\mathcal{H}_{2}$-norm for both linear and bilinear deterministic systems. We then developed optimality conditions for minimizing the error bounds computing reduced order models for stochastic systems. This approach revealed that modified versions of iterative rational Krylov methods are in fact natural schemes for reducing large scale stochastic systems with both additive and multiplicative noise. In addition, we have introduced a splitting method for linear systems with additive noise, where the deterministic and the noise part are treated independently. This is advantageous if one of the systems can be reduced easier than the other. It also allows for a different model order reduction method in one of the systems, which we did not discuss in this paper.

## A Ito product rule

In this section, we formulate an Ito product rule for semimartingales on a complete filtered probability space $\left(\Omega, \mathcal{F},\left(\mathcal{F}_{t}\right)_{t \geq 0}, \mathbb{P}\right)$. Semimartingales $Z$ are stochastic processes
that are càdlàg (right-continuous with exiting left limits) that have the representation

$$
Z(t)=M(t)+A(t), \quad t \geq 0,
$$

where $M$ is a càdlàg martingale with respect to $\left(\mathcal{F}_{t}\right)_{t \geq 0}$ and $A$ is a càdlàg process with bounded variation.
Now, let $Z_{1}, Z_{2}$ be scalar semimartingales with jumps $\Delta Z_{i}(s):=Z_{i}(s)-Z_{i}(s-)(i=1,2)$. Then, the Ito product formula is given as follows:

$$
\begin{equation*}
Z_{1}(t) Z_{2}(t)=Z_{1}(0) Z_{2}(0)+\int_{0}^{t} Z_{1}(s-) d Z_{2}(s)+\int_{0}^{t} Z_{2}(s-) d Z_{1}(s)+\left[Z_{1}, Z_{2}\right]_{t} \tag{27}
\end{equation*}
$$

for $t \geq 0$, see [21]. By [16, Theorem 4.52], the compensator process $\left[Z_{1}, Z_{2}\right.$ ] is given by

$$
\begin{equation*}
\left[Z_{1}, Z_{2}\right]_{t}=\left\langle M_{1}^{c}, M_{2}^{c}\right\rangle_{t}+\sum_{0 \leq s \leq t} \Delta Z_{1}(s) \Delta Z_{2}(s) \tag{28}
\end{equation*}
$$

for $t \geq 0$, where $M_{1}^{c}$ and $M_{2}^{c}$ are the square integrable continuous martingale parts of $Z_{1}$ and $Z_{2}$ (cf. [16, Theorem 4.18]). The process $\left\langle M_{1}^{c}, M_{2}^{c}\right\rangle$ is the uniquely defined angle bracket process that guarantees that $M_{1}^{c} M_{2}^{c}-\left\langle M_{1}^{c}, M_{2}^{c}\right\rangle$ is an $\left(\mathcal{F}_{t}\right)_{t \geq 0}$ - martingale, see [21, Proposition 17.2]. As a consequence of (27), we obtain the following product rule in the vector valued case.

Lemma A.1. Let $Y$ be an $\mathbb{R}^{d}$-valued and $Z$ be an $\mathbb{R}^{n}$-valued semimartingale, then we have
$Y(t) Z^{T}(t)=Y(0) Z^{T}(0)+\int_{0}^{t} d Y(s) Z^{T}(s-)+\int_{0}^{t} Y(s-) d Z^{T}(s)+\left(\left[Y^{(i)}, Z^{(j)}\right]_{t}\right)_{\substack{i=1, \ldots, d \\ j=1, \ldots, n}}$ for all $t \geq 0$, where $Y^{(i)}$ and $Z^{(j)}$ are the ith and $j$ th components of $Y$ and $Z$, respectively.

## B Proof of Lemma 2.1

In order to ease notation, we prove the result for $s=0$. Let us assume that $L$ and $\hat{L}$ have $m$ columns denoted by $l_{k}, \hat{l}_{k}$, such that we can decompose $L=\left[l_{1}, \ldots, l_{m}\right]$ and $\hat{L}=\left[\hat{l}_{1}, \ldots, \hat{l}_{m}\right]$. we obtain

$$
\begin{equation*}
\Phi(t) L \hat{L}^{T} \hat{\Phi}^{T}(t)=\sum_{k=1}^{m} Y_{k}(t) Z_{k}^{T}(t), \tag{29}
\end{equation*}
$$

where we set $Y_{k}(t)=\Phi(t) l_{k}$ and $Z_{k}(t)=\hat{\Phi}(t) \hat{l}_{k}$. We apply Corollary A.1 to every summand of (29). This yields
$Y_{k}(t) Z_{k}^{T}(t)=l_{k} \hat{l}_{k}^{T}+\int_{0}^{t} d\left(Y_{k}(s)\right) Z_{k}^{T}(s-)+\int_{0}^{t} Y_{k}(s-) d Z_{k}^{T}(s)+\left(\left[e_{i_{1}}^{T} Y_{k}, e_{i_{2}}^{T} Z_{k}\right]_{t}\right)_{\substack{i_{1}=1, \ldots, n \\ i_{2}=1, \ldots, r}}^{\substack{ \\\hline}}$
where $e_{i_{1}}$ and $e_{i_{2}}$ are unit vectors of suitable dimension. We determine the expected value of the compensator process of $e_{i_{1}}^{T} Y_{k}$ and $e_{i_{2}}^{T} Z_{k}$. Using (28), it can be seen that this process only depends on the jumps and the continuous martingale parts of $Y_{k}$ and $Z_{k}$. Taking (4) and the corresponding equation for the fundamental solution of the reduced system into account, we see that

$$
\mathcal{M}_{k}(t):=\sum_{i=1}^{m_{2}} \int_{0}^{t} N_{i} Y_{k}(s) d M_{i}(s), \quad \hat{\mathcal{M}}_{k}(t):=\sum_{i=1}^{m_{2}} \int_{0}^{t} \hat{N}_{i} Z_{k}(s) d M_{i}(s)
$$

are the martingale parts of $Y_{k}$ and $Z_{k}$ which furthermore contain all the jumps of these processes. This gives

$$
\left[e_{i_{1}}^{T} Y_{k}, e_{i_{2}}^{T} Z_{k}\right]_{t}=\left[e_{i_{1}}^{T} \mathcal{M}_{k}, e_{i_{2}}^{T} \hat{\mathcal{M}}_{k}\right]_{t}
$$

We apply Corollary A. 1 to $\mathcal{M}_{k} \hat{\mathcal{N}}_{k}^{T}$ and obtain
$\mathcal{M}_{k}(t) \hat{\mathcal{M}}_{k}^{T}(t)=\int_{0}^{t} d\left(\mathcal{M}_{k}(s)\right) \hat{\mathcal{M}}_{k}^{T}(s-)+\int_{0}^{t} \mathcal{M}_{k}(s-) d \hat{\mathcal{M}}_{k}^{T}(s)+\left(\left[e_{i_{1}}^{T} \mathcal{M}_{k}, e_{i_{2}}^{T} \hat{\mathcal{M}}_{k}\right]_{t}\right)_{\substack{i_{1}=1, \ldots, n \\ i_{2}=1, \ldots, r}}$.
Since $\mathcal{M}_{k}$ and $\hat{\mathcal{M}}_{k}$ are mean zero martingales [23], the above integrals with respect to these processes have mean zero as well [18]. Hence, we have

$$
\mathbb{E}\left[\mathcal{M}_{k}(t) \hat{\mathcal{M}}_{k}^{T}(t)\right]=\mathbb{E}\left(\left[e_{i_{1}}^{T} \mathcal{M}_{k}, e_{i_{2}}^{T} \hat{\mathcal{M}}_{k}\right]_{t}\right)_{\substack{i_{1}=1, \ldots, n \\ i_{2}=1, \ldots, r}}=\mathbb{E}\left(\left[e_{i_{1}}^{T} Y_{k}, e_{i_{2}}^{T} Z_{k}\right]_{t}\right)_{\substack{i_{1}=1, \ldots, n \\ i_{2}=1, \ldots, r}}
$$

We apply the expected value to both sides of (30) leading to
$\mathbb{E}\left[Y_{k}(t) Z_{k}^{T}(t)\right]=l_{k} \hat{l}_{k}^{T}+\mathbb{E}\left[\int_{0}^{t} d\left(Y_{k}(s)\right) Z_{k}^{T}(s-)\right]+\mathbb{E}\left[\int_{0}^{t} Y_{k}(s-) d Z_{k}^{T}(s)\right]+\mathbb{E}\left[\mathcal{M}_{k}(t) \hat{\mathcal{M}}_{k}^{T}(t)\right]$.
We insert $d Y_{k}$ and $d Z_{k}$ (given through (4)) into the above equation and exploit that an Ito integral has mean zero, and obtain
$\mathbb{E}\left[Y_{k}(t) Z_{k}^{T}(t)\right]=l_{k} \hat{l}_{k}^{T}+\int_{0}^{t} A \mathbb{E}\left[Y_{k}(s) Z_{k}^{T}(s)\right] d s+\int_{0}^{t} \mathbb{E}\left[Y_{k}(s) Z_{k}^{T}(s)\right] \hat{A}^{T} d s+\mathbb{E}\left[\mathcal{M}_{k}(t) \hat{\mathcal{M}}_{k}^{T}(t)\right]$.
Notice that we replaced the left limits by the function values above and hence changed the integrand only on Lebesgue zero sets since the processes have only countably many jumps on bounded time intervals [3]. The Ito isometry [23] now yields

$$
\mathbb{E}\left[\mathcal{M}_{k}(t) \hat{\mathcal{M}}_{k}^{T}(t)\right]=\sum_{i, j=1}^{m_{2}} \int_{0}^{t} N_{i} \mathbb{E}\left[Y_{k}(s) Z_{k}^{T}(s)\right] \hat{N}_{j}^{T} k_{i j} d s
$$

Combining this result with (29) proves the claim of this lemma.

## C Proof of Theorem 2.5

We only show the result for the optimality condition (c) in (13). All the other optimality conditions (a), (b) and (d) are derived similarly. We first reformulate optimality condition (c). Defining $\hat{\Psi}_{i}:=\sum_{j=1}^{m_{2}} \hat{N}_{j} k_{i j}$ and $\Psi_{i}:=\sum_{j=1}^{m_{2}} N_{j} k_{i j}$ for $i=1, \ldots, m_{2}$, we have that

$$
\begin{aligned}
& \hat{Q} \hat{\Psi}_{i} \hat{P}=Q_{2} \Psi_{i} P_{2} \Leftrightarrow\left(S^{-T} \hat{Q}\right) \hat{\Psi}_{i}\left(\hat{P} S^{T}\right)=\left(S^{-T} Q_{2}\right) \Psi_{i}\left(P_{2} S^{T}\right) \\
& \Leftrightarrow \operatorname{tr}\left(\left(S^{-T} \hat{Q}\right) \hat{\Psi}_{i}\left(\hat{P} S^{T}\right) e_{m} e_{k}^{T}\right)=\operatorname{tr}\left(\left(S^{-T} Q_{2}\right) \Psi_{i}\left(P_{2} S^{T}\right) e_{m} e_{k}^{T}\right) \quad \forall k, m=1, \ldots, r,
\end{aligned}
$$

where $S$ is the factor of the spectral decomposition of $\hat{A}$. Using the relation between the trace and the vectorization $\operatorname{vec}(\cdot)$ of a matrix as well as the Kronecker product $\otimes$ of two matrices, we obtain an equivalent formulation of the optimality condition (c) in (13), i.e. $\hat{Q} \hat{\Psi}_{i} \hat{P}=Q_{2} \Psi_{i} P_{2}$ is equivalent to

$$
\begin{equation*}
\Leftrightarrow \operatorname{vec}^{T}\left(\hat{Q} S^{-1}\right)\left(e_{k} e_{m}^{T} \otimes \hat{\Psi}_{i}\right) \operatorname{vec}\left(\hat{P} S^{T}\right)=\operatorname{vec}^{T}\left(Q_{2}^{T} S^{-1}\right)\left(e_{k} e_{m}^{T} \otimes \Psi_{i}\right) \operatorname{vec}\left(P_{2} S^{T}\right) \tag{31}
\end{equation*}
$$

for all $k, m=1, \ldots, r$. We now use Algorithm $\square$ to show that this equality holds. We multiply (10) with $S^{T}$ and (14) with $S^{-1}$ from the right to find the equations for $\hat{P} S^{T}$ and $\hat{Q} S^{-1}$. Subsequently, we vectorize these equations and get

$$
\begin{align*}
\operatorname{vec}\left(\hat{Q} S^{-1}\right) & =-\hat{\mathcal{K}}^{-T} \operatorname{vec}\left(\hat{C}^{T} \tilde{C}\right) \quad \text { and }  \tag{32}\\
\operatorname{vec}\left(\hat{P} S^{T}\right) & =-\hat{\mathcal{K}}^{-1} \operatorname{vec}\left(\hat{B}_{1} \mathcal{W}\left(\tilde{B}_{1} \mathcal{W}\right)^{T}\right), \tag{33}
\end{align*}
$$

where $\hat{\mathcal{K}}:=(I \otimes \hat{A})+(D \otimes I)+\sum_{i, j=1}^{m_{2}}\left(\tilde{N}_{i} \otimes \hat{N}_{j}\right) k_{i j}$ and recalling that $D=S \hat{A} S^{-1}, \tilde{B}_{1}=$ $S \hat{B}_{1}, \tilde{C}=\hat{C} S^{-1}, \tilde{N}_{i}=S \hat{N}_{i} S^{-1}$.
With $\mathcal{K}:=(I \otimes A)+(D \otimes I)+\sum_{i, j=1}^{m_{2}}\left(\tilde{N}_{i} \otimes N_{j}\right) k_{i j}$, and the definition of the reduced matrices $\hat{A}=\left(W^{T} V\right)^{-1} W^{T} A V, \hat{B}_{1}=\left(W^{T} V\right)^{-1} W^{T} B_{1}, \hat{C}=C V$ and $\hat{N}_{i}=\left(W^{T} V\right)^{-1} W^{T} N_{i} V$ in Algorithm $\rceil$ we furthermore have that

$$
\begin{aligned}
-\operatorname{vec}\left(\hat{C}^{T} \tilde{C}\right) & =-\operatorname{vec}\left(V^{T} C^{T} \tilde{C}\right)=-\left(I \otimes V^{T}\right) \operatorname{vec}\left(C^{T} \tilde{C}\right)=\left(I \otimes V^{T}\right) \mathcal{K}^{T} \operatorname{vec}(W) \\
& =\left(I \otimes V^{T}\right) \mathcal{K}^{T} \operatorname{vec}\left(W\left(V^{T} W\right)^{-1} V^{T} W\right) \\
& =\left(I \otimes V^{T}\right) \mathcal{K}^{T}\left(I \otimes W\left(V^{T} W\right)^{-1} V^{T}\right) \operatorname{vec}(W)=\hat{\mathcal{K}}^{T}\left(I \otimes V^{T}\right) \operatorname{vec}(W)
\end{aligned}
$$

and

$$
\begin{aligned}
-\operatorname{vec}\left(\hat{B}_{1} \mathcal{W}\left(\tilde{B}_{1} \mathcal{W}\right)^{T}\right) & =-\operatorname{vec}\left(\left(W^{T} V\right)^{-1} W^{T} B_{1} \mathcal{W}\left(\tilde{B}_{1} \mathcal{W}\right)^{T}\right) \\
& =-\left(I \otimes\left(W^{T} V\right)^{-1} W^{T}\right) \operatorname{vec}\left(B_{1} \mathcal{W}\left(\tilde{B}_{1} \mathcal{W}\right)^{T}\right) \\
& =\left(I \otimes\left(W^{T} V\right)^{-1} W^{T}\right) \mathcal{K} \operatorname{vec}(V) \\
& =\left(I \otimes\left(W^{T} V\right)^{-1} W^{T}\right) \mathcal{K} \operatorname{vec}\left(V\left(W^{T} V\right)^{-1} W^{T} V\right) \\
& =\left(I \otimes\left(W^{T} V\right)^{-1} W^{T}\right) \mathcal{K}\left(I \otimes V\left(W^{T} V\right)^{-1} W^{T}\right) \operatorname{vec}(V) \\
& =\hat{\mathcal{K}}\left(I \otimes\left(W^{T} V\right)^{-1} W^{T}\right) \operatorname{vec}(V)
\end{aligned}
$$

We insert both results into (32) and obtain expressions for $\operatorname{vec}\left(\hat{Q} S^{-1}\right)$ and $\operatorname{vec}\left(\hat{P} S^{T}\right)$ in terms of the projection matrices from Algorithm 1 .

$$
\operatorname{vec}\left(\hat{Q} S^{-1}\right)=\left(I \otimes V^{T}\right) \operatorname{vec}(W) \quad \text { and } \quad \operatorname{vec}\left(\hat{P} S^{T}\right)=\left(I \otimes\left(W^{T} V\right)^{-1} W^{T}\right) \operatorname{vec}(V)
$$

Hence, the left hand side of the optimality condition (31) can be written as

$$
\begin{aligned}
& \operatorname{vec}^{T}\left(\hat{Q} S^{-1}\right)\left(e_{k} e_{m}^{T} \otimes \hat{\Psi}_{i}\right) \operatorname{vec}\left(\hat{P} S^{T}\right) \\
& =\operatorname{vec}^{T}(W)(I \otimes V)\left(e_{k} e_{m}^{T} \otimes \hat{\Psi}_{i}\right)\left(I \otimes\left(W^{T} V\right)^{-1} W^{T}\right) \operatorname{vec}(V) \\
& \left.=\operatorname{vec}^{T}(W)\left(I \otimes V\left(W^{T} V\right)^{-1} W^{T}\right)\right)\left(e_{k} e_{m}^{T} \otimes \Psi_{i}\right)\left(I \otimes V\left(W^{T} V\right)^{-1} W^{T}\right) \operatorname{vec}(V) \\
& =\operatorname{vec}^{T}(W)\left(e_{k} e_{m}^{T} \otimes \Psi_{i}\right) \operatorname{vec}(V),
\end{aligned}
$$

where we have used properties of the Kronecker product again. It remains to show that $P_{2} S^{T}=V$ and $Q_{2}^{T} S^{-1}=W$ for the optimality condition to hold. This is obtained by multiplying (11) with $S^{T}$ from the right and (15) with $S^{-T}$ from the left. Hence (31) holds which concludes the proof.

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[^1]:    ${ }^{1}\left(\mathcal{F}_{t}\right)_{t \geq 0}$ is right continuous and complete.

