STABILITY EQUIVALENCE AMONG STOCHASTIC DIFFERENTIAL EQUATIONS AND STOCHASTIC DIFFERENTIAL EQUATIONS WITH PIECEWISE CONTINUOUS ARGUMENTS AND CORRESPONDING EULER-MARUYAMA METHODS*

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Abstract. In this paper, we consider the equivalence of the *p*th moment exponential stability for stochastic differential equations (SDEs), stochastic differential equations with piecewise continuous arguments (SDEPCAs) and the corresponding Euler-Maruyama methods EMSDEs and EMSDEP-CAs. We show that if one of the SDEPCAs, SDEs, EMSDEs and EMSDEPCAs is pth moment exponentially stable, then any of them is pth moment exponentially stable for a sufficiently small step size h and τ under the global Lipschitz assumption on the drift and diffusion coefficients

Key words. Exponential stability, Stochastic differential equations, Numerical solutions, Piecewise continuous arguments

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1. Introduction. Stochastic differential equations (SDEs) have been widely used in many branches of science and industry [1, 4, 8, 9, 28, 34]. There is an extensive literature in stochastic stability (e.g. the moment exponential stability or almost sure exponential stability [1, 5, 9, 18, 25, 36, 37]. One of the powerful techniques in the study of stochastic stability is the method of Lyapunov functions. In the absence of an appropriate Lyapunov function, we may carry out careful numerical simulations using a numerical method, say the Euler-Maruyama (EM) method [see e.g. 2, 12, 16, 17, 19, 26, 33, 39] with a small step size. Does the main question arise whether the numerical solutions can reproduce and predict the stability of the underlying solutions?

The case that stochastic stability of the general nonlinear equation and that of the numerical method are equivalent for a sufficiently small step size can be founded in [6, 13, 15, 22, 27, 30], while for the linear equation in [3, 11, 35]. Higham et al. in [14] showed that when the SDE obeys a linear growth condition, the EM method recovers almost surely exponential stability.

In this paper, we consider the following stochastic differential equation with piecewise continuous argument (SDEPCA)

(1.1)
$$dx(t) = (f(x(t)) + u_1(x([t/\tau)\tau))]dt + (g(x(t)) + u_2(x([t/\tau]\tau)))dw(t)$$

and the stochastic differential equation (SDE)

(1.2)
$$dy(t) = (f(y(t)) + u_1(y(t)))dt + (g(y(t)) + u_2(y(t)))dw(t).$$

We also consider the applications of EM method to SDEPCA (1.1) and SDE (1.2), respectively

(1.3)
$$X_{n+1} = X_n + (f(X_n) + u_1(X_{[n/m]m}))h + (g(X_n) + u_2(X_{[n/m]m}))\Delta w_n,$$

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(1.4)
$$Y_{n+1} = Y_n + (f(Y_n) + u_1(Y_n))h + (g(Y_n) + u_2(Y_n))\Delta w_n$$

where $h = \frac{\tau}{m}, m \in \mathbb{N}^+$. We refer to (1.3) and (1.4) by the terms EMSDEPCA (1.3) and EMSDE (1.4), respectively. The main purpose of the present paper is to show that if one of the SDEPCAs (1.1), SDEs (1.2), EMSDEPCA (1.3) and EMSDE (1.4) is *p*th moment exponential stable, then so are the others for a sufficiently small step size *h* and τ under a global Lipschitz assumption on the drift and diffusion coefficients. In order to do this, we shall concentrate on the following questions:

- (Q1) If for a sufficiently small τ , the SDEPCA (1.1) is *p*th moment exponentially stable, can we confidently infer that the SDE (1.2) is *p*th moment exponentially stable?
- (Q2) For a sufficiently small step size h, does the EMSDE (1.4) reproduce the pth moment exponential stability of the underlying SDE (1.2)?
- (Q3) For a sufficiently small τ , the EMSDEPCA (1.3) can preserve the *p*th moment exponential stability of EMSDE (1.4)?
- (Q4) If the EMSDEPCA (1.3) is *p*th moment exponentially stable, will the SDE-PCA (1.1) be the *p*th moment exponentially stable for a sufficiently small step size h?

It is known that the positive answer to (Q2) for SDE in case p = 2 can be founded in [13]. The stochastic differential equation with piecewise continuous arguments (SDEPCA) has been studied extensively [see e.g. 7, 21, 29, 31, 32, 40], and in the case of $\tau = 1$, we refer to [23, 24]. Mao in [29] is the first paper that investigated the mean square exponentially stable for SDEPCA. The positive answer to the converse problem of (Q1), we refer to [10, 29, 32, 38].

In this paper, we will give the positive answer for (Q1), (Q2), (Q3), (Q4). In section 2, we describe the SDEPCA and EM methods along with the definitions of *p*th moment exponential stability for SDE, SDEPCA, EMSDE, EMSDEPCA. Section 3, section 4, section 5, section 6 answer the questions (Q1), (Q3), (Q4), (Q2) respectively, the final conclusions are stated in the last section.

2. Perilimaries. Throughout this paper, unless otherwise specified, we will use the following notations. If **A** is a vector or matrix, its transpose is denoted by \mathbf{A}^{T} . If $x \in \mathbf{R}^n$, then |x| is the Euclidean norm. If **A** is a matrix, we let $|\mathbf{A}| = \sqrt{trace}(\mathbf{A}^{\mathrm{T}}\mathbf{A})$ be its trace norm. If *D* is a set, its indicator function is denoted by $\mathbf{1}_D$. Moreover, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (that is, it is right continuous and increasing while \mathcal{F}_0 contains all \mathbb{P} -null sets), and let \mathbb{E} denote the expectation corresponding to \mathbb{P} . Let B(t) be a *m*-dimensional Brownian motion defined on the space. Throughout this paper, we set $p \geq 2$.

In this paper, we deal with the following d-dimensional nonlinear stochastic differential equations with piecewise continuous arguments (SDEPCAs)

(2.1)
$$\begin{cases} dx(t) = [f(x(t)) + u_1(x([t/\tau] \tau))] dt + [g(x(t)) + u_2(x([t/\tau] \tau))] dw(t) \\ x(0) = x_0 \in \mathbb{R}^d \end{cases}$$

on $t \geq 0$, where w(t) is an *m*-dimensional Brownian motion, $f : \mathbb{R}^d \to \mathbb{R}^d$, $g : \mathbb{R}^d \to \mathbb{R}^{d \times m}$, $u_1 : \mathbb{R}^d \to \mathbb{R}^d$ and $u_2 : \mathbb{R}^d \to \mathbb{R}^{d \times m}$. τ is a positive constant, $[t/\tau]$ is the integer part of t/τ . We denote x(t) the solution of (2.1) with initial data $x(0) = x_0$ and y(t) the solution of the following SDEs

(2.2)
$$dy(t) = [f(y(t)) + u_1(y(t))]dt + [g(y(t)) + u_2(y(t))]dw(t)$$

on $t \ge 0$ with initial data $y(0) = x_0$.

In the present paper, we also deal with the application of EM method to SDEPCA (2.1) and SDE (2.2). We note that $[t/\tau]\tau = n\tau$ for $t \in [n\tau, (n+1)\tau)$, $n = 0, 1, 2, \cdots$, a natural choice for h is $h = \frac{\tau}{m}$, $m \in \mathbb{N}^+$. Hence, we have

(2.3) $X_{n+1} = X_n + (f(X_n) + u_1(X_{[n/m]m}))h + (g(X_n) + u_2(X_{[n/m]m}))\Delta w_n,$

(2.4) $Y_{n+1} = Y_n + (f(Y_n) + u_1(Y_n))h + (g(Y_n) + u_2(Y_n))\Delta w_n,$

where X_n and Y_n are the approximations of x(t) and y(t) at grid points $t = t_n = nh$, $n = 0, 1, 2, \cdots$, respectively, $\Delta w_n = w(t_{n+1}) - w(t_n)$. Let n = km + l, $k \in \mathbb{N}^+$, $l = 0, 1, \cdots, m - 1$. Then (2.3) and (2.4) would reduce to

$$(2.5) \ X_{km+l+1} = X_{km+l} + (f(X_{km+l}) + u_1(X_{km}))h + (g(X_{km+l}) + u_2(X_{km}))\Delta w_{km+l}$$

$$(2.6) Y_{km+l+1} = Y_{km+l} + (f(Y_{km+l}) + u_1(Y_{km+l}))h + (g(Y_{km+l}) + u_2(Y_{km+l}))\Delta w_{km+l}.$$

Remark 2.1. If we choose $h = \tau$, then (2.3) and (2.4) are the same and (2.5) and (2.6) are the same.

In spite of the simplicity of the EM method, explicit EM method is the most popular for approximating the solution of the SDE under global Lipschitz condition [see 12, 19, 33] and has often been used successfully in actual calculations. For further analysis it is more convenient to use continuous-time approximations, (2.7)

$$x_{\Delta}(t) = x_0 + \int_0^t f(\bar{x}_{\Delta}(s)) + u_1(\bar{x}_{\Delta}([s/\tau]\tau))ds + \int_0^t g(\bar{x}_{\Delta}(s)) + u_2(\bar{x}_{\Delta}([s/\tau]\tau))dw(s),$$

(2.8)
$$y_{\Delta}(t) = x_0 + \int_0^t f(\bar{y}_{\Delta}(s)) + u_1(\bar{y}_{\Delta}(s))ds + \int_0^t g(\bar{y}_{\Delta}(s)) + u_2(\bar{y}_{\Delta}(s))dw(s),$$

where

$$\bar{x}_{\Delta}(t) = \sum_{n=0}^{\infty} X_n \mathbf{1}_{[t_n, t_{n+1})}(t), \quad \bar{y}_{\Delta}(t) = \sum_{n=0}^{\infty} Y_n \mathbf{1}_{[t_n, t_{n+1})}(t), \quad \forall t \ge 0.$$

We observe that $x_{\Delta}(t_n) = \bar{x}_{\Delta}(t_n) = X_n$ and $y_{\Delta}(t_n) = \bar{y}_{\Delta}(t_n) = Y_n$. Consequently,

$$x_{\Delta}([t/\tau]\tau) - \bar{x}_{\Delta}([t/\tau]\tau) = 0, \quad y_{\Delta}([t/\tau]\tau) - \bar{y}_{\Delta}([t/\tau]\tau) = 0.$$

In this paper, we impose the following standing hypothesis.

Assumption 2.2. Assume that there exists a positive constant K such that

$$|f(x) - f(y)| \lor |g(x) - g(y)| \lor |u_1(x) - u_1(y)| \lor |u_2(x) - u_2(y)| \le K|x - y|,$$

for all $x, y \in \mathbb{R}^d$. Assume also that f(0) = 0, g(0) = 0, $u_1(0) = 0$ and $u_2(0) = 0$. Assumption 2.2 implies that

$$|f(x)| \lor |g(x)| \lor |u_1(x)| \lor |u_2(x)| \le K|x|$$

for all $x \in \mathbb{R}^d$.

We now give our basic definitions, which is cited from [28].

DEFINITION 2.3. The equations SDEPCA (2.1) and SDE (2.2) are said to be pth moment exponentially stable if there exist positive constants M_1 , γ_1 , M_2 and γ_2 such that

(2.9)
$$\mathbb{E}|x(t)|^p \le M_1 |x_0|^p e^{-\gamma_1 t}, \quad \forall t \ge 0,$$

and

(2.10)
$$\mathbb{E}|y(t)|^p \le M_2 |x_0|^p e^{-\gamma_2 t}, \quad \forall t \ge 0,$$

for any $x_0 \in \mathbb{R}^d$.

DEFINITION 2.4. For any given step size h > 0, the Euler-Maruyama numerical methods EMSDEPCA (2.3) and EMSDE (2.4) are said to be pth moment exponentially stable, if there exist positive constants λ_1 , L_1 , λ_2 and L_2 such that

(2.11)
$$\mathbb{E}|X_n|^p \le L_1|x_0|^p e^{-\lambda_1 nh},$$

(2.12)
$$\mathbb{E}|Y_n|^p \le L_2|x_0|^p e^{-\lambda_2 nh},$$

for any $x_0 \in \mathbb{R}^d$, $n \in \mathbb{N}$.

It is known that under Assumption 2.2, for any initial value x_0 given at time t = 0, the SDEPCA (2.1) and SDE (2.2) have a unique continuous solutions on $t \ge 0$ (see [28]). To emphasize the role of the initial value, we denote the solution x(t) and y(t) by $x(t; 0, x_0)$ and $y(t; 0, x_0)$, respectively. Of course, we may consider a more general case, for example, where the SDEs and the SDEPCAs have a random initial data $x(0) = \xi$ which is an \mathcal{F}_0 -measurable \mathbb{R}^d -valued random variable such that $\mathbb{E}|\xi|^p < \infty, \forall p \ge 0$. In this case, by the Markov property of the solution, we can easily see that the solution satisfies

$$\mathbb{E}|x(t)|^p = \mathbb{E}(\mathbb{E}(|x(t)|^p | \mathcal{F}_0)) \le \mathbb{E}(M_1|\xi|^p e^{-\gamma_1 t}) = M_1 \mathbb{E}|\xi|^p e^{-\gamma t}.$$

It is therefore clear why it is enough to consider only the deterministic initial value $x(0) = x_0$.

Let y(t; s, y(s)) be the solution of SDE (2.2) for t > s with initial value y(s). It is also known that the solutions to SDE (2.2) have the following flow property,

$$y(t; 0, x_0) = y(t; s, y(s)), \quad \forall t \ge s > 0.$$

Moreover, the solutions of SDE (2.2) also have the time-homegeneous Markov property. Hence (2.10) implies

$$\mathbb{E}|y(t;s,\xi)|^p \le M_2 \mathbb{E}|\xi|^p e^{-\gamma_2(t-s)}, \quad \forall \ t \ge s.$$

Given y_k for some $k \in \mathbb{N}^+$, the process $\{y_n\}_{n \geq k}$ can be regard as the process which is produced by EM method applied to the SDE (2.2) on $t \geq kh$ with the initial value $y(kh) = y_k$. In other words, the process $\{y_n\}_{n \geq k}$ is time-homogeneous Markov process. Hence, (2.12) is equivalent to the following more general form.

(2.13)
$$\mathbb{E}|y_n|^p \le L_2 \mathbb{E}|y_k|^p e^{-\lambda_2 (n-k)h}.$$

Due to the special feature of the SDEPCA (2.1), the solution x(t) has flow property and the Markov property at the discrete time $t = k\tau$ ($k \in \mathbb{N}^+$). Hence

$$x(t;0,x_0) = x(t;k\tau,x(k\tau))$$

and (2.9) implies

(2.14)
$$\mathbb{E}|x(t)|^p \le M_1 \mathbb{E}|x(k\tau)|^p e^{-\gamma_1(t-k\tau)}, \quad t \ge k\tau$$

Given x_{km} for some $k \in \mathbb{N}^+$, the process $\{x_n\}_{n \geq km}$ can be regard as the process which is produced by EM method applied to the SDEPCA (2.1) on $t \geq k\tau$ with the initial value $x(k\tau) = x_{km}$. The process $\{x_n\}_{n \geq km}$ is time-homogeneous Markov process. Hence, (2.11) is equivalent to the following more general form.

(2.15)
$$\mathbb{E}|x_n|^p \le L_1 \mathbb{E}|x_{km}|^p e^{-\lambda_1 (n-km)h}.$$

3. SDE (2.2) shares the stability with SDEPCA (2.1). In this section, we shall investigate that if the SDEPCA (2.1) is *p*th moment exponentially stable with a sufficiently small τ , then the SDE (2.2) is also *p*th moment exponentially stable, i.e. give the positive answer to (Q1). To show this, we need several lemmas. The last lemma estimates the difference in the *p*th moment between the solution of the SDE (2.2) and that of the SDEPCA (2.1).

LEMMA 3.1. Assume that Assumption 2.2 holds. Then for any given constant $T \ge 0$, we have

(3.1)
$$\sup_{0 \le t \le T} \mathbb{E} |x(t)|^p \le H_1(T, p, K) |x_0|^p,$$

where $H_1(T, p, K) = e^{2pK[1+(p-1)K]T}$.

Proof. In view of Itô formula and Assumption 2.2, we obtain

$$\begin{split} \mathbb{E}|x(v)|^{p} \leq &|x_{0}|^{p} + \mathbb{E}\int_{0}^{v} p|x(s)|^{p-1}|f(x(s)) + u_{1}(x([s/\tau]\tau))| \\ &+ \frac{p(p-1)}{2}|x(s)|^{p-2}|g(x(s)) + u_{2}(x([s/\tau]\tau))|^{2}ds \\ \leq &|x_{0}|^{p} + \mathbb{E}\int_{0}^{v} pK|x(s)|^{p-1}(|x(s)| + |x([s/\tau]\tau)|) \\ &+ p(p-1)K^{2}|x(s)|^{p-2}(|x(s)|^{2} + |x([s/\tau]\tau)|^{2}ds \\ \leq &|x_{0}|^{p} + 2pK[1 + (p-1)K]\int_{0}^{v} \sup_{0 \leq u \leq s} \mathbb{E}|x(u)|^{p}ds \end{split}$$

Taking the supremum value of both sides over $v \in [0, t]$, we have

$$\sup_{0 \le v \le t} \mathbb{E}|x(v)|^p \le |x_0|^p + 2pK[1 + (p-1)K] \int_0^t \sup_{0 \le u \le s} \mathbb{E}|x(u)|^p ds.$$

The desired result (3.1) follows from the well-known Gronwall inequality.

LEMMA 3.2. Assume that Assumption 2.2 holds. Then for any $t \ge 0$,

$$\mathbb{E}|x(t) - x([t/\tau]\tau)|^p \le C_1(K, p, \tau)\tau^{\frac{p}{2}}e^{2pK[1+(p-1)K]t}|x_0|^p.$$

where $C_1(K, p, \tau) = 2^{2p-1}K^p \left[\tau^{\frac{p}{2}} + (p(p-1)/2)^{\frac{p}{2}}\right].$

Proof. By basic inequality, Hölder inequality, moment inequality and Assumption 2.2, we obtain

$$\mathbb{E}|x(t) - x([t/\tau]\tau)|^{p} \leq 2^{p-1}\tau^{p-1}\mathbb{E}\int_{[t/\tau]\tau}^{t} |f(x(s)) + u_{1}(x([s/\tau]\tau))|^{p}ds + 2^{p-1}\left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}}\tau^{\frac{p-2}{2}}\mathbb{E}\int_{[t/\tau]\tau}^{t} |g(x(s)) + u_{2}(x([s/\tau]\tau))|^{p}ds \leq C_{1}(K, p, \tau)\tau^{\frac{p}{2}-1}\int_{[t/\tau]\tau}^{t}\left(\sup_{0\leq u\leq s}\mathbb{E}|x(u)|^{p}\right)ds$$

It comes from (3.1) that

$$\mathbb{E}|x(t) - x([t/\tau]\tau)|^p \leq C_1(K, p, \tau)\tau^{\frac{p}{2}-1} \int_{[t/\tau]\tau}^t e^{2pK[1+(p-1)K]s} |x_0|^p ds$$
$$\leq C_1(K, p, \tau)\tau^{\frac{p}{2}} e^{2pK[1+(p-1)K]t} |x_0|^p$$

The lemma is proved.

The following lemma estimates the difference in the *p*th moment between x(t) and y(t).

LEMMA 3.3. Let Assumption 2.2 hold. Then

$$\mathbb{E}|x(t) - y(t)|^{p} \leq C_{2}(K, p, \tau)\tau^{\frac{p}{2}}|x_{0}|^{p} \left(e^{C_{3}(p, K)t} - 1\right),$$

for all $x_0 \in \mathbb{R}^d$ and $t \ge 0$, where C_2 and C_3 are defined as (3.4) and (3.5), respectively.

Proof. Using Itô formula and Assumption 2.2, we have

$$\begin{split} \mathbb{E}|x(t) - y(t)|^{p} \\ &\leq \mathbb{E} \int_{0}^{t} pK|x(s) - y(s)|^{p-1} (|x(s) - y(s)| + |x([s/\tau]\tau) - y(s)|) \\ &+ p(p-1)K^{2}|x(s) - y(s)|^{p-2} (|x(s) - y(s)|^{2} + |x([s/\tau]\tau) - y(s)|^{2}) ds \\ &= \left(pK + p(p-1)K^{2} \right) \int_{0}^{t} \mathbb{E}|x(s) - y(s)|^{p} ds \\ &+ pK\mathbb{E} \int_{0}^{t} |x(s) - y(s)|^{p-1} |x([s/\tau]\tau) - y(s)| ds \\ &+ p(p-1)K^{2}\mathbb{E} \int_{0}^{t} |x(s) - y(s)|^{p-2} |x([s/\tau]\tau) - y(s)|^{2} ds \end{split}$$

By Young inequality, we have

$$\mathbb{E}|x(t) - y(t)|^{p} \leq \left[(2p - 1 + 2^{p-1})K + 2(p - 1)(p - 1 + 2^{p-1})K^{2}\right] \int_{0}^{t} \mathbb{E}|x(s) - y(s)|^{p} ds$$

$$(3.2) \qquad +2^{p-1}(K + 2(p - 1)K^{2}) \int_{0}^{t} \mathbb{E}|x([s/\tau]\tau) - x(s)|^{p} ds$$

In view of Lemma 3.2, we have

$$2^{p-1}(K+2(p-1)K^2) \int_0^t \mathbb{E}|x([s/\tau]\tau) - x(s)|^p ds$$

$$\leq 2^{p-1}(K+2(p-1)K^2) \int_0^t C_1(K,p,\tau)\tau^{\frac{p}{2}} e^{2pK[1+(p-1)K]s} |x_0|^p ds$$

$$= \frac{2^{p-1}(1+2(p-1)K)\tau^{\frac{p}{2}}C_1(K,p,\tau)|x_0|^p}{p[2+2(p-1)K]} \left(e^{2pK[1+(p-1)K]t} - 1\right)$$

(3.3)
$$\leq \frac{2^{p-1}\tau^{\frac{p}{2}}C_1(K,p,\tau)|x_0|^p}{p} \left(e^{2pK[1+(p-1)K]t} - 1\right)$$

Substituting (3.3) into (3.2) and using Gronwall inequality, we show that

$$\mathbb{E}|x(t) - y(t)|^{p} \le C_{2}(K, p, \tau)\tau^{\frac{p}{2}}|x_{0}|^{p} \left(e^{C_{3}(p, K)t} - 1\right),$$

where

(3.4)
$$C_2(K, p, \tau) = \frac{2^{p-1}C_1(K, p, \tau)}{p},$$

(3.5)
$$C_3(p,K) = [4p - 1 + 2^{p-1} + 2(p-1)(2p - 1 + 2^{p-1})K]K.$$

This completes the proof of Lemma 3.3.

Our positive answer to (Q1) is stated in the following theorem.

THEOREM 3.4. Let Assumption 2.2 hold and the SDEPCA (2.1) is pth moment exponentially stable, i.e. $\mathbb{E}|x(t)|^p \leq M_1 e^{-\gamma_1 t} |x_0|^p$. Choose $\delta \in (0, 1)$, if τ satisfies

(3.6)
$$R(\tau) = \delta + 2^{p-1} C_2(K, p, \tau) \tau^{\frac{p}{2}} \left(e^{C_3(p,K) \left(\frac{\ln\left(\frac{2^{p-1}M_1}{\delta}\right)}{\gamma_1} + \tau \right)} - 1 \right) < 1,$$

then the SDE (2.2) is also pth moment exponentially stable, where $C_2(K, p, \tau)$ and $C_3(p, K)$ are defined in Lemma 3.3.

Proof. Step1. Let us choose a positive integer \hat{n} such that

$$\frac{\ln\left(\frac{2^{p-1}M_1}{\delta}\right)}{\gamma_1\tau} \le \hat{n} < \frac{\ln\left(\frac{2^{p-1}M_1}{\delta}\right)}{\gamma_1\tau} + 1.$$

So $2^{p-1}M_1e^{-\gamma_1\hat{n}\tau} \leq \delta$. Hence,

(3.7)
$$2^{p-1}\mathbb{E}|x(\hat{n}\tau)|^p \le 2^{p-1}M_1e^{-\gamma_1\hat{n}\tau}|x_0|^p \le \delta|x_0|^p.$$

By virtue of Lemma 3.3, we obtain

$$\mathbb{E}|x(\hat{n}\tau) - y(\hat{n}\tau)|^{p} \le C_{2}(K, p, \tau)\tau^{\frac{p}{2}}|x_{0}|^{p}\left(e^{C_{3}(p, K)\hat{n}\tau} - 1\right),$$

which together with (3.7), we arrive at

$$\mathbb{E}|y(\hat{n}\tau)|^{p} \leq \left[\delta + 2^{p-1}C_{2}(K, p, \tau)\tau^{\frac{p}{2}}\left(e^{C_{3}(p, K)\hat{n}\tau} - 1\right)\right]|x_{0}|^{p} \leq R(\tau)|x_{0}|^{p}$$

In view of (3.6), there is a positive constant γ_2 such that $R(\tau) = e^{-\gamma_2 \hat{n}\tau}$. Consequently,

$$\mathbb{E}|y(\hat{n}\tau)|^p \le e^{-\gamma_2 \hat{n}\tau} |x_0|^p.$$

Step2. For any given $k \in \mathbb{N}^+$, let $\bar{x}(t)$ be the solution to the SDEPCA (2.1) for $t \ge k\hat{n}\tau$ with the initial value $\bar{x}(k\hat{n}\tau) = y(k\hat{n}\tau)$. We have from (2.14) that

(3.8)
$$\mathbb{E}|\bar{x}((k+1)\hat{n}\tau)|^p \le M_1 \mathbb{E}|y(k\hat{n}\tau)|^p e^{-\gamma_1 \hat{n}\tau}$$

In view of Lemma 3.3, we arrive at

(3.9)
$$\mathbb{E}|\bar{x}((k+1)\hat{n}\tau) - y((k+1)\hat{n}\tau)|^p \le C_2(K,p,\tau)\tau^{\frac{p}{2}}\mathbb{E}|y(k\hat{n}\tau)|^p \left(e^{C_3(p,K)\hat{n}\tau} - 1\right).$$

Using (3.8) and (3.9), we can show, in the same way as we did in *Step1*, that

$$\mathbb{E}|y((k+1)\hat{n}\tau)|^p \le \mathbb{E}|y(k\hat{n}\tau)|^p e^{-\gamma_2\hat{n}\tau}.$$

Consequently,

(3.10)
$$\mathbb{E}|y(k\hat{n}\tau)|^p \le e^{-\gamma_2\hat{n}\tau}\mathbb{E}|y((k-1)\hat{n}\tau)|^p \le \dots \le e^{-k\gamma_2\hat{n}\tau}|x_0|^p$$

Now, for any t > 0, there is a unique k such that $k\hat{n}\tau \leq t < (k+1)\hat{n}\tau$. In view of Itô formula and Assumption 2.2, similarly as the proof of Lemma 3.1, we arrive at

$$\mathbb{E}|y(t)|^p \le \mathbb{E}|y(k\hat{n}\tau)|^p + 2pK\left(1 + (p-1)K\right)\int_{k\hat{n}\tau}^t \mathbb{E}|y(s)|^p ds.$$

By the Gronwall inequality and (3.10), we can derive

$$\mathbb{E}|y(t)|^{p} \leq \mathbb{E}|y(k\hat{n}\tau)|^{p}e^{2pK(1+(p-1)K)(t-k\hat{n}\tau)} \\ \leq \mathbb{E}|y(k\hat{n}\tau)|^{p}e^{2pK(1+(p-1)K)\hat{n}\tau} \\ \leq e^{-k\gamma_{2}\hat{n}\tau}|x_{0}|^{p}e^{2pK(1+(p-1)K)\hat{n}\tau} \\ \leq M_{2}|x_{0}|^{p}e^{-\gamma_{2}t},$$

where $M_2 = e^{[\gamma_2 + 2pK(1+(p-1)K)]\hat{n}\tau}$. The proof is hence complete.

4. EMSDEPCA (2.3) shares the stability with EMSDE (2.4). In this section, we shall show that if the EMSDE (2.4) is *p*th moment exponentially stable, then the EMSDEPCAs (2.3) is also *p*th moment exponentially stable, i.e. give the positive answer to (Q3). It is known from Remark 2.1 that if $h = \tau$, then EMSDE (2.4) and EMSDEPCA (2.3) are the same, and the answer for (Q3) is obviously positive. So in this section, we assume $h \neq \tau$.

THEOREM 4.1. Assume that Assumption 2.2 holds. For a step size $h = \tau/m$, the EMSDE (2.4) is pth moment exponentially stable, i.e. $\mathbb{E}|Y_n|^p \leq L_2 e^{-\lambda_2 nh} |x_0|^p$. Choose $\delta \in (0, 1)$, if τ satisfies

(4.1)
$$2^{p-1}H_4\left(2\left(\frac{\ln(2^{p-1}L_2/\delta)}{\lambda_2}+\tau\right), K, \tau, p\right)\tau^{\frac{p}{2}}+\delta<1$$

where $H_4(T, K, \tau, p)$ is defined in Lemma 4.3, then the EMSDEPCA (2.3) is also pth moment exponentially stable.

The above theorem will be proved below by making use of the following lemmas.

LEMMA 4.2. Assume that Assumption 2.2 holds. Then for any given T > 0 such that

$$\sup_{0 \le t_n \le T} \mathbb{E} |X_n|^p \le H_3(T, p, K) |x_0|^p,$$

where $H_3(T, p, K) = e^{2pK(1+(p-1)K)T}$.

Proof. The proof follows from Lemma 3.1. But to highlight the importance of numerical solutions, it is given here. In view of Itô formula and Assumption 2.2, we have

$$\begin{split} \mathbb{E}|x_{\Delta}(v)|^{p} &= |x_{0}|^{p} + \mathbb{E}\int_{0}^{t} p|x_{\Delta}(s)|^{p-2}x_{\Delta}(s)^{T}(f(\bar{x}_{\Delta}(s)) + u_{1}(\bar{x}_{\Delta}([s/\tau]\tau))) \\ &\quad + \frac{p(p-1)}{2}|x_{\Delta}(s)|^{p-2}|g(\bar{x}_{\Delta}(s)) + u_{2}(\bar{x}_{\Delta}([s/\tau]\tau))|^{2}ds \\ &\leq |x_{0}|^{p} + \mathbb{E}\int_{0}^{v} pK|x_{\Delta}(s)|^{p-1}(|\bar{x}_{\Delta}(s)| + |\bar{x}_{\Delta}([s/\tau]\tau)|)ds \\ &\quad + p(p-1)K^{2}\mathbb{E}\int_{0}^{v}|x_{\Delta}(s)|^{p-2}\left(|\bar{x}_{\Delta}(s)|^{2} + |\bar{x}_{\Delta}([s/\tau]\tau)|^{2}\right)ds \\ &\leq |x_{0}|^{p} + 2pK\left(1 + (p-1)K\right)\int_{0}^{v}\sup_{0 \leq u \leq s}\mathbb{E}|x_{\Delta}(u)|^{p}ds \end{split}$$

According to the Gronwall inequality, we obtain

(4.2)
$$\sup_{0 \le t \le T} \mathbb{E} |x_{\Delta}(t)|^p \le |x_0|^p e^{2pK(1+(p-1)K)T}$$

The proof is completed by noting that $x_{\Delta}(t_n) = X_n$, i.e.

$$\sup_{0 \le t_n \le T} \mathbb{E} |X_n|^p \le |x_0|^p e^{2pK(1+(p-1)K)T}.$$

The following lemma estimates the difference in the pth moment between approximation of EMSDE (2.4) and that of EMSDEPCA (2.3).

LEMMA 4.3. Let Assumption 2.2 hold. Then for any given positive constant T > 0,

$$\sup_{0 \le t_n \le T} \mathbb{E} |X_n - Y_n|^p \le H_4(T, K, \tau, p) \tau^{\frac{p}{2}} |x_0|^p,$$

where $H_4(T, K, \tau, p) = C_2(K, p, \tau) \left(e^{C_3(p, K)T} - 1 \right)$, C_2 and C_3 are defined in Lemma 3.3.

Proof. According to (2.7), (2.8), Itô formula and Assumption 2.2, we have

$$\begin{split} \mathbb{E}|x_{\Delta}(v) - y_{\Delta}(v)|^{p} \\ \leq \mathbb{E} \int_{0}^{v} p|x_{\Delta}(s) - y_{\Delta}(s)|^{p-1} |f(\bar{x}_{\Delta}(s)) - f(\bar{y}_{\Delta}(s)) + u_{1}(\bar{x}_{\Delta}([s/\tau]\tau)) - u_{1}(\bar{y}_{\Delta}(s))| \\ + \frac{p(p-1)}{2} |x_{\Delta}(s) - y_{\Delta}(s)|^{p-2} |g(\bar{x}_{\Delta}(s)) - g(\bar{y}_{\Delta}(s)) + u_{2}(\bar{x}_{\Delta}([s/\tau]\tau)) - u_{2}(\bar{y}_{\Delta}(s))|^{2} ds \\ \leq \mathbb{E} \int_{0}^{v} pK |x_{\Delta}(s) - y_{\Delta}(s)|^{p-1} (|\bar{x}_{\Delta}(s) - \bar{y}_{\Delta}(s)| + |\bar{x}_{\Delta}([s/\tau]\tau) - \bar{y}_{\Delta}(s)|) \\ + p(p-1)K^{2} |x_{\Delta}(s) - y_{\Delta}(s)|^{p-2} (|\bar{x}_{\Delta}(s) - \bar{y}_{\Delta}(s)|^{2} + |\bar{x}_{\Delta}([s/\tau]\tau) - \bar{y}_{\Delta}(s)|^{2}) ds \\ \leq (pK + p(p-1)K^{2}) \int_{0}^{v} \sup_{0 \leq u \leq s} \mathbb{E} |x_{\Delta}(u) - y_{\Delta}(u)|^{p} ds \\ + \mathbb{E} \int_{0}^{v} pK |x_{\Delta}(s) - y_{\Delta}(s)|^{p-1} |\bar{x}_{\Delta}([s/\tau]\tau) - \bar{y}_{\Delta}(s)| ds \\ + \mathbb{E} \int_{0}^{v} p(p-1)K^{2} |x_{\Delta}(s) - y_{\Delta}(s)|^{p-2} |\bar{x}_{\Delta}([s/\tau]\tau) - \bar{y}_{\Delta}(s)|^{2} ds \\ (4.3) \end{split}$$

Similarly as in the proof of Lemma 3.2, we obtain

(4.4)
$$\mathbb{E}|\bar{x}_{\Delta}(t) - \bar{x}_{\Delta}([t/\tau]\tau)|^{p} = \mathbb{E}|X_{km+l} - X_{km}|^{p} \leq C_{1}(K, p, \tau)\tau^{\frac{p}{2}}|x_{0}|^{p}e^{2pK(1+(p-1)K)t}$$

Substituting (4.4) into (4.3), we have

$$\begin{split} & \mathbb{E}|x_{\Delta}(v) - y_{\Delta}(v)|^{2} \\ & \leq \left[(2p - 1 + 2^{p-1}) + 2(p - 1)(p - 1 + 2^{p-1})K \right] K \int_{0}^{v} \sup_{0 \leq u \leq s} \mathbb{E}|x_{\Delta}(u) - y_{\Delta}(u)|^{p} ds \\ & + 2^{p-1} \left(1 + 2(p - 1)K \right) K \int_{0}^{v} C_{1}(K, p, \tau) \tau^{\frac{p}{2}} |x_{0}|^{p} e^{2pK(1 + (p - 1)K)t} ds \end{split}$$

Applying the Gronwall inequality, we have

$$\sup_{0 \le v \le T} \mathbb{E} |x_{\Delta}(v) - y_{\Delta}(v)|^p \le C_2(K, p, \tau) \tau^{\frac{p}{2}} |x_0|^p \left(e^{C_3(p, K)T} - 1 \right),$$

where $C_2(K, p, \tau)$ and $C_3(p, K)$ are defined in Lemma 3.3. For ease of notations, set $H_4(T, K, p, \tau) = C_2(K, p, \tau) \left(e^{C_3(p, K)T} - 1 \right)$. The proof is completed by noting that $x_{\Delta}(t_n) = X_n$ and $y_{\Delta}(t_n) = Y_n$, i.e.

$$\sup_{0 \le t_n \le T} \mathbb{E} |X_n - Y_n|^p \le H_4(T, K, p, \tau) \tau^{\frac{p}{2}} |x_0|^p.$$

The proof of Theorem 4.1. Let

$$\hat{n} = \left[\frac{\ln(\frac{2^{p-1}L_2}{\delta})}{\lambda_2 \tau}\right] + 1,$$

which implies that

$$2^{p-1}L_2e^{-\lambda_2\hat{n}\tau} \le \delta, \quad and \quad \hat{n}\tau \le \frac{\ln(\frac{2^{p-1}L_2}{\delta})}{\lambda_2} + \tau.$$

By $|a+b|^p \le 2^{p-1}|a|^p + 2^{p-1}|b|^p$, we have

$$\mathbb{E}|X_n|^p \le 2^{p-1}\mathbb{E}|X_n - Y_n|^p + 2^{p-1}\mathbb{E}|Y_n|^p.$$

According to the *p*th moment exponentially stability of EMSDE (2.4) and Lemma 4.3, we have

$$\sup_{\hat{n}\tau \le t_n \le 2\hat{n}\tau} \mathbb{E}|X_n|^p \le 2^{p-1} \sup_{0 \le t_n \le 2\hat{n}\tau} \mathbb{E}|X_n - Y_n|^p + 2^{p-1} \sup_{\hat{n}\tau \le t_n \le 2\hat{n}\tau} \mathbb{E}|Y_n|^p$$

$$\le \left(2^{p-1}H_4(2\hat{n}\tau, K, \tau, p)\tau^{\frac{p}{2}} + 2^{p-1}L_2e^{-\lambda_2\hat{n}\tau}\right)|x_0|^p$$

$$\le \left(2^{p-1}H_4(2\hat{n}\tau, K, \tau, p)\tau^{\frac{p}{2}} + \delta\right)|x_0|^p$$

$$\le \left(2^{p-1}H_4\left(2\left(\frac{\ln(2^{p-1}L_2/\delta)}{\lambda_2} + \tau\right), K, \tau, p\right)\tau^{\frac{p}{2}} + \delta\right)|x_0|^p$$

Let $R(\tau) = 2^{p-1}H_4\left(2\left(\frac{\ln(2^{p-1}L_2/\delta)}{\lambda_2} + \tau\right), K, \tau, p\right)\tau^{\frac{p}{2}} + \delta$. It is known from (4.1) that $R(\tau) < 1$. Therefore, we can find a positive constant λ_1 such that

(4.5)
$$R(\tau) < e^{-\lambda_1 \hat{n}\tau},$$

and

(4.6)
$$\sup_{\hat{n}\tau \le t_n \le 2\hat{n}\tau} \mathbb{E}|X_n|^p \le e^{-\lambda_1 \hat{n}\tau} |x_0|^p.$$

Let $\{\bar{Y}_n\}_{t_n \ge \hat{n}\tau}$ be the solution of EMSDE (2.4) with initial data $\bar{Y}_{\hat{n}m} = X_{\hat{n}m}$ at initial time $t = \hat{n}\tau$. According to Lemma 4.3, we have

$$\sup_{\hat{n}\tau \le t_n \le 3\hat{n}\tau} \mathbb{E}|X_n - \bar{Y}_n|^p \le H_4\left(2\left(\frac{\ln(2^{p-1}L_2/\delta)}{\lambda_2} + \tau\right), K, \tau, p\right)\tau^{\frac{p}{2}} \mathbb{E}|X_{\hat{n}m}|^p$$

It comes from (2.13) that

$$\mathbb{E}|\bar{Y}_n|^p \le L_2 e^{-\lambda_2(nh-\hat{n}mh)} \mathbb{E}|X_{\hat{n}m}|^p$$

Using similar arguments that produced (4.6), we obtain

$$\sup_{2\hat{n}\tau \le t_n \le 3\hat{n}\tau} \mathbb{E}|X_n|^p \le R(\tau)\mathbb{E}|X_{\hat{n}m}|^p \le e^{-\lambda_1\hat{n}\tau}\mathbb{E}|X_{\hat{n}m}|^p \le e^{-\lambda_1\hat{n}\tau} \sup_{\hat{n}\tau \le t_n \le 2\hat{n}\tau} \mathbb{E}|X_n|^p$$

By (4.6), we obtain

$$\sup_{2\hat{n}\tau \le t_n \le 3\hat{n}\tau} \mathbb{E}|X_n|^p \le e^{-2\lambda_1 \hat{n}\tau} |x_0|^p$$

Continuing this approach and using (4.5), we have, for any $i = 1, 2, \cdots$,

(4.7)
$$\sup_{i\hat{n}\tau \le t_n \le (i+1)\hat{n}\tau} \mathbb{E}|X_n|^p \le e^{-i\lambda_1\hat{n}\tau}|x_0|^p \le \bar{L}_1 e^{-\lambda_1 nh}|x_0|^p$$

where $\bar{L}_1 = e^{\lambda_1 \hat{n} \tau}$. For i = 0, by using Lemma 4.2, we get

$$\sup_{0 \le t_n \le \hat{n}\tau} \mathbb{E}|X_n|^p \le H_3(\hat{n}\tau, p, K)|x_0|^p \le L_1|x_0|^p e^{-\lambda_1 nh},$$

where $L_1 = H_3(\hat{n}\tau, p, K)e^{\lambda_1\hat{n}\tau} > e^{\lambda_1\hat{n}\tau} = \bar{L}_1$. This, together with (4.7), we arrive at for all $n \in \mathbb{N}$

$$\mathbb{E}|X_n|^p \le L_1 |x_0|^p e^{-\lambda_1 nh}.$$

5. SDEPCA (2.1) shares the stability with EMSDEPCA (2.3). In this section, we shall show that for a given step size h, if the EMSDEPCA (2.3) is pth moment exponentially stable, then the SDEPCA (2.1) is also pth moment exponentially stable with some restriction with h, i.e. give the positive answer to (Q4). The first lemma shows that the EMSDEPCA (2.3) is convergent in the pth moment to SDEPCA (2.1).

LEMMA 5.1. Assume that Assumption 2.2 holds. For T > 0,

$$\sup_{0 \le t_n \le T} \mathbb{E} |x(t_n) - X_n|^p \le H_6(T, K, p) h^{\frac{p}{2}} |x_0|^p,$$

where $H_6(T, K, p)$ is defined as (5.3).

Proof. For any $t \geq 0,$ by Itô formula, Assumption 2.2 and Young inequality, we obtain

$$\begin{split} \mathbb{E}|x(t) - x_{\Delta}(t)|^{p} \\ &\leq \mathbb{E}\int_{0}^{t} pK|x(s) - x_{\Delta}(s)|^{p-1}(|x(s) - \bar{x}_{\Delta}(s)| + |x([s/\tau]\tau)) - \bar{x}_{\Delta}([s/\tau]\tau)|) \\ &+ p(p-1)K^{2}|x(s) - x_{\Delta}(s)|^{p-2} \left(|x(s) - \bar{x}_{\Delta}(s)|^{2} + |x([s/\tau]\tau) - \bar{x}_{\Delta}([s/\tau]\tau)|^{2}\right) ds \\ &\leq 2pK(1+2(p-1)K)\int_{0}^{t} \mathbb{E}\sup_{0 \leq u \leq s} |x(s) - x_{\Delta}(s)|^{p} ds \\ &+ pK\mathbb{E}\int_{0}^{t} |x(s) - x_{\Delta}(s)|^{p-1}|x_{\Delta}(s) - \bar{x}_{\Delta}(s)| ds \\ &+ pK\mathbb{E}\int_{0}^{t} |x(s) - x_{\Delta}(s)|^{p-1}|x_{\Delta}([s/\tau]\tau) - \bar{x}_{\Delta}([s/\tau]\tau)| ds \\ &+ 2p(p-1)K^{2}\mathbb{E}\int_{0}^{t} |x(s) - x_{\Delta}(s)|^{p-2}|x_{\Delta}(s) - \bar{x}_{\Delta}(s)|^{2} ds \\ &+ 2p(p-1)K^{2}\mathbb{E}\int_{0}^{t} |x(s) - x_{\Delta}(s)|^{p-2}|x_{\Delta}([s/\tau]\tau) - \bar{x}_{\Delta}([s/\tau]\tau)|^{2}) ds \end{split}$$

By noting $x_{\Delta}([s/\tau]\tau) - \bar{x}_{\Delta}([s/\tau]\tau) = 0$, we have

$$\mathbb{E}|x(t) - x_{\Delta}(t)|^{p} \leq K[3p - 1 + 2(p - 1)(3p - 2)K] \int_{0}^{t} \mathbb{E} \sup_{0 \leq u \leq s} |x(u) - x_{\Delta}(u)|^{p} ds$$
(5.1)
$$+ K[1 + 4(p - 1)K] \int_{0}^{t} \mathbb{E}|x_{\Delta}(s) - \bar{x}_{\Delta}(s)|^{p} ds$$

Now, we shall give the estimation of the second term of the right hand. For any t > 0, there exists k and l such that $t_{km+l} \leq t < t_{km+l+1}$. Then $\bar{x}_{\Delta}(t) = X_{km+l} =$

 $x_{\Delta}(t_{km+l})$. Hence from (2.5) we have

$$\mathbb{E}|x_{\Delta}(t) - \bar{x}_{\Delta}(t)|^{p}$$

$$= \mathbb{E}|(t - t_{km+l}) \left(f(X_{km+l}) + u_{1}(X_{km}) \right) + \left(g(X_{km+l}) + u_{2}(X_{km}) \right) \left(w(t) - w(t_{km+l}) \right)|^{p}$$

$$\leq 2^{2p-1} h^{\frac{p}{2}} K^{p} \left(\mathbb{E}|X_{km+l}|^{p} + \mathbb{E}|X_{km}|^{p} \right)$$

Applying (4.2), we obtain

(5.2)
$$\mathbb{E}|x_{\Delta}(t) - \bar{x}_{\Delta}(t)|^{p} \leq 2^{2p} h^{\frac{p}{2}} K^{p} |x_{0}|^{p} e^{2pK(1 + (p-1)K)t}$$

Substituting (5.2) into (5.1), we obtain

$$\mathbb{E}|x(v) - x_{\Delta}(v)|^{p} \leq K[3p - 1 + 2(p - 1)(3p - 2)K] \int_{0}^{t} \mathbb{E} \sup_{0 \le u \le s} |x(u) - x_{\Delta}(u)|^{p} ds + K[1 + 4(p - 1)K] \int_{0}^{t} 2^{2p} h^{\frac{p}{2}} K^{p} |x_{0}|^{p} e^{2pK(1 + (p - 1)K)s} ds$$

By Gronwall inequality, we have

$$\sup_{0 \le t \le T} \mathbb{E} |x(t) - x_{\Delta}(t)|^p \le H_6(T, p, K) h^{\frac{p}{2}} |x_0|^p,$$

where

(5.3)
$$H_6(T, p, K) = [1 + 4(p-1)K]2^{2p}K^{p+1}e^{KT[5p-1+4(p-1)(2p-1)K]}T.$$

By noting $x_{\Delta}(t_n) = X_n$, we get for $t = t_n$

$$\sup_{0 \le t_n \le T} \mathbb{E} |x(t_n) - X_n|^p \le H_6(T, p, K) h^{\frac{p}{2}} |x_0|^p$$

The proof is completed.

LEMMA 5.2. Assume that Assumption 2.2 holds. Then for any $0 \le t_n \le t \le t_{n+1} \le T$,

$$\sup_{0 \le t \le T} \mathbb{E} |x(t) - x(t_n)|^p \le H_7(T, K, p) h^{\frac{p}{2}} |x_0|^p,$$

where $H_7(T, K, p) = 2^{2p-1} K^p \left[T^{\frac{p}{2}} + \left(\frac{p(p-1)}{2} \right)^{\frac{p}{2}} \right] e^{2pK[1+(p-1)K]T}.$

Proof. For any $0 \le t_n \le t \le t_{n+1} \le T$, we have

$$x(t) - x(t_n) = \int_{t_n}^t f(x(s)) + u_1(x([s/\tau]\tau))ds + \int_{t_n}^t g(x(s)) + u_2(x([s/\tau]\tau))dw(s).$$

In view of Hölder inequality, Assumption 2.2 as well as moment inequality, we have

$$\begin{split} & \mathbb{E}|x(t) - x(t_n)|^p \\ & \leq 2^{2p-2}(t-t_n)^{\frac{p}{2}-1}K^p \left[(t-t_n)^{\frac{p}{2}} + \left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}} \right] \int_{t_n}^t \mathbb{E}|x(s)|^p + \mathbb{E}|x([s/\tau]\tau)|^p ds \\ & \leq 2^{2p-1}(t-t_n)^{\frac{p}{2}-1}K^p \left[(t-t_n)^{\frac{p}{2}} + \left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}} \right] \int_{t_n}^t \sup_{0 \leq u \leq s} \mathbb{E}|x(u)|^p ds \end{split}$$

It follows from (3.1) that

$$\begin{split} \mathbb{E}|x(t) - x(t_n)|^p &\leq 2^{2p-1}(t - t_n)^{\frac{p}{2}} K^p \left[(t - t_n)^{\frac{p}{2}} + \left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}} \right] e^{2pK[1 + (p-1)K]t} |x_0|^p \\ &\leq 2^{2p-1} h^{\frac{p}{2}} K^p \left[t^{\frac{p}{2}} + \left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}} \right] e^{2pK[1 + (p-1)K]t} |x_0|^p \end{split}$$

Hence,

$$\sup_{0 \le t \le T} \mathbb{E} |x(t) - x(t_n)|^p \le 2^{2p-1} K^p \left[T^{\frac{p}{2}} + \left(\frac{p(p-1)}{2} \right)^{\frac{p}{2}} \right] e^{2pK[1+(p-1)K]T} h^{\frac{p}{2}} |x_0|^p.$$

The proof is complete.

THEOREM 5.3. Assume that Assumption 2.2 holds. For a step size $h = \frac{\tau}{m}$, the EMSDEPCA (2.3) is pth moment exponentially stable, i.e. $\mathbb{E}|X_n|^p \leq L_1 e^{-\lambda_1 n h} |x_0|^p$. If the step size h satisfies

(5.4)
$$3^{p-1}H_8(2\hat{n}\tau, K, p)h^{\frac{p}{2}} + e^{-\frac{3}{4}\lambda_1\hat{n}\tau} \le e^{-\frac{1}{2}\lambda_1\hat{n}\tau}$$

where $\hat{n} = \left[\frac{4\ln(3^{p-1}L_1)}{\lambda_1\tau}\right] + 1$ and $H_8(2\hat{n}\tau, K, p) = H_7(2\hat{n}\tau, K, p) + H_6(2\hat{n}\tau, K, p)$, then the SDEPCA (2.1) is also pth moment exponentially stable, where $H_6(2\hat{n}\tau, K, p)$ is defined in Lemma 5.1 and $H_7(2\hat{n}\tau, K, p)$ in Lemma 5.2.

Proof. For any $t \ge 0$, there exist $n \in \mathbb{N}$ such that $t_n \le t < t_{n+1}$,

$$\mathbb{E}|x(t)|^{p} \leq 3^{p-1}\mathbb{E}|x(t) - x(t_{n})|^{p} + 3^{p-1}\mathbb{E}|x(t_{n}) - X_{n}|^{p} + 3^{p-1}\mathbb{E}|X_{n}|^{p}$$

According to Lemma 5.1, we have

$$\sup_{0 \le t_n \le 2\hat{n}\tau} \mathbb{E} |x(t_n) - X_n|^p \le H_6(2\hat{n}\tau, K, p) h^{\frac{p}{2}} |x_0|^p.$$

By Lemma 5.2, we have

$$\sup_{0 \le t \le 2\hat{n}\tau} \mathbb{E} |x(t) - x(t_n)|^p \le H_7(2\hat{n}\tau, K, p) h^{\frac{p}{2}} |x_0|^p.$$

Since
$$\hat{n} = \left[\frac{4\ln(3^{p-1}L_1)}{\lambda_1\tau}\right] + 1$$
, we have $3^{p-1}L_1e^{-\lambda_1\hat{n}\tau} \le e^{-\frac{3}{4}\lambda_1\hat{n}\tau}$, Therefore,

$$\sup_{\hat{n}\tau \le t \le 2\hat{n}\tau} \mathbb{E}|x(t)|^p$$

$$\le 3^{p-1} \sup_{0\le t\le 2\hat{n}\tau} \mathbb{E}|x(t) - x(t_n)|^p + 3^{p-1} \sup_{0\le t_n\le 2\hat{n}\tau} \mathbb{E}|x(t_n) - X_n|^p + 3^{p-1} \sup_{\hat{n}\tau\le t_n\le 2\hat{n}\tau} \mathbb{E}|X_n|^p$$

$$\le \left(3^{p-1}H_7(2\hat{n}\tau, K, p)h^{\frac{p}{2}} + 3^{p-1}H_6(2\hat{n}\tau, K, p)h^{\frac{p}{2}} + 3^{p-1}L_1e^{-\lambda_1\hat{n}\tau}\right)|x_0|^p$$

$$\le \left(3^{p-1}H_8(2\hat{n}\tau, K, p)h^{\frac{p}{2}} + e^{-\frac{3}{4}\lambda_1\hat{n}\tau}\right)|x_0|^p$$

where $H_8(2\hat{n}\tau, K, p) = H_7(2\hat{n}\tau, K, p) + H_6(2\hat{n}\tau, K, p)$. Recalling (5.4), we have

(5.5)
$$\sup_{\hat{n}\tau \le t \le 2\hat{n}\tau} \mathbb{E}|x(t)|^p \le e^{-\frac{1}{2}\lambda_1 \hat{n}\tau} |x_0|^p.$$

Denote by $\{\bar{X}_n\}_{nh \ge \hat{n}\tau}$ the numerical solution of (2.3) with initial data $\bar{X}_{\hat{n}m} = x(\hat{n}\tau)$ at $t = \hat{n}\tau$. Then from (2.15), we have

$$\mathbb{E}|\bar{X}_n|^p \le L_1 e^{-\lambda_1 (n-\hat{n}m)h} \mathbb{E}|x(\hat{n}\tau)|^p.$$

Using Lemma 5.1 and Lemma 5.2, we get

$$\sup_{\substack{\hat{n}\tau \leq t_n \leq 3\hat{n}\tau}} \mathbb{E}|x(t_n) - \bar{X}_n|^p \leq H_6(2\hat{n}\tau, K, p)h^{\frac{p}{2}} \mathbb{E}|x(\hat{n}\tau)|^p.$$
$$\sup_{\substack{\hat{n}\tau \leq t \leq 3\hat{n}\tau}} \mathbb{E}|x(t) - x(t_n)|^p \leq H_7(2\hat{n}\tau, K, p)h^{\frac{p}{2}} \mathbb{E}|x(\hat{n}\tau)|^p.$$

Therefore,

$$\sup_{2\hat{n}\tau \le t \le 3\hat{n}\tau} \mathbb{E}|x(t)|^{p} \le \left(3^{p-1}H_{7}(2\hat{n}\tau, K, p)h^{\frac{p}{2}} + 3^{p-1}H_{6}(2\hat{n}\tau, K, p)h^{\frac{p}{2}} + L_{1}e^{-\lambda_{1}\hat{n}\tau}\right) \mathbb{E}|x(\hat{n}\tau)|^{p}$$
$$\le \left(3^{p-1}H_{8}(2\hat{n}\tau, K, p)h^{\frac{p}{2}} + e^{-\frac{3}{4}\lambda_{1}\hat{n}\tau}\right) \mathbb{E}|x(\hat{n}\tau)|^{p}$$
$$\le e^{-\frac{1}{2}\lambda_{1}\hat{n}\tau} \sup_{\hat{n}\tau \le t \le 2\hat{n}\tau} \mathbb{E}|x(t)|^{p}$$

By (5.5), we obtain

$$\sup_{2\hat{n}\tau \le t \le 3\hat{n}\tau} \mathbb{E}|x(t)|^p \le e^{-\frac{\lambda_1}{2}2\hat{n}\tau}|x_0|^p.$$

Repeating this procedure, we find for $i = 1, 2, \cdots$,

(5.6)
$$\sup_{i\hat{n}\tau \le t \le (i+1)\hat{n}\tau} \mathbb{E}|x(t)|^p \le e^{-\frac{\lambda_1}{2}i\hat{n}\tau}|x_0|^p \le \bar{M}_1 e^{-\frac{\lambda_1}{2}t}|x_0|^p,$$

where $\bar{M}_1 = e^{\frac{\lambda_1}{2}\hat{n}\tau}$. On the other hand, by means of Lemma 3.1, we can show that

$$\sup_{0 \le t \le \hat{n}\tau} \mathbb{E}|x(t)|^p \le H_1(\hat{n}\tau, p, K)|x_0|^p \le M_1|x_0|^p e^{-\frac{\lambda_1}{2}t}$$

where $M_1 = H_1(\hat{n}\tau, p, K)e^{\frac{\lambda_1}{2}\hat{n}\tau} > e^{\frac{\lambda_1}{2}\hat{n}\tau} = \bar{M}_1$, this, together with (5.6), we arrive at for any $t \ge 0$,

$$\mathbb{E}|x(t)|^p \le M_1 |x_0|^p e^{-\frac{1}{2}\lambda_1 t}$$

This completes the proof.

6. EMSDE (2.4) shares the stability with SDE (2.2). [13] gives the positive answer to (Q2) only for the case p = 2. In this section, we shall show that for p > 2, if the SDE (2.2) is *p*th moment exponentially stable, then the EMSDE (2.4) is also *p*th moment exponentially stable with some restriction on *h*, i.e. give the positive answer to (Q2). The first lemma shows that the EMSDE (2.4) is convergent in the *p*th moment to SDE (2.2).

LEMMA 6.1. Assume that Assumption 2.2 holds. For any T > 0,

$$\sup_{0 \le t_n \le T} \mathbb{E} |y(t_n) - Y_n|^p \le H_9(T, K, p) h^{\frac{p}{2}} |x_0|^p,$$

where $H_9(T, K, p)$ is defined as (6.4).

 $\mathit{Proof.}$ For any $t \geq 0,$ by Itô formula, Assumption 2.2 and Young inequality, we obtain

$$\begin{split} \mathbb{E}|y(t) - y_{\Delta}(t)|^{p} \\ &\leq \mathbb{E} \int_{0}^{t} p|y(s) - y_{\Delta}(s)|^{p-1} |f(y(s)) - f(\bar{y}_{\Delta}(s)) + u_{1}(y(s)) - u_{1}(\bar{y}_{\Delta}(s))| \\ &\quad + \frac{p(p-1)}{2} |y(s) - y_{\Delta}(s)|^{p-2} |g(y(s)) - g(\bar{y}_{\Delta}(s)) + u_{2}(y(s)) - u_{2}(\bar{y}_{\Delta}(s))|^{2} ds \\ &\leq \mathbb{E} \int_{0}^{t} 2pK|y(s) - y_{\Delta}(s)|^{p-1} |y(s) - \bar{y}_{\Delta}(s)| \\ &\quad + 2p(p-1)K^{2}|y(s) - y_{\Delta}(s)|^{p-2} |y(s) - \bar{y}_{\Delta}(s)|^{2} ds \\ &\leq \left(2K(2p-1) + 8(p-1)^{2}K^{2}\right) \int_{0}^{t} \mathbb{E}|y(s) - y_{\Delta}(s)|^{p} ds \\ &\leq (6.1) + \left(2K + 8(p-1)K^{2}\right) \int_{0}^{t} \mathbb{E}|y_{\Delta}(s) - \bar{y}_{\Delta}(s)|^{p} ds \end{split}$$

Now, we shall give the estimation of the second term of the right hand. For any t > 0, there exists n such that $t_n \leq t < t_{n+1}$, and $\bar{y}_{\Delta}(t) = Y_n = y_{\Delta}(t_n)$. Hence from (2.4) we have

$$\mathbb{E}|y_{\Delta}(t) - \bar{y}_{\Delta}(t)|^{p}$$

$$= \mathbb{E}|(t - t_{n}) (f(Y_{n}) + u_{1}(Y_{n})) + (g(Y_{n}) + u_{2}(Y_{n})) (W(t) - W(t_{n}))|^{p}$$

$$\le 2^{p-1} (\mathbb{E}|(t - t_{n}) (f(Y_{n}) + u_{1}(Y_{n}))|^{p} + \mathbb{E}|(g(Y_{n}) + u_{2}(Y_{n})) (W(t) - W(t_{n}))|^{p})$$

$$\le 2^{2p} K^{p} \mathbb{E}|Y_{n}|^{p} h^{\frac{p}{2}}$$

Similarly as the proof of Lemma 3.1, we have

(6.2)
$$\sup_{0 \le s \le t} \mathbb{E} |y_{\Delta}(s)|^p \le |x_0|^p e^{2pK(1+(p-1)K)t}$$

Applying (6.2), we obtain

(6.3)
$$\mathbb{E}|y_{\Delta}(t) - \bar{y}_{\Delta}(t)|^{p} \leq 2^{2p} K^{p} e^{2pK(1 + (p-1)K)t} h^{\frac{p}{2}} |x_{0}|^{p}$$

Substituting (6.3) into (6.1), we obtain

$$\mathbb{E}|y(t) - y_{\Delta}(t)|^{p} \leq \left(2K(2p-1) + 8(p-1)^{2}K^{2}\right) \int_{0}^{t} \mathbb{E}|y(s) - y_{\Delta}(s)|^{p} ds + \left(2K + 8(p-1)K^{2}\right) \int_{0}^{t} 2^{2p}K^{p} e^{2pK(1+(p-1)K)s} h^{\frac{p}{2}} |x_{0}|^{p} ds$$

By Gronwall inequality, we have

$$\sup_{0 \le t \le T} \mathbb{E}|y(t) - y_{\Delta}(t)|^p \le H_9(T, p, K)h^{\frac{p}{2}}|x_0|^p,$$

where

(6.4)
$$H_9(T, p, K) = [1 + 4(p-1)K]2^{2p+1}K^{p+1}e^{2KT(3p-1+(p-1)(5p-4)K)}T.$$

By noting $y_{\Delta}(t_n) = Y_n$, we get for $t = t_n$

$$\sup_{0 \le t_n \le T} \mathbb{E}|y(t_n) - Y_n|^p \le H_9(T, p, K)h^{\frac{p}{2}}|x_0|^p$$

The proof is completed.

THEOREM 6.2. Let Assumption 2.2 hold. Assume that the SDE (2.2) is pth moment exponentially stable and satisfies (2.10). Let $T = 1 + 4 \ln(2^{p-1}M_2)/\gamma_2$. If h satisfies

(6.5)
$$2^{p-1}H_9(2T, p, K)h^{\frac{p}{2}} + e^{-\frac{3}{4}\gamma_2 T} \le e^{-\frac{1}{2}\gamma_2 T}.$$

Then the EMSDE (2.4) is pth moment exponentially stable.

Proof. Since $T = 1 + 4 \ln(2^{p-1}M_2)/\gamma_2$, we have

$$2^{p-1}M_2e^{-\gamma_2T} < e^{-\frac{3}{4}\gamma_2T}.$$

Now, for any given $i \in \mathbb{N}$, let $\{\hat{y}(t)\}_{t \geq iT}$ be the solution to the SDE (2.2) for $t \in [iT, \infty)$, with the initial condition $y_{\Delta}(iT)$. Then using basic inequality, Lemma 6.1, (2.10) and (6.5), we have

$$\sup_{\substack{(i+1)T \leq t \leq (i+2)T \\ iT \leq (i+2)T \\ iT \leq (i+2)T \\ iT \leq (i+2)T \\ iT \leq (i+2)T \\$$

According to (6.2),

(6.7)
$$\sup_{0 \le t \le T} \mathbb{E} |y_{\Delta}(t)|^p \le |x_0|^p e^{2pK(1+(p-1)K)T} \le L_2 e^{-\frac{1}{2}\gamma_2 t} |x_0|^p,$$

where $L_2 = e^{\frac{1}{2}\gamma_2 T + 2pK(1+(p-1)K)T}$. Combining (6.7) and (6.6), we obtain that

(6.8)

$$\begin{aligned} \sup_{\substack{(i+1)T \le t \le (i+2)T}} \mathbb{E} |y_{\Delta}(t)|^p \le e^{-\frac{1}{2}(i+1)\gamma_2 T} \sup_{\substack{0 \le t \le T}} \mathbb{E} |y_{\Delta}(t)|^p \\ \le e^{-\frac{1}{2}(i+1)\gamma_2 T} |x_0|^p e^{2pK(1+(p-1)K)T} \\ \le L_2 e^{-\frac{1}{2}\gamma_2 t} |x_0|^p. \end{aligned}$$

Due to (6.8) and (6.7), the proof is completed by using $t = t_n$.

7. Conclusion. In this paper, we have shown from Theorem 3.4, Theorem 4.1, Theorem 5.3 and Theorem 6.2 that, under the standing Assumption 2.2,

 $SDE(2.2) \xrightarrow{Q2} EMSDE(2.4) \xrightarrow{Q3} EMSDEPCA(2.3) \xrightarrow{Q4} SDEPCA(2.1) \xrightarrow{Q1} SDE(2.2).$ Hence we have the following theorem.

THEOREM 7.1. Under Assumption 2.2, if one of SDEPCA (2.1), SDE (2.2), EMSDEPCA (2.3) and EMSDE (2.4) is pth moment exponentially stable, then the other three are also pth moment exponentially stable for sufficiently small step size h and τ .

By examming the proof of the Theorem 3.4, Theorem 4.1, Theorem 5.3 and Theorem 6.2, we see that the *p*th moment exponential stability of SDEPCA (2.1), SDE (2.2), EMSDEPCA (2.3) and EMSDE (2.4) are equivalent as long as their solutions are *p*th moment bounded and arbitrarily close for sufficiently small τ and *h*. Let

$$F(y(t)) = f(y(t)) + u_1(y(t))$$
 and $G(y(t)) = g(y(t)) + u_2(y(t))$

For $V \in C^{2,1}(\mathbb{R}^d \times \mathbb{R}_+; \mathbb{R}_+)$, we define an operator $\mathcal{L}V$ by

$$\mathcal{L}V(y,t) = V_t(y,t) + V_y(y,t)F(y(t)) + \frac{1}{2}trace\left[G^T(y)V_{yy}(y,t)G(y)\right].$$

The sufficient criterion for pth moment exponential stability via a Lyapunov function is given by Theorem 4.4 in [28, P130]. Now we quote it here.

THEOREM 7.2. Assume that there is a function $V(y,t) \in C^{2,1}(\mathbb{R}^d \times \mathbb{R}_+; \mathbb{R}_+)$, and positive constants c_1, c_2, c_3 such that

$$c_1|y|^p \leq V(y,t) \leq c_2|y|^p$$
 and $\mathcal{L}V(y,t) \leq -c_3V(y,t)$

for all $(y,t) \in \mathbb{R}^d \times \mathbb{R}_+$. Then for the SDE (2.2), we have

$$\mathbb{E}|y(t)|^{p} \le \frac{c_{2}}{c_{1}}|x_{0}|^{p}e^{-c_{3}t},$$

for all $x_0 \in \mathbb{R}^d$. In other words, the SDE (2.2) is pth moment exponentially stable. For convenience, we impose the following hypothesis.

ASSUMPTION 7.3. There exists a pair of positive constants p and λ such that

$$|y|^2 \left(2y^T F(y) + |G(y)|^2 \right) - (2-p)|y^T G(y)|^2 \le -\lambda |y|^4, \quad \forall \ y \in \mathbb{R}^d.$$

Applying the Theorem 7.2 with $V(y,t) = |y|^p$, we easily obtain the following theorem [see 20].

THEOREM 7.4. Under Assumption 7.3, the SDE (2.2) is pth moment exponentially stable, i.e.

$$\mathbb{E}|y(t)|^p \le |x_0|^p e^{-\frac{\lambda}{2}pt}, \quad \forall t > 0,$$

where p and λ are given in Assumption 7.3.

In combination with Theorem 7.1, the following theorem provides an interesting result.

THEOREM 7.5. Assume that Assumption 2.2 and Assumption 7.3 hold, then SDE (2.2) is pth moment exponentially stable and SDEPCA (2.1), EMSDEPCA (2.3), EMSDE (2.4) are also pth moment exponentially stable as long as step size h and τ are sufficiently small.

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