# STABILITY EQUIVALENCE AMONG STOCHASTIC DIFFERENTIAL EQUATIONS AND STOCHASTIC DIFFERENTIAL EQUATIONS WITH PIECEWISE CONTINUOUS ARGUMENTS AND CORRESPONDING EULER-MARUYAMA METHODS* 


#### Abstract

In this paper, we consider the equivalence of the $p$ th moment exponential stability for stochastic differential equations (SDEs), stochastic differential equations with piecewise continuous arguments (SDEPCAs) and the corresponding Euler-Maruyama methods EMSDEs and EMSDEPCAs. We show that if one of the SDEPCAs, SDEs, EMSDEs and EMSDEPCAs is $p$ th moment exponentially stable, then any of them is $p$ th moment exponentially stable for a sufficiently small step size $h$ and $\tau$ under the global Lipschitz assumption on the drift and diffusion coefficients


Key words. Exponential stability, Stochastic differential equations, Numerical solutions, Piecewise continuous arguments

AMS subject classifications. $60 \mathrm{H} 10,65 \mathrm{C} 20,65 \mathrm{~L} 20,60 \mathrm{H} 35$

1. Introduction. Stochastic differential equations (SDEs) have been widely used in many branches of science and industry $[1,4,8,9,28,34]$. There is an extensive literature in stochastic stability (e.g. the moment exponential stability or almost sure exponential stability) $[1,5,9,18,25,36,37]$. One of the powerful techniques in the study of stochastic stability is the method of Lyapunov functions. In the absence of an appropriate Lyapunov function, we may carry out careful numerical simulations using a numerical method, say the Euler-Maruyama (EM) method [see e.g. $2,12,16,17,19,26,33,39$ ] with a small step size. Does the main question arise whether the numerical solutions can reproduce and predict the stability of the underlying solutions?

The case that stochastic stability of the general nonlinear equation and that of the numerical method are equivalent for a sufficiently small step size can be founded in $[6,13,15,22,27,30]$, while for the linear equation in $[3,11,35]$. Higham et al. in [14] showed that when the SDE obeys a linear growth condition, the EM method recovers almost surely exponential stability.

In this paper, we consider the following stochastic differential equation with piecewise continuous argument (SDEPCA)

$$
\begin{equation*}
d x(t)=\left(f(x(t))+u_{1}(x([t / \tau) \tau))\right] d t+\left(g(x(t))+u_{2}(x([t / \tau] \tau))\right) d w(t) \tag{1.1}
\end{equation*}
$$

and the stochastic differential equation (SDE)

$$
\begin{equation*}
d y(t)=\left(f(y(t))+u_{1}(y(t))\right) d t+\left(g(y(t))+u_{2}(y(t))\right) d w(t) . \tag{1.2}
\end{equation*}
$$

We also consider the applications of EM method to SDEPCA (1.1) and SDE (1.2), respectively

$$
\begin{equation*}
X_{n+1}=X_{n}+\left(f\left(X_{n}\right)+u_{1}\left(X_{[n / m] m}\right)\right) h+\left(g\left(X_{n}\right)+u_{2}\left(X_{[n / m] m}\right)\right) \Delta w_{n}, \tag{1.3}
\end{equation*}
$$

[^0]\[

$$
\begin{equation*}
Y_{n+1}=Y_{n}+\left(f\left(Y_{n}\right)+u_{1}\left(Y_{n}\right)\right) h+\left(g\left(Y_{n}\right)+u_{2}\left(Y_{n}\right)\right) \Delta w_{n}, \tag{1.4}
\end{equation*}
$$

\]

where $h=\frac{\tau}{m}, m \in \mathbb{N}^{+}$. We refer to (1.3) and (1.4) by the terms EMSDEPCA (1.3) and EMSDE (1.4), respectively. The main purpose of the present paper is to show that if one of the SDEPCAs (1.1), SDEs (1.2), EMSDEPCA (1.3) and EMSDE (1.4) is $p$ th moment exponential stable, then so are the others for a sufficiently small step size $h$ and $\tau$ under a global Lipschitz assumption on the drift and diffusion coefficients. In order to do this, we shall concentrate on the following questions:
(Q1) If for a sufficiently small $\tau$, the SDEPCA (1.1) is $p$ th moment exponentially stable, can we confidently infer that the SDE (1.2) is $p$ th moment exponentially stable?
(Q2) For a sufficiently small step size $h$, does the EMSDE (1.4) reproduce the $p$ th moment exponential stability of the underlying SDE (1.2)?
(Q3) For a sufficiently small $\tau$, the EMSDEPCA (1.3) can preserve the $p$ th moment exponential stability of EMSDE (1.4)?
(Q4) If the EMSDEPCA (1.3) is $p$ th moment exponentially stable, will the SDEPCA (1.1) be the $p$ th moment exponentially stable for a sufficiently small step size $h$ ?
It is known that the positive answer to (Q2) for SDE in case $p=2$ can be founded in [13]. The stochastic differential equation with piecewise continuous arguments (SDEPCA) has been studied extensively [see e.g. 7, 21, 29, 31, 32, 40], and in the case of $\tau=1$, we refer to [23,24]. Mao in [29] is the first paper that investigated the mean square exponentially stable for SDEPCA. The positive answer to the converse problem of (Q1), we refer to $[10,29,32,38]$.

In this paper, we will give the positive answer for (Q1), (Q2), (Q3), (Q4). In section 2, we describe the SDEPCA and EM methods along with the definitions of $p$ th moment exponential stability for SDE, SDEPCA, EMSDE, EMSDEPCA. Section 3, section 4 , section 5 , section 6 answer the questions (Q1), (Q3), (Q4), (Q2) respectively, the final conclusions are stated in the last section.
2. Perilimaries. Throughout this paper, unless otherwise specified, we will use the following notations. If $\mathbf{A}$ is a vector or matrix, its transpose is denoted by $\mathbf{A}^{\mathrm{T}}$. If $x \in \mathbf{R}^{n}$, then $|x|$ is the Euclidean norm. If $\mathbf{A}$ is a matrix, we let $|\mathbf{A}|=\sqrt{\operatorname{trace}\left(\mathbf{A}^{\mathrm{T}} \mathbf{A}\right)}$ be its trace norm. If $D$ is a set, its indicator function is denoted by $\mathbf{1}_{D}$. Moreover, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\left\{\mathcal{F}_{t}\right\}_{t \geq 0}$ satisfying the usual conditions (that is, it is right continuous and increasing while $\mathcal{F}_{0}$ contains all $\mathbb{P}$-null sets), and let $\mathbb{E}$ denote the expectation corresponding to $\mathbb{P}$. Let $B(t)$ be a $m$-dimensional Brownian motion defined on the space. Throughout this paper, we set $p \geq 2$.

In this paper, we deal with the following $d$-dimensional nonlinear stochastic differential equations with piecewise continuous arguments (SDEPCAs)

$$
\left\{\begin{array}{l}
d x(t)=\left[f(x(t))+u_{1}(x([t / \tau] \tau))\right] d t+\left[g(x(t))+u_{2}(x([t / \tau] \tau))\right] d w(t)  \tag{2.1}\\
x(0)=x_{0} \in \mathbb{R}^{d}
\end{array}\right.
$$

on $t \geq 0$, where $w(t)$ is an $m$-dimensional Brownian motion, $f: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}, g: \mathbb{R}^{d} \rightarrow$ $\mathbb{R}^{d \times m}, u_{1}: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}$ and $u_{2}: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d \times m} . \tau$ is a positive constant, $[t / \tau]$ is the integer part of $t / \tau$. We denote $x(t)$ the solution of (2.1) with initial data $x(0)=x_{0}$ and $y(t)$ the solution of the following SDEs

$$
\begin{equation*}
d y(t)=\left[f(y(t))+u_{1}(y(t))\right] d t+\left[g(y(t))+u_{2}(y(t))\right] d w(t) \tag{2.2}
\end{equation*}
$$

on $t \geq 0$ with initial data $y(0)=x_{0}$.
In the present paper, we also deal with the application of EM method to SDEPCA (2.1) and $\operatorname{SDE}(2.2)$. We note that $[t / \tau] \tau=n \tau$ for $t \in[n \tau,(n+1) \tau), n=0,1,2, \cdots$, a natural choice for $h$ is $h=\frac{\tau}{m}, m \in \mathbb{N}^{+}$. Hence, we have

$$
\begin{gather*}
X_{n+1}=X_{n}+\left(f\left(X_{n}\right)+u_{1}\left(X_{[n / m] m}\right)\right) h+\left(g\left(X_{n}\right)+u_{2}\left(X_{[n / m] m}\right)\right) \Delta w_{n}  \tag{2.3}\\
Y_{n+1}=Y_{n}+\left(f\left(Y_{n}\right)+u_{1}\left(Y_{n}\right)\right) h+\left(g\left(Y_{n}\right)+u_{2}\left(Y_{n}\right)\right) \Delta w_{n} \tag{2.4}
\end{gather*}
$$

where $X_{n}$ and $Y_{n}$ are the approximations of $x(t)$ and $y(t)$ at grid points $t=t_{n}=n h$, $n=0,1,2, \cdots$, respectively, $\Delta w_{n}=w\left(t_{n+1}\right)-w\left(t_{n}\right)$. Let $n=k m+l, k \in \mathbb{N}^{+}$, $l=0,1, \cdots, m-1$. Then (2.3) and (2.4) would reduce to

$$
\begin{align*}
& X_{k m+l+1}=X_{k m+l}+\left(f\left(X_{k m+l}\right)+u_{1}\left(X_{k m}\right)\right) h+\left(g\left(X_{k m+l}\right)+u_{2}\left(X_{k m}\right)\right) \Delta w_{k m+l}  \tag{2.5}\\
& Y_{k m+l+1}=Y_{k m+l}+\left(f\left(Y_{k m+l}\right)+u_{1}\left(Y_{k m+l}\right)\right) h+\left(g\left(Y_{k m+l}\right)+u_{2}\left(Y_{k m+l}\right)\right) \Delta w_{k m+l} \tag{2.6}
\end{align*}
$$

Remark 2.1. If we choose $h=\tau$, then (2.3) and (2.4) are the same and (2.5) and (2.6) are the same.

In spite of the simplicity of the EM method, explicit EM method is the most popular for approximating the solution of the SDE under global Lipschitz condition [see $12,19,33$ ] and has often been used successfully in actual calculations. For further analysis it is more convenient to use continuous-time approximations,
$x_{\Delta}(t)=x_{0}+\int_{0}^{t} f\left(\bar{x}_{\Delta}(s)\right)+u_{1}\left(\bar{x}_{\Delta}([s / \tau] \tau)\right) d s+\int_{0}^{t} g\left(\bar{x}_{\Delta}(s)\right)+u_{2}\left(\bar{x}_{\Delta}([s / \tau] \tau)\right) d w(s)$,

$$
\begin{equation*}
y_{\Delta}(t)=x_{0}+\int_{0}^{t} f\left(\bar{y}_{\Delta}(s)\right)+u_{1}\left(\bar{y}_{\Delta}(s)\right) d s+\int_{0}^{t} g\left(\bar{y}_{\Delta}(s)\right)+u_{2}\left(\bar{y}_{\Delta}(s)\right) d w(s) \tag{2.8}
\end{equation*}
$$

where

$$
\bar{x}_{\Delta}(t)=\sum_{n=0}^{\infty} X_{n} \mathbf{1}_{\left[t_{n}, t_{n+1}\right)}(t), \quad \bar{y}_{\Delta}(t)=\sum_{n=0}^{\infty} Y_{n} \mathbf{1}_{\left[t_{n}, t_{n+1}\right)}(t), \quad \forall t \geq 0
$$

We observe that $x_{\Delta}\left(t_{n}\right)=\bar{x}_{\Delta}\left(t_{n}\right)=X_{n}$ and $y_{\Delta}\left(t_{n}\right)=\bar{y}_{\Delta}\left(t_{n}\right)=Y_{n}$. Consequently,

$$
x_{\Delta}([t / \tau] \tau)-\bar{x}_{\Delta}([t / \tau] \tau)=0, \quad y_{\Delta}([t / \tau] \tau)-\bar{y}_{\Delta}([t / \tau] \tau)=0
$$

In this paper, we impose the following standing hypothesis.
Assumption 2.2. Assume that there exists a positive constant $K$ such that

$$
|f(x)-f(y)| \vee|g(x)-g(y)| \vee\left|u_{1}(x)-u_{1}(y)\right| \vee\left|u_{2}(x)-u_{2}(y)\right| \leq K|x-y|
$$

for all $x, y \in \mathbb{R}^{d}$. Assume also that $f(0)=0, g(0)=0, u_{1}(0)=0$ and $u_{2}(0)=0$.
Assumption 2.2 implies that

$$
|f(x)| \vee|g(x)| \vee\left|u_{1}(x)\right| \vee\left|u_{2}(x)\right| \leq K|x|
$$

for all $x \in \mathbb{R}^{d}$.
We now give our basic definitions, which is cited from [28].

DEfinition 2.3. The equations $S D E P C A$ (2.1) and $S D E$ (2.2) are said to be pth moment exponentially stable if there exist positive constants $M_{1}, \gamma_{1}, M_{2}$ and $\gamma_{2}$ such that

$$
\begin{equation*}
\mathbb{E}|x(t)|^{p} \leq M_{1}\left|x_{0}\right|^{p} e^{-\gamma_{1} t}, \quad \forall t \geq 0 \tag{2.9}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathbb{E}|y(t)|^{p} \leq M_{2}\left|x_{0}\right|^{p} e^{-\gamma_{2} t}, \quad \forall t \geq 0 \tag{2.10}
\end{equation*}
$$

for any $x_{0} \in \mathbb{R}^{d}$.
Definition 2.4. For any given step size $h>0$, the Euler-Maruyama numerical methods EMSDEPCA (2.3) and EMSDE (2.4) are said to be pth moment exponentially stable, if there exist positive constants $\lambda_{1}, L_{1}, \lambda_{2}$ and $L_{2}$ such that

$$
\begin{align*}
& \mathbb{E}\left|X_{n}\right|^{p} \leq L_{1}\left|x_{0}\right|^{p} e^{-\lambda_{1} n h},  \tag{2.11}\\
& \mathbb{E}\left|Y_{n}\right|^{p} \leq L_{2}\left|x_{0}\right|^{p} e^{-\lambda_{2} n h}, \tag{2.12}
\end{align*}
$$

for any $x_{0} \in \mathbb{R}^{d}, n \in \mathbb{N}$.
It is known that under Assumption 2.2, for any initial value $x_{0}$ given at time $t=0$, the SDEPCA (2.1) and SDE (2.2) have a unique continuous solutions on $t \geq 0$ (see [28]). To emphasize the role of the initial value, we denote the solution $x(t)$ and $y(t)$ by $x\left(t ; 0, x_{0}\right)$ and $y\left(t ; 0, x_{0}\right)$, respectively. Of course, we may consider a more general case, for example, where the SDEs and the SDEPCAs have a random initial data $x(0)=\xi$ which is an $\mathcal{F}_{0}$-measurable $\mathbb{R}^{d}$-valued random variable such that $\mathbb{E}|\xi|^{p}<\infty, \forall p \geq 0$. In this case, by the Markov property of the solution, we can easily see that the solution satisfies

$$
\mathbb{E}|x(t)|^{p}=\mathbb{E}\left(\mathbb{E}\left(|x(t)|^{p} \mid \mathcal{F}_{0}\right)\right) \leq \mathbb{E}\left(M_{1}|\xi|^{p} e^{-\gamma_{1} t}\right)=M_{1} \mathbb{E}|\xi|^{p} e^{-\gamma t}
$$

It is therefore clear why it is enough to consider only the deterministic initial value $x(0)=x_{0}$.

Let $y(t ; s, y(s))$ be the solution of $\operatorname{SDE}(2.2)$ for $t>s$ with initial value $y(s)$. It is also known that the solutions to $\operatorname{SDE}$ (2.2) have the following flow property,

$$
y\left(t ; 0, x_{0}\right)=y(t ; s, y(s)), \quad \forall t \geq s>0
$$

Moreover, the solutions of $\operatorname{SDE}$ (2.2) also have the time-homegeneous Markov property. Hence (2.10) implies

$$
\mathbb{E}|y(t ; s, \xi)|^{p} \leq M_{2} \mathbb{E}|\xi|^{p} e^{-\gamma_{2}(t-s)}, \quad \forall t \geq s
$$

Given $y_{k}$ for some $k \in \mathbb{N}^{+}$, the process $\left\{y_{n}\right\}_{n \geq k}$ can be regard as the process which is produced by EM method applied to the $\operatorname{SDE}$ (2.2) on $t \geq k h$ with the initial value $y(k h)=y_{k}$. In other words, the process $\left\{y_{n}\right\}_{n \geq k}$ is time-homogeneous Markov process. Hence, (2.12) is equivalent to the following more general form.

$$
\begin{equation*}
\mathbb{E}\left|y_{n}\right|^{p} \leq L_{2} \mathbb{E}\left|y_{k}\right|^{p} e^{-\lambda_{2}(n-k) h} \tag{2.13}
\end{equation*}
$$

Due to the special feature of the SDEPCA (2.1), the solution $x(t)$ has flow property and the Markov property at the discrete time $t=k \tau\left(k \in \mathbb{N}^{+}\right)$. Hence

$$
x\left(t ; 0, x_{0}\right)=x(t ; k \tau, x(k \tau))
$$

and (2.9) implies

$$
\begin{equation*}
\mathbb{E}|x(t)|^{p} \leq M_{1} \mathbb{E}|x(k \tau)|^{p} e^{-\gamma_{1}(t-k \tau)}, \quad t \geq k \tau \tag{2.14}
\end{equation*}
$$

Given $x_{k m}$ for some $k \in \mathbb{N}^{+}$, the process $\left\{x_{n}\right\}_{n \geq k m}$ can be regard as the process which is produced by EM method applied to the SDEPCA (2.1) on $t \geq k \tau$ with the initial value $x(k \tau)=x_{k m}$. The process $\left\{x_{n}\right\}_{n \geq k m}$ is time-homogeneous Markov process. Hence, (2.11) is equivalent to the following more general form.

$$
\begin{equation*}
\mathbb{E}\left|x_{n}\right|^{p} \leq L_{1} \mathbb{E}\left|x_{k m}\right|^{p} e^{-\lambda_{1}(n-k m) h} \tag{2.15}
\end{equation*}
$$

3. SDE (2.2) shares the stability with SDEPCA (2.1). In this section, we shall investigate that if the SDEPCA (2.1) is $p$ th moment exponentially stable with a sufficiently small $\tau$, then the $\operatorname{SDE}(2.2)$ is also $p$ th moment exponentially stable, i.e. give the positive answer to (Q1). To show this, we need several lemmas. The last lemma estimates the difference in the $p$ th moment between the solution of the SDE (2.2) and that of the SDEPCA (2.1).

Lemma 3.1. Assume that Assumption 2.2 holds. Then for any given constant $T \geq 0$, we have

$$
\begin{equation*}
\sup _{0 \leq t \leq T} \mathbb{E}|x(t)|^{p} \leq H_{1}(T, p, K)\left|x_{0}\right|^{p} \tag{3.1}
\end{equation*}
$$

where $H_{1}(T, p, K)=e^{2 p K[1+(p-1) K] T}$.
Proof. In view of Itô formula and Assumption 2.2, we obtain

$$
\begin{aligned}
\mathbb{E}|x(v)|^{p} \leq & \left|x_{0}\right|^{p}+\mathbb{E} \int_{0}^{v} p|x(s)|^{p-1}\left|f(x(s))+u_{1}(x([s / \tau] \tau))\right| \\
& +\frac{p(p-1)}{2}|x(s)|^{p-2}\left|g(x(s))+u_{2}(x([s / \tau] \tau))\right|^{2} d s \\
\leq & \left|x_{0}\right|^{p}+\mathbb{E} \int_{0}^{v} p K|x(s)|^{p-1}(|x(s)|+|x([s / \tau] \tau)|) \\
& +p(p-1) K^{2}|x(s)|^{p-2}\left(|x(s)|^{2}+|x([s / \tau] \tau)|^{2} d s\right. \\
\leq & \left|x_{0}\right|^{p}+2 p K[1+(p-1) K] \int_{0}^{v} \sup _{0 \leq u \leq s} \mathbb{E}|x(u)|^{p} d s
\end{aligned}
$$

Taking the supremum value of both sides over $v \in[0, t]$, we have

$$
\sup _{0 \leq v \leq t} \mathbb{E}|x(v)|^{p} \leq\left|x_{0}\right|^{p}+2 p K[1+(p-1) K] \int_{0}^{t} \sup _{0 \leq u \leq s} \mathbb{E}|x(u)|^{p} d s
$$

The desired result (3.1) follows from the well-known Gronwall inequality.
Lemma 3.2. Assume that Assumption 2.2 holds. Then for any $t \geq 0$,

$$
\mathbb{E}|x(t)-x([t / \tau] \tau)|^{p} \leq C_{1}(K, p, \tau) \tau^{\frac{p}{2}} e^{2 p K[1+(p-1) K] t}\left|x_{0}\right|^{p}
$$

where $C_{1}(K, p, \tau)=2^{2 p-1} K^{p}\left[\tau^{\frac{p}{2}}+(p(p-1) / 2)^{\frac{p}{2}}\right]$.

Proof. By basic inequality, Hölder inequality, moment inequality and Assumption 2.2, we obtain

$$
\begin{aligned}
\mathbb{E}|x(t)-x([t / \tau] \tau)|^{p} \leq & 2^{p-1} \tau^{p-1} \mathbb{E} \int_{[t / \tau] \tau}^{t}\left|f(x(s))+u_{1}(x([s / \tau] \tau))\right|^{p} d s \\
& +2^{p-1}\left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}} \tau^{\frac{p-2}{2}} \mathbb{E} \int_{[t / \tau] \tau}^{t}\left|g(x(s))+u_{2}(x([s / \tau] \tau))\right|^{p} d s \\
\leq & C_{1}(K, p, \tau) \tau^{\frac{p}{2}-1} \int_{[t / \tau] \tau}^{t}\left(\sup _{0 \leq u \leq s} \mathbb{E}|x(u)|^{p}\right) d s
\end{aligned}
$$

It comes from (3.1) that

$$
\begin{aligned}
\mathbb{E}|x(t)-x([t / \tau] \tau)|^{p} & \leq C_{1}(K, p, \tau) \tau^{\frac{p}{2}-1} \int_{[t / \tau] \tau}^{t} e^{2 p K[1+(p-1) K] s}\left|x_{0}\right|^{p} d s \\
& \leq C_{1}(K, p, \tau) \tau^{\frac{p}{2}} e^{2 p K[1+(p-1) K] t}\left|x_{0}\right|^{p}
\end{aligned}
$$

The lemma is proved.
The following lemma estimates the difference in the $p$ th moment between $x(t)$ and $y(t)$.

Lemma 3.3. Let Assumption 2.2 hold. Then

$$
\mathbb{E}|x(t)-y(t)|^{p} \leq C_{2}(K, p, \tau) \tau^{\frac{p}{2}}\left|x_{0}\right|^{p}\left(e^{C_{3}(p, K) t}-1\right)
$$

for all $x_{0} \in \mathbb{R}^{d}$ and $t \geq 0$, where $C_{2}$ and $C_{3}$ are defined as (3.4) and (3.5), respectively.
Proof. Using Itô formula and Assumption 2.2, we have

$$
\begin{aligned}
& \mathbb{E}|x(t)-y(t)|^{p} \\
\leq & \mathbb{E} \int_{0}^{t} p K|x(s)-y(s)|^{p-1}(|x(s)-y(s)|+|x([s / \tau] \tau)-y(s)|) \\
& +p(p-1) K^{2}|x(s)-y(s)|^{p-2}\left(|x(s)-y(s)|^{2}+|x([s / \tau] \tau)-y(s)|^{2}\right) d s \\
= & \left(p K+p(p-1) K^{2}\right) \int_{0}^{t} \mathbb{E}|x(s)-y(s)|^{p} d s \\
& +p K \mathbb{E} \int_{0}^{t}|x(s)-y(s)|^{p-1}|x([s / \tau] \tau)-y(s)| d s \\
& +p(p-1) K^{2} \mathbb{E} \int_{0}^{t}|x(s)-y(s)|^{p-2}|x([s / \tau] \tau)-y(s)|^{2} d s
\end{aligned}
$$

By Young inequality, we have

$$
\begin{align*}
& \mathbb{E}|x(t)-y(t)|^{p} \\
\leq & {\left[\left(2 p-1+2^{p-1}\right) K+2(p-1)\left(p-1+2^{p-1}\right) K^{2}\right] \int_{0}^{t} \mathbb{E}|x(s)-y(s)|^{p} d s } \\
& +2^{p-1}\left(K+2(p-1) K^{2}\right) \int_{0}^{t} \mathbb{E}|x([s / \tau] \tau)-x(s)|^{p} d s \tag{3.2}
\end{align*}
$$

In view of Lemma 3.2, we have

$$
\begin{align*}
& 2^{p-1}\left(K+2(p-1) K^{2}\right) \int_{0}^{t} \mathbb{E}|x([s / \tau] \tau)-x(s)|^{p} d s \\
\leq & 2^{p-1}\left(K+2(p-1) K^{2}\right) \int_{0}^{t} C_{1}(K, p, \tau) \tau^{\frac{p}{2}} e^{2 p K[1+(p-1) K] s}\left|x_{0}\right|^{p} d s \\
= & \frac{2^{p-1}(1+2(p-1) K) \tau^{\frac{p}{2}} C_{1}(K, p, \tau)\left|x_{0}\right|^{p}}{p[2+2(p-1) K]}\left(e^{2 p K[1+(p-1) K] t}-1\right) \\
\leq & \frac{2^{p-1} \tau^{\frac{p}{2}} C_{1}(K, p, \tau)\left|x_{0}\right|^{p}}{p}\left(e^{2 p K[1+(p-1) K] t}-1\right) \tag{3.3}
\end{align*}
$$

Substituting (3.3) into (3.2) and using Gronwall inequality, we show that

$$
\mathbb{E}|x(t)-y(t)|^{p} \leq C_{2}(K, p, \tau) \tau^{\frac{p}{2}}\left|x_{0}\right|^{p}\left(e^{C_{3}(p, K) t}-1\right),
$$

where

$$
\begin{gather*}
C_{2}(K, p, \tau)=\frac{2^{p-1} C_{1}(K, p, \tau)}{p},  \tag{3.4}\\
C_{3}(p, K)=\left[4 p-1+2^{p-1}+2(p-1)\left(2 p-1+2^{p-1}\right) K\right] K . \tag{3.5}
\end{gather*}
$$

This completes the proof of Lemma 3.3.
Our positive answer to (Q1) is stated in the following theorem.
Theorem 3.4. Let Assumption 2.2 hold and the SDEPCA (2.1) is pth moment exponentially stable, i.e. $\mathbb{E}|x(t)|^{p} \leq M_{1} e^{-\gamma_{1} t}\left|x_{0}\right|^{p}$. Choose $\delta \in(0,1)$, if $\tau$ satisfies

$$
\begin{equation*}
R(\tau)=\delta+2^{p-1} C_{2}(K, p, \tau) \tau^{\frac{p}{2}}\left(e^{C_{3}(p, K)\left(\frac{\ln \left(\frac{2^{p-1} M_{1}}{\delta}\right)}{\gamma_{1}}+\tau\right)}-1\right)<1 \tag{3.6}
\end{equation*}
$$

then the $\operatorname{SDE}(2.2)$ is also pth moment exponentially stable, where $C_{2}(K, p, \tau)$ and $C_{3}(p, K)$ are defined in Lemma 3.3.

Proof. Step1. Let us choose a positive integer $\hat{n}$ such that

$$
\frac{\ln \left(\frac{2^{p-1} M_{1}}{\delta}\right)}{\gamma_{1} \tau} \leq \hat{n}<\frac{\ln \left(\frac{2^{p-1} M_{1}}{\delta}\right)}{\gamma_{1} \tau}+1
$$

So $2^{p-1} M_{1} e^{-\gamma_{1} \hat{n} \tau} \leq \delta$. Hence,

$$
\begin{equation*}
2^{p-1} \mathbb{E}|x(\hat{n} \tau)|^{p} \leq 2^{p-1} M_{1} e^{-\gamma_{1} \hat{n} \tau}\left|x_{0}\right|^{p} \leq \delta\left|x_{0}\right|^{p} . \tag{3.7}
\end{equation*}
$$

By virtue of Lemma 3.3, we obtain

$$
\mathbb{E}|x(\hat{n} \tau)-y(\hat{n} \tau)|^{p} \leq C_{2}(K, p, \tau) \tau^{\frac{p}{2}}\left|x_{0}\right|^{p}\left(e^{C_{3}(p, K) \hat{n} \tau}-1\right),
$$

which together with (3.7), we arrive at

$$
\mathbb{E}|y(\hat{n} \tau)|^{p} \leq\left[\delta+2^{p-1} C_{2}(K, p, \tau) \tau^{\frac{p}{2}}\left(e^{C_{3}(p, K) \hat{n} \tau}-1\right)\right]\left|x_{0}\right|^{p} \leq R(\tau)\left|x_{0}\right|^{p}
$$

In view of (3.6), there is a positive constant $\gamma_{2}$ such that $R(\tau)=e^{-\gamma_{2} \hat{n} \tau}$. Consequently,

$$
\mathbb{E}|y(\hat{n} \tau)|^{p} \leq e^{-\gamma_{2} \hat{n} \tau}\left|x_{0}\right|^{p}
$$

Step2. For any given $k \in \mathbb{N}^{+}$, let $\bar{x}(t)$ be the solution to the $\operatorname{SDEPCA}(2.1)$ for $t \geq k \hat{n} \tau$ with the initial value $\bar{x}(k \hat{n} \tau)=y(k \hat{n} \tau)$. We have from (2.14) that

$$
\begin{equation*}
\mathbb{E}|\bar{x}((k+1) \hat{n} \tau)|^{p} \leq M_{1} \mathbb{E}|y(k \hat{n} \tau)|^{p} e^{-\gamma_{1} \hat{n} \tau} \tag{3.8}
\end{equation*}
$$

In view of Lemma 3.3, we arrive at
(3.9) $\mathbb{E}|\bar{x}((k+1) \hat{n} \tau)-y((k+1) \hat{n} \tau)|^{p} \leq C_{2}(K, p, \tau) \tau^{\frac{p}{2}} \mathbb{E}|y(k \hat{n} \tau)|^{p}\left(e^{C_{3}(p, K) \hat{n} \tau}-1\right)$.

Using (3.8) and (3.9), we can show, in the same way as we did in Step1, that

$$
\mathbb{E}|y((k+1) \hat{n} \tau)|^{p} \leq \mathbb{E}|y(k \hat{n} \tau)|^{p} e^{-\gamma_{2} \hat{n} \tau}
$$

Consequently,

$$
\begin{equation*}
\mathbb{E}|y(k \hat{n} \tau)|^{p} \leq e^{-\gamma_{2} \hat{n} \tau} \mathbb{E}|y((k-1) \hat{n} \tau)|^{p} \leq \cdots \leq e^{-k \gamma_{2} \hat{n} \tau}\left|x_{0}\right|^{p} \tag{3.10}
\end{equation*}
$$

Now, for any $t>0$, there is a unique $k$ such that $k \hat{n} \tau \leq t<(k+1) \hat{n} \tau$. In view of Itô formula and Assumption 2.2, similarly as the proof of Lemma 3.1, we arrive at

$$
\mathbb{E}|y(t)|^{p} \leq \mathbb{E}|y(k \hat{n} \tau)|^{p}+2 p K(1+(p-1) K) \int_{k \hat{n} \tau}^{t} \mathbb{E}|y(s)|^{p} d s
$$

By the Gronwall inequality and (3.10), we can derive

$$
\begin{aligned}
\mathbb{E}|y(t)|^{p} & \leq \mathbb{E}|y(k \hat{n} \tau)|^{p} e^{2 p K(1+(p-1) K)(t-k \hat{n} \tau)} \\
& \leq \mathbb{E}|y(k \hat{n} \tau)|^{p} e^{2 p K(1+(p-1) K) \hat{n} \tau} \\
& \leq e^{-k \gamma_{2} \hat{n} \tau}\left|x_{0}\right|^{p} e^{2 p K(1+(p-1) K) \hat{n} \tau} \\
& \leq M_{2}\left|x_{0}\right|^{p} e^{-\gamma_{2} t}
\end{aligned}
$$

where $M_{2}=e^{\left[\gamma_{2}+2 p K(1+(p-1) K)\right] \hat{n} \tau}$. The proof is hence complete.
4. EMSDEPCA (2.3) shares the stability with EMSDE (2.4). In this section, we shall show that if the EMSDE (2.4) is $p$ th moment exponentially stable, then the EMSDEPCAs (2.3) is also $p$ th moment exponentially stable, i.e. give the positive answer to (Q3). It is known from Remark 2.1 that if $h=\tau$, then EMSDE (2.4) and EMSDEPCA (2.3) are the same, and the answer for (Q3) is obviously positive. So in this section, we assume $h \neq \tau$.

Theorem 4.1. Assume that Assumption 2.2 holds. For a step size $h=\tau / m$, the EMSDE (2.4) is pth moment exponentially stable, i.e. $\mathbb{E}\left|Y_{n}\right|^{p} \leq L_{2} e^{-\lambda_{2} n h}\left|x_{0}\right|^{p}$. Choose $\delta \in(0,1)$, if $\tau$ satisfies

$$
\begin{equation*}
2^{p-1} H_{4}\left(2\left(\frac{\ln \left(2^{p-1} L_{2} / \delta\right)}{\lambda_{2}}+\tau\right), K, \tau, p\right) \tau^{\frac{p}{2}}+\delta<1 \tag{4.1}
\end{equation*}
$$

where $H_{4}(T, K, \tau, p)$ is defined in Lemma 4.3, then the EMSDEPCA (2.3) is also pth moment exponentially stable.

The above theorem will be proved below by making use of the following lemmas.
Lemma 4.2. Assume that Assumption 2.2 holds. Then for any given $T>0$ such that

$$
\sup _{0 \leq t_{n} \leq T} \mathbb{E}\left|X_{n}\right|^{p} \leq H_{3}(T, p, K)\left|x_{0}\right|^{p}
$$

where $H_{3}(T, p, K)=e^{2 p K(1+(p-1) K) T}$.
Proof. The proof follows from Lemma 3.1. But to highlight the importance of numerical solutions, it is given here. In view of Itô formula and Assumption 2.2, we have

$$
\begin{aligned}
\mathbb{E}\left|x_{\Delta}(v)\right|^{p}= & \left|x_{0}\right|^{p}+\mathbb{E} \int_{0}^{t} p\left|x_{\Delta}(s)\right|^{p-2} x_{\Delta}(s)^{T}\left(f\left(\bar{x}_{\Delta}(s)\right)+u_{1}\left(\bar{x}_{\Delta}([s / \tau] \tau)\right)\right) \\
& +\frac{p(p-1)}{2}\left|x_{\Delta}(s)\right|^{p-2}\left|g\left(\bar{x}_{\Delta}(s)\right)+u_{2}\left(\bar{x}_{\Delta}([s / \tau] \tau)\right)\right|^{2} d s \\
\leq & \left|x_{0}\right|^{p}+\mathbb{E} \int_{0}^{v} p K\left|x_{\Delta}(s)\right|^{p-1}\left(\left|\bar{x}_{\Delta}(s)\right|+\left|\bar{x}_{\Delta}([s / \tau] \tau)\right|\right) d s \\
& +p(p-1) K^{2} \mathbb{E} \int_{0}^{v}\left|x_{\Delta}(s)\right|^{p-2}\left(\left|\bar{x}_{\Delta}(s)\right|^{2}+\left|\bar{x}_{\Delta}([s / \tau] \tau)\right|^{2}\right) d s \\
\leq & \left|x_{0}\right|^{p}+2 p K(1+(p-1) K) \int_{0}^{v} \sup _{0 \leq u \leq s} \mathbb{E}\left|x_{\Delta}(u)\right|^{p} d s
\end{aligned}
$$

According to the Gronwall inequality, we obtain

$$
\begin{equation*}
\sup _{0 \leq t \leq T} \mathbb{E}\left|x_{\Delta}(t)\right|^{p} \leq\left|x_{0}\right|^{p} e^{2 p K(1+(p-1) K) T} \tag{4.2}
\end{equation*}
$$

The proof is completed by noting that $x_{\Delta}\left(t_{n}\right)=X_{n}$, i.e.

$$
\sup _{0 \leq t_{n} \leq T} \mathbb{E}\left|X_{n}\right|^{p} \leq\left|x_{0}\right|^{p} e^{2 p K(1+(p-1) K) T}
$$

The following lemma estimates the difference in the $p$ th moment between approximation of EMSDE (2.4) and that of EMSDEPCA (2.3).

Lemma 4.3. Let Assumption 2.2 hold. Then for any given positive constant $T>$ 0 ,

$$
\sup _{0 \leq t_{n} \leq T} \mathbb{E}\left|X_{n}-Y_{n}\right|^{p} \leq H_{4}(T, K, \tau, p) \tau^{\frac{p}{2}}\left|x_{0}\right|^{p}
$$

where $H_{4}(T, K, \tau, p)=C_{2}(K, p, \tau)\left(e^{C_{3}(p, K) T}-1\right), C_{2}$ and $C_{3}$ are defined in Lemma 3.3.

Proof. According to (2.7), (2.8), Itô formula and Assumption 2.2, we have

$$
\begin{align*}
& \mathbb{E}\left|x_{\Delta}(v)-y_{\Delta}(v)\right|^{p} \\
\leq & \mathbb{E} \int_{0}^{v} p\left|x_{\Delta}(s)-y_{\Delta}(s)\right|^{p-1}\left|f\left(\bar{x}_{\Delta}(s)\right)-f\left(\bar{y}_{\Delta}(s)\right)+u_{1}\left(\bar{x}_{\Delta}([s / \tau] \tau)\right)-u_{1}\left(\bar{y}_{\Delta}(s)\right)\right| \\
& +\frac{p(p-1)}{2}\left|x_{\Delta}(s)-y_{\Delta}(s)\right|^{p-2}\left|g\left(\bar{x}_{\Delta}(s)\right)-g\left(\bar{y}_{\Delta}(s)\right)+u_{2}\left(\bar{x}_{\Delta}([s / \tau] \tau)\right)-u_{2}\left(\bar{y}_{\Delta}(s)\right)\right|^{2} d s \\
\leq & \mathbb{E} \int_{0}^{v} p K\left|x_{\Delta}(s)-y_{\Delta}(s)\right|^{p-1}\left(\left|\bar{x}_{\Delta}(s)-\bar{y}_{\Delta}(s)\right|+\left|\bar{x}_{\Delta}([s / \tau] \tau)-\bar{y}_{\Delta}(s)\right|\right) \\
& +p(p-1) K^{2}\left|x_{\Delta}(s)-y_{\Delta}(s)\right|^{p-2}\left(\left|\bar{x}_{\Delta}(s)-\bar{y}_{\Delta}(s)\right|^{2}+\left|\bar{x}_{\Delta}([s / \tau] \tau)-\bar{y}_{\Delta}(s)\right|^{2}\right) d s \\
\leq & \left(p K+p(p-1) K^{2}\right) \int_{0}^{v} \sup _{0 \leq u \leq s} \mathbb{E}\left|x_{\Delta}(u)-y_{\Delta}(u)\right|^{p} d s \\
& +\mathbb{E} \int_{0}^{v} p K\left|x_{\Delta}(s)-y_{\Delta}(s)\right|^{p-1}\left|\bar{x}_{\Delta}([s / \tau] \tau)-\bar{y}_{\Delta}(s)\right| d s \\
& +\mathbb{E} \int_{0}^{v} p(p-1) K^{2}\left|x_{\Delta}(s)-y_{\Delta}(s)\right|^{p-2}\left|\bar{x}_{\Delta}([s / \tau] \tau)-\bar{y}_{\Delta}(s)\right|^{2} d s \tag{4.3}
\end{align*}
$$

Similarly as in the proof of Lemma 3.2, we obtain

$$
\begin{align*}
\mathbb{E}\left|\bar{x}_{\Delta}(t)-\bar{x}_{\Delta}([t / \tau] \tau)\right|^{p} & =\mathbb{E}\left|X_{k m+l}-X_{k m}\right|^{p} \\
& \leq C_{1}(K, p, \tau) \tau^{\frac{p}{2}}\left|x_{0}\right|^{p} e^{2 p K(1+(p-1) K) t} \tag{4.4}
\end{align*}
$$

Substituting (4.4) into (4.3), we have

$$
\begin{aligned}
& \mathbb{E}\left|x_{\Delta}(v)-y_{\Delta}(v)\right|^{2} \\
\leq & {\left[\left(2 p-1+2^{p-1}\right)+2(p-1)\left(p-1+2^{p-1}\right) K\right] K \int_{0}^{v} \sup _{0 \leq u \leq s} \mathbb{E}\left|x_{\Delta}(u)-y_{\Delta}(u)\right|^{p} d s } \\
& +2^{p-1}(1+2(p-1) K) K \int_{0}^{v} C_{1}(K, p, \tau) \tau^{\frac{p}{2}}\left|x_{0}\right|^{p} e^{2 p K(1+(p-1) K) t} d s
\end{aligned}
$$

Applying the Gronwall inequality, we have

$$
\sup _{0 \leq v \leq T} \mathbb{E}\left|x_{\Delta}(v)-y_{\Delta}(v)\right|^{p} \leq C_{2}(K, p, \tau) \tau^{\frac{p}{2}}\left|x_{0}\right|^{p}\left(e^{C_{3}(p, K) T}-1\right)
$$

where $C_{2}(K, p, \tau)$ and $C_{3}(p, K)$ are defined in Lemma 3.3. For ease of notations, set $H_{4}(T, K, p, \tau)=C_{2}(K, p, \tau)\left(e^{C_{3}(p, K) T}-1\right)$. The proof is completed by noting that $x_{\Delta}\left(t_{n}\right)=X_{n}$ and $y_{\Delta}\left(t_{n}\right)=Y_{n}$, i.e.

$$
\sup _{0 \leq t_{n} \leq T} \mathbb{E}\left|X_{n}-Y_{n}\right|^{p} \leq H_{4}(T, K, p, \tau) \tau^{\frac{p}{2}}\left|x_{0}\right|^{p}
$$

The proof of Theorem 4.1. Let

$$
\hat{n}=\left[\frac{\ln \left(\frac{2^{p-1} L_{2}}{\delta}\right)}{\lambda_{2} \tau}\right]+1
$$

which implies that

$$
2^{p-1} L_{2} e^{-\lambda_{2} \hat{n} \tau} \leq \delta, \quad \text { and } \quad \hat{n} \tau \leq \frac{\ln \left(\frac{2^{p-1} L_{2}}{\delta}\right)}{\lambda_{2}}+\tau
$$

By $|a+b|^{p} \leq 2^{p-1}|a|^{p}+2^{p-1}|b|^{p}$, we have

$$
\mathbb{E}\left|X_{n}\right|^{p} \leq 2^{p-1} \mathbb{E}\left|X_{n}-Y_{n}\right|^{p}+2^{p-1} \mathbb{E}\left|Y_{n}\right|^{p}
$$

According to the $p$ th moment exponentially stability of EMSDE (2.4) and Lemma 4.3, we have

$$
\begin{aligned}
\sup _{\hat{n} \tau \leq t_{n} \leq 2 \hat{n} \tau} \mathbb{E}\left|X_{n}\right|^{p} & \leq 2^{p-1} \sup _{0 \leq t_{n} \leq 2 \hat{n} \tau} \mathbb{E}\left|X_{n}-Y_{n}\right|^{p}+2^{p-1} \sup _{\hat{n} \tau \leq t_{n} \leq 2 \hat{n} \tau} \mathbb{E}\left|Y_{n}\right|^{p} \\
& \leq\left(2^{p-1} H_{4}(2 \hat{n} \tau, K, \tau, p) \tau^{\frac{p}{2}}+2^{p-1} L_{2} e^{-\lambda_{2} \hat{n} \tau}\right)\left|x_{0}\right|^{p} \\
& \leq\left(2^{p-1} H_{4}(2 \hat{n} \tau, K, \tau, p) \tau^{\frac{p}{2}}+\delta\right)\left|x_{0}\right|^{p} \\
& \leq\left(2^{p-1} H_{4}\left(2\left(\frac{\ln \left(2^{p-1} L_{2} / \delta\right)}{\lambda_{2}}+\tau\right), K, \tau, p\right) \tau^{\frac{p}{2}}+\delta\right)\left|x_{0}\right|^{p}
\end{aligned}
$$

Let $R(\tau)=2^{p-1} H_{4}\left(2\left(\frac{\ln \left(2^{p-1} L_{2} / \delta\right)}{\lambda_{2}}+\tau\right), K, \tau, p\right) \tau^{\frac{p}{2}}+\delta$. It is known from (4.1) that $R(\tau)<1$. Therefore, we can find a positive constant $\lambda_{1}$ such that

$$
\begin{equation*}
R(\tau)<e^{-\lambda_{1} \hat{n} \tau} \tag{4.5}
\end{equation*}
$$

and

$$
\begin{equation*}
\sup _{\hat{n} \tau \leq t_{n} \leq 2 \hat{n} \tau} \mathbb{E}\left|X_{n}\right|^{p} \leq e^{-\lambda_{1} \hat{n} \tau}\left|x_{0}\right|^{p} \tag{4.6}
\end{equation*}
$$

Let $\left\{\bar{Y}_{n}\right\}_{t_{n} \geq \hat{n} \tau}$ be the solution of EMSDE (2.4) with initial data $\bar{Y}_{\hat{n} m}=X_{\hat{n} m}$ at initial time $t=\hat{n} \tau$. According to Lemma 4.3, we have

$$
\sup _{\hat{n} \tau \leq t_{n} \leq 3 \hat{n} \tau} \mathbb{E}\left|X_{n}-\bar{Y}_{n}\right|^{p} \leq H_{4}\left(2\left(\frac{\ln \left(2^{p-1} L_{2} / \delta\right)}{\lambda_{2}}+\tau\right), K, \tau, p\right) \tau^{\frac{p}{2}} \mathbb{E}\left|X_{\hat{n} m}\right|^{p}
$$

It comes from (2.13) that

$$
\mathbb{E}\left|\bar{Y}_{n}\right|^{p} \leq L_{2} e^{-\lambda_{2}(n h-\hat{n} m h)} \mathbb{E}\left|X_{\hat{n} m}\right|^{p}
$$

Using similar arguments that produced (4.6), we obtain

$$
\sup _{2 \hat{n} \tau \leq t_{n} \leq 3 \hat{n} \tau} \mathbb{E}\left|X_{n}\right|^{p} \leq R(\tau) \mathbb{E}\left|X_{\hat{n} m}\right|^{p} \leq e^{-\lambda_{1} \hat{n} \tau} \mathbb{E}\left|X_{\hat{n} m}\right|^{p} \leq e^{-\lambda_{1} \hat{n} \tau} \sup _{\hat{n} \tau \leq t_{n} \leq 2 \hat{n} \tau} \mathbb{E}\left|X_{n}\right|^{p}
$$

By (4.6), we obtain

$$
\sup _{2 \hat{n} \tau \leq t_{n} \leq 3 \hat{n} \tau} \mathbb{E}\left|X_{n}\right|^{p} \leq e^{-2 \lambda_{1} \hat{n} \tau}\left|x_{0}\right|^{p}
$$

Continuing this approach and using (4.5), we have, for any $i=1,2, \cdots$,

$$
\begin{equation*}
\sup _{i \hat{n} \tau \leq t_{n} \leq(i+1) \hat{n} \tau} \mathbb{E}\left|X_{n}\right|^{p} \leq e^{-i \lambda_{1} \hat{n} \tau}\left|x_{0}\right|^{p} \leq \bar{L}_{1} e^{-\lambda_{1} n h}\left|x_{0}\right|^{p} \tag{4.7}
\end{equation*}
$$

where $\bar{L}_{1}=e^{\lambda_{1} \hat{n} \tau}$. For $i=0$, by using Lemma 4.2, we get

$$
\sup _{0 \leq t_{n} \leq \hat{n} \tau} \mathbb{E}\left|X_{n}\right|^{p} \leq H_{3}(\hat{n} \tau, p, K)\left|x_{0}\right|^{p} \leq L_{1}\left|x_{0}\right|^{p} e^{-\lambda_{1} n h}
$$

where $L_{1}=H_{3}(\hat{n} \tau, p, K) e^{\lambda_{1} \hat{n} \tau}>e^{\lambda_{1} \hat{n} \tau}=\bar{L}_{1}$. This, together with (4.7), we arrive at for all $n \in \mathbb{N}$

$$
\mathbb{E}\left|X_{n}\right|^{p} \leq L_{1}\left|x_{0}\right|^{p} e^{-\lambda_{1} n h}
$$

5. SDEPCA (2.1) shares the stability with EMSDEPCA (2.3). In this section, we shall show that for a given step size $h$, if the EMSDEPCA (2.3) is $p$ th moment exponentially stable, then the SDEPCA (2.1) is also $p$ th moment exponentially stable with some restriction with $h$, i.e. give the positive answer to (Q4). The first lemma shows that the EMSDEPCA (2.3) is convergent in the $p$ th moment to SDEPCA (2.1).

Lemma 5.1. Assume that Assumption 2.2 holds. For $T>0$,

$$
\sup _{0 \leq t_{n} \leq T} \mathbb{E}\left|x\left(t_{n}\right)-X_{n}\right|^{p} \leq H_{6}(T, K, p) h^{\frac{p}{2}}\left|x_{0}\right|^{p}
$$

where $H_{6}(T, K, p)$ is defined as (5.3).
Proof. For any $t \geq 0$, by Itô formula, Assumption 2.2 and Young inequality, we obtain

$$
\begin{aligned}
& \mathbb{E}\left|x(t)-x_{\Delta}(t)\right|^{p} \\
\leq & \left.\mathbb{E} \int_{0}^{t} p K\left|x(s)-x_{\Delta}(s)\right|^{p-1}\left(\left|x(s)-\bar{x}_{\Delta}(s)\right|+\mid x([s / \tau] \tau)\right)-\bar{x}_{\Delta}([s / \tau] \tau) \mid\right) \\
& +p(p-1) K^{2}\left|x(s)-x_{\Delta}(s)\right|^{p-2}\left(\left|x(s)-\bar{x}_{\Delta}(s)\right|^{2}+\left|x([s / \tau] \tau)-\bar{x}_{\Delta}([s / \tau] \tau)\right|^{2}\right) d s \\
\leq & 2 p K(1+2(p-1) K) \int_{0}^{t} \mathbb{E} \sup _{0 \leq u \leq s}\left|x(s)-x_{\Delta}(s)\right|^{p} d s \\
& +p K \mathbb{E} \int_{0}^{t}\left|x(s)-x_{\Delta}(s)\right|^{p-1}\left|x_{\Delta}(s)-\bar{x}_{\Delta}(s)\right| d s \\
& +p K \mathbb{E} \int_{0}^{t}\left|x(s)-x_{\Delta}(s)\right|^{p-1}\left|x_{\Delta}([s / \tau] \tau)-\bar{x}_{\Delta}([s / \tau] \tau)\right| d s \\
& +2 p(p-1) K^{2} \mathbb{E} \int_{0}^{t}\left|x(s)-x_{\Delta}(s)\right|^{p-2}\left|x_{\Delta}(s)-\bar{x}_{\Delta}(s)\right|^{2} d s \\
& \left.+2 p(p-1) K^{2} \mathbb{E} \int_{0}^{t}\left|x(s)-x_{\Delta}(s)\right|^{p-2}\left|x_{\Delta}([s / \tau] \tau)-\bar{x}_{\Delta}([s / \tau] \tau)\right|^{2}\right) d s
\end{aligned}
$$

By noting $x_{\Delta}([s / \tau] \tau)-\bar{x}_{\Delta}([s / \tau] \tau)=0$, we have

$$
\begin{align*}
\mathbb{E}\left|x(t)-x_{\Delta}(t)\right|^{p} \leq & K[3 p-1+2(p-1)(3 p-2) K] \int_{0}^{t} \mathbb{E} \sup _{0 \leq u \leq s}\left|x(u)-x_{\Delta}(u)\right|^{p} d s \\
5.1) & +K[1+4(p-1) K] \int_{0}^{t} \mathbb{E}\left|x_{\Delta}(s)-\bar{x}_{\Delta}(s)\right|^{p} d s \tag{5.1}
\end{align*}
$$

Now, we shall give the estimation of the second term of the right hand. For any $t>0$, there exists $k$ and $l$ such that $t_{k m+l} \leq t<t_{k m+l+1}$. Then $\bar{x}_{\Delta}(t)=X_{k m+l}=$
$x_{\Delta}\left(t_{k m+l}\right)$. Hence from (2.5) we have

$$
\begin{aligned}
& \mathbb{E}\left|x_{\Delta}(t)-\bar{x}_{\Delta}(t)\right|^{p} \\
= & \mathbb{E}\left|\left(t-t_{k m+l}\right)\left(f\left(X_{k m+l}\right)+u_{1}\left(X_{k m}\right)\right)+\left(g\left(X_{k m+l}\right)+u_{2}\left(X_{k m}\right)\right)\left(w(t)-w\left(t_{k m+l}\right)\right)\right|^{p} \\
\leq & 2^{2 p-1} h^{\frac{p}{2}} K^{p}\left(\mathbb{E}\left|X_{k m+l}\right|^{p}+\mathbb{E}\left|X_{k m}\right|^{p}\right)
\end{aligned}
$$

Applying (4.2), we obtain

$$
\begin{equation*}
\mathbb{E}\left|x_{\Delta}(t)-\bar{x}_{\Delta}(t)\right|^{p} \leq 2^{2 p} h^{\frac{p}{2}} K^{p}\left|x_{0}\right|^{p} e^{2 p K(1+(p-1) K) t} \tag{5.2}
\end{equation*}
$$

Substituting (5.2) into (5.1), we obtain

$$
\begin{aligned}
& \mathbb{E}\left|x(v)-x_{\Delta}(v)\right|^{p} \\
\leq & K[3 p-1+2(p-1)(3 p-2) K] \int_{0}^{t} \mathbb{E} \sup _{0 \leq u \leq s}\left|x(u)-x_{\Delta}(u)\right|^{p} d s \\
& +K[1+4(p-1) K] \int_{0}^{t} 2^{2 p} h^{\frac{p}{2}} K^{p}\left|x_{0}\right|^{p} e^{2 p K(1+(p-1) K) s} d s
\end{aligned}
$$

By Gronwall inequality, we have

$$
\sup _{0 \leq t \leq T} \mathbb{E}\left|x(t)-x_{\Delta}(t)\right|^{p} \leq H_{6}(T, p, K) h^{\frac{p}{2}}\left|x_{0}\right|^{p}
$$

where

$$
\begin{equation*}
H_{6}(T, p, K)=[1+4(p-1) K] 2^{2 p} K^{p+1} e^{K T[5 p-1+4(p-1)(2 p-1) K]} T \tag{5.3}
\end{equation*}
$$

By noting $x_{\Delta}\left(t_{n}\right)=X_{n}$, we get for $t=t_{n}$

$$
\sup _{0 \leq t_{n} \leq T} \mathbb{E}\left|x\left(t_{n}\right)-X_{n}\right|^{p} \leq H_{6}(T, p, K) h^{\frac{p}{2}}\left|x_{0}\right|^{p}
$$

The proof is completed.
Lemma 5.2. Assume that Assumption 2.2 holds. Then for any $0 \leq t_{n} \leq t \leq$ $t_{n+1} \leq T$,

$$
\sup _{0 \leq t \leq T} \mathbb{E}\left|x(t)-x\left(t_{n}\right)\right|^{p} \leq H_{7}(T, K, p) h^{\frac{p}{2}}\left|x_{0}\right|^{p}
$$

where $H_{7}(T, K, p)=2^{2 p-1} K^{p}\left[T^{\frac{p}{2}}+\left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}}\right] e^{2 p K[1+(p-1) K] T}$.
Proof. For any $0 \leq t_{n} \leq t \leq t_{n+1} \leq T$, we have

$$
x(t)-x\left(t_{n}\right)=\int_{t_{n}}^{t} f(x(s))+u_{1}(x([s / \tau] \tau)) d s+\int_{t_{n}}^{t} g(x(s))+u_{2}(x([s / \tau] \tau)) d w(s)
$$

In view of Hölder inequality, Assumption 2.2 as well as moment inequality, we have

$$
\begin{aligned}
& \mathbb{E}\left|x(t)-x\left(t_{n}\right)\right|^{p} \\
\leq & 2^{2 p-2}\left(t-t_{n}\right)^{\frac{p}{2}-1} K^{p}\left[\left(t-t_{n}\right)^{\frac{p}{2}}+\left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}}\right] \int_{t_{n}}^{t} \mathbb{E}|x(s)|^{p}+\mathbb{E}|x([s / \tau] \tau)|^{p} d s \\
\leq & 2^{2 p-1}\left(t-t_{n}\right)^{\frac{p}{2}-1} K^{p}\left[\left(t-t_{n}\right)^{\frac{p}{2}}+\left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}}\right] \int_{t_{n}}^{t} \sup _{0 \leq u \leq s} \mathbb{E}|x(u)|^{p} d s
\end{aligned}
$$

It follows from (3.1) that

$$
\begin{aligned}
\mathbb{E}\left|x(t)-x\left(t_{n}\right)\right|^{p} & \leq 2^{2 p-1}\left(t-t_{n}\right)^{\frac{p}{2}} K^{p}\left[\left(t-t_{n}\right)^{\frac{p}{2}}+\left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}}\right] e^{2 p K[1+(p-1) K] t}\left|x_{0}\right|^{p} \\
& \leq 2^{2 p-1} h^{\frac{p}{2}} K^{p}\left[t^{\frac{p}{2}}+\left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}}\right] e^{2 p K[1+(p-1) K] t}\left|x_{0}\right|^{p}
\end{aligned}
$$

Hence,

$$
\sup _{0 \leq t \leq T} \mathbb{E}\left|x(t)-x\left(t_{n}\right)\right|^{p} \leq 2^{2 p-1} K^{p}\left[T^{\frac{p}{2}}+\left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}}\right] e^{2 p K[1+(p-1) K] T} h^{\frac{p}{2}}\left|x_{0}\right|^{p} .
$$

The proof is complete.
Theorem 5.3. Assume that Assumption 2.2 holds. For a step size $h=\frac{\tau}{m}$, the EMSDEPCA (2.3) is pth moment exponentially stable, i.e. $\mathbb{E}\left|X_{n}\right|^{p} \leq L_{1} e^{-\lambda_{1} n h}\left|x_{0}\right|^{p}$. If the step size $h$ satisfies

$$
\begin{equation*}
3^{p-1} H_{8}(2 \hat{n} \tau, K, p) h^{\frac{p}{2}}+e^{-\frac{3}{4} \lambda_{1} \hat{n} \tau} \leq e^{-\frac{1}{2} \lambda_{1} \hat{n} \tau} \tag{5.4}
\end{equation*}
$$

where $\hat{n}=\left[\frac{4 \ln \left(3^{p-1} L_{1}\right)}{\lambda_{1} \tau}\right]+1$ and $H_{8}(2 \hat{n} \tau, K, p)=H_{7}(2 \hat{n} \tau, K, p)+H_{6}(2 \hat{n} \tau, K, p)$, then the SDEPCA (2.1) is also pth moment exponentially stable, where $H_{6}(2 \hat{n} \tau, K, p)$ is defined in Lemma 5.1 and $H_{7}(2 \hat{n} \tau, K, p)$ in Lemma 5.2.

Proof. For any $t \geq 0$, there exist $n \in \mathbb{N}$ such that $t_{n} \leq t<t_{n+1}$,

$$
\mathbb{E}|x(t)|^{p} \leq 3^{p-1} \mathbb{E}\left|x(t)-x\left(t_{n}\right)\right|^{p}+3^{p-1} \mathbb{E}\left|x\left(t_{n}\right)-X_{n}\right|^{p}+3^{p-1} \mathbb{E}\left|X_{n}\right|^{p}
$$

According to Lemma 5.1, we have

$$
\sup _{0 \leq t_{n} \leq 2 \hat{n} \tau} \mathbb{E}\left|x\left(t_{n}\right)-X_{n}\right|^{p} \leq H_{6}(2 \hat{n} \tau, K, p) h^{\frac{p}{2}}\left|x_{0}\right|^{p}
$$

By Lemma 5.2, we have

$$
\sup _{0 \leq t \leq 2 \hat{n} \tau} \mathbb{E}\left|x(t)-x\left(t_{n}\right)\right|^{p} \leq H_{7}(2 \hat{n} \tau, K, p) h^{\frac{p}{2}}\left|x_{0}\right|^{p}
$$

Since $\hat{n}=\left[\frac{4 \ln \left(3^{p-1} L_{1}\right)}{\lambda_{1} \tau}\right]+1$, we have $3^{p-1} L_{1} e^{-\lambda_{1} \hat{n} \tau} \leq e^{-\frac{3}{4} \lambda_{1} \hat{n} \tau}$, Therefore,

$$
\begin{aligned}
& \sup _{\hat{n} \tau \leq t \leq 2 \hat{n} \tau} \mathbb{E}|x(t)|^{p} \\
\leq & 3^{p-1} \sup _{0 \leq t \leq 2 \hat{n} \tau} \mathbb{E}\left|x(t)-x\left(t_{n}\right)\right|^{p}+3^{p-1} \sup _{0 \leq t_{n} \leq 2 \hat{n} \tau} \mathbb{E}\left|x\left(t_{n}\right)-X_{n}\right|^{p}+3^{p-1} \sup _{\hat{n} \tau \leq t_{n} \leq 2 \hat{n} \tau} \mathbb{E}\left|X_{n}\right|^{p} \\
\leq & \left(3^{p-1} H_{7}(2 \hat{n} \tau, K, p) h^{\frac{p}{2}}+3^{p-1} H_{6}(2 \hat{n} \tau, K, p) h^{\frac{p}{2}}+3^{p-1} L_{1} e^{-\lambda_{1} \hat{n} \tau}\right)\left|x_{0}\right|^{p} \\
\leq & \left(3^{p-1} H_{8}(2 \hat{n} \tau, K, p) h^{\frac{p}{2}}+e^{-\frac{3}{4} \lambda_{1} \hat{n} \tau}\right)\left|x_{0}\right|^{p}
\end{aligned}
$$

where $H_{8}(2 \hat{n} \tau, K, p)=H_{7}(2 \hat{n} \tau, K, p)+H_{6}(2 \hat{n} \tau, K, p)$. Recalling (5.4), we have

$$
\begin{equation*}
\sup _{\hat{n} \tau \leq t \leq 2 \hat{n} \tau} \mathbb{E}|x(t)|^{p} \leq e^{-\frac{1}{2} \lambda_{1} \hat{n} \tau}\left|x_{0}\right|^{p} \tag{5.5}
\end{equation*}
$$

Denote by $\left\{\bar{X}_{n}\right\}_{n h \geq \hat{n} \tau}$ the numerical solution of (2.3) with initial data $\bar{X}_{\hat{n} m}=x(\hat{n} \tau)$ at $t=\hat{n} \tau$. Then from (2.15), we have

$$
\mathbb{E}\left|\bar{X}_{n}\right|^{p} \leq L_{1} e^{-\lambda_{1}(n-\hat{n} m) h} \mathbb{E}|x(\hat{n} \tau)|^{p}
$$

Using Lemma 5.1 and Lemma 5.2, we get

$$
\begin{aligned}
& \sup _{\hat{n} \tau \leq t_{n} \leq 3 \hat{n} \tau} \mathbb{E}\left|x\left(t_{n}\right)-\bar{X}_{n}\right|^{p} \leq H_{6}(2 \hat{n} \tau, K, p) h^{\frac{p}{2}} \mathbb{E}|x(\hat{n} \tau)|^{p} . \\
& \sup _{\hat{n} \tau \leq t \leq 3 \hat{n} \tau} \mathbb{E}\left|x(t)-x\left(t_{n}\right)\right|^{p} \leq H_{7}(2 \hat{n} \tau, K, p) h^{\frac{p}{2}} \mathbb{E}|x(\hat{n} \tau)|^{p} .
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\sup _{2 \hat{n} \tau \leq t \leq 3 \hat{n} \tau} \mathbb{E}|x(t)|^{p} & \leq\left(3^{p-1} H_{7}(2 \hat{n} \tau, K, p) h^{\frac{p}{2}}+3^{p-1} H_{6}(2 \hat{n} \tau, K, p) h^{\frac{p}{2}}+L_{1} e^{-\lambda_{1} \hat{n} \tau}\right) \mathbb{E}|x(\hat{n} \tau)|^{p} \\
& \leq\left(3^{p-1} H_{8}(2 \hat{n} \tau, K, p) h^{\frac{p}{2}}+e^{-\frac{3}{4} \lambda_{1} \hat{n} \tau}\right) \mathbb{E}|x(\hat{n} \tau)|^{p} \\
& \leq e^{-\frac{1}{2} \lambda_{1} \hat{n} \tau} \sup _{\hat{n} \tau \leq t \leq 2 \hat{n} \tau} \mathbb{E}|x(t)|^{p}
\end{aligned}
$$

By (5.5), we obtain

$$
\sup _{2 \hat{n} \tau \leq t \leq 3 \hat{n} \tau} \mathbb{E}|x(t)|^{p} \leq e^{-\frac{\lambda_{1}}{2} 2 \hat{n} \tau}\left|x_{0}\right|^{p} .
$$

Repeating this procedure, we find for $i=1,2, \cdots$,

$$
\begin{equation*}
\sup _{i \hat{n} \tau \leq t \leq(i+1) \hat{n} \tau} \mathbb{E}|x(t)|^{p} \leq e^{-\frac{\lambda_{1}}{2} i \hat{n} \tau}\left|x_{0}\right|^{p} \leq \bar{M}_{1} e^{-\frac{\lambda_{1}}{2} t}\left|x_{0}\right|^{p} \tag{5.6}
\end{equation*}
$$

where $\bar{M}_{1}=e^{\frac{\lambda_{1}}{2} \hat{n} \tau}$. On the other hand, by means of Lemma 3.1, we can show that

$$
\sup _{0 \leq t \leq \hat{n} \tau} \mathbb{E}|x(t)|^{p} \leq H_{1}(\hat{n} \tau, p, K)\left|x_{0}\right|^{p} \leq M_{1}\left|x_{0}\right|^{p} e^{-\frac{\lambda_{1}}{2} t}
$$

where $M_{1}=H_{1}(\hat{n} \tau, p, K) e^{\frac{\lambda_{1}}{2} \hat{n} \tau}>e^{\frac{\lambda_{1}}{2} \hat{n} \tau}=\bar{M}_{1}$, this, together with (5.6), we arrive at for any $t \geq 0$,

$$
\mathbb{E}|x(t)|^{p} \leq M_{1}\left|x_{0}\right|^{p} e^{-\frac{1}{2} \lambda_{1} t}
$$

This completes the proof.
6. EMSDE (2.4) shares the stability with SDE (2.2). [13] gives the positive answer to (Q2) only for the case $p=2$. In this section, we shall show that for $p>2$, if the SDE (2.2) is $p$ th moment exponentially stable, then the EMSDE (2.4) is also $p$ th moment exponentially stable with some restriction on $h$, i.e. give the positive answer to (Q2). The first lemma shows that the EMSDE (2.4) is convergent in the $p$ th moment to SDE (2.2).

Lemma 6.1. Assume that Assumption 2.2 holds. For any $T>0$,

$$
\sup _{0 \leq t_{n} \leq T} \mathbb{E}\left|y\left(t_{n}\right)-Y_{n}\right|^{p} \leq H_{9}(T, K, p) h^{\frac{p}{2}}\left|x_{0}\right|^{p}
$$

where $H_{9}(T, K, p)$ is defined as (6.4).

Proof. For any $t \geq 0$, by Itô formula, Assumption 2.2 and Young inequality, we obtain

$$
\begin{aligned}
& \mathbb{E}\left|y(t)-y_{\Delta}(t)\right|^{p} \\
\leq & \mathbb{E} \int_{0}^{t} p\left|y(s)-y_{\Delta}(s)\right|^{p-1}\left|f(y(s))-f\left(\bar{y}_{\Delta}(s)\right)+u_{1}(y(s))-u_{1}\left(\bar{y}_{\Delta}(s)\right)\right| \\
& +\frac{p(p-1)}{2}\left|y(s)-y_{\Delta}(s)\right|^{p-2}\left|g(y(s))-g\left(\bar{y}_{\Delta}(s)\right)+u_{2}(y(s))-u_{2}\left(\bar{y}_{\Delta}(s)\right)\right|^{2} d s \\
\leq & \mathbb{E} \int_{0}^{t} 2 p K\left|y(s)-y_{\Delta}(s)\right|^{p-1}\left|y(s)-\bar{y}_{\Delta}(s)\right| \\
& +2 p(p-1) K^{2}\left|y(s)-y_{\Delta}(s)\right|^{p-2}\left|y(s)-\bar{y}_{\Delta}(s)\right|^{2} d s \\
\leq & \left(2 K(2 p-1)+8(p-1)^{2} K^{2}\right) \int_{0}^{t} \mathbb{E}\left|y(s)-y_{\Delta}(s)\right|^{p} d s \\
(6.1) & +\left(2 K+8(p-1) K^{2}\right) \int_{0}^{t} \mathbb{E}\left|y_{\Delta}(s)-\bar{y}_{\Delta}(s)\right|^{p} d s
\end{aligned}
$$

Now, we shall give the estimation of the second term of the right hand. For any $t>0$, there exists $n$ such that $t_{n} \leq t<t_{n+1}$, and $\bar{y}_{\Delta}(t)=Y_{n}=y_{\Delta}\left(t_{n}\right)$. Hence from (2.4) we have

$$
\begin{aligned}
& \mathbb{E}\left|y_{\Delta}(t)-\bar{y}_{\Delta}(t)\right|^{p} \\
= & \mathbb{E}\left|\left(t-t_{n}\right)\left(f\left(Y_{n}\right)+u_{1}\left(Y_{n}\right)\right)+\left(g\left(Y_{n}\right)+u_{2}\left(Y_{n}\right)\right)\left(W(t)-W\left(t_{n}\right)\right)\right|^{p} \\
\leq & 2^{p-1}\left(\mathbb{E}\left|\left(t-t_{n}\right)\left(f\left(Y_{n}\right)+u_{1}\left(Y_{n}\right)\right)\right|^{p}+\mathbb{E}\left|\left(g\left(Y_{n}\right)+u_{2}\left(Y_{n}\right)\right)\left(W(t)-W\left(t_{n}\right)\right)\right|^{p}\right) \\
\leq & 2^{2 p} K^{p} \mathbb{E}\left|Y_{n}\right|^{p} h^{\frac{p}{2}}
\end{aligned}
$$

Similarly as the proof of Lemma 3.1, we have

$$
\begin{equation*}
\sup _{0 \leq s \leq t} \mathbb{E}\left|y_{\Delta}(s)\right|^{p} \leq\left|x_{0}\right|^{p} e^{2 p K(1+(p-1) K) t} \tag{6.2}
\end{equation*}
$$

Applying (6.2), we obtain

$$
\begin{equation*}
\mathbb{E}\left|y_{\Delta}(t)-\bar{y}_{\Delta}(t)\right|^{p} \leq 2^{2 p} K^{p} e^{2 p K(1+(p-1) K) t} h^{\frac{p}{2}}\left|x_{0}\right|^{p} \tag{6.3}
\end{equation*}
$$

Substituting (6.3) into (6.1), we obtain

$$
\begin{aligned}
\mathbb{E}\left|y(t)-y_{\Delta}(t)\right|^{p} \leq & \left(2 K(2 p-1)+8(p-1)^{2} K^{2}\right) \int_{0}^{t} \mathbb{E}\left|y(s)-y_{\Delta}(s)\right|^{p} d s \\
& +\left(2 K+8(p-1) K^{2}\right) \int_{0}^{t} 2^{2 p} K^{p} e^{2 p K(1+(p-1) K) s} h^{\frac{p}{2}}\left|x_{0}\right|^{p} d s
\end{aligned}
$$

By Gronwall inequality, we have

$$
\sup _{0 \leq t \leq T} \mathbb{E}\left|y(t)-y_{\Delta}(t)\right|^{p} \leq H_{9}(T, p, K) h^{\frac{p}{2}}\left|x_{0}\right|^{p}
$$

where

$$
\begin{equation*}
H_{9}(T, p, K)=[1+4(p-1) K] 2^{2 p+1} K^{p+1} e^{2 K T(3 p-1+(p-1)(5 p-4) K)} T \tag{6.4}
\end{equation*}
$$

By noting $y_{\Delta}\left(t_{n}\right)=Y_{n}$, we get for $t=t_{n}$

$$
\sup _{0 \leq t_{n} \leq T} \mathbb{E}\left|y\left(t_{n}\right)-Y_{n}\right|^{p} \leq H_{9}(T, p, K) h^{\frac{p}{2}}\left|x_{0}\right|^{p}
$$

The proof is completed.
Theorem 6.2. Let Assumption 2.2 hold. Assume that the $\operatorname{SDE}$ (2.2) is pth moment exponentially stable and satisfies (2.10). Let $T=1+4 \ln \left(2^{p-1} M_{2}\right) / \gamma_{2}$. If $h$ satisfies

$$
\begin{equation*}
2^{p-1} H_{9}(2 T, p, K) h^{\frac{p}{2}}+e^{-\frac{3}{4} \gamma_{2} T} \leq e^{-\frac{1}{2} \gamma_{2} T} \tag{6.5}
\end{equation*}
$$

Then the EMSDE (2.4) is pth moment exponentially stable.
Proof. Since $T=1+4 \ln \left(2^{p-1} M_{2}\right) / \gamma_{2}$, we have

$$
2^{p-1} M_{2} e^{-\gamma_{2} T}<e^{-\frac{3}{4} \gamma_{2} T}
$$

Now, for any given $i \in \mathbb{N}$, let $\{\hat{y}(t)\}_{t \geq i T}$ be the solution to the $\operatorname{SDE}$ (2.2) for $t \in$ $[i T, \infty)$, with the initial condition $y_{\Delta}(\bar{i} T)$. Then using basic inequality, Lemma 6.1, (2.10) and (6.5), we have

$$
\begin{align*}
& \sup _{(i+1) T \leq t \leq(i+2) T} \mathbb{E}\left|y_{\Delta}(t)\right|^{p} \\
\leq & 2^{p-1} \sup _{i T \leq t \leq(i+2) T} \mathbb{E}\left|y_{\Delta}(t)-\hat{y}(t)\right|^{p}+2^{p-1} \sup _{(i+1) T \leq t \leq(i+2) T} \mathbb{E}|\hat{y}(t)|^{p} \\
\leq & \left(2^{p-1} H_{9}(2 T, p, K) h^{\frac{p}{2}}+2^{p-1} M_{2} e^{-\gamma_{2} T}\right) \mathbb{E}\left|y_{\Delta}(i T)\right|^{p} \\
\leq & \left(2^{p-1} H_{9}(2 T, p, K) h^{\frac{p}{2}}+e^{-\frac{3}{4} \gamma_{2} T}\right) \mathbb{E}\left|y_{\Delta}(i T)\right|^{p} \\
\leq & e^{-\frac{1}{2} \gamma_{2} T} \sup _{i T \leq t \leq(i+1) T} \mathbb{E}\left|y_{\Delta}(t)\right|^{p} . \tag{6.6}
\end{align*}
$$

According to (6.2),

$$
\begin{equation*}
\sup _{0 \leq t \leq T} \mathbb{E}\left|y_{\Delta}(t)\right|^{p} \leq\left|x_{0}\right|^{p} e^{2 p K(1+(p-1) K) T} \leq L_{2} e^{-\frac{1}{2} \gamma_{2} t}\left|x_{0}\right|^{p} \tag{6.7}
\end{equation*}
$$

where $L_{2}=e^{\frac{1}{2} \gamma_{2} T+2 p K(1+(p-1) K) T}$. Combining (6.7) and (6.6), we obtain that

$$
\begin{align*}
\sup _{(i+1) T \leq t \leq(i+2) T} \mathbb{E}\left|y_{\Delta}(t)\right|^{p} & \leq e^{-\frac{1}{2}(i+1) \gamma_{2} T} \sup _{0 \leq t \leq T} \mathbb{E}\left|y_{\Delta}(t)\right|^{p} \\
& \leq e^{-\frac{1}{2}(i+1) \gamma_{2} T}\left|x_{0}\right|^{p} e^{2 p K(1+(p-1) K) T} \\
& \leq L_{2} e^{-\frac{1}{2} \gamma_{2} t}\left|x_{0}\right|^{p} \tag{6.8}
\end{align*}
$$

Due to (6.8) and (6.7), the proof is completed by using $t=t_{n}$.
7. Conclusion. In this paper, we have shown from Theorem 3.4, Theorem 4.1, Theorem 5.3 and Theorem 6.2 that, under the standing Assumption 2.2,
$S D E(2.2) \xrightarrow{Q 2} E M S D E(2.4) \xrightarrow{Q 3} E M S D E P C A(2.3) \xrightarrow{Q 4} S D E P C A(2.1) \xrightarrow{Q 1} S D E(2.2)$.
Hence we have the following theorem.

Theorem 7.1. Under Assumption 2.2, if one of $\operatorname{SDEPCA}$ (2.1), SDE (2.2), EMSDEPCA (2.3) and EMSDE (2.4) is pth moment exponentially stable, then the other three are also pth moment exponentially stable for sufficiently small step size $h$ and $\tau$.

By examming the proof of the Theorem 3.4, Theorem 4.1, Theorem 5.3 and Theorem 6.2, we see that the $p$ th moment exponential stability of SDEPCA (2.1), SDE (2.2), EMSDEPCA (2.3) and EMSDE (2.4) are equivalent as long as their solutions are $p$ th moment bounded and arbitrarily close for sufficiently small $\tau$ and $h$. Let

$$
F(y(t))=f(y(t))+u_{1}(y(t)) \quad \text { and } \quad G(y(t))=g(y(t))+u_{2}(y(t))
$$

For $V \in C^{2,1}\left(\mathbb{R}^{d} \times \mathbb{R}_{+} ; \mathbb{R}_{+}\right)$, we define an operator $\mathcal{L} V$ by

$$
\mathcal{L} V(y, t)=V_{t}(y, t)+V_{y}(y, t) F(y(t))+\frac{1}{2} \operatorname{trace}\left[G^{T}(y) V_{y y}(y, t) G(y)\right]
$$

The sufficient criterion for $p$ th moment exponential stability via a Lyapunov function is given by Theorem 4.4 in [28, P130]. Now we quote it here.

Theorem 7.2. Assume that there is a function $V(y, t) \in C^{2,1}\left(\mathbb{R}^{d} \times \mathbb{R}_{+} ; \mathbb{R}_{+}\right)$, and positive constants $c_{1}, c_{2}, c_{3}$ such that

$$
c_{1}|y|^{p} \leq V(y, t) \leq c_{2}|y|^{p} \quad \text { and } \quad \mathcal{L} V(y, t) \leq-c_{3} V(y, t)
$$

for all $(y, t) \in \mathbb{R}^{d} \times \mathbb{R}_{+}$. Then for the $S D E$ (2.2), we have

$$
\mathbb{E}|y(t)|^{p} \leq \frac{c_{2}}{c_{1}}\left|x_{0}\right|^{p} e^{-c_{3} t}
$$

for all $x_{0} \in \mathbb{R}^{d}$. In other words, the $S D E$ (2.2) is pth moment exponentially stable.
For convenience, we impose the following hypothesis.
Assumption 7.3. There exists a pair of positive constants $p$ and $\lambda$ such that

$$
|y|^{2}\left(2 y^{T} F(y)+|G(y)|^{2}\right)-(2-p)\left|y^{T} G(y)\right|^{2} \leq-\lambda|y|^{4}, \quad \forall y \in \mathbb{R}^{d}
$$

Applying the Theorem 7.2 with $V(y, t)=|y|^{p}$, we easily obtain the following theorem [see 20].

Theorem 7.4. Under Assumption 7.3, the SDE (2.2) is pth moment exponentially stable, i.e.

$$
\mathbb{E}|y(t)|^{p} \leq\left|x_{0}\right|^{p} e^{-\frac{\lambda}{2} p t}, \quad \forall t>0
$$

where $p$ and $\lambda$ are given in Assumption 7.3.
In combination with Theorem 7.1, the following theorem provides an interesting result.
Theorem 7.5. Assume that Assumption 2.2 and Assumption 7.3 hold, then $S D E$ (2.2) is pth moment exponentially stable and SDEPCA (2.1), EMSDEPCA (2.3), EMSDE (2.4) are also pth moment exponentially stable as long as step size $h$ and $\tau$ are sufficiently small.

## References.

[1] L. Arnold, Stochastic differential equations: theory and applications, Wiley, New York, 1974.
[2] C. T. H. Baker and E. Buckwar, Numerical analysis of explicit one-step methods for stochastic delay differential equations, LMS J. Comput. Math., 3 (2000), pp. 315-335.
[3] C. T. H. Baker and E. Buckwar, Exponential stability in pth mean of solutions, and of convergent Euler-type solutions, of stochastic delay differential equations, J. Comput. Appl. Math., 184 (2005), pp. 404-427.
[4] C. A. Braumann, Introduction to Stochastic Differential Equations with Applications to Modelling in Biology and Finance, University of Évora Portugal, 2019.
[5] E. Buckwar, R. Horváth-Bokor, and R. Winkler, Asymptotic meansquare stability of two-step methods for stochastic ordinary differential equations, BIT, 46 (2006), pp. 261-282.
[6] S. Deng, C. Fei, W. Fei, and X. Mao, Stability equivalence between the stochastic differential delay equations driven by $G$-Brownian motion and the EulerMaruyama method, Appl. Math. Lett., 96 (2019), pp. 138-146.
[7] R. Dong and X. Mao, On pth moment stabilization of hybrid systems by discrete-time feedback control, Stoch. Anal. Appl., 35 (2017), pp. 803-822.
[8] L. C. Evans, An introduction to stochastic differential equations, American Mathematical Society, 2013.
[9] A. Friedman, Stochastic differential equations and applications, Academic Press, New York, 1976.
[10] Q. Guo, X. Mao, and R. Yue, Almost sure exponential stability of stochastic differential delay equations, SIAM J. Control Optim., 54 (2016), pp. 1919-1933.
[11] D. J. Higham, Mean-square and asymptotic stability of the stochastic theta method, SIAM J. Numer. Anal., 38 (2000), pp. 753-769.
[12] D. J. Higham, X. Mao, and A. M. Stuart, Strong convergence of Euler-type methods for nonlinear stochastic differential equations, SIAM J. Numer. Anal., 40 (2002), pp. 1041-1063.
[13] D. J. Higham, X. Mao, and A. M. Stuart, Exponential mean-square stability of numerical solutions to stochastic differential equations, LMS J. Comput. Math., 6 (2003), pp. 297-313.
[14] D. J. Higham, X. Mao, and C. Yuan, Almost sure and moment exponential stability in the numerical simulation of stochastic differential equations, SIAM J. Numer. Anal., 45 (2007), pp. 592-609.
[15] D. J. Higham, X. Mao, and C. Yuan, Preserving exponential mean-square stability in the simulation of hybrid stochastic differential equations, Numer. Math., 108 (2007), pp. 295-325.
[16] M. Hutzenthaler and A. Jentzen, Convergence of the stochastic Euler scheme for locally Lipschitz coefficients, Found. Comput. Math., 11 (2011), pp. 657-706.
[17] M. Hutzenthaler, A. Jentzen, and P. E. Kloeden, Divergence of the multilevel Monte Carlo Euler method for nonlinear stochastic differential equations, Ann. Appl. Probab., 23 (2013), pp. 1913-1966.
[18] R. Khasminskir, Stochastic stability of differential equations, Springer, Heidelberg, second ed., 2012.
[19] P. E. Kloeden and E. Platen, Numerical solution of stochastic differential equations, vol. 23, Springer, Berlin, 1992.
[20] X. Li, X. Mao, and G. Yin, Explicit numerical approximations for stochastic differential equations in finite and infinite horizons: truncation methods, convergence in pth moment and stability, IMA J. Numer. Anal., 39 (2019), pp. 847-892.
[21] Y. Li, J. Lu, X. Mao, and Q. Qiu, Stabilization of hybrid systems by feedback control based on discrete-time state and mode observations, Asian J. Control, 19 (2017), pp. 1943-1953.
[22] L. Liu, M. Li, and F. Deng, Stability equivalence between the neutral delayed stochastic differential equations and the Euler-Maruyama numerical scheme, Appl. Numer. Math., 127 (2018), pp. 370-386.
[23] Y. Lu, M. Song, and M. Liu, Convergence and stability of the compensated split-step theta method for stochastic differential equations with piecewise continuous arguments driven by Poisson random measure, J. Comput. Appl. Math., 340 (2018), pp. 296-317.
[24] Y. Lu, M. Song, and M. Liu, Convergence rate and stability of the splitstep theta method for stochastic differential equations with piecewise continuous arguments, Discrete Contin. Dyn. Syst. Ser. B, 24 (2019), pp. 695-717.
[25] X. Mao, Exponential stability of stochastic differential equations, vol. 182, Marcel Dekker, New York, 1994.
[26] X. MaO, Numerical solutions of stochastic functional differential equations, LMS J. Comput. Math., 6 (2003), pp. 141-161.
[27] X. MaO, Exponential stability of equidistant Euler-Maruyama approximations of stochastic differential delay equations, J. Comput. Appl. Math., 200 (2007), pp. 297-316.
[28] X. Mao, Stochastic differential equations and applications, Horwood, Chichester, second ed., 2007.
[29] X. Mao, Stabilization of continuous-time hybrid stochastic differential equations by discrete-time feedback control, Automatica., 49 (2013), pp. 3677-3681.
[30] X. MaO, Almost sure exponential stability in the numerical simulation of stochastic differential equations, SIAM J. Numer. Anal., 53 (2015), pp. 370-389.
[31] X. MaO, Almost sure exponential stabilization by discrete-time stochastic feedback control, IEEE Trans. Automat. Control, 61 (2016), pp. 1619-1624.
[32] X. Mao, W. Liu, L. Hu, Q. Luo, and J. Lu, Stabilization of hybrid stochastic differential equations by feedback control based on discrete-time state observations, Systems Control Lett., 73 (2014), pp. 88-95.
[33] G. N. Milstein, Numerical integration of stochastic differential equations, vol. 313, Kluwer Academic Publishers Group, Dordrecht, 1995.
[34] B. Øksendal, Stochastic differential equations, Springer-Verlag, Berlin, sixth ed., 2003.
[35] Y. Saito and T. Mitsui, Stability analysis of numerical schemes for stochastic differential equations, SIAM J. Numer. Anal., 33 (1996), pp. 2254-2267.
[36] H. Schurz, Stability, stationarity, and boundedness of some implicit numerical methods for stochastic differential equations and applications, Logos Verlag Berlin, Berlin, 1997.
[37] L. Shaikhet, Lyapunov functionals and stability of stochastic functional differential equations, Springer, Cham, 2013.
[38] M. Song and X. Mao, Almost sure exponential stability of hybrid stochastic functional differential equations, J. Math. Anal. Appl., 458 (2018), pp. 1390-1408.
[39] C. Tudor and M. Tudor, On approximation of solutions for stochastic delay equations, Stud. Cerc. Mat., 39 (1987), pp. 265-274.
[40] S. You, W. Liu, J. Lu, X. Mao, and Q. Qiu, Stabilization of hybrid systems by feedback control based on discrete-time state observations, SIAM J. Control Optim., 53 (2015), pp. 905-925.


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